Problem 1. [20 pts] Define the term topological property. Give three examples. Prove the topological invariance of the one of your choice.

Problem 2. [20 pts] Let \( f : A \to B \) be a continuous function between topological spaces.

(1) If \( A \) is compact, prove that \( f(A) \) is compact.

(2) If \( A \) is connected, prove that \( f(A) \) is connected.

Problem 3. [20 pts]

(1) Let \( \pi_j : X_1 \times X_2 \to X_j \) be the projection onto the \( j \)-th factor. Show that the product topology on \( X_1 \times X_2 \) is the coarsest topology making \( \pi_1 \) and \( \pi_2 \) continuous.

(2) Compute the fundamental group of \( X_1 \times X_2 \) in terms of the \( \pi_1(X_j) \).

Problem 4. [30 pts] Let \( A \) be the subset of the real numbers with two elements, \( \{1, 2\} \).

(1) Find all the connected subsets containing \( A \).

(2) Which of these connected subsets are compact?

(3) Give a proof of the connectedness of one of the sets in your answer.

Problem 5. [20 pts] Let \( X \) be the real numbers with the finite complement topology.

(1) Prove that the integers are dense in \( X \).

(2) Discuss convergence of sequences in this space.

Problem 6. [40 pts] Let \( X \) be the union of a regular hexagon and an inscribed equilateral triangle.

(1) State and use van Kampen’s theorem to compute the fundamental group of \( X \).

(2) Construct a 3-sheeting covering map from \( X \) to the join of two circles.

(3) Prove a result relating the fundamental groups of these two spaces.

(4) Verify this result for your 3-sheeted covering.

Problem 7. [10 pts] Prove that every map from the 2-disc to itself has a fixed point.

Problem 8. [20 pts] Compute the fundamental group of the real projective plane with one point removed.

Problem 9. [20 pts] Let \( X(\neq \emptyset) \) be a compact Hausdorff space, and let \( F : X \to X \) be a continuous map. Let \( A = \bigcap_n F^n(X) \). Show that:

(1) \( A \) is closed.

(2) \( A \) is compact.

(3) \( F(A) \subseteq A \).