Suppose that \( f : X \to S^2 \) is a continuous mapping that is not onto. Show that \( f \) is homotopic to a constant mapping.

**Solution:** Let \( f : X \to S^2 \) be the given map. We consider \( S^2 \) as a unit sphere, imbedded into \( \mathbb{R}^3 \). Since \( f(X) \neq S^2 \), we can fix a point \( P \) in \( S^2 \setminus f(X) \). Consider the stereographic projection \( \pi : S^2 \setminus \{P\} \to T \), where \( T \) is the linear subspace of \( \mathbb{R}^3 \) (i.e. 2-dimensional plane, containing the origin 0), orthogonal to the vector \( 0P \).

Consider the map \( G : T \times I \to T \), given by \( G(x,t) = (1-t)x \) for any \( t \in I = [0,1] \), and any \( x \in T \subset \mathbb{R}^3 \). Then \( G(-,0) = id_T \) and \( G(-,1) = c_0 \), the constant map with image in the origin.

We know that the stereographic projection \( \pi : S^2 \setminus \{s'\} \to T \) is a homeomorphism (in fact, diffeomorphism) from \( S^2 \setminus \{s'\} \) onto the plane \( T \). Therefore there exists the inverse map \( \pi^{-1} : T \to S^2 \setminus \{P\} \subset S^2 \) that also is continuous.

Consider the following diagram:

\[
\begin{array}{cccc}
X \times I & \xrightarrow{H} & S^2 & \xleftarrow{\text{inclusion}} S^2 \setminus P \\
\downarrow f \times id_I & & & \uparrow \pi^{-1} \\
(S^2 \setminus P) \times I & \xrightarrow{\pi \times id_I} & T \times I & \xrightarrow{G} & T \\
\end{array}
\]

Here \( f \times id_I \) is the induced map (product of \( f : X \to S^2 \) and \( id_I : [0,1] \to [0,1] \)). The map \( \pi \times id_I : (S^2 \setminus \{P\}) \times I \to T \times I \) is defined similarly. The composition \( H = \pi^{-1} \circ G \circ (\pi \times id_I) \circ (f \times id_I) \) is the map from \( X \times I \) to \( S^2 \). We would like to show that \( H \) is the desired homotopy between \( f \) and constant map \( c_0 \).

We need to show that \( H \) is continuous. For this purpose it is enough to show continuity of each function in the composition. The maps \( \pi^{-1} \) and \( G \) are clearly continuous. For the remaining two we recall the following fact about product maps:

If there are two continuous functions \( f_i : X_i \to Y_i \) for \( i = 1, 2 \), then the map \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is defined by \( f_1 \times f_2 : (x_1, x_2) \mapsto (f(x_1), f(x_2)) \). Using standard technique (using the standard subbasis of the product topology, and continuity by preimages of subbasis), one can show that \( f_1 \times f_2 \) is continuous.
Using this result, continuity of \( f \times \text{id}_I \) and \( \pi \times \text{id}_I \) follows immediately (since \( \text{id}_I \) is continuous). Therefore, the map \( H \) is continuous as a composition of continuous maps. So, it remains to check that \( H(-,0) = f \) and \( H(-,1) = c_{-P} \), where \(-P\) is the antipodal point of the point \( P \) in the unit sphere \( S^2 \).

For \( t = 0 \) and for any \( x \in X \), we have

\[
H(x,0) = \pi^{-1} \circ G \circ (\pi \times \text{id}_I) \circ (f \times \text{id}_I)(x,0) \\
= \pi^{-1} \circ G \circ (\pi \times \text{id}_I)(f(x),0) \\
= (\pi^{-1} \circ G)(\pi(f(x)),0) = \pi^{-1}((1-0)\pi(f(x))) \\
= \pi^{-1}(\pi(f(x))) = (\pi^{-1} \circ \pi)(f(x)) = f(x).
\]

Similarly, if \( t = 1 \), we get

\[
H(x,1) = \pi^{-1} \circ G \circ (\pi \times \text{id}_I) \circ (f \times \text{id}_I)(x,1) \\
= \pi^{-1} \circ G \circ (\pi \times \text{id}_I)(f(x),1) \\
= (\pi^{-1} \circ G)(\pi(f(x)),1) = \pi^{-1}((1-1)\pi(f(x))) \\
= \pi^{-1}(0) = -P
\]

for any \( x \in X \).

