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Optimal Control of a Degenerate PDE for Surface Shape

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Abstract Controlling the shapes of surfaces provides a novel way to direct self-assembly of colloidal particles *on those surfaces* and may be useful for material design. This motivates the investigation of an optimal control problem for surface shape in this paper. Specifically, we consider an objective (tracking) functional for surface shape with the prescribed mean curvature equation in graph form as a state constraint. The control variable is the prescribed curvature. We prove existence of an optimal control, and using improved regularity estimates, we show sufficient differentiability to make sense of the first order optimality conditions. This allows us to rigorously compute the gradient of the objective functional for both the continuous and discrete (finite element) formulations of the problem. Numerical results are shown to illustrate the minimizers and optimal controls on different domains.

Keywords Locally elliptic nonlinear PDE · $L^p - L^2$ norm discrepancy · Mean curvature

Mathematics Subject Classification 49J20 · 35Q35 · 35R35 · 65N30

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1 Introduction

Directed and self-assembly of micro and nano-structures is a growing research area with applications in material design [18,32,34]. Controlling surface geometry can be beneficial for directing the assembly of micro-structures (colloidal particles) [23]. This is because there is a coupling between the geometry of surfaces/interfaces and the arrangements of charged colloidal particles, or polymers, on those curved surfaces [26,36]; in particular, the presence of defects can seriously affect the surface geometry [22,23] and vice-versa. Moreover, experimental techniques have been developed for creating "custom shapes" (from swell gels) by encoding a desired surface metric [35].

With the above motivation, we investigate an optimal PDE control problem which controls the surface shape by prescribing the mean curvature. We consider an open, bounded, $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$ for an embedded surface in $\mathbb{R}^{n+1}$, with boundary of Ω denoted by $\partial\Omega$ and $n \geq 1$. If X_1 and X_2 are two Banach spaces, then $X_1 \hookrightarrow X_2$ and $X_1 \subset\subset X_2$ denote the continuous and compact embeddings of X_1 in X_2 respectively. $W_p^1(\Omega)$, $1 \leq p \leq \infty$ defines the standard Sobolev space with corresponding norm $\|\cdot\|_{W_p^1(\Omega)}$. Moreover, $\mathring{W}_p^1(\Omega)$ indicates the Sobolev space with zero trace and $W_{p'}^{-1}(\Omega)$ is the canonical dual of $\mathring{W}_p^1(\Omega)$, for $1 \leq p < \infty$, such that $1/p + 1/p' = 1$. In deriving various inequalities and estimates, we pay special attention to the constants, C, involved.

Then we are interested in solving the following PDE-constrained optimization problem:

$$\inf \mathcal{J}(y, u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \quad \text{over } y - v \in \mathring{W}^1_\infty(\Omega), u \in U_{ad}, \tag{1.1}$$

subject to

$$-\operatorname{div} \frac{\nabla y}{\mathcal{Q}(y)} - u = 0 \quad \text{in } \Omega. \tag{1.2}$$

The second order nonlinear operator in (1.2) describes the mean curvature in graph form, where y is the height function, and $\mathcal{Q}(y) = (1 + |\nabla y|^2)^{1/2}$ denotes the surface measure. Moreover, we have an integral constraint on u: for some fixed $p > n$ and fixed $\theta > 0$, u is in the convex set

$$U_{ad} := \left\{ u \in L^2(\Omega) : \int_\Omega |u|^p \leq \theta^p \right\}, \quad \text{(see Definition 2.12).}$$

Eventually, see Remark 2.5 and Theorem 2.8, we will show there exists a value of θ for which U_{ad} is not empty. Note: throughout the entire paper, we now fix p to a value strictly greater than n. In principle, either u or v (boundary value) may act as a control variable, but in this work we will assume that v is a fixed given function and u is the control variable. The set U_{ad} induces a "smallness" condition on the control, but this is still relevant for applications. Indeed, in [13,18] they are able to drive colloidal particles to aggregate on an immiscible interface using *very* mild distortions of the interface (see Fig. 1).

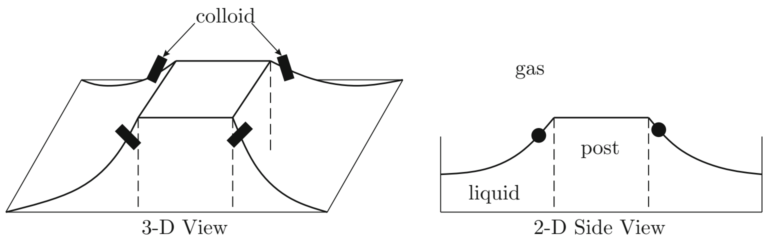

Fig. 1 *Left* 3-D perspective of colloidal rod-like particles (*thick bars*) on an immiscible interface. *Right* 2-D view showing the liquid-gas (immiscible) interface pinned to a central post. Colloidal rods shown as circular cylinders (viewed along their axes). The curvature of the interface drives the aggregation of the colloidal particles

We emphasize that the mean curvature operator in (1.2) is only locally coercive [25, P. 104], which makes this problem harder than it appears. For instance, a compatibility condition between the domain Ω and right-hand-side u must hold for (1.2) to have a solution [19]. For instance, integrating both sides of (1.2) leads to

$$\left|\int_\Omega u\right| = \left|\int_\Omega \operatorname{div}\frac{\nabla y}{\mathcal{Q}(y)}\right| = \left|\int_{\partial\Omega} \nu\cdot\frac{\nabla y}{\mathcal{Q}(y)}\right| \le \int_{\partial\Omega}\left|\nu\cdot\frac{\nabla y}{\mathcal{Q}(y)}\right| \le \int_{\partial\Omega} 1,$$

where ν is the outer unit normal of $\partial\Omega$. Clearly u cannot be too large if (1.2) is to be meaningful; in fact, the compatibility condition is even more involved [19]. Thus, (1.2) is intricate, even for "nice" domains.

