# TWO-DIMENSIONAL STOCHASTIC NAVIER-STOKES EQUATIONS WITH FRACTIONAL BROWNIAN NOISE

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Abstract. We study the perturbation of the two-dimensional stochastic Navier-Stokes equation by a Hilbert-space-valued fractional Brownian noise. Each Hilbert component is a scalar fractional Brownian noise in time, with a common Hurst parameter $H$ and a specific intensity. Because the noise is additive, simple Wiener-type integrals are sufficient for properly defining the problem. It is resolved by separating it into a deterministic nonlinear PDE, and a linear stochastic PDE. Existence and uniqueness of mild solutions are established under suitable conditions on the noise intensities for all Hurst parameter values. Almost surely, the solution's paths are shown to be quartically integrable in time and space. Whether this integrability extends to the random parameter is an open question. An extension to a multifractal model is given.

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## 1. Introduction

The stochastic Navier-Stokes system has been an important and active area of research, and has received considerable attention in recent years. The introduction of randomness in the Navier-Stokes equations arises from a need to understand (i) the velocity fluctuations observed in wind tunnels under identical experimental conditions, and (ii) the onset of turbulence. Random body forces also arise as control terms, or from random disturbances such as structural vibrations that act on the fluid. It was originally the idea of Kolmogorov (see Vishik and Fursikov [18]) to introduce white noise in the Navier-Stokes system in order to obtain an invariant measure for the system.

Of late, stochasticians have embraced this white-noise forcing for the 2-D Navier-Stokes system (see [4] and references therein), and in certain inertial scales, this is justifiable: in [6], Kuppiainen has shown that it is reasonable to model the uncertainty in velocity profiles by white noise. The independence of increments inherent in white noise is key to all these studies: this has been confirmed by the discovery and analysis of the solution's Markov semigroup (again see references listed in [4]). The only published article we have found in which there is a deviation from white noise for stochastic Navier-Stokes equations is [19]: a Levy process is used, but this is still confined to the realm of processes with independent increments.

In this paper, we study a case where the time-scaling in the random forces is not related in the usual manner to the state-space scaling, and where random forcing does not renew dynamically at every instant, so that white noise is not appropriate. We consider the stochastic Navier-Stokes equation (NSE) on a bounded open domain $G$ in $\mathbf{R}^2$, with an infinite-dimensional fractional Brownian noise

$\mathbf{W}^H$. Writing it in the abstract evolution setup, this is:

$$\frac{\partial \mathbf{u}}{\partial t} + \nu \mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}) = \Phi \, \frac{d\mathbf{W}^H(t)}{dt} \tag{1.1}$$

with $\mathbf{u}(t, x) = 0$ for all $x \in \partial G$, with $\mathbf{u}(0, x) = \mathbf{u}_0(x)$ for all $x \in G$, with $\mathbf{A}$ being the so-called Stokes operator, and $\mathbf{B}(\mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{u}$. In the next section we will define suitable Hilbert spaces in which to find mild-sense (evolution) solutions of this equation; we will consider incompressible flows with no-slip condition at the boundary. The process $\mathbf{W}^H = \left\{ \mathbf{W}^H(t, x) : t \geq 0; x \in G \right\}$ is a space-time fractional Brownian motion (fBm) in a suitable Hilbert space of functions, implying that for fixed space parameter $x$, it is a scalar fBm with Hurst parameter $H \in (0, 1)$. Tindel, Tudor and Viens [16] can be consulted for information on this type of noise in a square-integrable stochastic evolution context; the noise $\mathbf{W}^H$ is defined formally below in Section 3.2.

Fractional Brownian motion is not a semimartingale, nor a Markov process, and its increments have medium- or long-range dependence. The usual methods for solving stochastic Navier-Stokes equations such as energy equality, local monotonicity, Markov semigroups, and martingale problems, do not apply to the present system. In addition to noise memory length, fBm is self-similar with parameter $H$; when combining this with possible scaling in the space parameter, infinite-dimensional noises with prescribed multiscale or other regularity properties can be realized (see [17] for details).

The theory of stochastic integration with respect to fBm is in sharp contrast to the Itô theory of integrals; it has been developed by several authors (see Nualart [12, Chapter 5] and the references therein). However, infinite-dimensional equations with additive noise, even ones with non-linearities such as the one we consider here, can typically be expressed in a mild sense, making the required integration theory rather elementary, as we will see in Section 3.1. In particular, we certainly bypass the need for the noise term to be a semimartingale.

Stochastic partial differential equations of parabolic type perturbed by an fBm noise have been studied in recent years by several authors (in addition to [16], see Maslowski and Schmalfuss [8], Nualart [10], [11], and the references therein). A major question solved in these works is to identify sharp sufficient conditions on the noise coefficient $\Phi$ that guarantee the existence and uniqueness of solutions.

The Navier-Stokes system is quite distinct from all these works since it is a nonlinear system with an unbounded, non-Lipschitz term $\mathbf{B}$. Because of this, calculations in $L^2(\Omega)$, which are typical of the above works (see [16]), are insufficient in our case. A major quantitative difference between our system and some mainstream problems such as heat equations is its linear second-order operator $\mathbf{A}$, which, because it is restricted to a divergence-free domain, has unbounded eigenfunctions.

In this article, we take advantage of the fact that the noise term in (1.1) is additive; using a fixed point argument, the existence and uniqueness of a mild solution is established by combining a solution of the non-linear equation with no noise, and a solution of the stochastic equation without the non-linearity using properties of the semigroup of the Stokes operator. The question of finding conditions on $\Phi$ guaranteeing existence and uniqueness is dealt with in the linear stochastic portion of the analysis. These conditions are fully explicit, insofar as they depend in an elementary way on the eigenstructure of the Stokes operator. Our main result (Theorem 5.1 on page 12) states that under these conditions, almost-surely w.r.t. the randomness of $\mathbf{W}^H$, there is a unique solution in $L^4([0, T] \times G)$ to the stochastic Navier-Stokes system.

Our method leaves open the question of whether the solution has any moment with respect to $\mathbf{W}^H$'s randomness. We conjecture that the solution is square-integrable w.r.t $\mathbf{W}^H$, but no more (see for instance the results in the white-noise case [9]). This issue appears to be non-trivial, and is beyond the scope of this concise article; we will investigate it, and its connections to path regularity of the solution, in a separate paper.

The organization of the paper is as follows. In Section 2, the evolution equation setup of the Navier-Stokes equations is presented. The fractional Brownian motion and its basic integration theory are briefly presented in Section 3. The $L^4(\Omega)$-integrability of a convolution Wiener integral is proved in Section 4 under suitable conditions on the noise coefficient. The solvability of the stochastic Navier-Stokes system is proved in the final Section 5.

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## 2. Navier-Stokes equations

In this section, we express the NSE using appropriate function spaces. Let $G$ be a bounded open domain in $\mathbf{R}^2$ with a smooth boundary $\partial G$. For $t \in [0,T]$, consider the stochastic NSE for a viscous incompressible flow with no-slip condition at the boundary:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \Phi \, \frac{d\mathbf{W}^H(t)}{dt} \tag{2.1}$$

and

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

with initial and boundary data

$$\mathbf{u}(t,x) = 0 \ \forall x \in \partial G, \forall t \geq 0$$
$$\mathbf{u}(0,x) = \mathbf{u}_0(x) \ \forall \ x \in G.$$

In the above, $p$ denotes the pressure field and is a scalar-valued function. The deterministic noise coefficient $\Phi$ is assumed to be a (possibly generalized) operator that is deterministic and independent of $t$. This $\Phi$ is defined below. The cylindrical fBm $\mathbf{W}^H$ is constructed in Section 3. These modeling assumptions ensure that the driving noise has the power-decay memory length and self-similarity properties of fBm, in the parameter $t$.

2.1. **Functional analysis background for the 2-D Navier-Stokes equation.** To study the stochastic Navier-Stokes system (2.1), (2.2), we first write these stochastic partial differential equations in the abstract (variational, or evolution) form on suitable function spaces. For the functional analytic set up and the mathematical details, one can consult Ladyzhenskaya [7] and Temam [15]. Let $\mathcal{U}$ be the space of 2-dimensional vector functions $\mathbf{u}$ on $G$ which are infinitely differentiable with compact support strictly contained in $G$, satisfying $\nabla \cdot \mathbf{u} = 0$. For any fixed $\alpha \in \mathbf{R}$, we can define the restriction of the standard Sobolev space $W^{\alpha,2}$ to those divergence-free 2-vectors by letting $\mathcal{V}_\alpha$ denote the closure of $\mathcal{U}$ in $W^{\alpha,2}$.

