The Enskog Process

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Abstract

The existence of a weak solution to a McKean-Vlasov type stochastic differential system corresponding to the Enskog equation of the kinetic theory of gases is established under suitable hypotheses. The distribution of any solution to the system at each fixed time is shown to be unique. The existence of a probability density for the time-marginals of the velocity is verified in the case where the initial condition is Gaussian, and is shown to be the density of an invariant measure.

KEY WORDS: Boltzmann and Enskog equations; McKean-Vlasov stochastic differential system; invariant measure.

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1 Introduction

The Boltzmann equation describes the time evolution of the density function in a phase (position-velocity) space for a classical particle (molecule) under the influence of other particles in a diluted (or rarified) gas [10] (evolving in vacuum for a given initial distribution). It forms the basis for the kinetic theory of gases, see, for e.g. [14].

If f is the density function, which depends on time $t \geq 0$, the space variable $x \in \mathbb{R}^d$, and the velocity variable $u \in \mathbb{R}^d$ of the point particle, then $f(t, x, u) dx\, du$ is by definition the probability for the particle to have position x in a volume element dx around x and velocity u within the volume element du around u. For a single type of particles all of mass $m > 0$, in the absence of external forces, the Boltzmann equation has the general form

$$\frac{\partial f}{\partial t}(t, x, u) + u \cdot \nabla_x f(t, x, u) = Q(f, f)(t, x, u), \tag{1.1}$$

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where Q is a certain quadratic form in f, called collision operator (or integral).

Set $\Lambda := \mathbb{R}^3 \times (0,\pi] \times [0,2\pi)$. Then Q can be written in the general form

$$Q(f,f)(t,x,u) = \int_{\Lambda} \{f(t,x,u^\star)f(t,x,v^\star) - f(t,x,u)f(t,x,v)\}B(u,dv,d\theta)d\phi. \tag{1.2}$$

It is assumed that any gas particle travels straight until an elastic collision occurs with another particle. Each $v \in \mathbb{R}^3$ in (1.2) denotes the velocity of an incoming particle which may hit, at the fixed location $x \in \mathbb{R}^3$, particles whose velocity is fixed as $u \in \mathbb{R}^3$. Let $u^\star \in \mathbb{R}^3$ and $v^\star \in \mathbb{R}^3$ denote the resulting outgoing velocities corresponding to the incoming velocities u and v respectively. $\theta \in (0,\pi]$ denotes the azimuthal or colatitude angle of the deflected velocity, $v^\star$ (see [42]). Having determined θ, the longitude angle $\phi \in [0,2\pi)$ measures in polar coordinates, the location of v^*, and hence that of $u^\star$, as explained below.

In the Boltzmann model as the collisions are assumed to be elastic, conservation of kinetic energy as well as momentum of the molecules holds, i.e. considering particles of mass $m = 1$, the following equalities hold:

$$\begin{cases} u^\star + v^\star = u + v \\ (u^\star)^2 + (v^\star)^2 = u^2 + v^2 \end{cases} \tag{1.3}$$

$$\begin{cases} v^\star = v + (\mathbf{n}, u - v)\mathbf{n} \\ u^\star = u - (\mathbf{n}, u - v)\mathbf{n} \end{cases} \tag{1.4}$$

where

$$\mathbf{n} = \frac{v^\star - v}{|v^\star - v|} \tag{1.5}$$

where $(\cdot,\cdot)$ denotes the scalar product, and $|\cdot|$, the Euclidean norm in $\mathbb{R}^3$.

Remark 1.1. *The Jacobian of the transformation (1.4) has determinant 1 and* $(u^\star)^\star = u$ *since the collision dynamics are reversible.*

The outgoing velocity u^* is then uniquely determined in terms of the colatitude angle $\theta \in (0,\pi]$ measured from the center, and longitude angle $\phi \in [0,2\pi)$ of the deflection vector $\mathbf{n}$ in the sphere with northpole u and southpole v centered at $\frac{u+v}{2}$, which are used in equation (1.1) and (1.2) (see e.g. [12], [24], [44]).

$B(u,dv,d\theta)$ is a σ-additive positive measure defined on the Borel σ-field $\mathcal{B}(\mathbb{R}^3) \times \mathcal{B}((0,\pi])$, depending (Lebesgue) measurably on $u \in \mathbb{R}^3$. The form of B depends on the version of Boltzmann equation one has in mind. In Boltzmann's original work [10],

$$\begin{aligned} B(u,dv,d\theta) &= |(u-v)\cdot n|dvd\theta \\ &= |u-v|dv\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)d\theta, \end{aligned} \tag{1.6}$$

where in (1.6) we used that $\frac{\pi}{2} - \frac{\theta}{2}$ is the angle between $u - v$ and $\mathbf{n}$, so that

$$|(u - v, \mathbf{n})| = |u - v| \cos(\frac{\pi}{2} - \frac{\theta}{2}) = |u - v| \sin(\frac{\theta}{2}), \tag{1.7}$$

$$\text{and} \quad |(u - v, \mathbf{n})| d\mathbf{n} = |u - v| \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) d\theta d\phi = B(u, dv, d\theta) d\phi \tag{1.8}$$

is the differential cross section scattering the velocities v of incoming particles colliding with the particle with velocity u, written in polar coordinates.

In the case where the molecules interact by a force which varies as the nth inverse power of the distance between their centers, one has [14],

$$B(u, dv, d\theta) = |(u - v)|^{\frac{n-5}{n-1}} \beta(\theta) dv d\theta \tag{1.9}$$

where β is a Lebesgue measurable positive function of θ. In particular, for $n = 5$, one has the case of "Maxwellian molecules", where

$$B(u, dv, d\theta) = \beta(\theta) dv d\theta.$$

The function $\beta(\theta)$ decreases and behaves like $\theta^{-3/2}$ for $\theta \downarrow 0$, see, for e.g. [14]. Note that in the latter case $\int_0^\pi \beta(\theta) d\theta = +\infty$. We note that for Maxwellian particles the cross section $B(u, dv, d\theta) d\phi$ does not depend on the modulus $|u - v|$ of the velocity difference between the velocity u of the particle and the velocities v of incoming particles.

In the present paper, we shall mainly assume that

$$B(u, dv, d\theta) = \sigma(|u - v|) dv Q(d\theta) \tag{1.10}$$

where Q is a σ-finite measure on $\mathcal{B}((0, \pi])$, and σ is a bounded, Lipschitz continuous, positive function on $\mathbb{R}^+$. The assumptions on σ, though an improvement on the existing results for the case $\sigma = 1$, do restrict applicability to physically realizable molecules. When $Q(d\theta)$ is taken to be integrable, one speaks of a cut-off function.

Remark 1.2. *Both the rigorous derivation of Boltzmann equation from a microscopic model, and the study of existence, uniqueness and properties of solutions of the Boltzmann equation still present many challenging and open problems. For the derivation problem, see, for e.g., [17].*

Morgenstern [32] "mollified" $Q(f, f)$ by replacing it by

$$\begin{aligned} &Q_M(f, f)(t, x, u) \\ &= \int \{f(t, x, u^\star) f(t, y, v^\star) - f(t, x, u) f(t, y, v)\} K_M(x, y) B(u, dv, d\theta) dy d\phi \end{aligned}$$

with some measurable K_M and B such that $K_M(x,y)B(u,dv,d\theta)$ has a bounded density with respect to Lebesgue measure $dv \times d\theta$, and obtaining a global existence theorem in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. Povzner [35] obtained existence and uniqueness in the space of Borel measures in x, with the term $K_M(x,y)B(u,dv,d\theta)dyd\phi$ replaced by $K_P(x-y,u-v)dvdy$ (with a suitable reinterpretation of the relations between x, u^*, v^* and y, u, v, and suitable moments assumptions on K_P).

According to [14] (p. 399), this modification of Boltzmann's equation by Povzner is "*close to physical reality*". Cercignani also notes that Povzner equation has a form similar to the *Enskog equation* for dense gases, which we shall discuss below. Modification in another direction consists in taking the space of velocities as discrete, and is discussed in [14] (pp. 399-401).

The majority of further mathematical results concerns the spatially homogeneous case, where the initial condition on f is assumed to be independent of the space variable x, so that at all times f itself does not depend on x. For such results see, e.g. [14], [16].

Let us now associate to (1.1), (1.2) its weak (in the functional analytic sense) version. The following proposition is instrumental in this direction.

Using also Remark 1.1, Tanaka [42] proved the following result that is important for the weak formulation of the equation.

Proposition 1.1. *Let $\Psi(x,u) \in C_0(\mathbb{R}^6)$, as a function of $x \in \mathbb{R}^3$, $u \in \mathbb{R}^3$. With B as in* (1.10)*, we have*

$$\begin{aligned}\int_{\mathbb{R}^9\times(0,\pi]\times[0,2\pi)} &\Psi(x,u)f(t,x,u^\star)f(t,x,v^\star)B(u,dv,d\theta)dxdud\phi \\ = \int_{\mathbb{R}^9\times(0,\pi]\times[0,2\pi)} &\Psi(x,u^\star)f(t,x,u)f(t,x,v)B(u,dv,d\theta)dxdud\phi\end{aligned} \tag{1.11}$$

The above result is proven using equation (1.4) and Remark 1.1 [42]. From now on, we will assume that B is as in (1.10).

Weak formulation of the Boltzmann equation

Consider the Boltzmann equation (1.1) with collision operator (1.2). We multiply (1.1) by a function ψ (of $(x,u) \in \mathbb{R}^6$) belonging to $C_0^1(\mathbb{R}^6)$, and integrate with respect to x and u. Using integration by parts and Proposition (1.1), we arrive at the weak form of the Boltzmann equation:

$$\begin{aligned}&\int_{\mathbb{R}^6}\psi(x,u)\frac{\partial f}{\partial t}(t,x,u)dxdu - \int_{\mathbb{R}^6} f(t,x,u)(u,\nabla_x\psi(x,u))dxdu \\ &= \int_{\mathbb{R}^6} f(t,x,u)L_f\psi(x,u)dxdu\end{aligned} \tag{1.12}$$

for all $t \in \mathbb{R}_+$ with

$$L_f\psi(x,u) = \int_{\mathbb{R}^3\times(0,\pi]\times[0,2\pi)} \{\psi(x,u^\star) - \psi(x,u)\} f(t,x,v) B(u,dv,d\theta)d\phi,$$

where B is as in (1.10).

To proceed further, let us introduce an approximation to the weak form of the Boltzmann equation by introducing a smooth real-valued function β (which should not be confused with the one appearing in (1.9)) with compact support defined on $\mathbb{R}^1$:

$$\begin{aligned}&\int_{\mathbb{R}^6} \psi(x,u)\frac{\partial f}{\partial t}(t,x,u)dxdu - \int_{\mathbb{R}^6} f(t,x,u)(u, \nabla_x\psi(x,u))dxdu \\ &= \int_{\mathbb{R}^6} f(t,x,u)L_f^\beta\psi(x,u)dxdu \end{aligned}\tag{1.13}$$

for all $\psi \in C_0^1(\mathbb{R}^6)$ and for all $t \in \mathbb{R}_+$ with $L_f^\beta\psi(x,u)$

$$= \int_{\mathbb{R}^6\times(0,\pi]\times[0,2\pi)} \{\psi(x,u^\star) - \psi(x,u)\} f(t,y,v)\beta(|x-y|)dyB(u,dv,d\theta)d\phi.$$

Heuristically, when $\beta \to \delta_0$, then any solution of (1.13) tends to a solution of Boltzmann's equation (1.12), so that β can be seen as a regularization for (1.12).

Equation (1.13) is thus the (functional analytic) weak form of an equation closely related to the Boltzmann equation, which can be written as

$$\frac{\partial f}{\partial t}f(t,x,u) + u\cdot\nabla_x f(t,x,u) = Q_E^\beta(f,f)(t,x,u), \tag{1.14}$$

with

$$\begin{aligned}&Q_E^\beta(f,f)(t,x,u)\\ &= \int_\Lambda\int_{\mathbb{R}^3} \{f(t,y,u^*)f(t,x,v^*) - f(t,y,u)f(t,x,v)\}\beta(|x-y|)dyB(u,dv,d\theta)d\phi.\end{aligned}$$

In the case where β is replaced by the characteristic function (or a smooth version of it like in [14]) of a ball of radius $\epsilon > 0$, this is Enskog's equation used for (moderately) "dense gases" taking into account interactions at distance ϵ between molecules. For Enskog's equation, see e.g. [14], [13], [37] (pp. 6, 14), [18], [19], [1], [4], [15], [34], [6], [8]. For versions of the equation in a bounded region, see [3], [33]. The relationship between the Enskog and the Boltzmann equation have been discussed in several publications. In particular, their asymptotic equivalence (with respect to the support of β shrinking to $\{0\}$) has been discussed in [6]. In [37], a pointwise limit has been established.

If μ_t denotes the Borel probability measure on $\mathbb{R}^6$ corresponding to a smooth density function $f(t,x,u)$, i.e.

$$\mu_t(dx,du) = f(t,x,u)dxdu,$$

then the equation (1.13) can be written as

$$\frac{\partial}{\partial t}\langle \mu_t, \psi\rangle - \langle \mu_t, (u, \nabla_x \psi(x,u))\rangle = \langle \mu_t, L^\beta_{\mu_t}\psi\rangle \tag{1.15}$$

where $L^\beta_{\mu_t}\psi(x,u)$

$$= \int_{\mathbb{R}^6\times(0,\pi]\times[0,2\pi)} \{\psi(x,u^\star) - \psi(x,u)\}\beta(|x-y|)\mu_t(dy,dv)B(u,dv,d\theta)d\phi.$$

In the above, we have used the sharp bracket $\langle\cdot,\cdot\rangle$ to denote integration with respect to μ_t while $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^3$. If μ_t satisfies (1.14), we say that μ_t is a weak solution of the Enskog equation.

Define the space $\mathbb{D} := \mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$ as the space of all right continuous functions with left limits defined on $[0,\infty)$ taking values in $\mathbb{R}^3$, and equipped with the topology induced by the Skorohod metric (see e.g. [9]). We denote the value of any $\omega \in \mathbb{D}$ at any time s by ω_s or $\omega(s)$. Likewise, the time marginal of a Borel probability measure μ on $\mathbb{D}$ will be denoted by μ_s for all $s \in [0,\infty)$. The measure μ_s will be a Borel probability measure on $\mathbb{R}^3$. We will use similar notations for functions in $\mathbb{D}\times\mathbb{D}$ and for Borel measures on $\mathbb{D}\times\mathbb{D}$.