We refer to [6,10,27,31] and references therein for a vast literature on the optimal control of semilinear PDEs. However, the control of mean curvature (1.2) and similar quasilinear operators [20, Chap. 10] in full generality has not been dealt with before. The closest approach is in [4,5] where they study the control of a Laplace free boundary problem with surface tension effect for $n = 1$. This amounts to solving a Laplace equation in the bulk which is a subset of $\mathbb{R}^2$ and the prescribed mean curvature equation (1.2) on $(0, 1) \subset \mathbb{R}^1$ for an embedded surface in $\mathbb{R}^2$. Furthermore, they replaced the curvature operator by a simpler version, i.e.

$$\frac{\Delta y}{\mathcal{Q}(y)}. \tag{1.3}$$

In the present paper, we work in domains $\Omega \subset \mathbb{R}^n$, with $n \ge 2$, and we do not use the simplified curvature operator (1.3), i.e. we consider the general nonlinear operator (1.2). The second novelty of this paper is the proof of the existence of a strong unique solution to (1.2): for a given $u \in L^p(\Omega)$, $p > n$, if $u \in U_{ad}$, and $v \in W_p^2(\Omega)$ is small, we prove that $y \in W_p^2(\Omega)$ (see, Theorem 2.8). We use an implicit function theorem (IFT) [28, 2.7.2] based framework to prove this result. This is an improvement over previous results in [1,2]. The improvement being that in [2, Theorem 1], Amster et al use the Schauder theorem to show the existence and therefore y may lack uniqueness. The implicit function theorem framework not only gives us the existence and uniqueness but also the Fréchet differentiability of our control to state map [21, Sect. 1.4.2]; the latter is crucial to derive the first order necessary optimality system. In addition, by further assuming a smallness condition on the data v, we derive a continuity (a piori) estimate for the solution to the state equation (1.2) in Theorem 2.11.

The importance of such a continuity estimate is well-known in the literature; see [25, p. 97] for the obstacle problem with locally coercive-operators where a similar result leads to well-posedness. This has been used in the numerical approximation of such PDEs to derive error estimates [29]. We will exploit this result to prove existence of a solution to the control problem in Theorem 3.2.

In Lemma 2.14 we establish a W_p^2-regularity result for general linear elliptic PDEs in non-divergence form. We only require that the lower order coefficient is in $L^p(\Omega)$, $p > n$. This result extends the classical result for linear PDEs where such coefficients are assumed to be in $L^\infty(\Omega)$. We use this new result to establish existence of strong solutions for the adjoint state equation.

Due to the topology mismatch between the regularization term in (1.1) and the constraint set U_{ad}, a subtle issue of 2-norm discrepancy arises. This has been well-studied for semilinear problems (see [31]). We extend these results to our quasilinear problem in Sect. 3.3.

To summarize, by using a smallness assumption on v and u we prove the existence and uniqueness of a W_p^2 solution to (1.2) within an IFT framework (Theorem 2.8). In addition, we derive an a priori bound on the solution to the state equation using a fixed point iteration. As pointed out earlier, such a smallness condition on u appears naturally due to the structure of (1.2). However, at first glance, the condition on v might seem unnecessary. We would like to stress that without this additional assumption on the data v, using the techniques developed in this paper, it is not possible to show the crucial W_p^2 a priori estimate for the solution to (1.2). We conclude the paper with several illustrative examples in Sect. 5.

2 The State Equation

2.1 Weak Solution

The state equation (1.2) has been studied by several authors; we only mention two approaches which are of interest to us. For a Lipschitz domain Ω and v in $L^1(\Omega)$, Giaquinta in [19] gives a necessary and sufficient condition for the existence of a solution y in the space of functions of bounded variation (BV). On the other hand, if $v \in W_1^1(\Omega)$, it is possible to show that $y - v \in \mathring{W}_1^1(\Omega) \subset BV$, see [7, P. 394] for the latter inclusion.

Theorem 2.1 (W_1^1 state) *Let Ω be Lipschitz and $v \in W_1^1(\Omega)$. Then there exists an open set $U_1 \subset W_\infty^{-1}(\Omega)$, with $0 \in U_1$, such that for every $u \in U_1$, there exists a unique solution $y - v \in \mathring{W}_1^1(\Omega)$ solving (1.2).*

Proof See [16, P. 351]. □

To this end, we rewrite (1.2) in a weak form as: Find $y - v \in \mathring{W}^1_\infty(\Omega)$ satisfying

$$\langle \mathcal{N}(y,u), w\rangle_{W^{-1}_\infty(\Omega),\mathring{W}^1_1(\Omega)} := \mathcal{B}(y,w) - \varphi(w) = 0 \quad \text{for all } w \in \mathring{W}^1_1(\Omega), \tag{2.1}$$

where $\mathcal{B}(y,w) := \int_\Omega \frac{\nabla y}{\mathcal{Q}(y)}\nabla w$ and $\varphi(w) := \langle u, w\rangle_{W^{-1}_\infty(\Omega),\mathring{W}^1_1(\Omega)}$, and $\langle\cdot,\cdot\rangle_{W^{-1}_\infty(\Omega),\mathring{W}^1_1(\Omega)}$ denotes the duality pairing. Moreover, $\mathcal{N}$ is a nonlinear map. Thus Theorem 2.1 states that for a given $u \in U_1$, there exists a unique $y - v \in \mathring{W}^1_1(\Omega)$ satisfying (2.1).

Next, we will develop an implicit function theorem based framework to show that in fact $y - v \in \mathring{W}^1_\infty(\Omega)$.

2.2 Differentiability of $\mathcal{N}$

We begin by studying some differentiability properties of $\mathcal{N}$, for the case when $v \in W^1_\infty(\Omega)$.

Lemma 2.2 *If $v \in W^1_\infty(\Omega)$, then for every $u \in U_1$, the operator $\mathcal{N}(\cdot,u) : v \oplus \mathring{W}^1_\infty(\Omega) \to W^{-1}_\infty(\Omega)$ is twice Fréchet differentiable with respect to y and the first order Fréchet derivative at $y \in v \oplus \mathring{W}^1_\infty(\Omega)$ satisfies*

$$\left\langle D_y\mathcal{N}(y,u)\langle h\rangle, w\right\rangle_{W^{-1}_\infty(\Omega),\mathring{W}^1_1(\Omega)} = \left\langle \left(\mathcal{I} - \frac{\nabla y \nabla y^T}{\mathcal{Q}(y)^2}\right)\frac{\nabla h}{\mathcal{Q}(y)}, \nabla w\right\rangle_{L^\infty(\Omega),L^1(\Omega)}.$$

Moreover, both the first and second order derivatives are Lipschitz continuous.