We will use the shorthand notation $\mathcal{H} := \mathcal{V}_0$ and $\mathcal{V} := \mathcal{V}_1$. We thus define the space $\mathcal{H}$ to be the closure of $\mathcal{U}$ in $L^2$, and the space $\mathcal{V}$ to be the closure of $\mathcal{U}$ in $W^{1,2}$. The notation $L^2(G)$, $W_0^{1,2}(G)$, etc.

denotes 2-vector functions on $G$ with each coordinate in the scalar versions of $L^2(G)$, $W_0^{1,2}(G)$, etc. For instance, we simply have

$$W_0^{1,2}(G) = \{\mathbf{u} : u_i \in L^2(G, \mathbf{R}) \text{ and } \nabla u_i \in L^2(G, \mathbf{R}^2) \text{ for } i = 1, 2, \text{ and } \mathbf{u}|_{\partial G} = 0\}.$$

Denoting by $\mathbf{n}$ the outward normal on $\partial G$, the following characterizations of the spaces $\mathcal{H}$ and $\mathcal{V}$ are well-known, and will be convenient:

$$\mathcal{H} = \{\mathbf{u} \in L^2(G); \nabla \cdot \mathbf{u} = 0, \ \mathbf{u} \cdot \mathbf{n}|_{\partial G} = 0\},$$
$$\mathcal{V} = \{\mathbf{u} \in W_0^{1,2}(G) : \nabla \cdot \mathbf{u} = 0\}.$$

For proofs of the above characterizations, one can refer to Temam [15], Theorems 1.4 and 1.6, pages 11-13. Here, we have used derivatives in the sense of distributions in $G$, and divergence of $\mathbf{u}$ belongs to $L^2(G)$. Hence, the elements of $\mathcal{H}$ can be evaluated at $x \in G$.

Let $\mathcal{V}'$ be the dual of $\mathcal{V}$. We will denote the norm in $\mathcal{H}$ by $|\cdot|$, and the inner product in $\mathcal{H}$ by $(\cdot, \cdot)$. We have the dense, continuous and compact embedding (see [15]):

$$\mathcal{V} \subset_{\rightarrow} \mathcal{H} = \mathcal{H}' \subset_{\rightarrow} \mathcal{V}'.$$

Let $\mathcal{D}(\mathbf{A}) = W^{2,2}(G) \cap \mathcal{V}$. Define the linear operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to \mathcal{H}$ by $\mathbf{Au} = -\Delta\mathbf{u}$. Since $\mathcal{V} = \mathcal{D}(\mathbf{A}^{1/2})$, we can endow $\mathcal{V}$ with the norm $\|\mathbf{u}\| = \left|\mathbf{A}^{1/2}\mathbf{u}\right|$. The $\mathcal{V}$-norm is equivalent to the $W^{1,2}$-norm by the Poincaré inequality. From now on, $\|\cdot\|$ will denote the $\mathcal{V}$-norm. The pairing between $\mathcal{V}$ and its dual $\mathcal{V}'$ is denoted by $\langle \cdot, \cdot \rangle$. The operator $\mathbf{A}$ is known as the Stokes operator and is positive, self-adjoint with compact resolvent. The eigenvalues of $\mathbf{A}$ will be denoted by $0 < \lambda_1 < \lambda_2 \leq \cdots$, and the corresponding eigenfunctions by $e_1, e_2, \cdots$. The eigenfunctions form a complete orthonormal system for $\mathcal{H}$. It is known that the eigenfunctions, $\{e_j\}$, are not bounded uniformly in $j$, and that the eigenvalues grow linearly. We record the corresponding quantitative facts here (see [7]).

**Lemma 2.1.** *There are values $c, c' > 0$ such that*

$$\lim_{j \to \infty} j/\lambda_j = c > 0 \text{ and } \|e_j\|_{L^4(G)} \leq c' \lambda_j^{1/4} \text{ for all } j.$$

Define $b(\cdot, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \mapsto \mathbf{R}$ by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_G u_i\,(x)\, \frac{\partial v_j}{\partial x_i}\,(x)\, w_j\,(x)\; dx.$$

This allows us to define $\mathbf{B} : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}'$ as the continuous bilinear operator such that

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{v}); \mathbf{w} \rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \text{ for all } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}.$$

Note that $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$. We will denote $\mathbf{B}(\mathbf{u}, \mathbf{u})$ by $\mathbf{B}(\mathbf{u})$. This $\mathbf{B}(\mathbf{u})$ satisfies the following estimate:

$$\|\mathbf{B}(\mathbf{u})\|_{\mathcal{V}'} \leq 2\,|\mathbf{u}|\,||\mathbf{u}|| \tag{2.3}$$

2.2. **The 2-D stochastic Navier-Stokes equation.** We assume that $\mathbf{u}_0$ is $\mathcal{H}$-valued. Let $\Pi$ denote the Leray projection of $L^2(G)$ into $\mathcal{H}$. By applying this projection to each term of the Navier-Stokes system, and invoking the Leray decomposition of $L^2(G)$ into divergence free and irrotational components, we can write the system (2.1) and (2.2) as

$$d\mathbf{u}\,(t) + [\nu \mathbf{A}\,\mathbf{u}\,(t) + \mathbf{B}(\mathbf{u}\,(t))]\,dt = \Phi\,d\mathbf{W}^H(t). \tag{2.4}$$

This is to be understood in the integral form $\mathbf{u}\,(t) = \mathbf{u}\,(0) - \nu \int_0^t \mathbf{A}\,\mathbf{u}\,(s)\,ds - \int_0^t \mathbf{B}(\mathbf{u}\,(s))ds + \int_0^t \Phi d\mathbf{W}^H(s)$.

Here $\Phi$ is a (possibly generalized-function-valued) linear operator from $\mathcal{H}$ to $\mathcal{H}$. We can define it by its action on the complete orthonormal basis $\{e_n\}$ in $\mathcal{H}$ mentioned above, formed by the eigenfunctions of the Stokes operator $\mathbf{A}$ on $G$: for any $n = 1, 2, \cdots$

$$\Phi e_n = \sum_{n=1}^{\infty} (\Phi e_n, e_j)\, e_j\,(\cdot)\,.$$

Assuming $\Phi$ is non-random and not dependent on time, the expression $\int_0^t \Phi d\mathbf{W}^H(s)$ should simply be given as $\Phi \mathbf{W}^H\,(t)$, whose interpretation as the action of $\Phi$ on the cylindrical $\mathbf{W}^H$ is straightforward. However, a bit more care must be taken because (2.4) should be written in its "semigroup" or "evolution" form.

To write (2.4) in its evolution form, we will need $S\,(t)$ the semigroup generated by $\mathbf{A}$. Assuming for simplicity that the diffusion constant $\nu = 1$, the stochastic NSE in $\mathcal{H}$ then writes as

$$\mathbf{u}\,(t) = S\,(t)\,\mathbf{u}\,(0) - \int_0^t S\,(t-s)\,\mathbf{B}(\mathbf{u}\,(s))ds + \int_0^t [S\,(t-s)\,\Phi]\,d\mathbf{W}^H(s). \tag{2.5}$$

This means we only need to explain how to define integrals of the form $\int_0^t \phi\,(s)\,d\mathbf{W}^H(s)$, where $\phi$ is a suitable non-random integrand. While the strategy to construct this type of integral is well-known, we detail it in the next section for completeness, along with some general information about fBm.

## 3. Fractional Brownian motion

The calligraphic letter $\mathcal{H}$ is used for the Stokes' operator's Hilbert space, and is not to be confused with the letter $H$, the the so-called *Hurst* or self-similarity parameter for our fBm, which is a number in $(0, 1)$.

