Problem 3, Page 100
Show that if \( A \) is closed in \( X \) and \( B \) is closed in \( Y \), then \( A \times B \) is closed in \( X \times Y \).

Proof: Since \( A, B \) are closed in \( X, Y \) respectively, \( X - A \) and \( Y - B \) are open in \( X, Y \) respectively. Then \((X - A) \times Y, X \times (Y - B)\) are open in \( X \times Y \). We have \( X \times Y - A \times B = (X - A) \times Y \cup X \times (Y - B) \) are open. Thus \( A \times B \) is closed in \( X \times Y \).

Problem 6, Page 101
Let \( A, B \) and \( A_\alpha \) denote subsets of a space \( X \). Prove following:
(a) If \( A \subset B \), then \( A \subset B \).
(b) \( A \cup B = A \cup B \).
(c) \( \bigcup A_\alpha \supset \bigcup A_\alpha \).

Proof: (a) Since \( A \subset B \) and \( B \subset B \), we have \( A \subset B \). Since \( B \) is closed and \( A \) is the smallest closed set containing \( A \), we have \( A \subset B \).
(b) Since \( A \subset B \) and \( B \subset B \), we have \( A \cup B \subset A \cup B \). Similarly, \( A \cup B \subset A \cup B \). Therefore, \( A \cup B \subset A \cup B \).
(c) Since \( A_\alpha \subset \bigcup A_\alpha \) and \( \bigcup A_\alpha \) is closed, we have \( A_\alpha \subset \bigcup A_\alpha \). Thus \( \bigcup A_\alpha \subset \bigcup A_\alpha \). Example: let \( A_i = \{1/i\} \), then \( A_i = A_i \) and \( \bigcup A_i = \bigcup A_i \). But \( \bigcup A_i = \bigcup A_i \cup \{0\} \). The equality fails in this case.

Problem 10, Page 101
Show that every order topology is Hausdorff.

Proof: Suppose \( X \) is a space with order topology. Let \( a \neq b \in X \). Without loss of generality, we assume \( a < b \).

1. If there is no \( c \), such that \( a < c < b \). Take \( A = \{x \in X \mid x < b\} \) and \( B = \{y \in X \mid y > a\} \). By the definition of order topology, \( A, B \) are open in \( X \). It is easy to see \( A \cap B = \emptyset \).
2. If there is \( c \) such that \( a < c < b \). Take \( A = \{x \in X \mid x < c\} \) and \( B = \{y \in X \mid y > c\} \). By the definition of order topology, \( A, B \) are open in \( X \). It is easy to see \( A \cap B = \emptyset \).

Above all, \( X \) is a Hausdorff space.

Problem 11, Page 101
Show that the product of two Hausdorff space is a Hausdorff space.

Proof: Let \( X, Y \) be two Hausdorff topological space and \( X \times Y \) be the product space. Suppose \((x_1, y_1), (x_2, y_2)\) are two different points in \( X \times Y \). Then \( x_1 \neq x_2 \) or \( y_1 \neq y_2 \). Without loss of generality, assume \( x_1 \neq x_2 \). Then since \( X \) is a Hausdorff space, we can choose two open sets \( V_1, V_2 \subset X \), such that \( x_1 \in V_1, x_2 \in V_2 \)
Problem 17, Page 101
Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $C$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in this two topologies.

Proof:
Case 1: Lower limit topology.
For any open set $U$ containing 0, there is a $\epsilon$ such that $[0, \epsilon) \subset U$ by the definition of Lower limit topology. Then we have $\emptyset \neq [0, \epsilon) \cap (0, \sqrt{2}) \subset U \cap (0, \sqrt{2})$. Thus $0 \in \overline{A}$. We have that $[0, \sqrt{2}) \subset \overline{A}$. Since $(-\infty, 0) = \bigcup_{n \in \mathbb{N}} \left(-n, 0\right)$, $n \in \mathbb{N}$ and $[\sqrt{2}, \infty)$ are open, $\mathbb{R} = [0, \sqrt{2}) = (-\infty, 0) \cup [\sqrt{2}, \infty)$ is open. Thus $[0, \sqrt{2})$ is closed. Above all, we have $[0, \sqrt{2}) = \overline{A}$. Use the exactly same discussion as above, we have $\overline{B} = [\sqrt{2}, 3)$.