Proof The derivation of $D_y\mathcal{N}$ is straightforward, so is omitted. We begin by first showing that $\mathcal{Q} : v\oplus\mathring{W}^1_\infty(\Omega) \to L^\infty(\Omega)$ is Fréchet differentiable. Let $y \in v\oplus\mathring{W}^1_\infty(\Omega)$ and $h \in \mathring{W}^1_\infty(\Omega)$ (note: $y + h \in v \oplus \mathring{W}^1_\infty(\Omega)$). To this end we need to show that for every $\epsilon > 0$ there exists a $\delta > 0$, such that for $\|h\|_{W^1_\infty(\Omega)} < \delta$

$$\frac{\|\mathcal{Q}(y+h) - \mathcal{Q}(y) - D_y\mathcal{Q}(y)\langle h\rangle\|_{L^\infty(\Omega)}}{\|h\|_{W^1_\infty(\Omega)}} < \epsilon, \quad \text{where} \quad D_y\mathcal{Q}(y)\langle h\rangle = \frac{\nabla y}{\mathcal{Q}(y)}\cdot\nabla h.$$

Define the residual $\mathcal{R} = \mathcal{Q}(y+h) - \mathcal{Q}(y) - D_y\mathcal{Q}(y)\langle h\rangle$. Using an algebraic manipulation, we get

$$\mathcal{Q}(y+h) - \mathcal{Q}(y) = \frac{\nabla(2y+h)\cdot\nabla h}{\mathcal{Q}(y+h)+\mathcal{Q}(y)}, \tag{2.2}$$

whence

$$\mathcal{R} = \left(\frac{\nabla(2y+h)}{\mathcal{Q}(y+h)+\mathcal{Q}(y)} - \frac{\nabla y}{\mathcal{Q}(y)}\right)\cdot\nabla h = \frac{(\mathcal{Q}(y)-\mathcal{Q}(y+h))\nabla y + \mathcal{Q}(y)\nabla h}{\mathcal{Q}(y)(\mathcal{Q}(y+h)+\mathcal{Q}(y))}\cdot\nabla h.$$

Invoking the L^∞ norm and using the necessary regularity of the underlying terms, we deduce

$$\|\mathcal{R}\|_{L^\infty(\Omega)} \le (\|\mathcal{Q}(y) - \mathcal{Q}(y+h)\|_{L^\infty(\Omega)} + \|h\|_{W^1_\infty(\Omega)})\|h\|_{W^1_\infty(\Omega)}. \tag{2.3}$$

It only remains to show that $\mathcal{Q}$ is a Lipschitz continuous function. In view of (2.2), for $y, z \in v \oplus \mathring{W}^1_\infty(\Omega)$, $y \neq z$ we get

$$\|\mathcal{Q}(y) - \mathcal{Q}(z)\|_{L^\infty(\Omega)} \le \left\| \frac{\nabla(y+z)}{\mathcal{Q}(y) + \mathcal{Q}(z)} \right\|_{L^\infty(\Omega)} \|y - z\|_{W^1_\infty(\Omega)} \le \|y - z\|_{W^1_\infty(\Omega)}. \tag{2.4}$$

Then from (2.3) and (2.4) we obtain that $\|\mathcal{R}\|_{L^\infty(\Omega)} \le 2\|h\|^2_{W^1_\infty(\Omega)}$, thus $\mathcal{Q}(\cdot)$ is Fréchet differentiable. Moreover,

$$\left\| \frac{1}{\mathcal{Q}(y)} - \frac{1}{\mathcal{Q}(z)} \right\|_{L^\infty(\Omega)} = \left\| \frac{\mathcal{Q}(z) - \mathcal{Q}(y)}{\mathcal{Q}(y)\mathcal{Q}(z)} \right\|_{L^\infty(\Omega)} \le \|y - z\|_{W^1_\infty(\Omega)}. \tag{2.5}$$

Using these properties of $\mathcal{Q}$, and using the definition of the Fréchet derivative, it is not hard to derive the assertions; we omit the details for brevity. □

2.3 Limiting Case of the Linear Calderón–Zygmund Theory

With the differentiability of $\mathcal{N}$ in hand it seems to be possible to use the Implicit Function Theorem (IFT) to prove that $y - v \in \mathring{W}^1_\infty(\Omega)$. This, however, is not true in general. We begin by recalling the IFT from [28, 2.7.2].

Theorem 2.3 (implicit function theorem) *Let X, Y, and Z be Banach spaces and f a continuous mapping of an open set $U \subset X \times Y \to Z$. Assume that f has a Fréchet derivative with respect to x, $D_x f(x, y)$, which is continuous in U. Let $(x_0, y_0) \in U$ and $f(x_0, y_0) = 0$. If $D_x f(x_0, y_0)$ is an isomorphism of X onto Z then:*

- *There is a ball $B_r(y_0) := \{y : \|y - y_0\| < r\} \subset Y$ and a unique continuous map $g : B_r(y_0) \to X$ such that $g(y_0) = x_0$ and $f(g(y), y) = 0$, for all y in $B_r(y_0)$.*
- *If f is of class C^1, then $g(y)$ is of class C^1 and*

$$D_y g(y) = -[D_x f(g(y), y)]^{-1} \circ D_y f(g(y), y).$$

- *$D_y g(y)$ belongs to C^p if f is in C^p, for $p > 1$.*

Setting $y = u$ and $x = y$, $X = \mathring{W}^1_\infty(\Omega)$, $Z = W^{-1}_\infty(\Omega)$ in Theorem 2.3 and further taking $(y_0, u_0) = (0, 0)$, we obtain that $\mathcal{N}(y_0, u_0) = 0$ and $D_u\mathcal{N}(y_0, u_0)\langle h\rangle = -\Delta h$, where we have assumed that $v = 0$ for simplicity. Indeed IFT requires $-\Delta$ to be an isomorphism from $\mathring{W}^1_\infty(\Omega)$ to $W^{-1}_\infty(\Omega)$. This result, however, is not true. We illustrate this next.

Consider the linear PDE

$$\operatorname{div}(\nabla \mathfrak{u}) = \operatorname{div} \mathfrak{f} \quad \text{in } \Omega, \qquad \mathfrak{u}_{|\partial\Omega} = 0.$$

The mapping $\mathfrak{f} \mapsto \nabla \mathfrak{u}$ is given by a singular integral operator [20]. Consequently, the Calderón-Zygmund theory yields that, for $p \in (1, \infty)$, it is bounded from $L^p(\Omega)^n$ to $L^p(\Omega)^n$. This result, however, is not true for $p = \infty$. In fact, if $\mathfrak{f} \in L^\infty(\Omega)^n$, we can only assert that $\nabla \mathfrak{u} \in \mathsf{BMO}(\Omega)$. See also [3,15].

In light of this negative result, in the next section, we will show that y belongs to $W_p^2(\Omega)$ with $p > n$. Indeed with this requirement on p we have $W_p^2(\Omega) \subset\subset W_\infty^1(\Omega)$.