If the measure μ_t in (1.15) is the marginal at time t of a Borel probability measure μ on $\mathbb{D}\times\mathbb{D}$, then we can write the Enskog equation (1.15) as follows:

$$\frac{\partial}{\partial t}\langle \mu, \psi(x_t,u_t)\rangle - \langle \mu, (u_t, \nabla_{x_t}\psi(x_t,u_t))\rangle = \langle \mu, L^\beta_{\mu}\psi(x_t,u_t)\rangle \tag{1.16}$$

where x_t, u_t are t-coordinates in $\mathbb{D}\times\mathbb{D}$ and $L^\beta_\mu\psi(x_t,u_t)$

$$= \int_{U_0} \{\psi(x_t,u_t^\star) - \psi(x_t,u_t)\}\sigma(|u_t-v_t|)\beta(|x_t-y_t|)\mu(dy,dv)Q(d\theta)d\phi,$$

where we used the form (1.10) of B and the notation U_0 for $\mathbb{D}\times\mathbb{D}\times(0,\pi]\times[0,2\pi)$. In the following, we formulate the connection between equation (1.16) and stochastic analysis.

Remark 1.3. *The idea of looking at solutions of certain deterministic nonlinear parabolic evolution equations in connection with probability measures describing the distributions of suitable associated Markov processes goes back to McKean [29]. For a spatial homogeneous version of our present context for the case $\sigma = 1$, this idea has been adapted and ingeniously implemented by Tanaka [42],[45], and successively developed, for this case, e.g., in [43], [44],[45], [23], [21], [24]. In our work, we avoid the assumption of spacial homogeneity, and we allow σ to depend on $|u-v|$.*

Let us first derive heuristically the evolution of the stochastic process $(X_s, Z_s)_{s\in\mathbb{R}}$, describing the evolution of position and velocity of a particle evolving according to the Enskog equation (1.16). In the present context, the evolution of the velocity $(Z_s)_{s\in\mathbb{R}}$ of one particle is obtained by integrating (or in other words "summing") the velocity displacements $\alpha(Z_s, v_s.\theta, \phi)$ with respect to a counting measure $N_{X,Z}(ds, dy, dv, d\theta, d\phi)$ which depends over a time interval ds on the distribution $\mu(dy, dx)$ in position and velocity of the gas particles, as well as the position and velocity (X_s, Z_s) of the particle itself because of the presence of particles being close enough to hit (guaranteed by the function β), and the scattering measure for the velocity $B(u, dv, d\theta)d\phi$, defined in (1.10). The position then evolves according to $X_t = X_0 + \int_0^t Z_s ds$. Let us introduce such a suitable jump-Markov process $(X_s, Z_s)_{s\in\mathbb{R}}$. Let $\mu(dx, dv)$ be a probability measure on $\mathbb{D} \times \mathbb{D}$. Let $\tilde{N}_{X,Z}(ds, dy, dv, d\theta, d\phi)$ be a compensated random measure (crm) defined on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathrm{P})$ with compensated measure (or simply, compensator),

$$d\Gamma := \Gamma(dy, dv, d\theta, d\phi, ds) = \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\theta)d\phi ds \tag{1.17}$$

on $\mathbb{D} \times \mathbb{D} \times (0, \pi] \times [0, 2\pi) \times \mathbb{R}_+$ (where we recall that $Q(d\theta)$ is a σ - finite measure on the Borel σ - algebra $\mathcal{B}((0, \pi])$, $d\phi$ is the Lebesgue measure on $\mathcal{B}([0, 2\pi))$, and β is a $C_0^\infty(\mathbb{R}^1)$ function with support near 0. Here, X, Z are elements of $\mathbb{D}$, and v_s, y_s are s-coordinates of v, y in $\mathbb{D}$.

Now, let $(X_s, Z_s)_{s\in\mathbb{R}_+}$ be the process defined below, taking values in the Skorohod space $\mathbb{D} \times \mathbb{D}$ with the joint distribution of $(X_s, Z_s)_{s\in\mathbb{R}_+}$ denoted by $\mu(dx, dz)$:

$$\begin{aligned} Z_t =& Z_0 + \int_0^t \int_{\mathbb{D}\times\mathbb{D}\times(0,\pi]\times[0,2\pi)} \alpha(Z_s, v_s, \theta, \phi)\tilde{N}_{X,Z}(ds, dy, dv, d\theta, d\phi) \\ &+ \int_0^t \int_{\mathbb{D}\times\mathbb{D}\times(0,\pi]\times[0,2\pi)} \alpha(Z_s, v_s, \theta, \phi)\sigma(|Z_s - v_s|)\beta(|X_s - y_s|) \\ &\qquad \mu(dy, dv)Q(d\theta)d\phi ds & (1.18) \\ X_t =& X_0 + \int_0^t Z_s ds & (1.19) \end{aligned}$$

The initial values X_0 and Z_0 are specified. We have set

$$\alpha(u, v, \theta, \phi) := (\mathbf{n}, u - v)\mathbf{n}, \tag{1.20}$$

where, as above, the deflection vector $\mathbf{n}$ is given in spherical coordinates, i.e. in terms of the colatitude angle $\theta \in (0, \pi]$ and longitude angle $\phi \in [0, 2\pi)$.

We have obtained such a process heuristically considering the physics governing the evolution of the particles and will prove in this article that this is the stochastic process whose law corresponds to the solution of the Enskog equation (1.16). We remark however that the stochastic equation (1.18),(1.19) is defined in terms of a counting measure $\tilde{N}_{X,Z}$ with random compensator $\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\theta)d\phi ds$. The mathematical theory of point processes with random compensator has been analyzed extensively in e.g. [25], or [27], but as the theory of Stochastic Differential equations with Poisson random measure is more developed and better known, we prefer here to rewrite (1.18),(1.19) in an equivalent stochastic equation written in terms of a stochastic integral w.r.t to a Poisson random measure associated to a Lévy process, which is the following:

$$\begin{aligned} Z_t = Z_0 + \int_0^t \int_U \alpha(Z_s, v_s, \theta, \phi) 1_{[0,\, \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)]}(r) d\tilde{N}^\mu \\ + \int_0^t \int_{U_0} \alpha(Z_s, v_s, \theta, \phi)\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dv, dy)Q(d\theta)d\phi ds \end{aligned} \tag{1.21}$$

$$X_t = X_0 + \int_0^t Z_s ds, \tag{1.22}$$

where μ is still the law of the process (Z_t, X_t), $t \geq 0$, but now $\tilde{N}^\mu(dy, dv, d\theta d\phi, dr, ds)$ is a compensated Poisson random measure (cPrm) with Poisson measure $N^\mu := N^\mu(dy, dv, d\theta d\phi, dr, ds)$ and compensator $\mu(dy, dv)Q(d\theta)d\phi dr ds$ on $\mathbb{D} \times \mathbb{D} \times (0, \pi] \times [0, 2\pi) \times [0, 1] \times \mathbb{R}_+$.

That (1.21),(1.22) and (1.18),(1.19) are equivalent equations can be shown with at least two different methods:

i) For each Borel-subset B of $\mathbb{D} \times \mathbb{D} \times (0, \pi] \times [0, 2\pi)$, the counting measure $N_{X,Z}(B \times [0, t))$ can be represented by

$$\int_0^t \int_B \int_{[0,1]} 1_{[0,\, \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)]}(r) dN^\mu .$$

This is a consequence of the following equation, which shows the relation between the random compensator Γ of the point measure $N_{X,Z}$ defined (1.17) and the compensator $\mu(dy, dv)Q(d\theta)d\phi ds$ of the Poisson random measure N^μ:

$$\begin{aligned} \Gamma(B \times [0, t)) &= \int_0^t \int_B \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\theta)d\phi ds \\ &= \int_0^t \int_B \int_{[0,1]} 1_{[0,\, \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)]}(r)\mu(dy, dv)Q(d\theta)d\phi ds dr \end{aligned}$$

ii) Given $\psi \in C_0^2(\mathbb{R}^6)$, for all $t \geq 0$, $\Delta t \geq 0$, the Itô formula $\psi(X_{t+\Delta t}, Z_{t+\Delta t}) - \psi(X_t, Z_t)$ for (1.21),(1.22) and (1.18),(1.19) can be proven to be exactly the same. This implies that (1.21), (1.22) and (1.18),(1.19) solve the same martingale problem and are the same process in weak sense. In particular the law of the corresponding process (Z_t, X_t), $t \geq 0$ solves in both cases the Enskog equation, as proven in Proposition 2.2 for (1.21),(1.22). The Itô formula for (1.21), (1.22) is computed in the proof of Proposition 2.2. It can be obtained in a similar way for (1.18),(1.19) by using Theorem 2.42, Ch. II in [27].

We call the process (Z_t, X_t), $t \geq 0$, given by (1.21),(1.22), (1.20)(resp. its law $\mu = \mu_t, t \geq 0$) as the Markov process (resp. law) associated with the Enskog equation described by (1.16). Its existence is proven in Theorem 2.1 in Section 2 under suitable conditions which are satisfied by some physical models. In Proposition 2.2 we will prove that for any finite fixed time $T > 0$, its law $\mu = \{\mu_t\}, 0 \leq t \leq T$ solves the Enskog equation (1.16). Uniqueness of the Markov process (Z_t, X_t), $t \geq 0$ solving (1.21),(1.22), (1.20), is proven in Theorem 7.1 in Section 3 for the time marginals. The existence of a density $f(t, x, z)$ for the distribution $\mu = \{\mu_t\}, 0 \leq t \leq T$ is proven in Section 4, for the particular case where the velocity marginals are time invariant. $f(t, x, z)$ solves then the Enskog equation (1.13)

It is worthwhile to mention that we have not made the assumption of space homogeneity. We allow σ that appears as the differential cross section (see equation (1.10)) to depend on $|u - v|$.

2 Existence Results

In this section we establish the existence of a solution of the system of stochastic equations (1.21),(1.22), with (θ, ϕ) denoted by ξ that takes values in the set $\Xi := (0, \pi] \times [0, 2\pi)$. Also, $Q(d\theta)d\phi$ is written as $Q(d\xi)$ for notational simplicity. From the physical model, we know that $Q(d\xi)$ should be a σ-finite measure, and hence taken as σ-finite.

Hypotheses A:

A1. The measure Q is finite outside any neighborhood of 0, and for all $\epsilon > 0$, Q satisfies

$$\int_0^\epsilon \theta Q(d\theta) < \infty.$$

A2. $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ (as entering (1.10)) is a bounded, Lipschitz continuous function on $\mathbb{R}^+$.

There are many useful consequences of **A1**. Recall that $\alpha(z, v, \xi) = (\mathbf{n} \cdot (z - v))\mathbf{n}$ with $\frac{\pi}{2} - \frac{\theta}{2}$ as the angle between the vectors $(z - v)$ and $\mathbf{n}$ (see (1.7)). Hence, condition **A1** implies that there exists a constant C such that the following estimates hold.

$$\int_\Xi |\alpha(z, v, \xi) - \alpha(z', v', \xi)|^2 Q(d\xi) \leq C(|z - z'|^2 + |v - v'|^2) \tag{2.1}$$

$$\int_\Xi |\alpha(z, v, \xi) - \alpha(z', v', \xi)| Q(d\xi) \leq C(|z - z'| + |v - v'|). \tag{2.2}$$

From (2.1), it follows, by setting z' and v' to be z, and using the fact that $\alpha(z, z, \xi) = 0$ for all $\xi \in \Xi$, $z \in \mathbb{R}^3$ that

$$\int_\Xi |\alpha(z, v, \xi)|^2 Q(d\xi) \leq C|z - v|^2, \tag{2.3}$$

and hence

$$\int_{\Xi} |\alpha(z, v, \xi)|^2 Q(d\xi) \leq C(|z|^2 + |v|^2). \tag{2.4}$$

In a similar way, from (2.2) it follows

$$\int_{\Xi} |\alpha(z, v, \xi)| Q(d\xi) \leq C|z - v|, \tag{2.5}$$

and hence

$$\int_{\Xi} |\alpha(z, v, \xi)| Q(d\xi) \leq C(|z| + |v|), \tag{2.6}$$

Condition **A2** on σ is required for mathematical reasons. It is worthwhile to note that for Maxwellian molecules, σ is the constant function identically equal to 1. Hence hypothesis **A2** leads to more mathematical generality but still falls short of physical reality.

Before we proceed further, we recall the following: Since the function β that appears in (1.13) is held fixed and has been assumed to be bounded, we will set $\|\beta\|_\infty = 1$. We will also take $\|\sigma\|_\infty = 1$. Besides, we take the constant C that appears in the estimates (2.1) - (2.6) to be greater than 1 in order to avoid writing $C \vee 1$ in many of the estimates in this paper. A generic constant will be denoted by K though it may vary from line to line.

Let us fix a finite time $T > 0$, and denote the Skorohod space $\mathbb{D}([0, T]; \mathbb{R}^3)$ by $\mathbb{D}$. We consider it here equipped with the Skorohod topology. Given a probability measure μ on $\mathbb{D} \times \mathbb{D}$, let μ_t denote its marginal at time t. We define

$$\hat{\alpha}(z, v, \xi) := \alpha(z, v, \xi)\sigma(|z - v|)$$

for all $z, v \in \mathbb{R}^3$ and $\xi \in \Xi$ (the function α was defined in (1.20)). The main result of this paper is stated below.

Theorem 2.1. *Suppose that σ is in $C_b^\infty(\mathbb{R})$. Suppose hypothesis* **A** *hold. Let X_0 and Z_0 be $\mathbb{R}^3$- valued random variables with finite second moments. For any fixed $T > 0$, there exists a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P})$, an adapted process $(X_t, Z_t)_{t\in[0,T]}$ with values on $\mathbb{D} \times \mathbb{D}$, and a compensated random measure (crm) $\tilde{N}^\mu$, with μ being the law of the stochastic process (X, Z), satisfying a.s. the following stochastic equation for $t \in [0, T]$:*

$$\begin{aligned} Z_t = Z_0 &+ \int_0^t \int_{\mathbb{D}\times\mathbb{D}\times\Xi\times[0,1]} \alpha(Z_s, v_s, \xi) 1_{[0,\, \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)]}(r) d\tilde{N}^\mu \\ &+ \int_0^t \int_{\mathbb{D}\times\mathbb{D}\times\Xi} \hat{\alpha}(Z_s, v_s, \xi)\beta(|X_s - y_s|) d\mu Q(d\xi) ds \end{aligned} \tag{2.7}$$

and

$$X_t = X_0 + \int_0^t Z_s ds, \tag{2.8}$$

where $d\tilde{N}^\mu := \tilde{N}^\mu(dy, dv, d\xi, dr, ds)$. For any $t \in [0, T]$, X_t and Z_t have finite second moments.