**Definition 3.1.** *A continuous centered Gaussian process $\{\beta_t^H\}$ is called a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if its covariance function is given by*

$$\mathbf{E}\left[\left(\beta_t^H \beta_s^H\right)\right] = R_H(t,s) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{3.1}$$

It is well-known that $\{\beta_t^H\}$ is not a semimartingale, and that it exhibits medium or long-range memory. However, the fBm with $H = 1/2$ coincides with the standard Brownian motion (a.k.a the Wiener process). Stochastic integration with respect to fBm cannot be treated using the same martingale tools as for Itô integration with respect to the Wiener process.

3.1. **Gaussian integral with respect to fBm.** Fortunately for our purposes, the distinctions between various ways of defining stochastic integrals for fBm will not be relevant, because, as seen in (2.5), we only need to explain how to integrate deterministic functions w.r.t. an fBm $\beta^H$ (albeit perhaps an infnite-dimensional one).

All Riemann-sum or Lebesgue-Stieltjes-type approximation constructions of integrals of deterministic functions against fBm are essentially equivalent. The following is a traditional construction. Let $S$ be the set of all step functions on $[0,T]$. For a step function $\phi = \sum_0^{n-1} a_j 1_{(t_j,t_{j+1}]}$ we define

$$\int_0^T \phi(s) d\beta_s^H := \sum_{j=0}^{n-1} a_j (\beta_{t_{j+1}}^H - \beta_{t_j}^H).$$

Let $\mathcal{K}$ be the Reproducing Kernel Hilbert Space of the process $\beta^H$. In other words, $K$ is the closure of $S$ w.r.t. the inner product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{K}} := R_H(t,s).$$

Then the map $1_{[0,t]} \mapsto \beta_t^H$ extends to an isometry between $\mathcal{K}$ and the $L^2(\Omega)$-closure of the linear span of $\{\beta_t^H : t \in [0,T]\}$. This extension is called the Wiener integral w.r.t. $\beta$, and can be denoted by

$$\phi \mapsto \int_0^T \phi(s)\, d\beta^H(s) \in L^2(\Omega).$$

Note that the Wiener integral of any function $\phi \in \mathcal{K}$ w.r.t. $\beta^H$ is a centered Gaussian random variable, and that for $\phi, \psi \in \mathcal{K}$ we have that $\int_0^T \phi d\beta^H$ and $\int_0^T \psi d\beta^H$ are jointly Gaussian with covariance equal to $\langle \phi, \psi \rangle_{\mathcal{K}}$, thereby extending the Wiener integral for standard Brownian motion.

There is a connection between the standard Wiener process and fractional Brownian motions. One begins by noting that $R_H$ is, by definition, a non-negative definite kernel, which means that there exists a kernel function $K_H$, the "square root" of $R_H$, such that $R_H(t,s) = \int_0^{t \wedge s} K_H(t,u) K_H(s,u) du$. In fact, its expression is explicit (see [12]):

$$K_H(t,s) = c_H \left(\frac{t}{s}\right)^{H-1/2} (t-s)^{H-1/2} + s^{1/2-H} F\left(\frac{t}{s}\right)$$

where $F(z) = c_H(1/2 - H) \int_0^{z-1} r^{H-3/2} \left(1 - (1+r)^{H-1/2}\right) dr$. Using these facts, one proves that there exists a standard Brownian motion $W$ such that $\beta_t^H = \int_0^t K_H(t,s) dW_s$. In the next paragraph, we will simply write $K$ for $K_H$ for notational simplicity.

For $s < T$, if we define the adjoint operator $K_T^*$ on a possible subset of $L^2([0,T])$ by

$$(K_T^* \phi)(s) = K(T,s)\phi(s) + \int_s^T (\phi(r) - \phi(s)) \frac{\partial K}{\partial r}(r,s) dr;$$

a result of Alos, Mazet, Nualart [1] then guarantees that $K_T^*$ is an isometry between $\mathcal{K}$ and $L^2[0,T]$, and that the Wiener integral w.r.t $\beta^H$ can be represented in the following convenient way: for all $\phi \in \mathcal{K}$, $K_T^* \phi \in L^2[0,T]$ and

$$\int_0^T \phi(s) d\beta^H(s) = \int_0^T (K_T^* \phi)(s) dW_s$$

where the last integral is a Wiener integral w.r.t. standard Brownian motion. It turns out that when $H > 1/2$, $\mathcal{K}$ contains distributions (generalized functions), although we will not need for $K^*$ to act so broadly in our applications. It is easy to check that $K_T^*[\phi\, 1_{[0,t]}] = K_t^*[\phi] 1_{[0,t]}$. Therefore,

$$\int_0^t \phi(s) d\beta^H(s) = \int_0^t (K_t^* \phi)(s) dW_s. \tag{3.2}$$

3.2. **Cylindrical fBm.** As announced at the end of the Introduction, we now only need to define integrals of the form $\int_0^t \phi(s) d\mathbf{W}^H(s)$ where $\mathbf{W}^H$ is a cylindrical $\mathcal{H}$-valued fBm and $\phi$ is a deterministic, Borel-measurable function on $[0,T]$ taking values in the space $\mathcal{L}$ of linear operators from $\mathcal{H}$ to $\mathcal{H}$. The process $\mathbf{W}^H$ is an infinite-dimensional stochastic process with an fBm behavior in time. To define $\mathbf{W}^H$, take any complete orthonormal basis $\{h_n\}$ in $\mathcal{H}$, and $\{\beta_n^H\}_n$, a family of IID scalar fBm's. Define

$$\mathbf{W}^H(t) := \sum_{j=1}^{\infty} h_n \, \beta_n^H(t).$$

Since the definition does not depend on the choice of the basis, we will take the complete orthonormal basis $\{e_n\}$ in $\mathcal{H}$ formed by the eigenfunctions of the Stokes operator $\mathbf{A}$ on $G$, and set

$$\mathbf{W}^H(t) := \sum_{j=1}^{\infty} e_n \, \beta_n^H(t).$$

For every $t$, our random vector $\mathbf{W}^H(t)$ is a generalized member of $L^2(\Omega, \mathcal{H})$, since its norm is infinite, but it will be easy to guarantee that an integral w.r.t. $\mathbf{W}^H$ will be in that space.

Indeed, let $\{\phi(s) : s \in [0,T]\}$ be a deterministic $\mathcal{L}$-valued measurable function (for every $s$, $\phi(s) \in \mathcal{L}$). We can write $\phi(s) e_n = \sum_m (\phi(s) e_n, e_m) e_m$, and this is a deterministic measurable function on $[0,T]$. We may now define

$$\begin{aligned}
\int_0^t \phi(s) \, d\mathbf{W}^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \phi(s) \, e_n \, d\beta_n^H(s) \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} e_j \int_0^t (\phi(s) \, e_n, e_j) \, d\beta_n^H(s) \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} e_j \int_0^t \left( K_t^* \left( (\phi(\cdot) \, e_n, e_j) \right) \right)(s) \; dW_n(s)
\end{aligned}$$

where the third line follows from the convenient representation (3.2), in which $W_n$ is the standard Brownian motion used to represent $\beta_n^H$, provided the last expression exists as a member of $L^2(\Omega, \mathcal{H})$.

Consider the last expression with the summations interchanged. Then, for each fixed $j$, we obtain the element $e_j \in \mathcal{H}$ multiplied by a series of independent centered Gaussian r.v.'s. This observation provides us immediately with a necessary and sufficient condition for the above integral to exist: it is a Gaussian random element in $L^2(\Omega, \mathcal{H})$ if and only if

$$\begin{aligned}
\mathbf{E}\left[ \left| \int_0^t \phi(s) \, d\mathbf{W}^H(s) \right|^2 \right] &= \mathbf{E}\left[ \sum_{j=1}^{\infty} \left| \sum_{n=1}^{\infty} \int_0^t \left( K_t^* \left( (\phi(\cdot) \, e_n, e_j) \right) \right)(s) \; dW_n(s) \right|^2 \right] \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \mathbf{E}\left[ \left| \int_0^t \left( K_t^* \left( (\phi(\cdot) \, e_n, e_j) \right) \right)(s) \; dW_n(s) \right|^2 \right]
\end{aligned}$$

$$= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \int_0^t \left| \left( K_t^* \left( \left( \phi\left(\cdot\right) e_n, e_j \right) \right) \right) (s) \right|^2 ds < \infty. \tag{3.3}$$