2.4 $W_p^2(\Omega)$-Strong Solution

For a fixed $p > n$, throughout this section, we assume that $v \in W_p^2(\Omega)$. We introduce the following space

$$Y := \big(v \oplus \mathring{W}_\infty^1(\Omega)\big) \cap W_p^2(\Omega),$$

so $y \in Y$ means $y - v \in \mathring{W}_\infty^1(\Omega) \cap W_p^2(\Omega)$. We next state a W_p^2-version of Lemma 2.2.

Lemma 2.4 *Let $U_2 \subset U_1 \cap L^p(\Omega)$ be open, then for every $u \in U_2$ and $v \in W_p^2(\Omega)$, the operator $\mathcal{N}(\cdot, u) : Y \to L^p(\Omega)$ is Fréchet differentiable and the Fréchet derivative is Lipschitz continuous and is given by*

$$D_y\mathcal{N}(y, u)\,\langle h\rangle = -div\left(\left(\mathcal{I} - \frac{\nabla y \nabla y^T}{\mathcal{Q}(y)^2}\right)\frac{\nabla h}{\mathcal{Q}(y)}\right).$$

Moreover, $\mathcal{N}$ is twice Fréchet differentiable with Lipschitz second order Fréchet derivative.

Proof For $p > n$, $W_p^1(\Omega)$ is a Banach algebra. Using this fact the proof is the same as in Lemma 2.2. □

Remark 2.5 Using Sobolev embedding theorem, for $p > n$, we have $\mathring{W}_1^1(\Omega) \subset\subset L^{p'}(\Omega)$, consequently $L^p(\Omega) \subset\subset W_\infty^{-1}(\Omega)$. Since $0 \in U_1$, we have that $U_1 \cap L^p(\Omega)$ in Lemma 2.4 is not empty. So we can set $U_2 = U_1 \cap L^p(\Omega) \neq \emptyset$.

This brings us to the main result of this section. Using the IFT we will prove that y solving (2.1) belongs to Y. Remarkably enough, we not only get the improved regularity for y but also the Fréchet differentiability of the control to state map (compare with [21, Sect. 1.4.2]). We recall that an application of the IFT requires an isomorphism of the Fréchet derivative at a point; we illustrate this in Lemma 2.7. In order to prove this, we make use of the following *Neumann* perturbation theorem [24, Chap. 4: Theorem 1.16].

Lemma 2.6 (Perturbation of identity) *Consider two Banach spaces X and Y, and two bounded linear operators $\mathcal{A}$ and $\mathcal{B}$ from X to Y. Suppose $\mathcal{A}$ has a bounded inverse from Y to X and that*

$$\|\mathcal{B}x\|_Y \leq C\|\mathcal{A}x\|_Y \quad \forall x \in X,$$

with a constant $0 < C < 1$. *Then* $\mathcal{A}+\mathcal{B} : X \to Y$ *is bijective with a bounded inverse.*

Lemma 2.7 *Let* Ω *be* $C^{1,1}$ *and* $v \in W_p^2(\Omega)$ *satisfies* $-\operatorname{div} \frac{\nabla v}{\mathcal{Q}(v)} = 0$ *in* Ω *then there exists a ball* $V \subset W_p^2(\Omega)$ *such that for every* $v \in V$, $D_y\mathcal{N}(v, 0)$ *is an isomorphism between* $W_p^2(\Omega)$ *and* $L^p(\Omega)$.

Proof We notice that $D_y\mathcal{N}(0, 0)$ coincides with the Laplace operator, which is an isomorphism between $W_p^2(\Omega)$ and $L^p(\Omega)$ for Ω of class $C^{1,1}$; see [20, Theorem 9.15]. Since $\mathcal{N}$ is twice continuously Fréchet differentiable, we obtain that

$$\begin{aligned} D_y\mathcal{N}(v, 0) &= D_y\mathcal{N}(0, 0) + (D_y\mathcal{N}(v, 0) - D_y\mathcal{N}(0, 0)) \\ &= D_y\mathcal{N}(0, 0) + \int_0^1 D_y^2\mathcal{N}(tv)\, \langle v\rangle \,\mathrm{d}t. \end{aligned}$$

Applying $D_y\mathcal{N}(0, 0)^{-1}$ to both sides yields

$$D_y\mathcal{N}(0, 0)^{-1} D_y\mathcal{N}(v, 0) = \mathcal{I} + D_y\mathcal{N}(0, 0)^{-1} \int_0^1 D_y^2\mathcal{N}(tv)\, \langle v\rangle \,\mathrm{d}t,$$

where $\mathcal{I}$ is the identity operator. We now employ Lemma 2.6 to conclude that the right-hand-side is an isomorphism provided

$$\left\| D_y\mathcal{N}(0, 0)^{-1} \int_0^1 D_y^2\mathcal{N}(tv)\, \langle v\rangle \,\mathrm{d}t \right\|_{\mathcal{L}(Y,Y)} < 1.$$

The above inequality can be guaranteed by taking $\|v\|_{W_p^2(\Omega)}$ sufficiently small, which also motivates the choice of the set V. □

We notice the above result is of relevance only when $y|_{\partial\Omega} \neq 0$, as otherwise $v = 0$ does not play any role and it is sufficient to show the isomorphism of $D_y\mathcal{N}(0, 0)$. We finally state the main result of this section.

Theorem 2.8 ($\mathring{W}_\infty^1(\Omega) \cap W_p^2(\Omega)$ state using IFT) *Let* Ω *be* $C^{1,1}$ *and* $v \in V$ *where* V *is defined in Lemma 2.7. There exists an open set* $U_3 \subset U_2$ *(recall Remark 2.5) such that* $0 \in U_3$ *and for all* $u \in U_3$, *there exists a unique solution map* $\mathcal{S} : U_3 \to Y$ *such that*

$$\mathcal{N}(\mathcal{S}(u), u) = 0, \quad \textit{for all } u \in U_3.$$

Furthermore, $\mathcal{S}$ *is twice continuously Fréchet differentiable as a function of* u *with first order derivative at* $u \in U_3$ *given by*

$$D_u\mathcal{S}(u) = -\left[D_y\mathcal{N}(y, u)\right]^{-1} \circ D_u\mathcal{N}(y, u).$$

Proof To this end it is sufficient to confirm the hypothesis of Theorem 2.3.