Proposition 2.2. *Let μ denote the law of the process $\{X_s, Z_s : 0 \leq s \leq T\}$, solving (2.7), (2.8). Then μ solves the Enskog equation (1.16) for any $\psi \in C_0^2(\mathbb{R}^6)$.*

Proof. As for any $t \in [0, T]$ X_t and Z_t have finite second moments, and due to the conditions (**A1**) and (2.4) we can apply the Itô formula to $(X_s, Z_s)_{s\in\mathbb{R}_+}$. In fact let t, $\Delta t > 0$, then

$$\begin{aligned}
&\psi(X_{t+\Delta t}, Z_{t+\Delta t})\\
&= \psi(X_t, Z_t) + \int_t^{t+\Delta t} (Z_s, \nabla_x \psi(X_s, Z_s))ds\\
&+ \int_t^{\Delta t} \int_{U_0\times[0,1]} \{\psi(X_s, Z_s + \alpha(Z_s, v_s, \xi)1_{[0,\, \sigma(|Z_s-v_s|)\beta(|X_s-y_s|)]}(r)) - \psi(X_s, Z_s)\\
&- \nabla_z \psi(X_s, Z_s)\alpha(Z_s, v_s, \theta, \phi)1_{[0,\, \sigma(|Z_s-v_s|)\beta(|X_s-y_s|)]}(r)\}\mu(dy, dv)Q(d\theta)d\phi dsdr +\\
&\int_t^{\Delta t} \int_{U_0} \alpha(Z_s, v_s, \theta, \phi), \nabla_z \psi(X_s, Z_s))\}\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\theta)d\phi ds\\
&+ M_t^{t+\Delta t}(\psi))\\
&= \psi(X_t, Z_t) + \int_t^{t+\Delta t} (Z_s, \nabla_x \psi(X_s, Z_s))ds\\
&+ \int_t^{\Delta t} \int_{U_0} \{\psi(X_s, Z_s + \alpha(Z_s, v_s, \xi)) - \psi(X_s, Z_s)\\
&- \nabla_z \psi(X_s, Z_s)\alpha(Z_s, v_s, \theta, \phi)\}\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\theta)d\phi dsdr +\\
&\int_t^{\Delta t} \int_{U_0} \alpha(Z_s, v_s, \theta, \phi), \nabla_z \psi(X_s, Z_s))\}\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\theta)d\phi ds\\
&+ M_t^{t+\Delta t}(\psi)
\end{aligned} \tag{2.9}$$

where $\{M_t^{t+\Delta t}(\psi)\}_{\Delta t\in(0,T]}\}$ is a martingale, for each $T \in \mathbb{R}$.

Taking the expectation $\mathbb{E}$ with respect to the measure $\mu(dx, dv)$ in (2.9), we get

$$\begin{aligned}
&\mathbb{E}[\psi(X_{t+\Delta t}, Z_{t+\Delta t}) - \psi(X_t, Z_t)]\\
&= \mathbb{E}[\int_t^{t+\Delta t} (Z_s, \nabla_x \psi(X_s, Z_s))ds]\\
&+ \mathbb{E}[\int_0^t \int_{U_0} \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)(\alpha(Z_s, v_s, \theta, \phi), \nabla_z \psi(X_s, Z_s))\mu(dy, dv)Q(d\theta)d\phi ds]\\
&+ \mathbb{E}[\int_0^t \int_{U_0} \{\psi(X_s, Z_s + \alpha(Z_s, v_s, \theta, \phi)) - \psi(X_s, Z_s)\\
&- (\nabla_z \psi(X_s, Z_s), \alpha(Z_s, v_s, \theta, \phi))\}\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\theta)d\phi ds]
\end{aligned}$$

Dividing by Δt and letting $\Delta t \to 0$ we obtain (1.16) by noting that μ is also the law of (Z, X). □

In Section 7 in Theorem 7.1 we will prove uniqueness of the law μ of the process $\{X_s, Z_s : 0 \leq s \leq T\}$, solving the McKean -Vlasov equation (2.7), (2.8) in the following sense: we prove that for any fixed t in the interval $[0,T]$, the t-marginal distribution of weak solutions of (2.7), (2.8) is unique within the class of Borel probability measures on $\mathbb{R}^6$ that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^6$.

The existence of a probability density for the time-marginals of the velocity is verified in the case where the initial condition is Gaussian, and is shown to be the density of an invariant measure in Section 8, Theorem 8.1.

3 Existence and uniqueness of a stochastic equation

Consider a given filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathrm{P})$ satisfying the usual conditions. Let $S_T := S^1_T(\mathbb{R}^d)$ denote the linear space of all adapted càdlàg processes $(X_t)_{t\in[0,T]}$ with values on $\mathbb{R}^d$ equipped with norm

$$\|X\|_{S^1_T} := \mathbb{E}[\sup_{s\in[0,T]} |X_s|]. \tag{3.1}$$

$S^1_T(\mathbb{R}^d)$ is a Banach space. This can be shown similar to the proof of Lemma 4.2.1, page 93 in [30]. We consider $S_T(\mathbb{R}^d)$ for $d = 6$.

Let $\mathbb{D}$ be equipped with the Skorohod topology, where $\mathbb{D}$ denotes $\mathbb{D}([0,T];\mathbb{R}^3)$. Given a probability measure λ on $\mathbb{D}\times\mathbb{D}$, let λ_t denote its marginal at time t. Let us assume

$$\int_0^T \int_{\mathbb{D}\times\mathbb{D}} (|v_t| + |y_t|)\lambda(dv, dy)dt \, < \infty \quad \forall T > 0. \tag{3.2}$$

Consider a Poisson random measure N^λ on $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathrm{P})$ with intensity measure $\lambda_t(dy, dv)Q(d\xi)drdt$ on the Borel subsets of $\mathbb{D}\times\mathbb{D}\times(0,\pi]\times[0,2\pi)\times[0,1]\times[0,T]$. We denote by $\tilde{N}^\lambda$ the corresponding compensated Poisson random measure. From the condition (3.2) and (2.6), and the hypothesis $\|\sigma\|_\infty = \|\beta\|_\infty = 1$, it follows that

$$\int_0^T \int_{\mathbb{D}\times\mathbb{D}\times\Xi} |\alpha(z, v_t, \xi)|\sigma(|z - v_t|)\beta(|x - y_t|)\lambda(dv, dy)dtQ(d\xi) \, < \infty\,, \tag{3.3}$$

$\forall z \in \mathbb{R}^3, x \in \mathbb{R}^3, \forall T > 0$.

Let us use the following notation:

$$U_0 = \mathbb{D}^2 \times \Xi$$

$$U = \mathbb{D}^2 \times \Xi \times [0,1]$$

where $\mathbb{D}^2 := \mathbb{D}\times\mathbb{D}$.

Theorem 3.1. *Let (Z_0, X_0) be a random vector with values on $\mathbb{R}^3 \times \mathbb{R}^3$ with*

$$\mathbb{E}[|Z_0|] < \infty, \quad \mathbb{E}[|X_0|] < \infty\,, \tag{3.4}$$

and assume (3.2)*. Then for all $T > 0$ there exists a unique strong solution of the stochastic equation*

$$\begin{aligned} Z_t^\lambda &= Z_0 \\ &\quad + \int_0^t \int_U \alpha(Z_s^\lambda, v_s, \xi) 1_{[0,\, \sigma(|Z_s^\lambda - v_s|)\beta(|X_s^\lambda - y_s|)]}(r) d\tilde{N}^\lambda \\ &\quad + \int_{U_0} \hat{\alpha}(Z_s^\lambda, v_s, \xi)\beta(|X_s^\lambda - y_s|)\lambda(dydv)Q(d\xi)ds \end{aligned} \tag{3.5}$$

$$X_t^\lambda = X_0 + \int_0^t Z_s^\lambda ds, \tag{3.6}$$

on S_T^1, where $d\tilde{N}^\lambda$ denotes $\tilde{N}^\lambda(dy, dv, d\xi, dr, ds)$.

We first introduce some notation and preliminary results.

Let $T > 0$ and $(Z, X)_{t \in [0,T]}$ be an adapted process with values in $\mathbb{D}^2$.

Lemma 3.2. *Assume $(Z, X)_{t\in[0,T]} \in S_T^1$. Then the stochastic integrals $(I(Z))_{t\in[0,T]}$ and $(\hat{I}(Z))_{t\in[0,T]}$, with*

$$I(Z)_t := \int_0^t \int_U \alpha(Z_{s_-}, v_s, \xi) 1_{[0,\, \sigma(|Z_{s_-} - v_s|)\beta(|X_s - y_s|)]}(r) dN^\lambda, \tag{3.7}$$

and

$$\hat{I}(Z)_t := \int_0^t \int_U |\alpha(Z_{s_-}, v_s, \xi)| 1_{[0,\, \sigma(|Z_{s_-} - v_s|)\beta(|X_s - y_s|)]}(r) dN^\lambda, \tag{3.8}$$

are well defined, and there exist constants $K > 0$ and $M_T > 0$ satisfying

$$\mathbb{E}[\hat{I}(Z))_T] \le K \int_0^T \mathbb{E}[\sup_{s\in[0,t]} |Z_s|]dt + M_T. \tag{3.9}$$

Proof. We need to prove only inequality (3.9). It then follows that the stochastic integrals $(\hat{I}(Z)))_{t\in[0,T]}$ and $(I(Z)))_{t\in[0,T]}$ are well defined (see Section 3.5, in particular Lemma 3.5.20 of [30], or Theorem 4.12 [39]).

Using (2.6) it follows

$$\begin{aligned}
\mathbb{E}[\hat{I}(Z)_T] &= \int_0^t \int_{U_0} \mathbb{E}[|\alpha(Z_s, v_s, \xi)|\sigma(|Z_{s_-} - v_s|)\beta(|X_s^\lambda - y_s|)]\lambda(dydv)Q(d\xi)ds \\
&\leq C\left(\int_0^T \mathbb{E}[|Z_s|]ds + \int_0^T \int_{\mathbb{D}\times\mathbb{D}} |v_s|\lambda(dv, dy)ds\right) \\
&\leq C\left(\int_0^T \mathbb{E}[\sup_{s\in[0,t]} |Z_s|]dt + \int_0^T \int_{\mathbb{D}\times\mathbb{D}} |v_s|\lambda(dv, dy)ds\right) < \infty
\end{aligned}$$

□

Let $(SZ, SX)_{t\in[0,T]}$ denote the process defined through

$$SZ_t := Z_0 + I(Z)_t \quad \text{and} \quad SX_t := X_0 + \int_0^t Z_s ds.$$

If the random vector (Z_0, X_0) satisfies (3.4) and $(Z, X) \in S_T^1$ then $(SZ, SX) \in S_T^1$.

Indeed

$$\sup_{t\in[0,T]} |SZ_t| \leq \hat{I}(Z)_T + |Z_0| \quad a.s. \tag{3.10}$$

and hence

$$\mathbb{E}[\sup_{t\in[0,T]} |SZ_t|] \leq \mathbb{E}[\hat{I}(Z)_T] + \mathbb{E}[|Z_0|] \tag{3.11}$$

The statement follows from the estimate (3.9).

Lemma 3.3. *For any $T > 0$ fixed, there exists a constant $K > 0$, such that for all $n \in \mathbb{N}$ and all $(Z, X) \in S_T^1$ satisfying $\sup_{s\in[0,T]} |Z_s| \leq n$, $\sup_{s\in[0,T]} |Z'_s| \leq n$, the following inequality holds*

$$\begin{aligned}
&\int_0^t \int_U |\alpha(Z_s, v_s, \xi)1_{[0,\, \sigma(|Z_s - v_s|)\beta([|X_s - y_s|]}(r) - \alpha(Z'_s, v_s, \xi)1_{[0,\, \sigma(|Z'_s - v_s|)\beta(|X'_s - y_s|)]}(r)| \\
&\qquad \lambda(dv, dy)Q(d\xi)drds \\
&\qquad \leq \int_0^t L_n(|Z_s - Z'_s| + |X_s - X'_s|)ds \quad P - a.s.,
\end{aligned}$$

with $L_n = Kn$.

Proof.

$$\begin{aligned}
&\int_0^t \int_U |\alpha(Z_s, v_s, \xi)1_{[0,\, \sigma(|Z_s - v_s|)\beta([|X_s - y_s|]}(r) - \alpha(Z'_s, v_s, \xi)1_{[0,\, \sigma(|Z'_s - v_s|)\beta(|X'_s - y_s|)]}(r)| \\
&\qquad \lambda(dv, dy)Q(d\xi)drds \leq I + II \quad P - a.s., \quad \text{with}
\end{aligned}$$

$$
\begin{aligned}
I &:= \int_0^t \int_U |\alpha(Z_s, v_s, \xi) - \alpha(Z'_s, v_s, \xi)| 1_{[0,\, \sigma(|Z_s - v_s|)\beta([|X_s - y_s|]]}(r)\lambda(dv, dy)Q(d\xi)drds \\
&\leq \int_0^t C|Z_s - Z'_s|ds
\end{aligned}
$$

where the last inequality follows from (2.2), and

$$
\begin{aligned}
&II := \int_0^t \int_U |\alpha(Z'_s, v_s, \xi)| \times \\
&\{1_{[0,\, \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)]}(r) - 1_{[0,\, \sigma(|Z'_s - v_s|)\beta(|X'_s - y_s|)]}(r)|\}\lambda(dv, dy)Q(d\xi)drds \\
&\leq \int_0^t \int_{U_0} |\alpha(Z'_s, v_s, \xi)| |\max(\sigma(|Z_s - v_s|)\beta(|X_s - y_s|), \sigma(|Z'_s - v_s|)\beta(|X'_s - y_s|)) \\
&- \min(\sigma(|Z_s - v_s|)\beta(|X_s - y_s|), \sigma(|Z'_s - v_s|)\beta(|X'_s - y_s|)|)\lambda(dv, dy)Q(d\xi)ds = \\
&\int_0^t \int_{U_0} |\alpha(Z'_s, v_s, \xi)| \, |\sigma(|Z_s - v_s|)\beta(|X_s - y_s|) - \sigma(|Z'_s - v_s|)\beta(|X'_s - y_s|)|\lambda(dv, dy)Q(d\xi)ds
\end{aligned}
$$

Using that σ and β are Lipschitz continuous functions bounded by 1, as well as (2.6), we get that there exists a constant $K > 0$, such that

$$
\begin{aligned}
II &\leq \int_0^t \int_{U_0} |\alpha(Z'_s, v_s, \xi)| \\
&\times (||Z'_s - v_s| - |Z_s - v_s|| + ||X'_s - y_s| - |X_s - y_s||)\lambda(dv, dy)Q(d\xi)ds \\
&\leq K \int_0^t \int_{U_0} |\alpha(Z'_s, v_s, \xi)|(|Z'_s - Z_s| + |X'_s - X_s|)\lambda(dv, dy)Q(d\xi)ds \\
&\leq K \int_0^t \int_{\mathbb{D}^2} (|Z'_s| + |v_s|)|(|Z'_s - Z_s| + |X'_s - X_s|)\lambda(dv, dy)ds
\end{aligned}
$$

□

In the next Lemma we will use the local Lipschitz condition stated in Lemma 3.3 to prove a local contraction property of S on S^1_T.