Case 2: $C$-basis topology.
For any open set $U$ containing 0, there is a $\epsilon \in \mathbb{Q}$ such that $[0, \epsilon) \subset U$ by the definition of $C$-basis topology. Then we have $\emptyset \neq [0, \epsilon) \cap (0, \sqrt{2}) \subset U \cap (0, \sqrt{2})$. Thus $0 \in \overline{A}$. For any open set $U$ containing $\sqrt{2}$, there is an interval $[a, b) \subset U$ containing $\sqrt{2}$ where $a, b \in \mathbb{Q}$ by the definition of $C$-basis topology. Then we have $\emptyset \neq [a, b) \cap (0, \sqrt{2}) \subset U \cap (0, \sqrt{2})$. Thus $\sqrt{2} \in \overline{A}$. Above all, we have $[0, \sqrt{2}] \subset \overline{B}$. Since $(-\infty, 0) = \bigcup_{n \in \mathbb{N}} \left(-n, 0\right)$, $n \in \mathbb{N}$ and $(\sqrt{2}, \infty) = \bigcup_{n \in \mathbb{N}} \left(a_n, \infty\right)$ are open where $\{a_n\} \subset \mathbb{Q}$ and $a_n \to \sqrt{2}$, we have $[0, \sqrt{2}]$ is closed. Thus $\overline{A} = [0, \sqrt{2}]$.

For any open set $U$ containing $\sqrt{2}$, there is an interval $[a, b) \subset U$ containing $\sqrt{2}$ where $a, b \in \mathbb{Q}$ by the definition of $C$-basis topology. Then we have $\emptyset \neq [a, b) \cap (\sqrt{2}, 3) \subset U \cap (\sqrt{2}, 3)$. Thus $\sqrt{2} \in \overline{A}$. Therefore, $(\sqrt{2}, 3) \subset \overline{B}$. Since $[3, \infty)$ and $(-\infty, \sqrt{2}) = \bigcup_{n \in \mathbb{N}} \left(-\infty, a_n\right)$ are open, where $\{a_n\} \subset \mathbb{Q}$ and $a_n \to \sqrt{2}$, we have $[\sqrt{2}, 3)$ is closed. Thus $\overline{B} = [\sqrt{2}, 3)$.

Problem 18, Page 101
Determine the closures of the following subsets of the ordered square:

$A = \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\}$
$B = \{(1 - 1/n) \times (1/2) \mid n \in \mathbb{Z}_+\}$
$C = \{x \times 0 \mid 0 < x < 1\}$
$D = \{x \times (1/2) \mid 0 < x < 1\}$
$E = \{(1/2) \times y \mid 0 < y < 1\}$

Proof:
$\overline{A} = A \cup \{0 \times 1\}$
$\overline{B} = B \cup \{1 \times 0\}$
We prove that if

\[ E = C \cup [0, 1] \times 1 \]
\[ D = D \cup (0, 1) \times 0 \cup [0, 1] \times 1 \]
\[ F = E \cup \{1/2 \times 0, 1/2 \times 1\} \]

**Problem 1, Page 111**
Prove that for functions \( f : \mathbb{R} \rightarrow \mathbb{R} \), the \( \epsilon - \delta \) definition of the continuity implies the open set definition.

**Proof:** We need to show that if \( f \) is continuous in sense of \( \epsilon - \delta \) definition, then it is continuous in sense of the open set definition.
If \( f \) is continuous in sense of \( \epsilon - \delta \) definition, then we have for any point \( x \in \mathbb{R} \) and \( \delta > 0 \), there is an \( \epsilon > 0 \) such that for any \( y \in (x-\epsilon, x+\epsilon) \) \( f(y) \in (f(x)-\delta, f(x)+\delta) \), i.e. \( (x-\epsilon, x+\epsilon) \subset f^{-1}((f(x)-\delta, f(x)+\delta)) \). Now we need to show that for any open set \( U \subset \mathbb{R} \), \( f^{-1}(U) \) is open in \( \mathbb{R} \). Since \( U \) is open, for any \( f(x) \in U \) we have an \( \delta_x \) such that \( (f(x)-\delta_x, f(x)+\delta_x) \subset U \). Since \( f \) is continuous in sense of \( \epsilon - \delta \) definition, we have that there is an \( \epsilon_x \) such that \( (x-\epsilon_x, x+\epsilon_x) \subset f^{-1}(U) \). Therefore, by exercise 1 in homework 1, we have \( f^{-1}(U) \) is open. Thus \( f \) is continuous in sense of the open set definition.

**Problem 2, Page 111**
Suppose that \( f : X \rightarrow Y \) is continuous. If \( x \) is a limit point of the subset \( A \) of \( X \), is it necessarily true that \( f(x) \) is a limit point of \( f(A) \).

**Proof:** No.
Let \( X = (0, 1) \subset \mathbb{R} \) with usual topology. And \( Y = \{0, 1\} \) discrete topology. Let \( f(x) = 0 \) for all \( x \). Then \( f(0) = 0 \) but \( f(0) \) is not the limit point of \( f([0, 1]) \).

**Problem 8, Page 111**
Let \( Y \) be an ordered set in the order topology. Let \( f, g : X \rightarrow Y \) be continuous.
(a)Show that the set \( \{x \mid f(x) \leq g(x)\} \) is closed in \( X \).
(b)Let \( h : X \rightarrow Y \) be the function
\[ h(x) = \min\{f(x), g(x)\} \]
Show that \( h \) is continuous.