So, \( H : X \times I \to S^2 \) is a homotopy between the given map \( f : X \to S^2 \) and the constant map \( c_{-P} : X \to \{-P\} \subset S^2 \).
**Problem 7.3.** If $X$ is a space, recall that the *cone on $X$* is the quotient space $CX = X \times [0, 1]/X \times \{1\}$. Suppose $f : X \to Y$ is a continuous function and $f$ is homotopic to a constant mapping $c_y : X \to Y$ for some $y \in Y$. Show that there is an extension of $f$, $\hat{f} : CX \to Y$ so that $f = \hat{f} \times i$ where $i : X \to CX$ is the inclusion, $i(x) = [(x, 0)]$.

**Solution:** Let $\pi : X \times I \to CX$, defined by $(x, t) \mapsto [(x, t)]$ be the natural projection (quotient map), where $[(x, t)]$ denotes an equivalence class of $(x, t)$ in $X \times I$, i.e. a point in $CX$.

Recall that $U \subseteq CX$ is open if and only if $\pi^{-1}(U) \subseteq X \times I$ is open.

Let $H : X \times I \to Y$ be the homotopy, connecting $f$ and $c_y$. That means, $H$ is continuous, and furthermore,

$$H(-, 0) = f : X \to Y,$$

$$H(-, 1) = c_y : X \to Y.$$ 

Define the map $\hat{f} : CX \to Y$ by $\hat{f}([(x, t)]) = H(x, t)$ for any $x \in X$ and $t \in I$. The map $\hat{f}$ is clearly well-defined. In fact, if $t \neq 1$, nothing to show. If we take $t = 1$, and $x_1, x_2 \in X$, then

$$\hat{f}([(x_1, 1)]) = H(x_1, 1) = y = H(x_2, 1) = \hat{f}([(x_2, 1)]).$$

In order to check the continuity of $\hat{f}$, consider the following commutative diagram

$$
\begin{array}{ccc}
X \times I & \xrightarrow{H} & Y \\
\downarrow{\pi} & & \downarrow{\hat{f}} \\
CX & & 
\end{array}
$$

where $\pi$ is the quotient map, and $H = \hat{f} \circ \pi$. Then by the Theorem 4.18, the composition $\hat{f} \circ \pi$ is continuous if and only if $\hat{f}$ is continuous. But we already know that $\hat{f} \circ \pi = H$ is continuous. Therefore $\hat{f}$ is continuous as well.

Now define the inclusions $j : X \to X \times I$, given by $x \mapsto (x, 0)$, and $i : X \to CX$, given by $x \mapsto [(x, 0)]$. Clearly, $i = \pi \circ j$, and by continuity of $j$ (inclusion into the first factor of the product topology), and by continuity of $\pi$ we get that $i$ is also continuous.

So, we have the following commutative diagram, where each triangle commutes:
Then
\[ \widehat{f} \circ i = \widehat{f} \circ (\pi \circ j) = (\widehat{f} \circ \pi) \circ j = H \circ j = f. \]
This means that \( \widehat{f} \) is an extension of \( f \) from \( X \) to \( CX \).
**Problem 7.5.** Prove that a disk minus two points is a deformation retract of a figure 8 (that is, $S^1 \vee S^1$).

**Solution:** Consider the following picture:

![Division of the disk in closed subsets.](image)

**Figure 1.** Division of the disk in closed subsets.

Here $l$ is the line (x-axis) on which centers of circles and center of the disk is located. The union of two circles with centers $P2$ and $P4$ build the figure 8, i.e. space $Y = S^1 \vee S^1$. The center of the given disk is located in $P3 = 0$, the origin. The space $X = D^2 \setminus \{P1, P2\}$ is the given disk with punctured two points. We divide $X$ into the closed regions $A, B, C, D, E$ and $F$. Regions $E$ and $F$ are small closed disks with removed centers. Line segments $l1$ and $l2$ are vertical, touching small circles at $P1$ and $P2$ correspondingly. $A$ is the closed circular segment, the region cutted out by $l1$ from $X$. Similarly, $D$ contains all the points of $X$, that are located to the right side of $l2$, including
this segment. Finally, $B$ and $C$ are upper and lower parts of the remaining regions (again, we take them to be closed in $X$). For example, $B \cap C = \{P1, P3, P5\}$.

Our goal is to construct a deformation retraction of $X$, i.e. a homotopy of $X$ into itself, that contracts this space onto $Y$, fixing $Y$ pointwise under this homotopy. Idea is to build a retraction $r : X \to Y$, and construct $H$ in a path-linear way, connecting $id_X$ with $r$.