Lemma 3.4. *For each $n \in \mathbb{N}$ there exists constant $L_n > 0$, such that*

$$
\mathbb{E}[\sup_{s\in[0,t]} |SZ_s - SZ'_s|] \leq L_n \int_0^t \mathbb{E}[\sup_{s'\in[0,s]} \{|Z_{s'} - Z'_{s'}| + |X_{s'} - X'_{s'}|\}]ds
$$
$$
\forall (Z, X)_{s\in[0,T]}\,, (Z', X')_{s\in[0,T]} \in S^1_T, \text{ with } \sup_{s\in[0,T]} |Z_s| \leq n\,, \sup_{s\in[0,T]} |Z'_s| \leq n.
$$

Proof.

$$
\begin{aligned}
&\mathbb{E}[\sup_{s\in[0,t]} |SZ_s - SZ'_s|] \le \\
&\mathbb{E}[\sup_{s\in[0,t]} \int_0^s \int_U |\alpha(Z_{s'\_}, v_{s'}, \xi) 1_{[0,\ \sigma(|Z_{s'\_} - v_{s'}|)\beta(|X_{s'} - y_{s'}|)]}(r) \\
&- \alpha(Z'_{s'\_}, v_{s'}, \xi) 1_{[0,\ \sigma(|Z'_{s'\_} - v_{s'}|)\beta(|X'_{s'} - y_{s'}|)]}(r)| dN_\lambda] \\
&\le \mathbb{E}[\int_0^t \int_U |\alpha(Z_{s'\_}, v_{s'}, \xi) 1_{[0,\ \sigma(|Z_{s'\_} - v_{s'}|)\beta(|X_{s'} - y_{s'}|)]}(r) \\
&- \alpha(Z'_{s'\_}, v_{s'}, \xi) 1_{[0,\ \sigma(|Z'_{s'\_} - v_{s'}|)\beta(|X'_{s'} - y_{s'}|)]}(r)| dN_\lambda] \\
&= \mathbb{E}[\int_0^t \int_U |\alpha(Z_{s'}, v_{s'}, \xi) 1_{[0,\ \sigma(|Z_{s'} - v_{s'}|)\beta(|X_{s'} - y_{s'}|)]}(r) \\
&- \alpha(Z'_{s'}, v_{s'}, \xi) 1_{[0,\ \sigma(|Z'_{s'} - v_{s'}|)\beta(|X'_{s'} - y_{s'}|)]}(r)| \lambda(dydv) Q(d\xi) dr ds']
\end{aligned}
$$

$$
\begin{aligned}
&\le L_n \int_0^t \mathbb{E}[|Z'_{s'} - Z_{s'}| + |X'_{s'} - X_{s'}|] ds' \\
&\le L_n \int_0^t \mathbb{E}[\sup_{s'\in[0,s]} \{|Z'_{s'} - Z_{s'}| + |X'_{s'} - X_{s'}|\}] ds
\end{aligned}
$$

where we have used Lemma 3.3. □

In the proof of the next theorem we will use the local contraction property in Lemma 3.4 to prove existence and uniqueness of a modification of the stochastic equation defined through (3.5), (3.6). The modified stochastic equation satisfies global growth and Lipschitz conditions.

Let $j \in \mathbb{N}$, $B_j := \{z \in \mathbb{R}^3 : |z| \le j\}$ and

$$
\alpha_j(z, v, \xi) := \frac{\alpha(z, v, \xi)}{1 + d(z, B_j)} \tag{3.12}
$$

where $d(z, B_j)$ denotes the distance of $z \in \mathbb{R}^3$ from B_j.

Theorem 3.5. *Let the random vector (Z_0, X_0) satisfy (3.4). For all $T > 0$ there exists a unique solution on S_T^1 of the stochastic equation*

$$
\begin{aligned}
Z_t^{\lambda,j} &= Z_0 + \int_0^t \int_U \alpha_j(Z_{s-}^{\lambda,j}, v_s, \xi) \\
&\quad \times 1_{[0,\ \sigma(|Z_{s-}^{\lambda,j} - v_s|)\beta(|X_s^{\lambda,j} - y_s|)]}(r) N^\lambda(dy, dv, d\xi, dr, ds) \qquad (3.13) \\
X_t^{\lambda,j} &= X_0 + \int_0^t Z_s^{\lambda,j} ds. \qquad (3.14)
\end{aligned}
$$

It then follows directly the statement of the following corollary:

Corollary 3.6. *Let the random vector (Z_0, X_0) satisfy* (3.4). *For all $T > 0$ there exists a unique solution on S^1_T of the stochastic equation*

$$
\begin{aligned}
Z_t^{\lambda,j} &= Z_0 \\
&+ \int_0^t \int_U \alpha_j(Z_s^{\lambda,j}, v_s, \xi) 1_{[0,\, \sigma(|Z_s^{\lambda,j} - v_s|)\beta(|X_s^{\lambda,j} - y_s|)]}(r) d\tilde{N}^\lambda \\
&+ \int_{U_0} \alpha_j(Z_s^{\lambda,j}, v_s, \xi)\sigma(|Z_s^{\lambda,j} - v_s|)\beta(|X_s^{\lambda,j} - y_s|) d\lambda(dydv) Q(d\xi) ds \qquad (3.15)\\
X_t^{\lambda,j} &= X_0 + \int_0^t Z_s^{\lambda,j} ds. \qquad (3.16)
\end{aligned}
$$

Proof. If the stochastic integrals in the stochastic equation (3.13), (3.14) are well-defined then the stochastic equation (3.13), (3.14) is equivalent to the stochastic equation (3.15), (3.16). (See e.g. Chapter 5 [30]). $\square$

Proof of Theorem 3.5

Proof. We start by remarking that

$$\frac{|z|}{1 + d(z, B_j)} \leq \min(j, |z|)$$

and there exists a constant $K_j > 0$, such that

$$\left| \frac{z}{1 + d(z, B_j)} - \frac{z'}{1 + d(z', B_j)} \right| \leq K_j |z - z'|. \qquad (3.17)$$

Assume $(Z, X)_{t \in [0,T]} \in S^1_T$. As $|\alpha_j(z, v, \xi)| \leq |\alpha(z, v, \xi)|$ it follows from Lemma 3.2 that the stochastic integrals $(I(Z))_{t \in [0,T]}$ and $(\hat{I}(Z))_{t \in [0,T]}$, with

$$\hat{I}_j(Z)_t := \int_0^t \int_U |\alpha_j(Z_{s_-}, v_s, \xi)| 1_{[0,\, \sigma(|Z_{s\_} - v_s|)\beta(|X_s - y_s|)]}(r) dN^\lambda,$$

and

$$I_j(Z))_t := \int_0^t \int_U \alpha_j(Z_{s_-}, v_s, \xi) 1_{[0,\, \sigma(|Z_{s\_} - v_s|)\beta(|X_s - y_s|)]}(r) dN^\lambda,$$

are well defined. Moreover, using $(\hat{I}_j(Z))_t \leq (\hat{I}(Z))_t \ \forall t \in [0, T]$ and (3.9), it follows that

$$\mathbb{E}[\hat{I}_j(Z))_T] \leq K \int_0^T \mathbb{E}[\sup_{s \in [0,t]} |Z_s|] dt + M_T\,, \qquad (3.18)$$

with $K > 0$, $M_T > 0$. Let $(S_jZ, S_jX)_{t\in[0,T]}$ denote the process defined through

$$S_jZ_t := Z_0 + I_j(Z)_t \ \text{ and } \ S_jX_t := X_0 + \int_0^t Z_s ds\,.$$

If the random vector (Z_0, X_0) satisfies (3.4), then

$$\mathbb{E}[\sup_{t\in[0,T]} |S_jZ_t|] \leq \mathbb{E}[\hat{I}_j(Z)_T] + \mathbb{E}[|Z_0|] \tag{3.19}$$

and, due to the growth condition (3.18), $(S_jZ, S_jX)_{t\in[0,T]} \in S^1_T$.

From (3.17) and Lemma 3.3 it follows that there is a constant $L_j > 0$ such that

$$\begin{aligned}\int_0^t \int_U &|\alpha_j(z, v_s, \xi)1_{[0,\, \sigma(|z-v_s|)\beta([|x-y_t|]}(r) - \alpha_j(z', v_s, \xi)1_{[0,\, \sigma(|z'-v_s|)\beta(|x-y_s|)]}(r)| \\ &\lambda(dv, dy)Q(d\xi)drds \\ &\leq L_j(|z-z'| + |x-x'|) \quad \text{for all } \ z, z', x, x' \in \mathbb{R}^3,\end{aligned}$$

Similar to Lemma 3.4 it can then be proven that the following inequality holds:

$$\mathbb{E}[\sup_{s\in[0,t]} |S_jZ_s - S_jZ'_s|] \leq L_j \int_0^t \mathbb{E}[\sup_{s'\in[0,s]} \{|Z_{s'} - Z'_{s'}| + |X_{s'} - X'_{s'}|\}]ds$$

It follows that there exists $n \in \mathbb{N}$ such that $(S^n_jZ, S^n_jX)_{t\in[0,T]}$ is a contraction from S^1_T to S^1_T. It then follows, that the mapping S_j has a unique fixed point on S^1_T. □

Proof of Theorem 3.1:

Proof. It is sufficient to prove that for all $T > 0$ there exists a unique process $(Z, X)_{t\in[0,T]} \in S^1_T$ satisfying a.s. the following stochastic equation

$$\begin{aligned}Z^\lambda_t &= Z_0 + \\ &\int_0^t \int_U \alpha(Z^\lambda_s, v_s, \xi)1_{[0,\, \sigma(|Z^\lambda_s-v_s|)\beta(|X^\lambda_s-y_s|)]}(r)N^\lambda(dy, dv, d\xi, dr, ds)\end{aligned} \tag{3.20}$$

$$X^\lambda_t = X_0 + \int_0^t Z^\lambda_s ds, \tag{3.21}$$

(3.20), (3.21) is then equivalent to (3.5), (3.6).

We will follow the strategy of the proof of Theorem 4.3.1 in [30]. Let $(Z^{\lambda,j}, X^{\lambda,j})_{t\in[0,T]} \in S^1_T$ be the unique solution of (3.13), (3.14). Let

$$\tau_j := \inf\{t \in [0, T] : |Z^{\lambda,j}_t| > j\}$$

By uniqueness of the solution of (3.13), (3.14) it follows that

$$Z_t^{\lambda,j} = Z_t^{\lambda,j+1} \quad \text{a.s., for } t \in [0,\tau_j]\,,$$

giving $\mathrm{P}(\tau_j \leq \tau_{j+1}) = 1 \ \forall j \in \mathbb{N}$.

We will prove

$$\mathrm{P}(\cup_{j\in\mathbb{N}}\{\tau_j = T\}) = 1. \tag{3.22}$$

It then follows that the a.s. limit process $(Z^\lambda, X^\lambda)_{t\in[0,T]} = \lim_{j\to\infty}(Z^{\lambda,j}, X^{\lambda,j})_{t\in[0,T]}$ is the solution of (3.20), (3.21), and hence (3.5), (3.6).

It follows from (3.19) and (3.18) that

$$\mathbb{E}[\sup_{t\in[0,T]} |Z_t^{\lambda,j}|] \leq K \int_0^T \mathbb{E}[\sup_{s\in[0,t]} |Z_s^{\lambda,j}|]dt + M_T + \mathbb{E}[|Z_0|] \tag{3.23}$$

so that by Gronwall's Lemma

$$\mathbb{E}[\sup_{t\in[0,T]} |Z_t^{\lambda,j}|] \leq \exp KT(M_T + \mathbb{E}[|Z_0|]) \tag{3.24}$$

It follows

$$\begin{aligned}\mathrm{P}(\tau_j < T) &= \mathrm{P}(\sup_{t\in[0,T]} |Z_t^{\lambda,j}| > j)\\ &\leq \frac{1}{j}\mathbb{E}[\sup_{t\in[0,T]} |Z_t^{\lambda,j}|] \leq \frac{1}{j}\exp KT(M_T + \mathbb{E}[|Z_0|])\end{aligned}$$

so that

$$\mathrm{P}(\cap_{j\in\mathbb{N}}\{\tau_j < T\}) = \lim_{j\to\infty} \mathrm{P}(\{\tau_j < T)\} = 0. \tag{3.25}$$

$\square$

4 Tightness

In this section, we formulate an approximating sequence $\{Z^{(n)}, X^{(n)})$ for the McKean-Vlasov limit, and prove the tightness of this sequence by Kurtz's criterion on the Skorohod space $\mathbb{D}\times\mathbb{D}$ with the Skorohod topology.

