Final Exam - 4055

1. (5 points each) A card is drawn from a deck of 52 cards. What is the probability that
   (a) it is a face card or a black card? Note: Face card refers to one of Jack, Queen or King.
   (b) it is neither a heart nor a queen? 

   \[
   \begin{align*}
   & (a) \quad \frac{32}{52} \\
   & (b) \quad \frac{36}{52}
   \end{align*}
   \]

2. (6 points) At a party, \( n \) men and \( m \) women put their drinks on a table and go out on
   the floor to dance. When they return, none of them recognizes his or her drink, so
   everyone takes a drink at random. What is the probability that each man selects his
   own drink?

   \[
   \frac{m!}{(n+m)!}
   \]

3. (5 points) Five boys and five girls sit in a row at random. What is the probability
   that the boys are together and the girls are together?

   \[
   \frac{5! 5! 2!}{10!}
   \]

4. (8 points) A poker hand consists of 5 cards dealt from a deck of 52 cards. Find the
   probability that the hand has a distribution of suits 2, 2, 1, 0.

   \[
   \binom{4}{2} 2! \left( \binom{13}{2} \right)^2 \binom{13}{1} \binom{13}{0}
   \]

5. (8 points) An urn contains 5 white and 3 red chips. Each time we draw a chip, we
   look at its color. If it is red, we replace it along with 2 new red chips, and if it is white,
   we replace it along with 3 new white chips. What is the probability that in successive
   drawing of chips, the colors of the first 4 chips alternate?

   \[
   \frac{\frac{3 \cdot 5 \cdot 5 \cdot 8}{8 \cdot 10 \cdot 13 \cdot 15} + \frac{5 \cdot 3 \cdot 8 \cdot 5}{8 \cdot 11 \cdot 13 \cdot 16}}{\binom{52}{5}}
   \]

6. (8 points) In a study, it was discovered that 25% of the paintings of a certain gallery
   are not original. A collector makes a 15% error by judging an original as a fake. He
   makes an equal error in judging a fake as an original. If she buys a piece thinking
   that it is an original, what is the probability that it is not?

   Use Bayes' formula

7. (5 points each) Suppose that 2.5% of the population of a border town are illegal
   immigrants.

   (i) Find the probability that in a theater of this town with 80 random viewers, there are
   at least 2 illegal immigrants.

   \[
   1 - (0.975)^{80} - 80 (0.975)^{79} (0.025)
   \]

   (ii) What is the Poisson approximation to the above question?

   \[
   \lambda = 2 \\
   1 - e^{-2} - 2 e^{-2}
   \]

8. (6 points) Let \( X \) be an exponential random variable with parameter 1. Find the
   distribution function and the density function of \( Y = -\ln X \).

   \[
   \begin{align*}
   F_Y(y) &= e^{-y} \\
   f_Y(y) &= e^{-y}
   \end{align*}
   \]

9. (8 points) An expert witness in a paternity suit testifies that the length (in days) of
   a pregnancy, from conception to delivery, is approximately normally distributed, with
   parameters \( \mu = 270 \) and \( \sigma = 10 \). The defendant in the suit is able to prove that he
   was out of the country during the period from 290 to 240 days before the birth of the
   child. What is the probability that the defendant was in the country when the child
   was conceived? HINT: Find the probability that from conception to delivery, it took
   either more than 290 days OR less than 240 days.

   \[
   D \cdot D \cdot 2 \cdot 4 \cdot 1
   \]

10. (8 points) Let \( X \) and \( Y \) be independent random variables with uniform density
    functions on \([0, 1]\). Find \( E(\min(X, Y)) \) and \( E((X + Y)^2) \).

    \[
    \begin{align*}
    & E(\min(X, Y)) = \frac{1}{3} \\
    & E((X + Y)^2) = \frac{7}{6}
    \end{align*}
    \]
11. (5 points each) Suppose that \( n \) people have their hats returned at random. Let \( X_i = 1 \) if the \( i \)th person gets his or her own hat back and 0 otherwise. Let \( S_n = \sum_{i=1}^{n} X_i \). Then \( S_n \) is the total number of people who get their own hats back. Find

(a) \( E(X_i^2) \). 

(b) \( E(X_i \cdot X_j) \) for \( i \neq j \).

(c) \( E(S_n^2) \) using parts (a) and (b).

12. (8 points) Let \( U \) and \( V \) be random numbers chosen independently from the interval [0,1] with uniform distribution. Find the distribution and density of the random variable \( Y = \max(U,V) \).

\[ f_Y(x) = 2x \quad 0 \leq x \leq 1. \]