**Proof:**

a) We prove that \( \{x : f(x) > g(x)\} = X - \{x : f(x) \leq g(x)\} \) is open in \( X \).
Given \( x_0 \in \{x : f(x) > g(x)\} \)
1) If \( \{y : f(x_0) > y > g(x_0)\} \) is not empty, then choose \( y_0 \in \{y : f(x_0) > y > g(x_0)\} \). Thus, the set \( U_1 = f^{-1}((y_0, \infty)) \cap g^{-1}((-\infty, y_0)) \) contains \( x_0 \). Since \( f \) and \( g \) are both continuous, \( U_1 \) is open in \( X \). For \( \forall x \in U_1 \), we have \( f(x) > y_0 > g(x) \).
   So \( U_1 \subset \{x : f(x) > g(x)\} \)
2) If \( \{y : f(x_0) > y > g(x_0)\} \) is empty.
Set $U_2 = f^{-1}((g(x_0), \infty)) \cap g^{-1}((-\infty, f(x_0)))$. So $U_2$ is an open set in $X$, containing $x_0$. For $\forall x \in U_2$, $f(x) > g(x_0)$ and $g(x) < f(x_0)$. Since \{\{y : f(x_0) > y > g(x_0)\}\} is empty, we have $f(x) \geq f(x_0)$. It follows $f(x) \geq g(x)$. So $x \in \{x : f(x) > g(x)\}$. Therefore, $\{x : f(x) > g(x)\}$ is open in $X$. So $x | f(x) \leq g(x)$ is closed in $X$.

b) Set $X_1 = \{x | f(x) \geq g(x)\}$, $X_2 = \{x | f(x) \leq g(x)\}$. So $X = X_1 \cup X_2$. By the conclusion of Part a, we have both $X_1$ and $X_2$ are closed. Moreover, $h|_{X_1} = g(x)$ and $h|_{X_2} = f(x)$, both of which are continuous. Additionally, $X_1 \cap X_2 = \{x | f(x) = g(x)\}$, so $h|_{X_1 \cap X_2} = g|_{X_1 \cap X_2}$. By pasting Lemma, $h$ is continuous.

**Problem 10, Page 111**

Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be continuous functions. Let us define a map $f \times g : A \times B \rightarrow C \times D$ by equation:

$$(f \times g)(a \times c) = f(a) \times g(c)$$

Show that $f \times g$ is continuous.

**Proof:** Given $U \times V \subset B \times D$, where $U$ is open in $B$ and $V$ is open in $D$. Then $(f \times g)^{-1}(U \times V) = \{(a, c) \in A \times C \mid f(a) \in U \text{ and } g(c) \in V\}$. That is, $(f \times g)^{-1}(U \times V) = \{(a, c) \in A \times C \mid a \in f^{-1}(U) \text{ and } c \in g^{-1}(V)\}$. So $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$. Since both $f$ and $g$ are continuous, we have $f^{-1}(U)$ and $g^{-1}(V)$ are both open. So $f^{-1}(U) \times g^{-1}(V)$ are open in $A \times C$. So $f \times g$ is continuous.

**Problem 12, Page 111**

Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2), & x \times y \neq 0 \times 0, \\ 0, & x \times y = 0 \times 0. \end{cases}$$

(a) Show that $F$ is continuous in each variable separately.

(b) Compute the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = F(x \times y)$.

(c) Show that $F$ is not continuous.

**Proof:**

a) Fix $y_0 \in \mathbb{R}$. Prove that $F(x, y_0)$ as a map from $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.

- If $y_0 = 0$ then $F(x, y_0) = 0$ for $\forall x \in \mathbb{R}$. So $F(x, y_0)$ is continuous.

- If $y_0 \neq 0$, then $F(x, y_0) = \frac{xy_0}{x^2 + y_0^2}$ for $\forall x \in \mathbb{R}$. Since $xy_0$ and $x^2 + y_0^2$ are both continuous in $\mathbb{R}$, we have $F(x, y_0) = \frac{xy_0}{x^2 + y_0^2}$ is continuous. Similarly, if fix $x_0 \in \mathbb{R}$. $F(x_0, y)$ as a map from $\mathbb{R}$ to $\mathbb{R}$ is continuous as well.

Therefore, $F$ is continuous in each variable separately.
b) As $y = x$

\[ g(x) = F(x, y) = \begin{cases} 
\frac{1}{2} & x \neq 0 \\
0 & x = 0. 
\end{cases} \]

c) By the part b, we have along line $x = y$, $F(x, y)$ is not continuous at $(0, 0)$. So $F(x, y)$ is not continuous at $(0, 0)$. 