## 4. Quartic integrability of the convolution integral

In equation (2.5), we announced that we only need to be able to define the convolution integral $\mathbf{z}(t) = \int_0^t [S(t-s)\Phi] \, d\mathbf{W}^H(s)$ where $S(t)$ is the semigroup of the Stokes operator $\mathbf{A}$, and $\Phi$ is a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$. In other words, we apply the result of the end of the previous section with the function $\phi(s) = S(t-s)\Phi$. Using the eigenstructure of $\mathbf{A}$ and the Fubini theorem, we thus have

$$\begin{aligned}
\mathbf{z}(t) &:= \int_0^t S(t-s)\Phi d\mathbf{W}_s^H = \sum_{n=1}^{\infty} \int_0^t S(t-s)\Phi e_n d\beta_n^H(s) \\
&= \int_0^t \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (\Phi e_n, e_j) S(t-s) e_j d\beta_n^H(s) \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (\Phi e_n, e_j) \int_0^t e^{-(t-s)\lambda_j} d\beta_n^H(s) e_j
\end{aligned} \tag{4.1}$$

It is elementary to check that if $\mathbf{Az}$ happens to be defined as an $\mathcal{H}$-valued function on $[0,T]$, then $\mathbf{z}$ is the solution of

$$\mathbf{z}(t) - \mathbf{z}(0) + \int_0^t \mathbf{Az}(s) \, ds = \Phi \mathbf{W}^H(t) \tag{4.2}$$

with $\mathbf{z}(0) = 0$. This justifies our claim that $\mathbf{z}$ in (4.1) is the mild- (evolution-) sense solution to the stochastic evolution equation (4.2), modulo finiteness of the expression in (3.3), even if $\mathbf{Az}$ is not defined, i.e. even if $\mathbf{z}$ does not take values in the domain of $\mathbf{A}$.

We now compute the fourth moments of $\mathbf{z}$ in order to see under what conditions on $\Phi$ we may have

$$\mathbf{z} \in L^4(\Omega \times [0,T] \times G).$$

This restriction will be needed for our main theorem, when we construct the evolution solution to the full NSE (2.5). Let $H \in (0,1)$. Recall that $\mathcal{K}$ is the canonical Hilbert space of fBm $\beta^H$, the space such that for any scalar-valued functions $g, h \in \mathcal{K}$, $\mathbf{E}\left[\left(\int_0^\infty h(s) d\beta^H(s)\right)\left(\int_0^\infty g(s) d\beta^H(s)\right)\right] = \langle g, h \rangle_{\mathcal{K}}$. The following lemma is a key estimate.

**Lemma 4.1.** *Fix $H \in (0,1)$. There is a constant $c_H$ depending only on $H$ such that for any $\lambda, t \geq 0$*

$$\mathbf{E}\left[\left(\int_0^t e^{-\lambda(t-s)} d\beta^H(s)\right)^2\right] = \left|\mathbf{1}_{[0,t]}(\cdot) \, e^{-\lambda(t-\cdot)}\right|_{\mathcal{K}}^2 \leq c_H \lambda^{-2H}.$$

*Proof.* Let $I_t := \int_0^t e^{-\lambda(t-s)} d\beta^H(s)$.

We claim first that the Gaussian centered distribution of this Wiener integral for any $H \in (0,1)$ is invariant by time reversal in the integrand, for fixed $t$. Indeed, $I_t$ can be written, for instance, as the $L^2(\Omega)$-limit as $n \to \infty$ of the following Riemann sum:

$$I_{t,n} := \sum_{k=1}^{n-1} e^{-\lambda(t-kt/n)} \Delta_{k,n} \beta$$

where the increment of $\beta$ used above is defined as

$$\Delta_{k,n}\beta = \beta^H\left((2k+1)t/2n\right) - \beta^H\left((2k-1)t/2n\right).$$

The distribution of the centered Gaussian vector $G_n := \{\Delta_{k,n}\beta : k = 1, \cdots, n-1\}$ is determined by the covariance

$$\mathbf{E}\left[\Delta_{k,n}\beta\Delta_{\ell,n}\beta\right] : k, \ell = 1 \cdots, n-1,$$

which, by the stationarity of the increments of $\beta^H$, depends only on $|k-\ell|$, as an elementary computation using formula (3.1) readily shows. This implies that the Gaussian vector $G_n$ has the same distribution as the vector

$$\check{G}_n := \{\Delta_{n-k,n}\beta : k = 1, \cdots, n-1\}.$$

Now by a simple change of index (replace $k$ by $n-k$) , the Riemann sum

$$J_{t,n} := \sum_{k=1}^{n-1} e^{-\lambda(t-kt/n)}\Delta_{n-k,n}\beta$$

is immediately seen to converge in $L^2(\Omega)$ to $\int_0^t e^{-\lambda s} d\beta^H(s)$ as $n \to \infty$. But since $G_n$ and $\check{G}_n$ have the same distribution and the coefficients in the Riemann sums are non-random, thus $I_{t,n}$ and $J_{t,n}$ have the same distribution, from which the claim follows.

Thus $I_t$ has the same centered Gaussian probability law as

$$\check{I}_t := \int_0^t e^{-\lambda s} d\beta^H(s).$$

By the $H$-self-similarity of fBm (or using Riemann sum arguments similar to the ones above), we can replace $d\beta^H(r/\lambda)$ by $\lambda^{-H} d\beta^H(r)$; this means that by the change of variable $r = \lambda s$, $I_t$ and $\check{I}_t$ have the same distribution as the centered Gaussian r.v.

$$K_t := \lambda^{-H} \int_0^{\lambda t} e^{-r} d\beta^H(r),$$

whose variance equals $\lambda^{-2H}\mathbf{E}\left[\left(\int_0^{\lambda t} e^{-r} d\beta^H(r)\right)^2\right]$.

To finish the proof of the lemma, we only need to bound the second factor in the variance of $K_t$ uniformly in $\lambda t \in \mathbf{R}_+$. By integration by parts, we get

$$\int_0^T e^{-r} d\beta^H(r) = e^{-T}\beta^H(T) + \int_0^T e^{-r}\beta^H(r)\, dr$$

and thus

$$\sqrt{\mathbf{E}\left[\left(\int_0^T e^{-r} d\beta^H(r)\right)^2\right]} \leq T^H e^{-T} + \int_0^T e^{-r} r^H dr.$$

which is bounded for all $T \in \mathbf{R}_+$. This finishes the proof of the lemma. □

**Remark 4.1.** *The above proof shows that the constant in the statement of Lemma 4.1 can be taken as*

$$c_H := \sup_{T \geq 0}\left\{T^H e^{-T} + \int_0^T e^{-r} r^H dr\right\}.$$

*Elementary calculus then shows that* $c_H = \int_0^\infty e^{-r} r^H dr$.

We now make calculations to ascertain conditions under which $\mathbf{z}$ belongs to $L^4(\Omega \times [0,T] \times G)$. With the notation

$$p_s^t(n,j) := \mathbf{1}_{[0,t]}(s) e^{-(t-s)\lambda_j} (\Phi e_n, e_j) e_j,$$

using standard Gaussian calculations, we get

$$\mathbf{E}[\mathbf{z}^4(t,x)] = 3 \sum_{n,m} \mathbf{E}\left[ \left( \int_0^t [\textstyle\sum_j p_s^t(n,j)(x)] d\beta_n^H(s) \right)^2 \right] \mathbf{E}\left[ \left( \int_0^t [\textstyle\sum_j p_s^t(m,j)(x)] d\beta_m^H(s) \right)^2 \right].$$

With the inner product notation in $\mathcal{K}$, this now reads as

$$\mathbf{E}[\mathbf{z}^4(t,x)] = 3 \sum_{n,m} \left| \textstyle\sum_j p_\cdot^t(n,j)(x) \right|_{\mathcal{K}}^2 \left| \textstyle\sum_j p_\cdot^t(m,j)(x) \right|_{\mathcal{K}}^2 = 3 \left( \sum_n \left| \textstyle\sum_j p_\cdot^t(n,j)(x) \right|_{\mathcal{K}}^2 \right)^2.$$

Now let us reintroduce the terms in the notation $p_\cdot^t(n,j)$. As a shorthand, we will omit the factor $\mathbf{1}_{[0,t]}$ in $p^t(n,j)$. We thus get

$$\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)}^4 = 3 \int_0^T \int_G dtdx \left( \sum_n \textstyle\sum_{j,k} \langle e^{-(t-\cdot)\lambda_j}; e^{-(t-\cdot)\lambda_k} \rangle_{\mathcal{K}} (\Phi e_n, e_j)(\Phi e_n, e_k) e_j(x) e_k(x) \right)^2.$$

