Problems in HW 2.

Problem 1.3.5:
It is given that \( a \leq b \forall a \in A \), \( b \in B \).

\[ \therefore \text{Any} \ b \in B \text{ is an upper bound for } A. \]

\[ \therefore A \text{ has a lub. Which is also the sup } A. \text{ Thus} \sup A \leq b \forall b \in B. \]

\[ \sup A < \infty, \text{ and hence } B \text{ has a lower bound, namely } \sup A. \text{ Therefore, it has a greatest lower bound. This glb is inf } B. \]

Thus \( \sup A \leq \inf B \).

Problem 1.3.10:
If \( \{ x_n \} = \{-1, 0, 1, 2, 3, \ldots \} \)

and \( \{ y_n \} = \{-2, -1, 0, 1, 2, \ldots \} \)

then

\[ \{ x_n y_n \} = \{ 2, 0, 0, 2, 6, \ldots \} \]

which is not monotone.

Problem 1.3.12: \( x_n = (-1)^n / n \). The full segment is \( \{ x_n \} = \{-1, 1/2, -1/3, 1/4, \ldots \} \).

\[ \therefore s_1 = 1/2, \ s_2 = 1/2, \ s_3 = 1/4, \ s_4 = 1/4, \ldots \]

et cetera.

where \( s_n = \sup \{ x_n, x_{n+1}, \ldots \} \).

Clearly, \( s_{2n-1} = 1/2n \) and \( s_{2n} = 1/2n \) for all \( n \).

We know that \( \{ s_n \} \) is always a
decreasing sequence. In this case, it is also bounded. \( n \to \infty \) \( \implies \lim s_n = \inf s_n \).

Clearly, \( \lim s_n = 0 \) \( \implies \inf s_n = 0 \).

i.e. \( \lim \sup x_n = 0 \).

Let us consider \( \{ i_n \} \).

\[ i_1 = -1, \quad i_2 = -\frac{1}{3}, \quad i_3 = -\frac{1}{3}, \text{ etc.} \]

In general,

\[ i_{2n} = -\frac{1}{2n+1}; \quad i_{2n+1} = -\frac{1}{2n+1} \]

\( \forall n \geq 1 \).

We know that \( \{ i_n \} \) is always an increasing sequence. Here \( \{ i_n \} \) is also bounded. Therefore,

\[ \implies \lim i_n = \sup i_n. \]

\( \lim i_n = 0 \) in this case. \( \implies \sup i_n = 0 \).

\( \forall n \to \infty \)

i.e. \( \lim \inf i_n = 0. \)

1.3.15: \( \implies \) part: Assume that \( x_n \to \infty \).

Then given any \( M > 0, \exists N \in \mathbb{N} \) such that \( \forall n > N, x_n > M \). \( \text{(1)} \).

\[ \forall n \geq M \implies n \geq N. \]

\( \{ s_n \} \) is a decreasing sequence.

\[ \forall n \geq M \implies n \geq 1 \text{ since } s_1 > s_2 > s_3 \geq s_n \]
\[ \inf_{n} \alpha_{n} \geq M \quad \text{i.e.} \quad \limsup_{n \to \infty} \alpha_{n} \geq M. \]

This is so for all \( M > 0 \).

\[ \limsup_{n \to \infty} \alpha_{n} = \infty. \]

Also from (1), we get \( \inf_{n} \geq M + n \geq N \).

\[ \sup_{n} \inf_{\infty} = \infty. \quad \text{This is so for any } M. \]

i.e. \( \liminf_{n \to \infty} \alpha_{n} = \infty. \)

\[ \leq \quad \text{part: Assume that } \liminf_{n \to \infty} \alpha_{n} = \infty. \]

Then

\[ \sup_{n} \inf_{\infty} = \infty. \quad \text{i.e. } \{i_{n}\} \text{ is not bounded above.} \]

\[ \therefore \text{Given any } M > 0, \exists N \Rightarrow i_{N} > M. \]

\[ \inf_{n} \geq M + n \geq N \quad \text{since } \{i_{n}\} \text{ is an increasing sequence.} \]

\[ \text{Note that } \alpha_{n} \geq i_{n} + n. \]

\[ \therefore \alpha_{n} \geq M + n \geq N. \]

i.e. \( \alpha_{n} \) diverges to \( \infty \). \( \square \).