Define the processes

$$\begin{aligned}Z_t^{(0)} &= Z_0\\ X_t^{(0)} &= X_0 + Z_0 t\end{aligned}$$

for all $t \in [0,T]$. Let $\mu^{(0)} := \mathcal{L}(Z^{(0)}, X^{(0)})$. Consider a Poisson random measure $N_{\mu^{(0)}}$ on $U \times [0,T]$ whose compensator measure is given by $d\mu^{(0)}Q(d\xi)drds$. Let $\tilde{N}_{\mu^{(0)}}$ denote the corresponding compensated Poisson random measure (cPrm). By the square integrability of Z_0 and X_0, one has

$$\mathbb{E}(\sup_{0\le t\le T}\left[|Z_t^{(0)}|^2 + |X_t^{(0)}|^2\right] < \infty\,.$$

This implies in particular that $\forall T > 0$ $(Z^{(0)}, X^{(0)}) \in S_T^2 \subset S_T^1$, where S_T^2 denotes here the Banach space of all adapted càdlàg processes $(X_t)_{t\in[0,T]}$ with values on $\mathbb{R}^6$ equipped with norm

$$\|X\|_{S_T^2} := (\mathbb{E}[\sup_{s\in[0,T]}|X_s|^2])^{1/2} \tag{4.1}$$

(see e.g. page 93 in [30]). In particular it implies that the measure $\mu^{(0)}$ is square integrable, and as a consequence satisfies the assumption (3.2). For all $n \ge 0$, define

$$\begin{aligned} Z_t^{(n+1)} &= Z_0 + \int_0^t\int_U \alpha(Z_s^{(n+1)}, v_s, \xi)1_{[0,\sigma(|Z_s^{(n+1)}-v_s|)\beta(|X_s^{(n+1)}-y_s|)]}(r)d\tilde{N}_{\mu^{(n)}} \\ &\quad + \int_0^t\int_{U_0}\hat{\alpha}(Z_s^{(n+1)}, v_s, \xi)\beta(|X_s^{(n+1)}-y_s|)d\mu^{(n)}Q(d\xi)ds \end{aligned} \tag{4.2}$$

and

$$X_t^{(n+1)} = X_0 + \int_0^t Z_s^{(n+1)}ds. \tag{4.3}$$

Here, $\mu^{(n)}$ is the law of $(Z^{(n)}, X^{(n)})$, and $\tilde{N}_{\mu^{(n)}}$ is the cPrm with compensator measure given by $d\mu^{(n)}Q(d\xi)drds$. Taking $n = 0$, it follows from Theorem 3.1 that there exists a unique solution of $(Z^{(1)}, X^{(1)})$ in S_T^1 solving (4.2), (4.3). Moreover,

$$\begin{aligned} &\mathbb{E}(|Z_t^{(1)}|^2) \\ &\le 3\mathbb{E}\Bigg[(|Z_0|^2) + \int_0^t\int_{U_0}|\alpha(Z_s^{(1)}, v_s, \xi)|^2\sigma(|Z_s^{(1)}-v_s|)\beta(|X_s^{(1)}-y_s|)d\mu^{(0)}Q(d\xi)ds \\ &+|\int_0^t\int_{U_0}\hat{\alpha}(Z_s^{(1)}, v_s, \xi)\beta(|X_s^{(1)}-y_s|)d\mu^{(0)}Q(d\xi)ds|^2\Bigg] \\ &\le 3\mathbb{E}\Bigg[(|Z_0|^2) + \int_0^t\int_{U_0}|\alpha(Z_s^{(1)}, v_s, \xi)|^2 d\mu^{(0)}Q(d\xi)ds\ + \\ &t\int_0^t\int_{\mathbb{D}^2}|\int_\Xi \alpha(Z_s^{(1)}, v_s, \xi)Q(d\xi)|^2 d\mu^{(0)}\int_{\mathbb{D}^2}|\sigma(|Z_s^{(1)}-v_s|)\beta(|X_s^{(1)}-y_s|)|^2 d\mu^{(0)}ds\Bigg] \end{aligned}$$

by Cauchy-Schwarz inequality; continuing, by a use of (2.1), (2.2), and (2.4), one obtains

$$\leq 3\mathbb{E}\left[(|Z_0|^2) + C\int_0^t [|Z_s^{(1)}|^2 + |v_s|^2] d\mu^{(0)} ds + Ct\int_0^t [|Z_s^{(1)}|^2 + |v_s|^2] d\mu^{(0)} ds\right]$$
$$\leq 3\left[\mathbb{E}|Z_0|^2 + C(1+T)\int_0^t \mathbb{E}|Z_s^{(1)}|^2 ds + Ct(1+T)\mathbb{E}|Z_0|^2\right]$$

Hence by the Gronwall inequality, we obtain

$$\mathbb{E}(|Z_t^{(1)}|^2) \leq K_1 E(|Z_0|^2)(1 + K_2 t) \tag{4.4}$$

with $K_1 = 3e^{3CT(1+T)}$ and $K_2 = C(1+T)$. It then follows for $n = 1$ that the law $\mu^{(1)}$ satisfies the assumption (3.2), so that there exists a unique strong solution $(Z^{(2)}, X^{(2)})$ solving (4.2), (4.3). Along similar lines, one obtains

$$\mathbb{E}(|Z_t^{(2)}|^2)$$
$$\leq 3\mathbb{E}\left[|Z_0|^2 + C\int_0^t (|Z_s^{(2)}|^2 + (|Z_s^{(1)}|^2)) ds + CT\int_0^t (|Z_s^{(2)}|^2 + (|Z_s^{(1)}|^2)) ds\right]$$

so that by the Gronwall inequality,

$$\mathbb{E}|Z_t^{(2)}|^2 \leq K_1 E(|Z_0|^2)(1 + K_1 K_2 t + \frac{(K_1 K_2 t)^2}{2}) \tag{4.5}$$

Further iterations result that (4.2), (4.3) has a unique strong solution $(Z^{(n)}, X^{(n)})$ and the bound

$$\mathbb{E}|Z_t^{(n)}|^2 \leq K_1 E(|Z_0|^2) \sum_{i=0}^{n} \frac{(K_1 K_2 t)^i}{i!},$$

so that for all $n \in \mathbb{N}$, we have

$$\mathbb{E}|Z_t^{(n)}|^2 \leq K_1 e^{K_1 K_2 t} \mathbb{E}|Z_0|^2. \tag{4.6}$$

By the definition of $X^{(n)}$, we obtain an upper bound uniformly in n for $\mathbb{E}\left[|Z_t^{(n)}|^2 + |X_t^{(n)}|^2\right]$. This is uniform boundedness of the sequence at each fixed $t \in [0,T]$. Using the Burkholder-Davis-Gundy inequality, and proceeding exactly as above, one obtains an upper bound K uniformly in n for $\mathbb{E}\left[\sup_{0\leq t\leq T}(|Z_t^{(n)}|^2 + |X_t^{(n)}|^2)\right]$. It follows in particular that $(Z^{(n)}, X^{(n)}) \in S_T^2 \subset S_T^1$. As S_T^p, $p = 1, 2$ are not separable Banach spaces, tightness has however to be proven on the Skorohod space $\mathbb{D} \times \mathbb{D}$ with the Skorohod topology:

in order to verify the second requirement in Kurtz's criterion, we consider for any fixed $\delta > 0$,

$$
\begin{aligned}
&\mathbb{E}\left[|Z_{t+\delta}^{(n)} - Z_t^{(n)}|^2 \,|\, \mathcal{F}_t\right] \\
&\le 2\mathbb{E}\left[\{|\int_t^{t+\delta}\int_U \alpha(Z_s^{(n)}, v_s, \xi) 1_{[0,\sigma(|Z_s^{(n)}-v_s|)\beta(|X_s^{(n)}-y_s|)]}(r) d\tilde{N}_{\mu^{(n-1)}}|^2 \right. \\
&\left. +|\int_t^{t+\delta}\int_{U_0} \hat{\alpha}(Z_s^{(n)}, v_s, \xi)\beta(|X_s^{(n)} - y_s|)Q(d\xi)d\mu^{(n-1)}ds|^2\} \,|\mathcal{F}_t\right] \\
&\le 2\mathbb{E}\left[\{|\int_t^{t+\delta}\int_{U_0} |\alpha(Z_s^{(n)}, v_s, \xi)|^2 \sigma(|Z_s^{(n)} - v_s|)\beta(|X_s^{(n)} - y_s|)d\mu^{(n-1)}Q(d\xi)ds \right. \\
&+ \delta\int_t^{t+\delta}\int_{\mathbb{D}^2} |\int_{\Xi} \alpha(Z_S^{(n)}, v_s, \xi)Q(d\xi)|^2 d\mu^{(n-1)} \times \\
&\left. \int_{\mathbb{D}^2} \sigma^2(|Z_s^{(n)} - v_s|)\beta^2(|X_s^{(n)} - y_s|)d\mu^{(n-1)}ds\} \,|\, \mathcal{F}_t\right].
\end{aligned}
$$

We will call the above expression on the right side as $2\mathbb{E}(A_\delta^{(n)} \,|\, \mathcal{F}_t)$. Then,

$$
\begin{aligned}
\mathbb{E}(A_\delta^{(n)} \,|\, \mathcal{F}_t) &\le 2C(1+\delta)\int_t^{t+\delta} \mathbb{E}(|Z_s^{(n)}|^2 + |Z_s^{(n-1)}|^2)ds \\
&\le K\delta
\end{aligned}
$$

for a suitable constant $K > 0$ which is independent of n. Hence,

$$
\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{E}(A_\delta^{(n)}) = 0. \tag{4.7}
$$

From (4.6) and (4.7), we conclude that $\{Z^{(n)}\}$ is tight in $\mathbb{D}$. By the definition of $X^{(n)}$, it follows that $\{Z^{(n)}, X^{(n)})$ is tight in $\mathbb{D}^2$.

5 Distance Between Successive Approximations

In this section, we give a result on the closeness of the measures $\mu^{(n+1)}$ and $\mu^{(n)}$ as $n \to \infty$.

Proposition 5.1. *Suppose that σ is in $C_b^\infty(\mathbb{R})$. Then for all $h \in C_b^\infty(\mathbb{R}^6)$ and for any fixed $t \in [0, T]$, we have*

$$
\mathbb{E}[h(Z_t^{(n+1)}, X_t^{(n+1)}) - h(Z_t^{(n)}, X_t^{(n)})] \to 0 \tag{5.1}
$$

as $n \to \infty$.

Proof. By the Itô formula, we write $\mathbb{E}[h(Z_t^{(n+1)}, X_t^{(n+1)}) - h(Z_t^{(n)}, X_t^{(n)})]$ as $A_1 + A_2$ where

$$
A_1 = \mathbb{E}[\int_0^t \{\nabla_x h(Z_s^{(n+1)}, X_s^{(n+1)}) \cdot Z_s^{(n+1)} - \nabla_x h(Z_s^{(n)}, X_s^{(n)}) \cdot Z_s^{(n)}\}ds].
$$

and

$$
\begin{aligned}
A_2 = \mathbb{E}[\int_0^t \int_{U_0} & \{h(Z_s^{(n+1)} + \alpha(Z_s^{(n+1)}, v_s, \xi), X_s^{(n+1)}) - h(Z_s^{(n+1)}, X_s^{(n+1)})\} \\
& \times \sigma(|Z_s^{(n+1)} - v_s|)\beta(|X_s^{(n+1)} - y_s|)d\mu^{(n)}Q(d\xi)ds] \\
& - \mathbb{E}[\int_0^t \int_{U_0} \{h(Z_s^{(n)} + \alpha(Z_s^{(n)}, v_s, \xi), X_s^{(n)}) - h(Z_s^{(n)}, X_s^{(n)})\} \\
& \times \sigma(|Z_s^{(n)} - v_s|)\beta(|X_s^{(n)} - y_s|)d\mu^{(n-1)}Q(d\xi)ds]
\end{aligned}
$$

By tightness, given any $\epsilon > 0$, there exists an $R > 0$ such that

$$
P\{\sup_{0 \le t \le T} |\max\{|Z_t^{(j)}| + |X_t^{(j)}| : j = n-1, n, n+1\} > R\} < \epsilon.
$$

This is a statement about the measures $\mu^{(n+1)}, \mu^{(n)}$ and $\mu^{(n-1)}$. Let B_R denote the R-ball in $\mathbb{R}^6$. First, we will deal with A_1. Clearly, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
& |\int_0^t \int_{B_R^C} \{\nabla_x h(z_s^{(n+1)}, x_s^{(n+1)}) \cdot z_s^{(n+1)}\} d\mu^{(n+1)} ds - \int_0^t \int_{B_R^C} \{\nabla_x h(z_s^{(n)}, x_s^{(n)}) \cdot z_s^{(n)}\} d\mu^{(n)}\} ds| \\
& \le \|\nabla_x h\|_\infty \sqrt{\epsilon t} \left[\int_{B_R^C} \int_0^t \{|z_s^{(n+1)}|^2 d\mu^{(n+1)} ds + |z_s^{(n)}|^2 d\mu^{(n)}\} ds\} \right]^{1/2} \\
& \le K_t \sqrt{\epsilon}
\end{aligned}
$$

for a suitable constant $K_t > 0$. Restricted to B_R, note that the function $g(z, x) = \nabla_x h(z, x) \cdot z$ can be uniformly approximated by functions in $C_b^\infty(\mathbb{R}^6)$, so that

$$
A_1 \le K\sqrt{\epsilon} + \int_0^t \sup_{\phi \in C_b^\infty(\mathbb{R}^6)} |\int_{\mathbb{R}^6} \phi(z, x)\{d\mu_s^{(n+1)} - d\mu_s^{(n)}\}| ds. \tag{5.2}
$$

In order to bound A_2, we will split A_2 into two parts so that $A_2 \le I_1 + I_2$ where

$$
\begin{aligned}
I_1 = \mathbb{E}[\int_0^t \int_{U_0} & \{h(Z_s^{(n+1)} + \alpha(Z_s^{(n+1)}, v_s, \xi), X_s^{(n+1)}) - h(Z_s^{(n+1)}, X_s^{(n+1)})\} \\
& \times \sigma(|Z_s^{(n+1)} - v_s|)\beta(|X_s^{(n+1)} - y_s|)d\mu^{(n)}Q(d\xi)ds] \\
& - \mathbb{E}[\int_0^t \int_{U_0} \{h(Z_s^{(n)} + \alpha(Z_s^{(n)}, v_s, \xi), X_s^{(n)}) - h(Z_s^{(n)}, X_s^{(n)})\} \\
& \times \sigma(|Z_s^{(n)} - v_s|)\beta(|X_s^{(n)} - y_s|)d\mu^{(n)}Q(d\xi)ds],
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 = \mathbb{E}[\int_0^t \int_{U_0} & \{h(Z_s^{(n)} + \alpha(Z_s^{(n)}, v_s, \xi), X_s^{(n)}) - h(Z_s^{(n)}, X_s^{(n)})\} \\
& \times \sigma(|Z_s^{(n)} - v_s|)\beta(|X_s^{(n)} - y_s|)d\mu^{(n)}Q(d\xi)ds] \\
& - \mathbb{E}[\int_0^t \int_{U_0} \{h(Z_s^{(n)} + \alpha(Z_s^{(n)}, v_s, \xi), X_s^{(n)}) - h(Z_s^{(n)}, X_s^{(n)})\} \\
& \times \sigma(|Z_s^{(n)} - v_s|)\beta(|X_s^{(n)} - y_s|)d\mu^{(n-1)}Q(d\xi)ds].
\end{aligned}
$$

To bound I_1, we will use the notation z, x instead of z_s, x_s in defining the function

$$
\psi(z,x) = \int_{\mathbb{R}^6 \times \Xi} \{h(z + \alpha(z,v,\xi), x) - h(z,x)\}\sigma(|z-v|)\beta(|x-y|)d\mu_s^{(n)}Q(d\xi)
$$

where s is fixed. Given $\epsilon > 0$, there exists an $R > 0$ such that for all n,

$$
\mathbb{E}\int_0^t 1_{\{|Z_s^{(n)}|>R\}}|Z_s^{(n)}|ds < \epsilon.
$$

A similar statement holds for $X^{(n)}$ in the place of $Z^{(n)}$. Therefore, we obtain

$$
\int_{B_R^C} |\psi(z,x)|(d\mu_s^{(n)} + \mu_s^{(n+1)}) < K\epsilon \tag{5.3}
$$

where K is a suitable constant, and B_R is the R-ball in $\mathbb{R}^6$.