At this point, one notes that to do this computation exactly, it would be necessary to evaluate an inner product in $\mathcal{H}$ of two exponentials relative to two different modes $\lambda_j$ and $\lambda_k$. There is a wide class of examples, that of noise spatial covariances which are co-diagonalizable with the Stokes operator, where this is unnecessary, since the sum over $j,k$ reduces to a single term where $j=k=n$. Therefore, there is not much loss of power in invoking the Cauchy-Schwarz inequality to write:

$$\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)}^4 \le 3 \int_0^T \int_G dtdx \left( \sum_n \textstyle\sum_{j,k} |e^{-(t-\cdot)\lambda_j}|_{\mathcal{K}} |e^{-(t-\cdot)\lambda_k}|_{\mathcal{K}} \, |(\Phi e_n, e_j)(\Phi e_n, e_k) e_j(x) e_k(x)| \right)^2.$$

Now, in order to reunite the space base functions $e_j$ etc... with their space integral, in principle it is necessary to expand the above square, resulting in terms of the form $\int_G e_i(x)e_j(x)e_k(x)e_\ell(x)dx$. Unfortunately, nothing is known about the values of the four-way integrals of the eigenfunctions for the Stokes operator. The best we can do is to invoke Hölder's inequality and say that the integral over $G$ which one obtains after expanding the square above, is bounded in absolute value by the product of the four $\|e_i\|_{L^4(G)}$'s. By Lemma 2.1 given in Section 2, we know that $\|e_j\|_{L^4(G)} \le c\lambda_j^{1/4}$ Using this and Lemma 4.1 yields:

$$\begin{aligned}
&\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)}^4 \\
&\le 3c^4 \int_0^T (c_H)^2 dt \sum_{n,m} \textstyle\sum_{i,j,k,\ell} (\lambda_i\lambda_j\lambda_k\lambda_\ell)^{-H+1/4} \, |(\Phi e_n, e_j)(\Phi e_n, e_k)(\Phi e_m, e_i)(\Phi e_m, e_\ell)| \\
&= 3c^4 \int_0^T (c_H)^2 dt \left( \sum_n (\textstyle\sum_j \lambda_j^{-H+1/4} \, |(\Phi e_n, e_j)|)^2 \right)^2
\end{aligned}$$

where the constant $(c_H)^2$ depends only on $H$. We have thus proved the following.

**Theorem 4.2.** *Assume $H \in (0,1)$. The evolution solution $\mathbf{z}$ of the stochastic parabolic equation (4.2) on $[0,T]\times G$ with the Stokes operator $\mathbf{A}$ and driven by the additive noise $\Phi dW^H$, which is given by the* *formula (4.1), satisfies* $\|\mathbf{z}\|_{L^4(\Omega\times[0,T]\times G)} < \infty$ *as soon as*

$$\sum_n \left(\sum_j \lambda_j^{-H+1/4} \left|(\Phi e_n, e_j)\right|\right)^2 < \infty, \tag{4.3}$$

*where* $(\lambda_j, e_j)_j$ *are the eigen-elements of* $\mathbf{A}$.

It is important to note that under the condition (4.3), the interchange of sums and integrals in the above calculation is justified by the Fubini theorem.

**Remark 4.2.** *Since* $\lambda_j$ *is asymptotically linear by Lemma 2.1, Condition (4.3) is equivalent to*

$$\sum_n \left(\sum_j j^{-H+1/4} \left|(\Phi e_n, e_j)\right|\right)^2 < \infty,$$

We can get a sense of what this condition means in terms of the spatial regularity of the noise $\Phi\mathbf{W}^H$ by looking specifically at the co-diagonalizable case, that is the case where the eigenfunctions of $\Phi$ are the $e_n$'s.

**Corollary 4.3.** *Under the assumptions of Theorem 4.2, if in addition* $\Phi$ *is co-diagonalizable with* $\mathbf{A}$ *in the sense that* $(\Phi e_n, e_j) = 0$ *if* $j \neq n$*, then, denoting* $q_n = |(\Phi e_n, e_n)|^2$ *the nth squared eigenvalue of* $\Phi$*, Condition (4.3) in Theorem 4.2 becomes* $\sum_n \lambda_n^{-2H+1/2} q_n < \infty$*, or equivalently*

$$\sum_n n^{-2H+1/2} q_n < \infty, \tag{4.4}$$

Condition (4.4) is not needed for the stochastic convolution $\mathbf{z}$ to exist. Since $\mathbf{z}$ is Gaussian, the existence of $\mathbf{z}(t,x)$ as a random variable is equivalent to the existence of its variance. In the co-diagonalizable case, we can compute, for instance, the variance of $\mathbf{z}$ as an element of $L^2([0,T]\times G)$: from formula (4.1) and Lemmas 2.1 and 4.1, we immediately get

$$\|\mathbf{z}\|^2_{L^2([0,T]\times G\times\Omega)} = \int_0^T \sum_n q_n |e^{-(t-\cdot)\lambda_n}|^2_{\mathcal{K}} dt \leq T c_H \sum_n n^{-2H} q_n$$

so that $\mathbf{z}$ exists as soon as $\sum_n n^{-2H} q_n < \infty$. This is evidently a weaker condition than (4.4). This discrepancy is due, again, to the difficulty in computing fourth-power integrals of the Stokes operator's eigenfunctions of the form $\int_G |e_n|^2 |e_m|^2$, combined with the fact that the norms of the $e_n$'s in $L^2(G)$ increase with $n$. These problems do not occur when computing second moments, since the eigenfunctions can be taken to be orthogonal in $L^2(G)$.

While Condition (4.4) guarantees the existence of a solution to the stochastic Navier-Stokes equation in the co-diagonalizable case as we are about to see in the next section, we suspect that existence does not hold under the weaker condition $\sum_n n^{-2H} q_n < \infty$.

## 5. Existence and uniqueness of solutions

The existence and uniqueness of mild solutions to Navier-Stokes evolution systems have been studied by a number of authors (cf. Da Prato and Zabczyk [2], [3], Sohr [14], Temam [15]). The method of solvability in the stochastic case, consists in breaking up the system (2.5) into a linear stochastic system and a nonlinear partial differential equation. Since our system is perturbed by an additive fractional noise term, this approach works in a straightforward way. The theorem below is this article's main result. Its proof is given in full detail, and divided into several steps for the reader's convenience. Note that this theorem falls short of proving that the solution exists in $L^4([0,T]\times G\times\Omega)$, showing only that almost surely, it belongs to the space $L^4([0,T]\times G)$. We will investigate the stronger, former statement in a separate publication, conjecturing here that the solution is only square-integrable with respect to $\Omega$.

**Theorem 5.1.** *Let $\{e_n : n\in\mathbf{N}\}$ be an orthonormal basis in the Hilbert space $\mathcal{H}$ of eigenfunctions of the Stokes operator $\mathbf{A}$. Let $\Phi$ be a linear operator from $\mathcal{H}$ to $\mathcal{H}$. For any $H\in(0,1)$, under the condition*

$$\sum_n \left(\sum_j j^{1/4-H}\,|(\Phi e_n, e_j)|\right)^2 < \infty,$$

*there exists a unique mild solution of the stochastic Navier-Stokes system, i.e. $\mathbf{P}$-almost surely, there is a unique solution in $L^4([0,T]\times G)$ to equation (2.5) driven by the infinite-dimensional fractional Brownian noise $\Phi\mathbf{W}^H$ where $\mathbf{W}^H$ is defined in Section 3.2.*

**Remark 5.1.** *In the case where $\Phi$ and $\mathbf{A}$ are co-diagonalizable (i.e. $(\Phi e_n, e_j)=0$ for $j\neq n$), letting $q_n := |(\Phi e_n, e_n)|^2$, the above condition reduces to $\sum_n n^{1/2-2H}\,q_n<\infty$.*

Before proceeding with the proof of the theorem, we present an extension to the case of inhomogeneous Hurst parameter, relative to the Stokes operator's eigenstructure.