1. In view of Lemma 2.4, $\mathcal{N}$ is continuously Fréchet differentiable with respect to y on an open subset of $W_p^2(\Omega)$.
2. At $(y_0, u_0) = (v, 0)$, Lemma 2.7 yields $\mathcal{N}(y_0, u_0) = 0$.
3. Again using Lemma 2.7 we deduce that $D_y\mathcal{N}(y_0, u_0)$ is a Banach space isomorphism between $W_p^2(\Omega)$ to $L^p(\Omega)$.

Using the implicit function theorem, we conclude. □

2.5 W_p^2-Continuity Estimate

Theorem 2.8 provides existence and uniqueness of $W_p^2(\Omega)$-solution to the state equation but not the continuity (a priori) estimate for the solution variable. We will develop a fixed point argument to show the existence and uniqueness of solution in a ball where such an estimate holds. The proof requires the data $v \in W_p^2(\Omega)$ to be small and u to be in an open subset of $L^p(\Omega)$ (see Definition 2.12).

We begin by defining a solution set

$$\mathbb{B} = \left\{ y \in Y \ : \ \|y\|_{W_p^2(\Omega)} \le B_1 \right\}, \tag{2.6}$$

for some $B_1 > 0$ as yet to be determined (see Remark 2.17). For a given $y \in \mathbb{B}$, and fixed $u \in L^p(\Omega)$, define a map $T : \mathbb{B} \to Y$ such that $T(y) = \tilde{y}$ solves

$$-(\mathcal{Q}(y)^2 I - \nabla y \nabla y^T) : \mathrm{D}^2 \tilde{y} = u\mathcal{Q}(y)^3 \quad \text{in } \Omega. \tag{2.7}$$

This is a linearization of the state equation (1.2) obtained by formally expanding the left-hand-side of (1.2) and evaluating the non-linear "coefficient" at $y \in \mathbb{B}$. The existence of the map T is asserted in Lemma 2.10.

Lemma 2.9 *The coefficient matrix* $(\mathcal{Q}(y)^2 I - \nabla y \nabla y^T)$ *in* (2.7) *is uniformly positive definite.*

Proof Let $\mathbf{b} \in \mathbb{R}^n$ be an arbitrary nonzero column vector with components $b_1, \ldots, b_n$ and set $E = \mathcal{Q}(y)^2 I - \nabla y \nabla y^T$. Then, using the definition of $\mathcal{Q}$, we obtain

$$\begin{aligned} \mathbf{b}^T E \mathbf{b} &= \mathbf{b}^T \mathbf{b} + (\nabla y^T \nabla y)(\mathbf{b}^T \mathbf{b}) - \mathbf{b}^T \nabla y \nabla y^T \mathbf{b} \\ &= \mathbf{b}^T \mathbf{b} + (\nabla y^T \nabla y)(\mathbf{b}^T \mathbf{b}) - (\nabla y^T \mathbf{b})^T (\nabla y^T \mathbf{b}) \\ &= \mathbf{b}^T \mathbf{b} + \left(\sum_{i=1}^n |\partial_i y|^2 \right) \left(\sum_{j=1}^n |b_j|^2 \right) - \left| \sum_{i=1}^n \partial_i y b_i \right|^2 \ge \mathbf{b}^T \mathbf{b} > 0, \end{aligned}$$

where we have used the Cauchy–Schwarz inequality. □

Lemma 2.10 (existence of) T *There exist constants* $C_\Omega > 0$*, and* $B_2(n, p, B_1, \Omega) > 0$*, such that if* $v \in W_p^2(\Omega)$ *and* $u \in L^p(\Omega)$ *satisfy*

$$C_\Omega \big(\|v\|_{W_p^2(\Omega)} + B_2 \|u\|_{L^p(\Omega)} \big) \le B_1, \tag{2.8}$$

then T maps $\mathbb{B}$ to $\mathbb{B}$.

Proof For a given $y \in \mathbb{B}$, $\mathcal{Q}(y) \in L^\infty(\Omega)$, whence the right hand side in (2.7) belongs to $L^p(\Omega)$. In view of [20, Theorem 9.15] in conjunction with Lemma 2.9, there exists a unique $\tilde{y}$ solving (2.7). Moreover [20, Lemma 9.17] implies there exists a constant C_Ω such that $\tilde{y}$ satisfies the a priori estimate:

$$\|\tilde{y}\|_{W_p^2(\Omega)} \le C_\Omega(\|v\|_{W_p^2(\Omega)} + \|u\|_{L^p(\Omega)}\|\mathcal{Q}(y)^3\|_{L^\infty(\Omega)}).$$

Since $y \in \mathbb{B}$ and $W_p^2(\Omega) \subset\subset W_\infty^1(\Omega)$ with embedding constant C_S we deduce

$$\|\tilde{y}\|_{W_p^2(\Omega)} \le C_\Omega(\|v\|_{W_p^2(\Omega)} + B_2\|u\|_{L^p(\Omega)}) \tag{2.9}$$

where the constant B_2 depends on B_1, p, n and the embedding constant C_S. Choosing $\|v\|_{W_p^2(\Omega)}$ and $\|u\|_{L^p(\Omega)}$ such that (2.8) hold, we conclude that T maps $\mathbb{B}$ to $\mathbb{B}$. □

Theorem 2.11 (T is a contraction) *If, in addition to* (2.8), $u \in L^p(\Omega)$ *and* $v \in W_p^2(\Omega)$ *further satisfy*

$$C_T := B_3(2\|u\|_{L^p(\Omega)} + \|v\|_{W_p^2(\Omega)}) < 1, \tag{2.10}$$

for some constant $B_3(n, p, B_1, B_2, \Omega) > 0$ *then the map* $T : \mathbb{B} \to \mathbb{B}$ *is a contraction with contraction constant* C_T.