Hence, we can focus our attention on I_1 when the processes are restricted to values in B_R. Next, let Ξ_δ denote the subset $(0,\delta] \times [0,\pi)$ of Ξ. Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all n and s,

$$
\begin{aligned}
& |\int_{\mathbb{R}^6}\int_{B_R \times \Xi_\delta} \{h(z + \alpha(z,v,\xi), x) - h(z,x)\}\sigma(|z-v|)\beta(|x-y|)\mu_s^{(n)}(dv,dy)Q(d\xi)| \\
& \qquad (d\mu_s^{(n)} + \mu_s^{(n+1)}) \\
& < \epsilon.
\end{aligned} \tag{5.4}
$$

With this estimate in hand, we observe that the function of (z,x) given by

$$
\int_{B_R \times (\delta, 2\pi] \times [0,\pi)} \{h(z + \alpha(z,v,\xi), x) - h(z,x)\}\sigma(|z-v|)\beta(|x-y|)d\mu_s^{(n)}Q(d\xi)
$$

is a function in $C_b^\infty(\mathbb{R}^6)$. This observation along with (5.3) and (5.4) yields

$$
|I_1| \leq K\epsilon + \int_0^t \sup_{\phi \in C_b^\infty(\mathbb{R}^6)} |\int_{\mathbb{R}^6} \phi(z,x)\{d\mu_s^{(n+1)} - d\mu_s^{(n)}\}|ds. \tag{5.5}
$$

For bounding I_2, we repeat a procedure similar to the one used for I_1 for the function

$$g(v,y) = \int_{\mathbb{R}^6\times\Xi}\{h(z+\alpha(z,v,\xi),x) - h(z,x)\}\sigma(|z-v|)\beta(|x-y|)Q(d\xi)\mu_s^{(n)}(dz,dx)$$

where s is fixed and the notation v, y is used instead of v_s, y_s. We obtain the estimate

$$|I_2| \leq K\epsilon + \int_0^t \sup_{\phi\in C_b^\infty(\mathbb{R}^6)} |\int_{\mathbb{R}^6}\phi(v,y)\{d\mu_s^{(n)} - d\mu_s^{(n-1)}\}|ds \tag{5.6}$$

for a suitable constant $K > 0$. Combining (5.2), (5.5), (5.6), we conclude that for suitable constants K, and C_1,

$$\begin{aligned}
&|\int_{\mathbb{R}^6} h(z,x)\{d\mu_t^{(n+1)} - d\mu_t^{(n)}\}| \\
&\leq K\epsilon + C_1\int_0^t \sup_{\phi\in C_b^\infty(\mathbb{R}^6)} |\int_{\mathbb{R}^6}\phi(z,x)\{d\mu_s^{(n+1)} - d\mu_s^{(n)}\}|ds \\
&+ \int_0^t \sup_{\phi\in C_b^\infty(\mathbb{R}^6)} |\int_{\mathbb{R}^6}\phi(z,x)\{d\mu_s^{(n)} - d\mu_s^{(n-1)}\}|ds.
\end{aligned}$$

We can take supremum on the left side over $h \in C_b^\infty(\mathbb{R}^6)$ and call the resulting expression as $J_{n+1}(t)$. Then we have

$$J_{n+1}(t) \leq K\epsilon + C_1\int_0^t J_{n+1}(s)ds + \int_0^t J_n(s)ds.$$

By the Gronwall inequality,

$$\begin{aligned}
J_{n+1}(t) &\leq K\epsilon e^{C_1 t} + C_1 e^{C_1 t}\int_0^t J_n(s)e^{-C_1 s}ds \\
&\leq K\epsilon e^{C_1 t} + C_1 e^{C_1 t}\int_0^t e^{-C_1 s}\left[K\epsilon e^{C_1 s} + C_1 e^{C_1 s}\int_0^s J_{n-1}(r)e^{-C_1 r}dr\right] \\
&\leq \cdots \\
&\leq K\epsilon e^{C_1 t}(1 + C_1 t + \frac{C_1^2 t^2}{2!} + \cdots + \frac{(C_1 t)^{n-1}}{(n-1)!}) \\
&+ C_1^n e^{C_1 t}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-1}} J_1(r)dr ds_{n-1}\cdots ds_1
\end{aligned}$$

which tends to zero as $n \to \infty$ and $\epsilon \to 0$. □

6 Identification of the Limit

In this section, we will conclude the proof of Theorem 2.1 by using tightness, Proposition 5.1 and convergence of martingale problems. From the tightness of $\{\mu^{(n)}\}$, we have the existence of a weakly convergent subsequence $\{\mu^{(n_k)}\}$. Let its weak limit be denoted by μ. Consider the associated subsequence $\{\mu^{(n_k+1)}\}$ which is also tight so that there exists a further subsequence $\{\mu^{(n_{k_j}+1)}\}$ which converges weakly. We will call its limit as ν. Clearly, $\{\mu^{n_{k_j}}\}$ being a subsequence of $\{\mu^{(n_k)}\}$ converges weakly to μ. Our aim in this section is to identify μ as a weak solution of the Enskog equation.

Let us denote a generic element of $\mathbb{D} \times \mathbb{D}$ as $\omega_1 \times \omega_2$. In the canonical setup on the path space $\mathbb{D} \times \mathbb{D}$, we recall that $\mu^{(n_{k_j}+1)}$ is the solution of the following martingale problem:

For any function $\phi \in C_b^2(\mathbb{R}^3 \times \mathbb{R}^3)$, and any $t \in [0, T]$,

(i) $\mu_0^{(n_{k_j}+1)} = \mathcal{L}(Z_0, X_0)$ (specified).

$$\begin{aligned}(ii) \quad & \phi(\omega_1(t), \omega_2(t)) - \phi(\omega_1(0), \omega_2(0)) - \int_0^t \nabla_x \phi(\omega_1(s), \omega_2(s)) \cdot \omega_1(s) ds \\ & - \int_0^t \int_{U_0} \{\phi(\omega_1(s) + \alpha(\omega_1(s), v(s), \xi), \omega_2(s)) - \phi(\omega_1(s), \omega_2(s))\} \\ & \qquad \sigma(|\omega_1(s) - v_s|)\beta(|\omega_2(s) - y_s|)Q(d\xi)\mu^{(n_{k_j})}(dv, dy)ds\end{aligned}$$

is a $\mu^{(n_{k_j}+1)}$-martingale.

While it is important to keep the above setup of martingale problems in mind, we will pass on to construct convenient random processes (on a possibly different probability space) for ease in calculations. Given that $\mu^{(n_{k_j}+1)} \to \nu$ and $\mu^{(n_{k_j})} \to \mu$, by the Skorohod representation theorem, we can construct random processes $(Z^{(n_{k_j}+1)}, X^{(n_{k_j}+1)})$ and (Z, X) such that $\mathcal{L}(Z^{(n_{k_j}+1)}, X^{(n_{k_j}+1)}) = \mu^{(n_{k_j}+1)}$ and $\mathcal{L}(Z, X) = \nu$, and

$$(Z^{(n_{k_j}+1)}, X^{(n_{k_j}+1)}) \to (Z, X) \text{ a.s.}$$

Independently of $(Z^{(n_{k_j}+1)}, X^{(n_{k_j}+1)})$, we construct processes $(\tilde{Z}^{(n_{k_j})}, \tilde{X}^{(n_{k_j})})$ and $(\tilde{Z}, \tilde{X})$ with

$$\mathcal{L}(\tilde{Z}^{(n_{k_j})}, \tilde{X}^{(n_{k_j})}) = \mu^{(n_{k_j})}; \;\; \mathcal{L}(\tilde{Z}, \tilde{X}) = \mu$$

such that $(\tilde{Z}^{(n_{k_j})}, \tilde{X}^{(n_{k_j})}) \to (\tilde{Z}, \tilde{X})$ a.s. The latter processes are used in this section only to shorten certain expressions and write them in more convenient forms to aid calculations. In terms of $(Z^{(n_{k_j}+1)}, X^{(n_{k_j}+1)})$, we are able to write the requirement (ii) in the statement of the martingale problem as follows: Fix any finite integer r. For any $0 \le s_1 \le \cdots \le s_r \le s < t \le T$, and any choice of bounded $\mathcal{F}_{s_i}$ functions g_i for all $i = 1, 2, \cdots, r$, we require

$$\mathbb{E}\Bigg[(\phi(Z_t^{(n_{k_j}+1)}, X_t^{(n_{k_j}+1)}) - \phi(Z_s^{(n_{k_j}+1)}, X_s^{(n_{k_j}+1)}) - \int_s^t \nabla_x \phi(Z_u^{(n_{k_j}+1)}, X_u^{(n_{k_j}+1)}) \cdot Z_u^{(n_{k_j}+1)} du$$
$$- \int_s^t \int_{U_0} \{\phi(Z_u^{(n_{k_j}+1)}) + \alpha(Z_u^{(n_{k_j}+1)}, v_u, \xi), X_u^{(n_{k_j}+1)}) - \phi(Z_u^{(n_{k_j}+1)}, X_u^{(n_{k_j}+1)})\}$$
$$\sigma(|Z_u^{(n_{k_j}+1)} - v_u|)\beta(|X_u^{(n_{k_j}+1)} - y_u|)Q(d\xi)\mu^{(n_{k_j})}(dv, dy)du)\Pi_{i=1}^r g_i\Bigg] = 0 \tag{6.1}$$

By letting $j \to \infty$, we know that

$$\mathbb{E}|\phi(Z_t^{(n_{k_j}+1)}, X_t^{(n_{k_j}+1)})] - \phi(Z_t, X_t)| \to 0 \tag{6.2}$$

if $t \in D$ where $D := \{a \in [0, T] : \nu(\Delta(Z_a, X_a) \neq (0, 0)) = 0\}$. A similar statement holds when t is replaced by s provided that $s \in D$. The statement that

$$\mathbb{E}|\int_s^t \nabla_x \phi(Z_u^{(n_{k_j}+1)}, X_u^{(n_{k_j}+1)}) \cdot Z_u^{(n_{k_j}+1)} du - \int_s^t \nabla_x \phi(Z_u, X_u) \cdot Z_u du| \to 0 \tag{6.3}$$

follows from the $L^2(P)$ boundedness of $\sup_{0 \le u \le T} |Z_u^{(n_{k_j}+1)}|$ indexed by j. Hence, our next objective is to show that the last term on the left side of (6.1) converges to a limit.

For $\phi \in C_b^2(\mathbb{R}^3 \times \mathbb{R}^3)$, we will use the notation

$$\|\phi_z'\|_\infty = \left(\sum_{i=1}^3 |\frac{\partial \phi}{\partial z_i}|^2\right)^{1/2} \quad ; \quad \|\phi_{zz}''\|_\infty = \left(\sum_{i,j=1}^3 |\frac{\partial^2 \phi}{\partial z_i \partial z_j}|^2\right)^{1/2}$$

with similar meanings for $\|\phi_x'\|_\infty, \|\phi_{xx}''\|_\infty$, and $\|\phi_{zx}''\|_\infty$.

Fix any function $\phi \in C_b^2(\mathbb{R}^3)$. On $\mathbb{R}^{12} \times \Xi$, define

$$G(z, x, v, y, \xi) = \{\phi(z + \alpha(z, v, \xi), x) - \phi(z, x)\}\sigma(|z - v|)\beta(|x - y|).$$

Claim:

$$\int_0^t |\int_{U_0} G(Z_s^{(n_{k_j}+1)}, X_s^{(n_{k_j}+1)}, v_s, y_s, \xi)\mu^{(n_{k_j})}(dv, dy)dQ$$
$$- \int_{U_0} G(Z_s, X_s, v_s, y_s, \xi)\mu(dv, dy)dQ|ds$$

converges to 0 almost surely.

Proof. Denoting the above expression by $D(Z_s^{(n_{k_j}+1)}, X_s^{(n_{k_j}+1)}, Z_s, X_s, \mu^{(n_{k_j})}, \mu)$, we have

$$
\begin{aligned}
&D(Z_s^{(n_{k_j}+1)}, X_s^{(n_{k_j}+1)}, Z_s, X_s, \mu^{(n_{k_j})}, \mu)\\
&\leq D(Z_s^{(n_{k_j}+1)}, X_s^{(n_{k_j}+1)}, Z_s, X_s, \mu^{(n_{k_j})}, \mu^{(n_{k_j})}) + D(Z_s, X_s, Z_s, X_s, \mu^{(n_{k_j})}, \mu)\\
&= I_1 + I_2
\end{aligned}
$$

for short. Then, $I_1 \leq J_1 + J_2$ where

$$
\begin{aligned}
J_1 = \int_0^t |\int_{U_0} &[\{\phi(Z_s^{(n_{k_j}+1)} + \alpha(Z_s^{(n_{k_j}+1)}, v_s, \xi), X_s^{(n_{k_j}+1)}) - \phi(Z_s^{(n_{k_j}+1)}, X_s^{(n_{k_j}+1)})\}\\
&- \{\phi(Z_s + \alpha(Z_s, v_s, \xi), X_s) - \phi(Z_s, X_s)\}]\\
&\sigma(|Z_s^{(n_{k_j}+1)} - v_s|)\beta(|X_s^{(n_{k_j}+1)} - y_s|)dQ\mu^{(n_{k_j})}(dv, dy)|ds, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
J_2 = \int_0^t |\int_{U_0} &\{\phi(Z_s + \alpha(Z_s, v_s, \xi), X_s) - \phi(Z_s, X_s)\}\\
&\{\sigma(|Z_s^{(n_{k_j}+1)} - v_s|)\beta(|X_s^{(n_{k_j}+1)} - y_s|) - \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\}dQd\mu^{(n_{k_j})}|ds
\end{aligned}
$$

In order to bound J_1, we bound σ and β by 1 in the above expression, and then, we break up the rest of the integrand appearing in J_1 as follows, and estimate each part separately.