Let $R$ be the radius of small circles. Define

$$r(x) = \begin{cases} 
P1 & \text{if } x \in A, 
P5 & \text{if } x \in D, 
p(x) & \text{if } x \in B \text{ or } x \in C, 
P2 + \frac{x-P2}{||x-P2||} \cdot R & \text{if } x \in E, 
P4 + \frac{x-P4}{||x-P4||} \cdot R & \text{if } x \in F 
\end{cases}$$

where $p$ is the vertical projection of region $B$ on the upper part of $Y$. Similarly, $p$ vertically projects $C$ onto the lower part of $Y = S^1 \vee S^1$. On $E$ and $F$ the map $r$ is a radial retraction inside corresponding punctured disks onto the boundary circles. Remark that $r$ is well-defined (easy check on intersections of regions). Next, $r$ is continuous on each of the closed regions $A, B, \ldots, F$. Therefore, by ”pasting lemma” (Theorem 4.4), the map $r$ is continuous. Furthermore, $r(y) = y$ for any $y \in Y$. Therefore, $r$ is a retraction of $X$ onto $Y$.

Let us now define $H : X \times I \to \mathbb{R}^2$ by $H(x, t) = (1 - t)x + t \cdot r(x)$. Then clearly, $H$ is continuous (multivariable calculus). Furthermore, it is also clear that the line segment, connecting $x$ to $r(x)$ is contained in $X$ for any $x \in X$. This can be checked by considering each region separately. Since $D^2$ is convex, we just need to make sure that line segment does not contain neither of $P2$ and $P4$. On $E$ and $F$ it is clear. On $A$ and $D$, corresponding segments are outside of interior of regions $E$ and $F$ (i.e. do not intersect the removed points $P2$ and $P4$), and on $B$ and $C$ the statement is also clearly true.

Thus, $H$ is a mapping from $X \times I$ to $X$, i.e. homotopy on $X$. Furthermore,

$$H(-, 0) = id_X : X \to X, \quad H(-, 1) = r : X \to X.$$  

For any $y \in Y$, have $r(y) = y$. Therefore, for any $y \in Y$ and $t \in I = [0, 1]$,

$$H(y, t) = (1 - t)y + tr(y) = (1 - t)y + ty = y.$$  

This means that the homotopy $H$ is a deformation retraction of $X = D^2 \setminus \{P2, P4\}$ onto the subspace $Y = S^1 \vee S^1$.  

P.S.
Check continuity of $p$ on $B$. Let $(u,v) \in B$. Then define the projection on the first coordinate $\pi(u,v) = (u,0)$, and the map $s : Z \to Y$ (here $Z = [-2R,2R] \times \{0\}$ is the line segment, connecting $P1$ to $P5$) via

\[
s(u,0) = \begin{cases} 
(u, \sqrt{R^2 - (u+R)^2}) & \text{if } -2R \leq u \leq 0, \\
(u, \sqrt{R^2 - (u-R)^2}) & \text{if } 0 \leq u \leq 2R,
\end{cases}
\]

for any point $(u,0) \in [-2R,2R] \times \{0\}$. Then $p = s \circ \pi : B \to Y$, and continuity of $\pi$ and $s$ (easy) implies continuity of $p$.
Problem 7.6. A starlike space is a slightly weaker notion than a convex space – in a starlike space \( X \subset \mathbb{R}^n \), there is a point \( x_0 \in X \) so that for any other point \( f \in X \) and any \( t \in [0,1] \) the point \( tx_0 + (1-t)y \) is in \( X \). Give an example of starlike space that is not convex. Show that a starlike space is a deformation retract of a point.

Solution: The simplest example is a union of two segments joined with the common vertex 0 in the plane, building a nonzero geometric angle of less than \( \pi \) radians. Denote this object by \( M \). Then \( M \subset \mathbb{R}^2 \) is a starlike space with respect to the point \( x_0 = 0 \). Clearly, \( M \) is not convex.

Now, let \( X \) be a starlike space with special point \( x_0 \). We will show that there is a deformation retraction of \( X \) onto the point \( \{x_0\} \subset X \).

Define the map \( H : X \times I \rightarrow \mathbb{R}^n \) via
\[
H(x, t) = tx_0 + (1-t)x
\]
for any \( x \in X \) and \( t \in [0,1] \). We need to check that \( H \) is a deformation retraction of \( X \) onto the point \( \{x_0\} \).

Since \( X \) is a starlike space, the image of \( H \) is actually in \( X \). So, have \( H : X \times I \rightarrow X \). Furthermore, \( H(x, 0) = 0 \cdot x_0 + (1-0) \cdot x = x \) for any \( x \in X \), and \( H(x, 1) = 1 \cdot x_0 + (1-1) \cdot x = x_0 \). The homotopy \( H \) fixes the space \( A = \{x_0\} \) pointwise, since for any \( t \in [0,1] \) have
\[
H(x_0, t) = tx_0 + (1-t)x_0 = x_0.
\]

This means that the homotopy \( H \) is a deformation retraction of \( X \) onto the point \( x_0 \).