Proof Take y_1, y_2 in $\mathbb{B}$, with $y_1 \ne y_2$, and let $\tilde{y}_i = T(y_i)$ (for $i = 1, 2$) solve the linearized system (2.7). Define $\delta y := y_1 - y_2$ and $\delta\tilde{y} := \tilde{y}_1 - \tilde{y}_2$. Computing the difference between the equations satisfied by $\tilde{y}_1$ and $\tilde{y}_2$ and after various algebraic manipulations we deduce

$$\begin{aligned} -\left(\mathcal{Q}(y_2)^2 I - \nabla y_2 \nabla y_2^T\right) : \mathrm{D}^2\delta\tilde{y} &= u\left(\mathcal{Q}(y_1)^3 - \mathcal{Q}(y_2)^3\right) \\ &- \left(\mathcal{Q}(y_2)^2 I - \mathcal{Q}(y_1)^2 I + \nabla\delta y \nabla y_1^T + \nabla y_2 \nabla \delta y^T\right) : \mathrm{D}^2\tilde{y}_1. \end{aligned}$$

Again using the Sobolev embedding theorem, and $p > n$, it is easy to check that the right-hand-side belongs to $L^p(\Omega)$. Toward this end, we invoke [20, Theorem 9.15] in conjunction with Lemma 2.9 and [20, Lemma 9.17], and find that there exists a constant C_Ω, such that

$$\begin{aligned} \|\delta\tilde{y}\|_{W_p^2(\Omega)} \le C_\Omega\bigg(&\left\|u(\mathcal{Q}(y_1)^3 - \mathcal{Q}(y_2)^3)\right\|_{L^p(\Omega)} + \left\|(\mathcal{Q}(y_2)^2 - \mathcal{Q}(y_1)^2)\Delta\tilde{y}_1\right\|_{L^p(\Omega)} \\ &+ \|\nabla\delta y \nabla y_1^T : \mathrm{D}^2\tilde{y}_1\|_{L^p(\Omega)} + \|\nabla y_2 \nabla\delta y^T : \mathrm{D}^2\tilde{y}_1\|_{L^p(\Omega)}\bigg). \end{aligned}$$

We further deduce

$$
\begin{aligned}
\|\delta\tilde{y}\|_{W_p^2(\Omega)} \leq C_\Omega \Big(& \|u\|_{L^p(\Omega)} \|\mathcal{Q}(y_1)^3 - \mathcal{Q}(y_2)^3\|_{L^\infty(\Omega)} \\
& + \|\mathcal{Q}(y_1)^2 - \mathcal{Q}(y_2)^2\|_{L^\infty(\Omega)} \|\tilde{y}_1\|_{W_p^2(\Omega)} \\
& + |y_1|_{W_\infty^1(\Omega)} \|\tilde{y}_1\|_{W_p^2(\Omega)} |\delta y|_{W_\infty^1(\Omega)} + |y_2|_{W_\infty^1(\Omega)} \|\tilde{y}_1\|_{W_p^2(\Omega)} |\delta y|_{W_\infty^1(\Omega)} \Big) \\
= \mathsf{I} + \mathsf{II} + \mathsf{III} + \mathsf{IV}.
\end{aligned}
$$

Regarding the terms III and IV, it suffices to estimate III. For every $h \in W_p^2(\Omega)$ we have $|h|_{W_\infty^1(\Omega)} \leq C_S \|h\|_{W_p^2(\Omega)}$, and $\tilde{y}_1$ satisfies (2.9), therefore

$$
\mathsf{III} \leq C_\Omega^2 C_S \|y_1\|_{W_p^2(\Omega)} (\|v\|_{W_p^2(\Omega)} + B_2 \|u\|_{L^p(\Omega)}) |\delta y|_{W_\infty^1(\Omega)}.
$$

To estimate I and II, we use the fact that $\mathcal{Q}$ is Lipschitz continuous (see the proof of Lemma 2.2), $\tilde{y}$ satisfies (2.9), and $y_1, y_2 \in \mathbb{B}$, to obtain

$$
\|\delta\tilde{y}\|_{W_p^2(\Omega)} \leq B_3 (\|u\|_{L^p(\Omega)} + \|v\|_{W_p^2(\Omega)}) \|\delta y\|_{W_p^2(\Omega)},
$$

where the constant B_3 depends on C_Ω, B_1, p, and C_S where the latter is the embedding constant for $W_p^2(\Omega)$ in $W_\infty^1(\Omega)$ for $p > n$. Choosing u and v such that (2.10) holds, we get the desired contraction. □

Definition 2.12 (Control sets U and U_{ad}) Recall that $p > n$ is fixed. We define an open set

$$
U(v) := \left\{ u \in L^p(\Omega) : v \in V \ and \ (2.8), (2.10) \ holds \right\} \cap U_3.
$$

Next, define the closed set of admissible controls

$$
U_{ad} := \left\{ u \in L^2(\Omega) : \|u\|_{L^p(\Omega)} \leq \theta, \ p > n \right\},
$$

where θ is chosen such that $U_{ad} \subset U(v)$.

Since v is fixed, for simplicity, from here on, we will simply use the notation U instead of $U(v)$.

Corollary 2.13 (Fixed point solves the state equation) *Let the assumptions of Theorem 2.11 hold. Then for every $u \in U$ (see Definition 2.12), there exists a unique solution $\mathcal{S}(u) = y \in \mathbb{B}$ to the state equation.*

Proof Let $u \in U$ be fixed but arbitrary. It now follows that T is a contraction in the closed convex set $\mathbb{B}$ (cf. Theorem 2.11). Now take an initial function $y_0 \in \mathbb{B}$ and apply the Banach fixed point theorem (i.e. the iteration algorithm) to obtain a unique $y \in \mathbb{B}$ such that $T(y) = y$. In view of (2.7), this is equivalent to saying that y is the solution to the state equation. □

Next we will generalize a result from Gilbarg-Trudinger [20, Theorem 9.15, Lemma 9.17] where the lower order coefficient is in $L^q(\Omega)$, for $q > n$, instead of being in $L^\infty(\Omega)$. This is a standalone result and it will be used to show the global uniqueness of solution to our state equation (1.2) and will later in Sect. 3.2 help us in deriving the regularity for the adjoint state and the optimal control.

Theorem 2.14 *Let $A = (a_{ij})_{i,j=1}^n \in C(\overline{\Omega})^{n\times n}$ satisfy the ellipticity condition, i.e., there exists a constant $\gamma > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^n.$$

If $\mathbf{b} \in L^q(\Omega)^n$, $n < q < \infty$, then for all $f \in L^r(\Omega)$ with $1 < r \leq q$, there exists a unique $w \in W_r^2(\Omega) \cap \mathring{W}_r^1(\Omega)$ solving

$$\begin{aligned} -A : \mathrm{D}^2 w - \mathbf{b}\cdot\nabla w &= f \quad \textit{in } \Omega, \\ w &= 0 \quad \textit{on } \partial\Omega, \end{aligned} \tag{2.11}$$

with

$$\|w\|_{W_r^2(\Omega)} \leq C_\Omega \|f\|_{L^r(\Omega)}. \tag{2.12}$$

Proof We prove the result in two steps.