**Corollary 5.2.** *We make the same assumptions as in Theorem 5.1, and replace $\mathbf{W}^H$ by a multifractal infinite-dimensional noise. Using the notation in Section 3.2, we define*

$$\bar{\mathbf{W}}(t) := \sum_{n=1}^{\infty} e_n\,\beta_n^{H_n}(t)$$

*where $(H_n)_{n\in\mathbf{N}}$ is a sequence of numbers in $(0,1)$. Then the conclusions of Theorem 5.1 remains unchanged if we replace $\mathbf{W}^H$ by $\bar{\mathbf{W}}$.*

*Proof of Theorem 5.1. Step 1.* Consider the system

$$d\mathbf{u}+[\nu\mathbf{A}\,\mathbf{u}+\mathbf{B}(\mathbf{u})]\,dt=\Phi d\mathbf{W}_t^H,$$

as in (2.1). In order to find the solution $\mathbf{u}$, we will use the previous theorems, which tell us how to find the unique evolution (mild) solution $\mathbf{z}(t)$ of

$$d\mathbf{z}\,(t)+\mathbf{Azdt}=\Phi d\mathbf{W}_t^H,$$

with $\mathbf{z}(0)=0$. If $\mathbf{u}$ existed, say in a strong sense, we would denote $\mathbf{v}:=\mathbf{u}-\mathbf{z}$, and notice that

$$\begin{aligned}\frac{\partial\mathbf{v}}{\partial t}&=\frac{\partial\mathbf{u}}{\partial t}-\frac{\partial\mathbf{z}}{\partial t}\\ &=(-\mathbf{Au}-\mathbf{B}(\mathbf{u})+\Phi\frac{d\mathbf{W}^H}{dt})-(-\mathbf{Az}+\Phi\frac{d\mathbf{W}^H}{dt})\\ &=-\mathbf{A}(\mathbf{u}-\mathbf{z})-\mathbf{B}(\mathbf{u})=-\mathbf{Av}-\mathbf{B}(\mathbf{v}+\mathbf{z})\end{aligned}$$

Therefore, with $\mathbf{z}$ given, solving for $\mathbf{u}$ in (2.5) would be equivalent to solving for $\mathbf{v}$ in

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{Av}+\mathbf{B}(\mathbf{v}+\mathbf{z})=0 \tag{5.1}$$

with initial data $\mathbf{v}(0) = \mathbf{u}_0 \in \mathcal{H}$.

More precisely, Theorem 4.2 guarantees the existence (and uniqueness) in $L^4(\Omega \times [0,T] \times G)$ of $\mathbf{z}$ as a mild solution of (4.2) given by formula (4.1); therefore the evolution equation (2.5) has a unique solution mild in that same space (starting from $\mathbf{u}_0$) if the evolution (mild) version of equation (5.1) admits a solution in $L^4(\Omega \times [0,T] \times G)$ as well. This evolution solution $\mathbf{v}$, when it exists in that space, satisfies

$$\mathbf{v}(t) = S(t)\mathbf{u}_0 - \int_0^t S(t-s)\mathbf{B}(\mathbf{v}(s) + \mathbf{z}(s))ds \tag{5.2}$$

where $S(t) = e^{-t\mathbf{A}}$ is the semigroup generated by the operator $\mathbf{A}$. Let us introduce notation meant to signify that equation (5.2) is a fixed point problem:

$$\Lambda(\mathbf{w}) := S(t)\mathbf{u}_0 - \int_0^t S(t-s)\mathbf{B}(\mathbf{w}(s) + \mathbf{z}(s))ds.$$

Studying the properties of this operator $\Lambda$ is the main subject of this proof.

*Step 2*: Let $\mathbf{w} \in L^4([0,T] \times G) \cap \mathcal{V}$. We will show that $\mathbf{B}(\mathbf{w} + \mathbf{z}) \in L^2(0,T;\mathcal{V}')$. Indeed, for any $\phi \in L^2(0,T;\mathcal{V})$, and suppressing time in the argument of functions, and denoting $\mathbf{w} + \mathbf{z}$ as $y$,

$$\begin{aligned} |\langle \mathbf{B}(y), \phi \rangle| &= |b(y, \phi, y)| \\ &= \left| \sum_{i=1}^2 \int_G y_i \frac{d\phi_j}{dx_i} y_j dx \right| \\ &\le |y|_{L^4(G)} |\nabla \phi|_{L^2(G)} |y|_{L^4(G)}. \end{aligned} \tag{5.3}$$

By the Poincaré inequality which applies by the boundedness of the domain $G$ and the zero boundary condition, we get the equivalence of $|\nabla\phi|_{L^2(G)}$ and $\|\phi\|_{\mathcal{V}}$. Hence,

$$\int_0^T |\langle \mathbf{B}(y), \phi \rangle| ds \le \int_0^T |y|^2_{L^4(G)} |\nabla \phi|_{L^2(G)} ds. \tag{5.4}$$

By the Schwarz inequality, the assertion of this step is obtained.0.1in

*Step 3*: We show here that $\Lambda(\mathbf{w}) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}) =: Y$, and in fact,

$$||\Lambda(\mathbf{w})||_Y \le 2||\mathbf{B}(\mathbf{w} + \mathbf{z})||_{L^2(0,T;\mathcal{V}')}.$$

Indeed, by the Sobolev embedding theorem $\mathcal{H}^{1/2} = W^{1/2,2}(G) \hookrightarrow L^4(G)$, there is a non-random constant $C$ depending only on the bounded domain $G$ such that $\|\mathbf{u}\|_{L^4(G)} \le C\|\mathbf{u}\|_{W^{1/2,2}(G)}$. Using this in (5.3),

$$\begin{aligned} |\langle \mathbf{B}(y), \phi \rangle| &\le C|y|^2_{W^{\frac{1}{2},2}} \|\phi\|_{\mathcal{V}} \\ &\le C|y|_{\mathcal{H}} \|y\|_{\mathcal{V}} \|\phi\|_{\mathcal{V}} \end{aligned}$$

by the interpolation theorem. Thus $\|\mathbf{B}(y)\|_{\mathcal{V}'} \le C|y|_{\mathcal{H}} \|y\|_{\mathcal{V}}$.

Now define $h(t) = -\int_0^t S(t-s)\mathbf{B}(y(s))ds$, and $y \in L^4([0,T] \times G)$. Then $h(0) = 0$, and by the energy equality,

$$\begin{aligned} |h(t)|^2_{L^2} &= -2\int_0^t |\nabla h|^2_{L^2} ds - 2\int_0^t \langle \mathbf{B}(y(s)), h(s) \rangle_{\mathcal{V}' \times \mathcal{V}} ds \\ &\le -2\int_0^t |h|^2_{\mathcal{V}} ds + 2\int_0^t |\mathbf{B}(y(s))|_{\mathcal{V}'} \cdot |h(s)|_{\mathcal{V}} ds \end{aligned}$$

$$\leq -2\int_0^t |h|_{\mathcal{V}}^2 ds + \int_0^t |\mathbf{B}(y(s))|_{\mathcal{V}'}^2 ds + \int_0^t |h(s)|_{\mathcal{V}}^2 ds.$$

So

$$|h(t)|_{\mathcal{H}}^2 + \int_0^t |h(s)|_{\mathcal{V}}^2 ds \leq \int_0^t |\mathbf{B}(y)|_{\mathcal{V}'}^2 ds,$$

and thus

$$\sup_{0\leq t\leq T} |h(t)|_{\mathcal{H}}^2 + \int_0^T |h(s)|_{\mathcal{V}}^2 ds \leq 2\int_0^T |\mathbf{B}(y)|_{\mathcal{V}'}^2 ds,$$

which is bounded. Therefore, $h(t) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V})$. Therefore, $\Lambda(\mathbf{w}) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V})$.

*Step 4*: Let $L^4$ denote $L^4([0,T]\times G) = L^4(0,T;L^4(G))$. We now show that for any $\mathbf{w}_1, \mathbf{w}_2 \in L^4([0,T]\times G) \cap \mathcal{V}$, we have

$$|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)|_{L^4} \leq CC' |\mathbf{w}_1 - \mathbf{w}_2|_{L^4} (|\mathbf{w}_1 + \mathbf{z}|_{L^4} + |\mathbf{w}_2 + \mathbf{z}|_{L^4}).$$

Here $C$ is the universal ($G$-dependent) constant from the Sobolev embedding theorem used in Step 3, and $C'$ is another constant which depends only on $G$.