$$
\begin{aligned}
J_1 \leq \int_0^t \int_{\mathbb{D}^2} \int_{\Xi} &|[\{\phi(Z_s^{(n_{k_j}+1)} + \alpha(Z_s^{(n_{k_j}+1)}, v_s, \xi), X_s^{(n_{k_j}+1)}) - \phi(Z_s^{(n_{k_j}+1)}, X_s^{(n_{k_j}+1)})\}\\
&- \{\phi(Z_s + \alpha(Z_s^{(n_{k_j}+1)}, v_s, \xi), X_s^{(n_{k_j}+1)}) - \phi(Z_s, X_s^{(n_{k_j}+1)})\}]\\
&+ [\{\phi(Z_s + \alpha(Z_s^{(n_{k_j}+1)}, v_s, \xi), X_s^{(n_{k_j}+1)}) - \phi(Z_s, X_s^{(n_{k_j}+1)})\}\\
&- \{\phi(Z_s + \alpha(Z_s, v_s, \xi), X_s^{(n_{k_j}+1)}) - \phi(Z_s, X_s^{(n_{k_j}+1)})\}]\\
&+ [\{\phi(Z_s + \alpha(Z_s, v_s, \xi), X_s^{(n_{k_j}+1)}) - \phi(Z_s, X_s^{(n_{k_j}+1)})\}\\
&- \{\phi(Z_s + \alpha(Z_s, v_s, \xi), X_s) - \phi(Z_s, X_s)\}]|dQd\mu^{(n_{k_j})}ds\\
\leq \int_0^t \int_{U_0} &[\|\phi_{zz}^{''}\|_\infty |\alpha(Z_s^{(n_{k_j}+1)}, v_s, \xi)||Z_s^{(n_{k_j}+1)} - Z_s|\\
&+ \|\phi_z^{'}\|_\infty |\alpha(Z_s^{(n_{k_j}+1)}, v_s, \xi) - a(Z_s, v_s, \xi)|\\
&+ \|\phi_{zx}^{''}\|_\infty |\alpha(Z_s, v_s, \xi)||X_s^{n_{k_j}+1} - X_s|]dQ\mu^{(n_{k_j})}(dv, dy)ds
\end{aligned}
$$

which tends to zero as $j \to \infty$, by restricting all processes to B_R as in the previous section, and using the bounded convergence theorem. Now, we consider J_2.

$$
\begin{aligned}
J_2 = & \int_0^t \int_{U_0} |\{\phi(Z_s + \alpha(Z_s, v_s, \xi), X_s) - \phi(Z_s, X_s)\} \\
& \{\sigma(|Z_s^{(n_{k_j}+1)} - v_s|)\beta(|X_s^{(n_{k_j}+1)} - y_s|) \\
& \quad - \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\}|dQd\mu^{(n_{k_j})}|ds \\
& \leq \int_0^t \int_{U_0} \|\phi_z'\|_\infty |\alpha(Z_s, v_s, \xi)| \{|Z_s^{(n_{k_j}+1)} - Z_s| + |X_s^{(n_{k_j}+1)} - X_s|\} dQd\mu^{(n_{k_j})}ds \\
& \leq C_1 \int_0^t \int_{\mathbb{D}^2} (|Z_s| + |v_s|)\{|Z_s^{(n_{k_j}+1)} - Z_s| + |X_s^{(n_{k_j}+1)} - X_s|\} d\mu^{(n_{k_j})}ds
\end{aligned}
$$

with C_1 as a suitable constant, and the last expression above $\to 0$ as $j \to \infty$ using arguments as before.

Next, we consider I_2 where

$$
\begin{aligned}
I_2 = \int_0^t | \int_{U_0} & \{\phi(Z_s + \alpha(Z_s, v_s, \xi), X_s) - \phi(Z_s, X_s)\} \\
& \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)dQ(d\mu^{(n_{k_j})} - d\mu)|ds.
\end{aligned}
$$

We write the expression within absolute value as

$$
\begin{aligned}
\tilde{\mathbb{E}} \int_{\Xi} & [\{\phi(Z_s + \alpha(Z_s, \tilde{Z}_s^{(n_{k_j})}, \xi), X_s) - \phi(Z_s, X_s)\}\sigma(|Z_s - \tilde{Z}_s^{(n_{k_j})}|)\beta(|X_s - \tilde{X}_s^{(n_{k_j})}|) \\
& - \{\phi(Z_s + \alpha(Z_s, \tilde{Z}_s, \xi), X_s) - \phi(Z_s, X_s)\}\sigma(|Z_s - \tilde{Z}_s|)\beta(|X_s - \tilde{X}_s|)]dQ.
\end{aligned}
$$

where $\tilde{\mathbb{E}}$ refers to expectation with respect to the random variables $\tilde{Z}_s^{(n_{k_j})} \tilde{X}_s^{(n_{k_j})}$, $\tilde{Z}_s$, and $\tilde{X}_s$. Using this,

$$
\begin{aligned}
I_2 \leq \int_0^t & \tilde{\mathbb{E}}[C\|\phi_z'\|_\infty \{|\tilde{Z}_s^{(n_{k_j})} - \tilde{Z}_s| + \|\phi\|_\infty (|Z_s| + |\tilde{Z}_s|) \\
& \times (|\tilde{Z}_s^{(n_{k_j})} - \tilde{Z}_s| + |\tilde{X}_s^{(n_{k_j})} - \tilde{X}_s|)\}]ds
\end{aligned}
$$

which goes to zero as $j \to \infty$ boundedly by restricting the processes appearing in the above expression to a compact set. By bounded convergence theorem, $I_2 \to 0$. This finishes the proof of Claim. $\square$

Next, we observe that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|\int_0^t\int_{U_0}\{\phi(Z_s^{(n_{k_j}+1)}+\alpha(Z_s^{(n_{k_j}+1)},v_s,\xi),X_s^{(n_{k_j}+1)})-\phi(Z_s^{(n_{k_j}+1)},X_s^{(n_{k_j}+1)})\}\\
&\qquad\sigma(|Z_s^{n_{k_j}+1}-v_s|)\beta(|X_s^{(n_{k_j}+1)}-y_s|)dQd\mu^{(n_{k_j})}|^2ds\\
&\le\|\phi_z^{'}\|_\infty^2\int_0^T\int_{\mathbb{D}^2}\mathbb{E}(|Z_s^{(n_{k_j}+1)}|^2+|v_s|^2)d\mu^{(n_{k_j})}ds\\
&\le C
\end{aligned}$$

for a suitable constant C since $\mathbb{E}\left(\sup_{0\le t\le T}[|Z_s^{(n_{k_j}+1)}|^2+|Z_s^{(n_{k_j})}|^2]\right)\le C_1$

for a constant C_1 which is independent of n_{k_j}.

$$\begin{aligned}
&\mathbb{E}[\int_0^t|\int_{U_0}G(Z_s^{(n_{k_j}+1)},X_s^{(n_{k_j}+1)},v_s,y_s,\xi)\mu^{(n_{k_j})}(dv,dy)dQ\\
&\quad-\int_{U_0}G(Z_s,X_s,v_s,y_s,\xi)\mu(dv,dy)dQ|ds]\ \to 0
\end{aligned}\tag{6.4}$$

By (6.2) - (6.4), we conclude that the martingale problem posed by the $(n_{k_j}+1)$st stochastic differential equation converges to a solution of the martingale problem posed by the stochastic system

$$\begin{aligned}
Z_t^\mu&=Z_0\\
&\quad+\int_0^t\int_U\alpha(Z_s^\mu,v_s,\xi)1_{[0,\ \sigma(|Z_s^\mu-v_s|)\beta(|X_s^\mu-y_s|)]}(r)d\tilde{N}^\mu\\
&\quad+\int_{U_0}\hat{\alpha}(Z_s^\mu,v_s,\xi)\beta(|X_s^\mu-y_s|)\mu(dydv)Q(d\xi)ds
\end{aligned}\tag{6.5}$$

$$X_t^\mu=X_0+\int_0^t Z_s^\mu ds,. \tag{6.6}$$

We know from Section 3 that the above system, with μ given, has a unique solution. Since $\mu^{(n_{k_j}+1)}$ converges in law to ν, and the law of (Z^μ,X^μ) is given by ν. By the result in Section 5, we know that if $\mu^{(n_{k_j})}$ converges in law, then $\mu^{(n_{k_j}+1)}$ also converges in law to the same limit. Hence $\nu=\mu$, and μ is a solution of the martingale problem posed by (6.5) and (6.6). The proof of Theorem 2.1 is thus completed.

7 Uniqueness of Solutions

In this section, we will study the uniqueness of solutions to the martingale problem posed by the Enskog equation. Uniqueness is taken in the sense of uniqueness of time marginals of the distribution of the processes (X, Z). We consider the Enskog equation under the additional hypothesis that the law of any weak solution of the limit equation admits at each time point t, a density with respect to the Lebesgue measure on $\mathbb{R}^6$. Such a hypothesis can likely be replaced by imposing certain conditions on the functions α and σ that are expedient though it would take us far away from the physics of the problem.

Theorem 7.1. *Let the hypotheses used in Theorem 2.1 hold. In addition, let σ be in $C_b^\infty(\mathbb{R}^1)$. Then, for any fixed t in the interval $[0, T]$, the t-marginal distribution of weak solutions of the limit equation (given below) is unique within the class of Borel probability measures on $\mathbb{R}^6$ that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^6$.*

$$\begin{aligned} Z_t = Z_0 &+ \int_0^t \int_{\mathbb{D}^2\times\Xi} \alpha(Z_s, v_s, \xi) 1_{[0,\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)]}(r)\tilde{N}(dv, dy, d\xi, ds) \\ &+ \int_0^t \int_{U_0} \alpha(Z_s, v_s, \xi)\sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\xi)ds \qquad (7.1) \\ X_t = X_0 &+ \int_0^t Z_s ds \qquad (7.2) \end{aligned}$$

wherein the law of (X, Z) is given by μ, and the compensator of $\tilde{N}$ is given by $\mu(dy, dv)Q(d\xi)dt$.

Proof. Let h be in the space $L := C_b^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. By the Itô formula, we have

$$\begin{aligned} \mathbb{E}[h(X_t, Z_t)] = \mathbb{E}[h(X_0, Z_0)] &+ \mathbb{E}\int_0^t \nabla_x h(X_s, Z_s)\cdot Z_s ds \\ &+ \mathbb{E}[\int_0^t \int_{\mathbb{D}^2\times\Xi} \{h(X_s, Z_s + \alpha(Z_s, v_s, \xi)) - h(X_s, Z_s)\} \\ &\qquad \sigma(|Z_s - v_s|)\beta(|X_s - y_s|)\mu(dy, dv)Q(d\xi)ds]. \qquad (7.3) \end{aligned}$$

If there exists another solution of (7.1) and (7.2), let us denote it as (X', Z') with its law denoted by ρ. For each $s \in [0, T]$, the measures μ_s and ρ_s denote the s-marginals of μ and ρ respectively.

Using the fact that both $(X_t, Z_t), (X'_t, Z'_t)$ are solutions of (7.1) and (7.2), and using $\mathbb{E}$ to denote expectation of functions of (X_t, Z_t) as well as (X'_t, Z'_t), we have

$$
\begin{aligned}
&\mathbb{E}h(X_t, Z_t) - \mathbb{E}h(X'_t, Z'_t) \\
&= \mathbb{E}\int_0^t \left[\nabla_x h(X_s, Z_s)\cdot Z_s - \nabla_x h(X'_s, Z'_s)\cdot Z'_s\right] ds \\
&+ \mathbb{E}[\int_0^t \int_{\mathbb{D}^2}\int_{\Xi}\{h(X_s, Z_s + \alpha(Z_s, v_s, \xi)) - h(X_s, Z_s)\}\sigma(|Z_s - v_s|) \\
&\qquad\qquad \beta(|X_s - y_s|)\mu(dy, dv)Q(d\xi)ds] \\
&\quad - \mathbb{E}[\int_0^t \int_{\mathbb{D}^2}\int_{\Xi}\{h(X'_s, Z'_s + \alpha(Z'_s, v'_s, \xi)) - h(X'_s, Z'_s)\}\sigma(|Z'_s - v'_s|) \\
&\qquad\qquad \beta(|X'_s - y'_s|)\rho(dy', dv')Q(d\xi)ds]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^t \left[\nabla_x h(X_s, Z_s)\cdot Z_s - \nabla_x h(X'_s, Z'_s)\cdot Z'_s\right] ds \\
&+ \int_0^t \int_{\mathbb{D}^2}\int_{\mathbb{D}^2}\int_{\Xi}\{h(x_s, z_s + \alpha(z_s, v_s, \xi)) - h(x_s, z_s)\} \\
&\qquad\qquad \sigma(|z_s - v_s|)\beta(|x_s - y_s|)\mu(dx, dz)\mu(dy, dv)Q(d\xi)ds \\
&\quad - \int_0^t \int_{\mathbb{D}^2}\int_{\mathbb{D}^2}\int_{\Xi}\{h(x'_s, z'_s + \alpha(z'_s, \Pi_s(v'), \xi)) - h(x'_s, z'_s)\} \\
&\qquad\qquad \sigma(|z'_s - v'_s|)\beta(|x'_s - y'_s|)\rho(dx', dz')\rho(dy', dv')Q(d\xi)ds \\
&= D_0 + D_1.
\end{aligned}
$$

where

$$
D_0 := \mathbb{E}\int_0^t \left[\nabla_x h(X_s, Z_s)\cdot Z_s - \nabla_x h(X'_s, Z'_s)\cdot Z'_s\right] ds \tag{7.4}
$$

$$
\begin{aligned}
D_1 := &\int_0^t \int_{\mathbb{D}^2}\int_{\mathbb{D}^2}\int_{\Xi}[h(z_s + \alpha(z_s, v_s, \xi))h(z_s)]\sigma(|z_s - v_s|) \\
&\qquad \beta(|x_s - y_s|)\mu(dy, dv)\mu(dx, dz)Q(d\xi)ds \\
&- \int_0^t \int_{\mathbb{D}^2}\int_{\mathbb{D}^2}\int_{\Xi}[h(z'_s + \alpha(z'_s, v'_s, \xi)) - h(z_s)]\sigma(|z'_s - v'_s|) \\
&\qquad \beta(|x'_s - y'_s|)\rho(dy', dv')\rho(dx', dz')Q(d\xi)ds.
\end{aligned} \tag{7.5}
$$

We will first obtain an upper bound for D_0. Define $f(x, z) := \nabla_x h(x, z)\cdot z$.