1. Existence and Uniqueness. As $L^\infty(\Omega)$ is dense in $L^q(\Omega)$, for $\mathbf{b} \in L^q(\Omega)^n$ there exists $\{\mathbf{b}_m\}_{m\in\mathbb{N}} \subset L^\infty(\Omega)^n$ such that $\mathbf{b}_m \to \mathbf{b}$ in $L^q(\Omega)^n$. Similarly as $C^\infty(\Omega)$ is dense in $L^r(\Omega)$, therefore there exists $\{f_m\}_{m\in\mathbb{N}} \subset C^\infty(\Omega)$ such that $f_m \to f$ in $L^r(\Omega)$. If we consider the auxiliary problem

$$\begin{aligned} -A : \mathrm{D}^2 w_m - \mathbf{b}_m\cdot\nabla w_m &= f_m \quad \text{in } \Omega \\ w_m &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

using [20, Lemma 9.17], we deduce

$$\|w_m\|_{W_r^2(\Omega)} \leq C_\Omega \|f_m\|_{L^r(\Omega)}, \quad \forall r \in (1,\infty),$$

and the right hand side converges to $\|f\|_{L^r(\Omega)}$. Since a unit ball in $W_r^2(\Omega)$ is weakly compact, there exists a subsequence, still labeled w_m, that converges weakly in $W_r^2(\Omega)$ and for $s = \frac{rq}{q-r}$ strongly in $W_s^1(\Omega)$ to a function $w \in W_r^2(\Omega) \cap \mathring{W}_r^1(\Omega)$. It remains to show that w satisfies (2.11). Because

$$\left|\int_\Omega v(\mathbf{b}_m\cdot\nabla w_m)\right| \leq \|v\|_{L^{r'}(\Omega)}\|\mathbf{b}_m\|_{L^q(\Omega)}\|w_m\|_{W_s^1(\Omega)},$$

we obtain

$$\int_\Omega f_m v = -\int_\Omega v(A : \mathrm{D}^2 w_m + \mathbf{b}_m \cdot \nabla w_m) \to \int_\Omega f v = -\int_\Omega v(A : \mathrm{D}^2 w + \mathbf{b} \cdot \nabla w),$$

for all $v \in L^{r'}(\Omega)$.

2. Continuity estimate. We first rewrite (2.11):

$$\begin{aligned} -A : \mathrm{D}^2 w &= f + \mathbf{b} \cdot \nabla w \quad && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

In view of the definition of $s = \frac{rq}{q-r}$, it immediately follows that $f + \mathbf{b} \cdot \nabla w \in L^r(\Omega)$, whence [20, Lemma 9.17] implies

$$\|w\|_{W_r^2(\Omega)} \le C_\Omega(\|f\|_{L^r(\Omega)} + \|\mathbf{b}\|_{L^q(\Omega)} \|w\|_{W_s^1(\Omega)}). \tag{2.13}$$

Toward this end, we will prove (2.12) by contradiction. Let $\{w_m\}_{m \in \mathbb{N}} \subset W_r^2(\Omega) \cap \mathring{W}_r^1(\Omega)$ be a sequence satisfying

$$\|w_m\|_{W_r^2(\Omega)} = 1, \quad \|f_m\|_{L^r(\Omega)} \to 0$$

as $m \to \infty$, where $f_m = -A : \mathrm{D}^2 w_m - \mathbf{b} \cdot \nabla w_m$. Since the unit ball of $W_r^2(\Omega)$ is weakly compact, there exists a subsequence, that converges weakly in $W_r^2(\Omega)$ and strongly in $W_s^1(\Omega)$ to a $w \in W_r^2(\Omega) \cap \mathring{W}_r^1(\Omega)$. Therefore,

$$\int_\Omega f_m v = -\int_\Omega v(A : \mathrm{D}^2 w_m + \mathbf{b} \cdot \nabla w_m) \to -\int_\Omega v(A : \mathrm{D}^2 w + \mathbf{b} \cdot \nabla w) = 0,$$

for all $v \in L^{r'}(\Omega)$, whence $-A : \mathrm{D}^2 w - \mathbf{b} \cdot \nabla w = 0$ and $w = 0$ by uniqueness. But from (2.13) we deduce

$$1 \le C_\Omega \|\mathbf{b}\|_{L^q(\Omega)} \|w\|_{W_s^1(\Omega)},$$

which is a contradiction. Thus, (2.12) holds. □

We next proceed to prove the global uniqueness of solution to our state equation (1.2). Consequently the control-to-state map $\mathcal{S} : U \to Y$ is well-defined.

Proposition 2.15 (global uniqueness of state solution) *Let the assumptions of Corollary 2.13 hold. If $y_1, y_2 \in Y$ be solutions of (1.2). Then $y_1 = y_2$.*

Proof We begin by rewriting the state equation for y_1 and y_2 in a form given in (2.7), i.e.,

$$-(\mathcal{Q}(y_1)^2 I - \nabla y_1 \nabla y_1^T) : \mathrm{D}^2 y_1 = u\mathcal{Q}(y_1)^3 \quad \text{in } \Omega,$$

and

$$-(\mathcal{Q}(y_2)^2 I - \nabla y_2 \nabla y_2^T) : \mathrm{D}^2 y_2 = u\mathcal{Q}(y_2)^3 \quad \text{in } \Omega.$$

Taking the difference, setting $\delta y = y_1 - y_2$, and rearranging terms we obtain

$$\begin{aligned} 0 = &- (\mathcal{Q}(y_2)^2 I - \nabla y_2 \nabla y_2^T) : \mathrm{D}^2 \delta y \\ &- \left(\mathcal{Q}(y_2)^2 I - \mathcal{Q}(y_1)^2 I + \nabla \delta y \nabla y_1^T + \nabla y_2 \nabla \delta y^T\right) : \mathrm{D}^2 y_1 \\ &- u(\mathcal{Q}(y_1)^3 - \mathcal{Q}(y_2)^3) \\ = &\ I_1 + I_2 + I_3, \end{aligned} \tag{2.14}$$

with $\delta y = 0$ on $\partial\Omega$. We inspect I_2 and I_3 separately. Since y_1 and y_2 belong to Y, it is straight-forward to see that

$$I_2 = -\mathbf{b} \cdot \nabla \delta y, \quad I_3 = -\mathbf{c} \cdot \nabla \delta y$$

with $\mathbf{b}, \mathbf{c} \in L^p(\Omega)^n$ where $p > n$. As a result, (2.14) becomes

$$-(\mathcal{Q}(y_2)^2 I - \nabla y_2 \nabla y_2^T) : \mathrm{D}^2 \delta y - (\mathbf{b} + \mathbf{c}) \cdot \nabla \delta y = 0$$

which according to Theorem 2.14 has a unique solution $\delta y \in W_p^2(\Omega) \cap \mathring{W}_\infty^1(\Omega)$. Finally using (2.12) we obtain $y_1 = y_2$. □