For any $\mathbf{u}_1$, $\mathbf{u}_2 \in L^4$, and $\phi \in \mathcal{V}$, we have

$$\begin{aligned}
&|\langle \mathbf{B}(\mathbf{u}_1) - \mathbf{B}(\mathbf{u}_2), \psi\rangle| \\
&= |b(\mathbf{u}_1, \mathbf{u}_1, \psi) - b(\mathbf{u}_2, \mathbf{u}_2, \psi)| \\
&\leq |b(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1, \psi)| + |b(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2, \psi)| \\
&= |b(\mathbf{u}_1 - \mathbf{u}_2, \psi, \mathbf{u}_1)| + |b(\mathbf{u}_2, \psi, \mathbf{u}_1 - \mathbf{u}_2)| \\
&\leq |\mathbf{u}_1 - \mathbf{u}_2|_{L^4} |\nabla\psi|_{\mathcal{H}} |\mathbf{u}_1|_{L^4} + |\mathbf{u}_2|_{L^4} |\nabla\psi|_{\mathcal{H}} |\mathbf{u}_1 - \mathbf{u}_2|_{L^4} \\
&= |\mathbf{u}_1 - \mathbf{u}_2|_{L^4} |\psi|_{\mathcal{V}} (|\mathbf{u}_1|_{L^4} + |\mathbf{u}_2|_{L^4}),
\end{aligned}$$

which implies, by Jensen's inequality for some $C'$ depending only on $G$,

$$|\mathbf{B}(\mathbf{u}_1) - \mathbf{B}(\mathbf{u}_2)|_{L^2(0,T;\mathcal{V}')} \leq C' |\mathbf{u}_1 - \mathbf{u}_2|_{L^4} (|\mathbf{u}_1|_{L^4} + |\mathbf{u}_2|_{L^4})$$

and thus

$$|\mathbf{B}(\Lambda(\mathbf{w}_1) + \mathbf{z}) - \mathbf{B}(\Lambda(\mathbf{w}_2) + \mathbf{z})|_{L^2(0,T;\mathcal{V}')} \leq C' |\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)|_{L^4} (|\Lambda(\mathbf{w}_1) + \mathbf{z}|_{L^4} + |\Lambda(\mathbf{w}_2) + \mathbf{z}|_{L^4}).$$

If we let $\mathbf{y}_j = \Lambda(\mathbf{w}_j) + \mathbf{z}$ for $j = 1, 2$, then, using again the Sobolev embedding of $L^4(G)$ in $W^{1/2,2}$,

$$|h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{L^4(G)} \leq C |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{W^{\frac{1}{2},2}} \leq C |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{\mathcal{H}}^{\frac{1}{2}} \cdot |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{\mathcal{V}}^{\frac{1}{2}}$$

Note that $h(\mathbf{y}_1) - h(\mathbf{y}_2) = \Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)$ so that by the above estimate

$$\begin{aligned}
|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)|_{L^4} &= |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{L^4} \\
&= \left(\int_0^T |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{L^4}^4 dt\right)^{\frac{1}{4}} \\
&\leq C\left(|h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{\mathcal{H}}^2 \cdot \int_0^T |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{\mathcal{V}}^2 dt\right)^{\frac{1}{4}} \\
&\leq C\left(\sup_{0\leq t\leq T} |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{\mathcal{H}}^2 \cdot \int_0^T |h(\mathbf{y}_1) - h(\mathbf{y}_2)|_{\mathcal{V}}^2 dt\right)^{\frac{1}{4}}
\end{aligned}$$

$$\begin{aligned}
&\leq C[(\int_0^T |\mathbf{B}(\mathbf{y}_1) - \mathbf{B}(\mathbf{y}_2)|^2_{\mathcal{V}'} dt)^2]^{\frac{1}{4}} \\
&\leq CC'|\mathbf{y}_1 - \mathbf{y}_2|_{L^4}(|\mathbf{y}_1|_{L^4} + |\mathbf{y}_2|_{L^4}) \\
&= CC'|\mathbf{w}_1 - \mathbf{w}_2|_{L^4}(|\mathbf{w}_1 + \mathbf{z}|_{L^4} + |\mathbf{w}_2 + \mathbf{z}|_{L^4}).
\end{aligned}$$

*Step 5*:

The previous step proves that the operator

$$\Lambda : \begin{cases} L^4 \mapsto L^4 \\ \mathbf{w} \ \mapsto \Lambda(\mathbf{w}) := S(\cdot)\mathbf{u}_0 - \int_0^{\cdot} S(\cdot - s)\mathbf{B}(\mathbf{w}(s) + \mathbf{z}(s))ds \end{cases}$$

is well-defined as mapping $L^4$ to itself. As mentioned in Step 1, from Theorem 4.2, $\mathbf{z}$ is in $L^4(\Omega \times [0,T] \times G)$, which implies that $\mathbf{z} \in L^4$ almost surely. Fix any $\omega$ in this almost sure set. Note that

$$\Lambda(0) = S(\cdot)\mathbf{u}_0 - \int_0^{\cdot} S(\cdot - s)\mathbf{B}(\mathbf{z}(s))ds$$

is then a fixed function on $[0,T] \times G$, and a member of $L^4$. Let $\eta = \frac{1}{4CC'}$. Replacing $T$ in the definition of $L^4$ by a smaller value, one can choose a time $T_1 > 0$ small enough, which may depend on $\omega$, so that

$$|\mathbf{\Lambda}(0)|_{L^4} \leq \frac{\eta}{2} \quad \text{and} \quad |\mathbf{z}|_{L^4} \leq \frac{\eta}{4}.$$

Define $L := \{\mathbf{w} \in L^4 : |\mathbf{w} + \mathbf{z}|_{L^4} \leq \eta\}$. Then $0 \in L$ and $\Lambda(0) \in L$. In fact we have $|\Lambda(0)| \leq \frac{\eta}{2}$. Therefore, denoting by $\Lambda^j$ the $j$th iteration of the map $\Lambda$, we get by the result of the previous step,

$$\begin{aligned}
|\Lambda^2(0)|_{L^4} &\leq |\Lambda(\Lambda(0)) - \Lambda(0)|_{L^4} + \frac{\eta}{2} \\
&\leq CC'|\Lambda(0)|_{L^4} (|\Lambda(0) + z|_{L^4} + |z|_{L^4}) + \frac{\eta}{2} \\
&\leq CC'|\Lambda(0)|_{L^4}\eta + \frac{\eta}{2} \\
&\leq \frac{\eta}{8} + \frac{\eta}{2}.
\end{aligned}$$

Thus $|\Lambda^2(0) + z|_{L^4} \leq \eta(1/2 + 1/4 + 1/8)$ which means $\Lambda^2(0) \in L$.

More generally we can prove by induction that $|\Lambda^n(0)|_{L^4} < \frac{3}{4}\eta$ for all $n \geq 1$. Indeed, this is true for $n = 1, 2$, and repeating the above calculation we get

$$\begin{aligned}
|\Lambda^n(0)|_{L^4} &\leq \left|\Lambda(\Lambda^{n-1}(0)) - \Lambda(0)\right|_{L^4} + |\Lambda(0)|_{L^4} \leq C\frac{3}{4}\eta(1 + \frac{1}{4})\eta + \frac{\eta}{2} \\
&= C\frac{3}{4}\eta(1 + \frac{1}{4})\frac{1}{4C} + \frac{\eta}{2} < \frac{\eta}{4} + \frac{\eta}{2} = \frac{3}{4}\eta.
\end{aligned}$$

This proves that $|\Lambda^n(0) + z|_{L^4} < \eta$ for all $n \geq 1$. Thus, $\{\Lambda^n(0)\}$ is a sequence that remains within the closed set $L$. With our choice of $\eta$, $\Lambda$ is a contraction in $L$: indeed, the result of Step 4 implies, for points $\mathbf{w}_1$ and $\mathbf{w}_2$ restricted to $L$, that

$$\begin{aligned}
|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)|_{L^4} &\leq CC'|\mathbf{w}_1 - \mathbf{w}_2|_{L^4}(|\mathbf{w}_1 + \mathbf{z}|_{L^4} + |\mathbf{w}_2 + \mathbf{z}|_{L^4}) \\
&\leq |\mathbf{w}_1 - \mathbf{w}_2|_{L^4} CC' \cdot 2 \cdot \eta = \frac{1}{2}|\mathbf{w}_1 - \mathbf{w}_2|_{L^4}.
\end{aligned}$$

Thus $\{\Lambda^n(0) : n \geq 1\}$ converges to a function $\mathbf{v} \in L$, which is the unique fixed point of the map $\Lambda$ in $L$; this is the unique solution in $L$ of equation (5.2) restricted to $[0, T_1]$, i.e. the unique evolution solution in $L$ of the stochastic Navier Stokes equation (2.1) on $[0, T_1]$.