Fix any $R > 0$. If $|z| < R$, then f is in the space L. By existence of the second moments of $\sup_{0\le t\le T}|Z_t|$ and $\sup_{0\le t\le T}|Z'_t|$, we obtain that for any fixed $t \in [0, T]$, given any $\epsilon > 0$, we can find a compact set K in $\mathbb{R}^6$ such that

$$\int_{K^c} |f(x,z)|\{\mu_t(dx,dz)+\rho_t(dx,dz)\} < \frac{\epsilon}{2T}$$

Therefore, we have

$$|D_0| \leq \epsilon/2 + \int_0^t |\int_K f(x,z)\{\mu_t(dx,dz)-\rho_t(dx,dz)\}|dt$$

Let R be large enough so that $K \subset B_R$ where B_R denotes the R-ball in $\mathbb{R}^6$. The function $f(x,z)1_K(x,z)$ can be approximated by a function f_δ in L by replacing 1_K by a smooth non-negative function which is identically equal to 1 on K and whose support is in the δ neighborhood of K, $\delta > 0$, which is contained in B_R. As δ tends to 0, $f_\delta \to f1_K$, pointwise. By the bounded convergence theorem,

$$\lim_{\delta\to 0}\int_0^t \int_K |f(x,z)-f_\delta(x,z)|\{\mu_t(dx,dz)+\rho_t(dx,dz)\}dt = 0.$$

Hence, for all δ small enough,

$$\int_0^t \int_K |f(x,z)-f_\delta(x,z)|\{\mu_t(dx,dz)+\rho_t(dx,dz)\}dt < \epsilon/2.$$

The above estimates allow us to conclude that

$$D_0 \leq \epsilon + \int_0^t |\int f_\delta(x,z)\{\mu_t(dx,dz)-\rho_t(dx,dz)\}|dt.$$

Since $f_\delta \in L$, the above inequality ipso facto yields

$$|D_0| \leq \epsilon + \int_0^t \sup_{\phi\in L} |\int \phi(x,z)\{\mu_t(dx,dz)-\rho_t(dx,dz)\}|dt.$$

Next, we proceed to estimate D_1. On the right side of equation (7.5), we add and subtract

$$\int_0^t \int_{\mathbb{D}^2}\int_{\mathbb{D}^2}\int_{\Xi}[h(x_s,z_s+\alpha(z_s,v_s,\xi))-h(x_s,z_s)]\sigma(|z_s-v_s|)\beta(|x_s-y_s|)$$
$$\mu(dy,dv)\rho(dx,dz)Q(d\xi)ds$$

This enables us to split D_1 and write it as G_1+G_2, where

$$\begin{aligned}G_1 := &\int_0^t \int_{\mathbb{D}^2}\int_{\mathbb{D}^2}\int_{\Xi}[h(x_s,z_s+\alpha(z_s,v_s,\xi))-h(x_s,z_s)]\sigma(|z_s-v_s|)\beta(|x_s-y_s|)\\ &\quad\mu(dx,dz)\mu(dy,dv)Q(d\xi)ds\\ &-\int_0^t \int_{\mathbb{D}^2}\int_{\mathbb{D}^2}\int_{\Xi}[h(x_s,z_s+\alpha(z_s,v_s,\xi))-h(x_s,z_s)]\sigma(|z_s-v_s|)\beta(|x_s-y_s|)\\ &\quad\mu(dy,dv)\rho(dx,dz)Q(d\xi)ds.\end{aligned}$$

and

$$G_2 := \int_0^t \int_{\mathbb{D}^2} \int_{\mathbb{D}^2} \int_{\Xi} [h(x_s, z_s + \alpha(z_s, v'_s, \xi)) - h(x_s, z_s)]\sigma(|z_s - v'_s|)\beta(|x_s - y'_s|)$$
$$\mu(dy', dv')\rho(dx, dz)Q(d\xi)ds$$
$$- \int_0^t \int_{\mathbb{D}^2} \int_{\mathbb{D}^2} \int_{\Xi} [h(x'_s, z'_s + \alpha(z'_s, v'_s, \xi)) - h(x'_s, z'_s)]\sigma(|z'_s - v'_s|)\beta(|x'_s - y'_s|)$$
$$\rho(dy', dv')\rho(dx', dz')Q(d\xi)ds.$$

To find an upper bound for G_1, first introduce, for each $s \in [0, T]$, the function f_s defined on $\mathbb{R}^6$ by

$$f_s(x, z) = \int_{\mathbb{R}^6 \times \Xi} [h(x, z + \alpha(z, v, \xi)) - h(x, z)]\sigma(|z - v|)\beta(|x - y|)Q(d\xi)\mu_s(dy, dv).$$

Then G_1 can be written as

$$\int_0^t \int_{\mathbb{D}^2} f_s(x_s, z_s)(\mu(dx, dz) - \rho(dx, dz)).$$

We will now use the assumption in the statement of the theorem that for each s, $\mu_s << \lambda$, and $\rho_s << \lambda$, where λ is the Lebesgue measure on $\mathbb{R}^6$. Hence the product measures $\mu_s \times \mu_s$ and $\mu_s \times \rho_s$ are both absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{12}$. Therefore, given $\epsilon > 0$, there exists a δ-neighborhood $G := G_\delta$ of the diagonal in $\mathbb{R}^6 \times \mathbb{R}^6$ such that

$$(\mu_s \times \mu_s + \mu_s \times \rho_s)(G) < \epsilon$$

For a suitable constant $C > 0$, we have

$$|G_1| \leq C\epsilon + \int_{\mathbb{R}^6} \int_{\mathbb{R}^6 \times \Xi} 1_{G^c}(x, z, y, v)[h(x, z + \alpha(z, v, \xi)) - h(x, z)]\sigma(|z - v|)$$
$$\beta(|x - y|)Q(d\xi)\mu_s(dy, dv)(\mu_s(dx, dz) - \rho_s(dx, dz)). \tag{7.6}$$

We approximate the function 1_{G^c} by a smooth bounded function ϕ which takes the value 1 in G^c, and zero in $G_{\delta/2}$. With such a choice of ϕ, we have

$$\int_{\mathbb{R}^{12} \times \Xi} |\{1_{G^c}(y, v) - \phi\}(x, z, y, v)[h(x, z + \alpha(z, v, \xi)) - h(x, z)]$$
$$s(|z - v|)\beta(|x - y|)Q(d\xi)|\mu_s(dy, dv)\{\mu_s(dx, dz) + \rho(dx, dz)\}$$
$$< C\epsilon. \tag{7.7}$$

By (7.6) and (7.7), it follows that

$$|G_1| \leq 2C\epsilon + |\int_{\mathbb{R}^6} \int_{\mathbb{R}^6 \times \Xi} \phi(x, z, y, v)[h(x, z + \alpha(z, v, \xi)) - h(x, z)] \tag{7.8}$$
$$\sigma(|z - v|)\beta(|x - y|)Q(d\xi)|\mu_s(dy, dv)(\mu_s(dx, dz) - \rho_s(dx, dz))|. \tag{7.9}$$

The integral

$$\int_{\mathbb{R}^6\times\Xi} \phi(x,z,y,v)[h(x,z+\alpha(z,v,\xi)) - h(x,z)]\sigma(|z-v|)\beta(|x-y|)Q(d\xi)|\mu_s(dy,dv)$$

is smooth and bounded as a function of x, z, and therefore it is in L. Hence, (7.9) yields

$$|G_1| \leq 2C\epsilon + \int_0^t \sup_{\psi\in L} |\int \psi(x,z)\{\mu_t(dx,dz) - \rho_t(dx,dz)\}|dt. \tag{7.10}$$

To bound G_2, we introduce the function g_s on $\mathbb{R}^6$ by

$$g_s(y,v) := \int [h(x,z+\alpha(z,v,\xi)) - h(x,z)]\sigma(|z-v|)\beta(|x-y|)Q(d\xi)\rho_s(dx,dz).$$

Following the ideas used in finding an upper bound for G_1, one can, mutatis mutandis, bound G_2, and conclude that

$$|G_2| \leq 2C\epsilon + \int_0^t \sup_{\psi\in L} |\int \psi(y,v)\{\mu_t(dy,dv) - \rho_t(dy,dv)\}|dt. \tag{7.11}$$

In summary, we have for any $\epsilon > 0$,

$$|E[h(X_t,Z_t) - h(X_t',Z_t')]| \leq 5C\epsilon + 3\int_0^t \sup_{\phi\in L} |\mathbb{E}[\phi(X_s,Z_s)] - \mathbb{E}[\phi(X_s',Z_s')]|\, ds \tag{7.12}$$

for all $h \in L$, for a suitable constant $C > 1$. With ϵ being arbitrary, let $\epsilon \to 0$. By the Gronwall Lemma, the uniqueness of the time-marginal distributions of weak solutions is obtained. $\square$

Remark 7.1. *In the context of martingale problems, uniqueness of distribution of the time marginals of solutions implies uniqueness of the law on the path space ([28], page 69). Hence, Theorem 7.1 gives us uniqueness in law of the solution of* (7.1) *and* (7.2) *on* $\mathbb{D}\times\mathbb{D}$.

8 Invariant Gaussian density for velocity

Let $\{X_s, Z_s\}_{s\in\mathbb{R}^+}$ be a solution of (2.7),(2.8), which corresponds to the Enskog equation in the kinetic theory of gases. Let MVN $(0, I)$ denote the standard normal distribution on $\mathbb{R}^3$, where 0 stands for the mean vector, and I, for the 3×3 identity matrix for the variance. In the following "density of measures" shall be understood relative to the underlying Lebesgue measure.

Theorem 8.1. *Let us assume that the law of the initial velocity Z_0 and that of the initial location X_0 of* (2.7),(2.8) *are independent. Let Z_0 have MVN $(0, I)$ distribution. Assume that the distribution η_0 of the initial location X_0 has density $h(x)$, $x \in \mathbb{R}^3$. Then the joint distribution $\mu(dx,dz)$ of $\{X_s, Z_s\}_{s\in\mathbb{R}^+}$ has for all $t \geq 0$ density $\rho_t(x,y) := h_t(x)g(y)$, where $g(y)$ is the density of the normal distribution MVN $(0, I)$, while $h_t(x)$ is the density of X_t.*

Remark 8.1. *In particular the marginal velocity Z_t at time t is distributed according to the MVN $(0, I)$ distribution for all $t \geq 0$, and is independent of the location X_t for all $t \geq 0$.*

Proof. Our method of proof relies on guessing the solution and proving that it is indeed the solution. We take

$$\mu_t(dz, dx) = \mu_t(dz \,|\, x)\eta_t(x). \tag{8.1}$$

with

$$\mu_t(dz \,|\, x) := g(z)dz \quad \forall t \geq 0 \tag{8.2}$$

and

$$\eta_t(dx) := h_t(x)dx \quad \forall t \geq 0, \tag{8.3}$$

where $h_t(x)$ is a probability density function on $\mathbb{R}^3$ which is ascertained below. We will then prove that $\mu_t(dz, dx)$ is the distribution of a process $\{X_s, Z_s\}_{s \in \mathbb{R}^+}$ which solves (2.7),(2.8) with Z_s having the same distribution as Z_0 and solving (2.7). It would then follow that $\int_0^t Z_s ds$ is a Gaussian random variable for all $t \geq 0$, and X_t has therefore a density function denoted by $h_t(x)$.

Consider $\phi_t(\lambda) := \mathbb{E}\left[e^{i(\lambda, Z_t)}\right]$ for any $\lambda \in \mathbb{R}^3$, and $t \geq 0$. It is enough to prove that $\phi_t(\lambda) = \phi_0(\lambda)$ for all λ. Using the Itô formula and taking expectation, one obtains

$$\begin{aligned} \phi_t(\lambda) = \phi_0(\lambda) + \int_0^t \int_{\mathbb{D} \times \mathbb{D} \times \Xi} &\left\{ e^{i(\lambda, z_s + \alpha(z_s, v_s, \xi))} - e^{i(\lambda, z_s)} \right\} \\ &\sigma(|z_s - v_s|)\beta(|x - y|)\mu_s(dz, dx)\mu_s(dv, dy)Q(d\xi)ds. \end{aligned} \tag{8.4}$$

This is an equation that is satisfied by the characteristic function of Z_t where Z is a solution of (2.7). If $\mu_t(dz, dx)$ is as specified above, then we can write

$$\begin{aligned} \phi_t(\lambda) = \phi_0(\lambda) + \int_0^t \int_{\mathbb{R}^6 \times \mathbb{R}^6 \times \Xi} &\left\{ e^{i(\lambda, z_s + \alpha(z_s, v_s, \xi))} - e^{i(\lambda, z_s)} \right\} \\ &\sigma(|z_s - v_s|)\beta(|x - y|)g(z)h_s(x)dzdxg(v)h_s(y)dvdyQ(d\xi)ds \end{aligned} \tag{8.5}$$

which we write as $\phi_0(\lambda) + I$. Let us write I as

$$I = \int_0^t \int_{\mathbb{R}^6 \times \Xi} \phi(s, x, y, \xi)h_s(x)h_s(y)Q(d\xi)dxdyds$$

where

$$\phi(s, x, y, \xi) := \int_{\mathbb{R}^6} \left\{ e^{i(\lambda, z_s + \alpha(z_s, v_s, \xi))} - e^{i(\lambda, z_s)} \right\} \sigma(|z_s - v_s|)\beta(|x - y|)g(z)dzg(v)dv.$$

Then $\phi(s,x,y,\xi)$ is

$$
\begin{aligned}
&= \int_{\mathbb{R}^3\times\mathbb{R}^3} (e^{i(\lambda, z_s+\alpha(z_s,v_s,\xi))} - e^{i(\lambda,z_s)})\sigma(|z_s-v_s|)\beta(|x-y|)\mu_s(dz\,|\,x)\mu_s(dv\,|\,y) \\
&= \frac{\beta(|x-y|)}{(2\pi)^{3/2}} \int_{\mathbb{R}^3\times\mathbb{R}^3} (e^{i(\lambda,z_s^*)} - e^{i(\lambda,z_s)})\sigma(|z_s^*-v_s^*|)\exp\{-1/2(|z_s|^2+|v_s|^2)\}dzdv \\
&= \frac{\beta(|x-y|)}{(2\pi)^{3/2}} \int_{\mathbb{R}^3\times\mathbb{R}^3} (e^{i(\lambda,z_s)} - e^{i(\lambda,z_s^*)})\sigma(|z_s-v_s|)\exp\{-1/2(|z_s^*|^2+|v_s^*|^2)\}dzdv
\end{aligned}
$$

by Proposition 1.1; continuing,

$$
\begin{aligned}
&= \frac{\beta(|x-y|)}{(2\pi)^{3/2}} \int_{\mathbb{R}^3\times\mathbb{R}^3} (e^{i(\lambda,z_s)} - e^{i(\lambda,z_s^*)})\sigma(|z_s-v_s|)\exp\{-1/2(|z_s|^2+|v_s|^2)\}dzdv \\
&= -\int_{\mathbb{R}^3\times\mathbb{R}^3} (e^{i(\lambda, z_s+\alpha(z_s,v_s,\xi))} - e^{i(\lambda,z_s)})\sigma(|z_s-v_s|)\beta(|x-y|)g(z)dzg(v)dv \qquad (8.6)
\end{aligned}
$$

by using conservation of energy in (1.3), and with z^* and v^* denoting the post-collision velocities corresponding to the pre-collision velocities z and v. It follows that $I=-I$ and hence $I=0$ so that $\phi_t(\lambda)=\phi_0(\lambda)$ for all λ at all times $t>0$. □

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