*Step 6*:

If there existed another distinct solution $\tilde{\mathbf{v}}$ to equation (5.2) on $[0, T_1]$, it would have to not be in $L$. Then by replacing $T_1$ by a sufficiently smaller time $T_0 < T_1$, the other solution $\tilde{\mathbf{v}}$ can be made to be in $L$ also. Therefore, $\mathbf{v}$ and $\tilde{\mathbf{v}}$ coincide on $[0, T_0]$. In other words, for almost every fixed $\omega$, we have existence and uniqueness of the solution to equation (5.2) up to some time $T_0$ which may depend on $\omega$.

Now considering $\mathbf{v}(T_0)$ instead of $\mathbf{u}_0$ as a new initial condition of the evolution equation (5.2), one can find $T_2 > T_0$ such that the time interval can be extended to $[0, T_2]$ on which the unique solution exists. Continuing this way, suppose $R$ is the supremum of all the times in $[0, T]$ up to which the unique solution exists using the above procedure. In other words, the equation has a unique solution in $[0, R)$. A standard argument can now be used to prove that $R = T$, as in [13, pages 186-187]. We include the details in our case for the sake of completeness.

We start by noting that $\mathbf{z}(s)$ is defined for all $s \leq T$. Also define $\mathbf{y}(s) = \mathbf{v}(s) + \mathbf{z}(s)$ for all $s < R$. Now we may define $h(R)$ as in Step 3, for which we note that $\mathbf{v}(R)$ is not needed. We also note that by the estimates from Step 3, and the inclusion $L^\infty(0, R; \mathcal{H}) \cap L^2(0, R; \mathcal{V}) \subset L^4(G)$,

$$\|h(R) - h(s)\|_{L^4(G)} \leq \left[\int_s^R |B(\mathbf{y}(r))|_{\mathcal{V}'}^2 \, dr\right]^{1/2}$$

for all $s < R$, and in particular $h(s)$ converges in $L^4(G)$ to $h(R)$ as $s \to R$. Trivially since $S(t)\,u_0 = \sum_k e^{-\lambda_k t}(u_0, e_k)\, e_k$, $S(s)\,u_0$ converges in $L^4(G)$ to $S(R)\,u_0$ as $s \to R$. Therefore, we can define $\mathbf{v}(R)$ by continuity in $L^4(G)$ as $S(R)\,u_0 + h(R)$, as it should be. This $\mathbf{v}(R)$ is thus such that $\mathbf{v}$ satisfies the equation (5.2) on the closed interval $[0, R]$ and is in $L^4([0, R] \times G)$. From this point, if $R$ were less than $T$, one would run the equation from $\mathbf{v}(R)$ instead of $u_0$, up to a latter time, contradicting $R$'s maximality, showing that $R$ cannot be less than $T$. □

*Proof of the Corollary.* The proof of the Theorem above shows that the value of $H$ only has an effect on whether the solution $\mathbf{z}$ to the stochastic parabolic equation $d\mathbf{z}(t) + \mathbf{Azdt} = \Phi d\mathbf{W}_t^H$ has a solution starting at 0 which is quartically integrable in space and time. This solution is the stochastic convolution studied in Section 4. We saw in Theorem 4.2 that this solution is in $L^4([0, T] \times G \times \Omega)$ as soon as Condition (4.3) is satisfied. This was established by controlling the solution's norm in this space via Lemma 2.1 and Lemma 4.1. These lemmas introduce the constants $c$ and $c_H$ respectively. The constant $c$ depends only on the geometry of $G$, and therefore is not effected by replacing $\mathbf{W}^H$ by $\bar{\mathbf{W}}$. The constant $c_H$ was identified in Remark 4.1, as

$$c_H = \int_0^\infty e^{-r} r^H dr.$$

As a function of $H$, $c_H$ is strictly convex and twice differentiable on the interval $[0, 1]$, since $dc_H/dH = \int_0^\infty e^{-r} r^H \log r \; dr$ and $d^2 c_H/dH^2 = \int_0^\infty e^{-r} r^H \log^2 r \; dr$. We calculate that $c'(0)$ is the opposite of the Euler constant $-\gamma \approx -0.58$, while $c'(1) = 1 - \gamma \approx 0.42$. We also calculate that $c_0 = c_1 = 1$. Therefore, as a function on $[0, 1]$, $c_H$ has a unique positive minimum, and its maximum is 1. [Numerical integration reveals that the minimum is reached near $H = 0.46$, and is near 0.886.] Thus, since $c_H$ is bounded above and below by positive universal constants, one may use the value $c_H = 1$ in applying Lemma 4.1 in the proof of Theorem 4.2, uniformly on all eigencomponents of $\bar{\mathbf{W}}$ even if the values of $H_n$ vary, and moreover, there is no loss of efficiency in doing so. □

## References


[1] Alos, E.; Mazet, O.; Nualart, D. (2001) Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, **29**, no 2, 766-801.

[2] Da Prato, G., and Zabczyk, J. (1992) *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge.

[3] Da Prato, G., and Zabczyk, J. (1996) *Ergodicity for infinite dimensional systems*, Cambridge Univ. Press.

[4] Da Prato, G., and Debussche, A. (2008). *On the martingale problem associated to the 2D and 3D Stochastic Navier-Stokes equations*. Preprint, at http://arxiv.org/abs/0805.1906 .

[5] Decreusefond, L. and Ustunel, A.-S. (1997) Stochastic Analysis of the fractional Brownian motion, *Poten. Anal.*, **10**, 177-214.

[6] Kuppiainen, A. (2000) Statistical theories of turbulence; Lecture Notes *Random Media 2000*, Madralin.

[7] Ladyzhenskaya, O.A. (1969) *The Mathematical Theory of Viscous Incompressible Flow*, Gordon Breach Science Publishers, New York.

[8] Maslowski, B. and Schmalfuss, B. (2004) Random dynamical systems and stationary solutions of differential equations driven by fractional Brownian motions; *Stoch. Anal. Appl.* **22**, 1577-1607.

[9] Menaldi, J.-L., and Sritharan, S.S. (2002). Stochastic 2-D Navier–Stokes Equation. *Appl Math Optim* **46**, 31-53.

[10] Nualart, D. (2003) Stochastic integration with respect to fractional Brownian motion and applications; *Stochastic models* (Mexico City, 2002), 3–39, Contemp. Math., 336, AMS, Providence, RI.

[11] Nualart, D. (2006) Stochastic calculus with respect to fractional Brownian motion; *Ann. Fac. Sci. Toulouse Math.*(6), **15**, 63–78.

[12] Nualart, D. (2006) *Malliavin Calculus and Related Topics. Second Edition.* Springer.

[13] Pazy, A. (1992) *Semigroups Of Linear Operators And Applications To Partial Differential Equations.* Springer.

[14] Sohr, H. (2001) *The Navier-Stokes Equations*, Birkhäuser Verlag, Basel.

[15] Temam, R. (1984) *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam.

[16] Tindel, S., Tudor, C., and Viens, F. (2003) Stochastic evolution equations with fractional Brownian motion; *Prob. Th. and Rel. Fields* 127, 186–204.

[17] Tindel, S., Tudor, C., and Viens, F. (2004) Sharp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation. *J. Funct. Analysis*, **217** (2004) no. 2, 280-313.

[18] Vishik, M. J., and Fursikov, A. V. (1988) *Mathematical Problems in Statistical Hydromechanics*, Kluwer Academic Publ., Boston.

[19] Xu, T., and Zhang, T. (2006) Large deviation principles for 2-D stochastic Navier–Stokes equations driven by Lévy processes. *Journal of Functional Analysis*, **257** (5) 1519-1545.



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