Lemma 2.16 ($\mathcal{S}$ Lipschitz) *Let $\mathcal{S}$ be the control to state map and B_1 be small enough so that*

$$\left(\frac{1}{\sqrt{1 + C_S^2 B_1^2}} - C_S^2 B_1^2\right) \geq \frac{1}{2}, \tag{2.15}$$

where C_S is the embedding constant for $W_p^2(\Omega)$ in $W_\infty^1(\Omega)$ for $p > n$. If $u_1, u_2 \in U$, then

$$\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{W_2^1(\Omega)} \leq C(n, p, B_1, \Omega)\|u_1 - u_2\|_{W_2^{-1}(\Omega)}, \tag{2.16}$$

$$\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{W_p^2(\Omega)} \leq C(n, p, B_1, \Omega)\|u_1 - u_2\|_{L^p(\Omega)}, \tag{2.17}$$

where the generic constant C only depends on its arguments.

Proof Set $y_1 = \mathcal{S}(u_1)$ and $y_2 = \mathcal{S}(u_2)$. Note that $y_1, y_2 \in \mathbb{B}$ by Corollary 2.13. Recall the equations satisfied by $y_1 \in Y$ and $y_2 \in Y$

$$-\operatorname{div}\left(\frac{1}{\mathcal{Q}(y_1)}\nabla y_1\right) = u_1, \qquad -\operatorname{div}\left(\frac{1}{\mathcal{Q}(y_2)}\nabla y_2\right) = u_2.$$

On subtracting and rearranging, we obtain

$$-\mathrm{div}\left(\frac{1}{\mathcal{Q}(y_1)}\nabla(y_1-y_2)\right)=\mathrm{div}\left(\left(\frac{1}{\mathcal{Q}(y_1)}-\frac{1}{\mathcal{Q}(y_2)}\right)\nabla y_2\right)+u_1-u_2.$$

Multiplying by (y_1-y_2) and integrating by parts, we arrive at

$$\int_\Omega \frac{1}{\mathcal{Q}(y_1)}|\nabla(y_1-y_2)|^2=-\int_\Omega\left(\frac{1}{\mathcal{Q}(y_1)}-\frac{1}{\mathcal{Q}(y_2)}\right)\nabla y_2\cdot\nabla(y_1-y_2) \\ +\langle u_1-u_2,y_1-y_2\rangle_{W_2^{-1}(\Omega),\mathring{W}_2^1(\Omega)}\,.$$

In view of

$$\mathcal{Q}(y_1)=\sqrt{1+|\nabla y_1|^2}\le\sqrt{1+\|y_1\|^2_{W^1_\infty(\Omega)}}\le\sqrt{1+C_S^2\|y_1\|^2_{W^2_p(\Omega)}}\le\sqrt{1+C_S^2B_1^2},$$

we have

$$\frac{1}{\sqrt{1+C_S^2B_1^2}}|y_1-y_2|^2_{W_2^1(\Omega)}\le\int_\Omega\frac{1}{\mathcal{Q}(y_1)}|\nabla(y_1-y_2)|^2.$$

Combining this with the algebraic identity

$$\frac{1}{\mathcal{Q}(y_1)}-\frac{1}{\mathcal{Q}(y_2)}=\frac{1}{\mathcal{Q}(y_1)\mathcal{Q}(y_2)}\frac{\nabla(y_2-y_1)\cdot\nabla(y_2+y_1)}{\mathcal{Q}(y_1)+\mathcal{Q}(y_2)},$$

and

$$\langle u_1-u_2,y_1-y_2\rangle_{W_2^{-1}(\Omega),\mathring{W}_2^1(\Omega)}\le C(\Omega)\|u_1-u_2\|_{W_2^{-1}(\Omega)}|y_1-y_2|_{W_2^1(\Omega)}$$

where $C(\Omega)$ is the Poincaré constant, we deduce

$$\frac{1}{\sqrt{1+C_S^2B_1^2}}|y_1-y_2|_{W_2^1(\Omega)}\le\left\|\frac{1}{\mathcal{Q}(y_1)}\right\|_{L^\infty(\Omega)}\left\|\frac{\nabla y_2}{\mathcal{Q}(y_2)}\right\|_{L^\infty(\Omega)} \\ \cdot\left\|\frac{\nabla(y_2+y_1)}{\mathcal{Q}(y_1)+\mathcal{Q}(y_2)}\right\|_{L^\infty(\Omega)}|y_2-y_2|_{W_2^1(\Omega)}+\|u_1-u_2\|_{W_2^{-1}(\Omega)}.$$

Since

$$\left\|\frac{1}{\mathcal{Q}(y_1)}\right\|_{L^\infty(\Omega)}<1,\quad\left\|\frac{\nabla y_2}{\mathcal{Q}(y_2)}\right\|_{L^\infty(\Omega)}<C_SB_1,\quad\left\|\frac{\nabla(y_2+y_1)}{\mathcal{Q}(y_1)+\mathcal{Q}(y_2)}\right\|_{L^\infty(\Omega)}<C_SB_1,$$

we arrive at

$$\left(\frac{1}{\sqrt{1 + C_S^2 B_1^2}} - C_S^2 B_1^2 \right) |y_1 - y_2|_{W_2^1(\Omega)} < \|u_1 - u_2\|_{W_2^{-1}(\Omega)},$$

which in combination with (2.15) yields (2.16).

To prove (2.17), we rewrite the state equation (1.2) into the non-divergence form as

$$-(\mathcal{Q}(y_1)^2 I - \nabla y_1 \nabla y_1^T) : \mathrm{D}^2 y_1 = u_1 \mathcal{Q}(y_1)^3 \ \text{ in } \Omega, \quad \text{and}$$
$$-(\mathcal{Q}(y_2)^2 I - \nabla y_2 \nabla y_2^T) : \mathrm{D}^2 y_2 = u_2 \mathcal{Q}(y_2)^3 \ \text{ in } \Omega.$$

After subtracting, rearranging, and setting $\delta y = y_1 - y_2$, we obtain

$$\begin{aligned} -(\mathcal{Q}(y_2)^2 I - \nabla y_2 \nabla y_2^T) : \mathrm{D}^2 \delta y &= (u_1 - u_2)\mathcal{Q}(y_1)^3 + u_2(\mathcal{Q}(y_1)^3 - \mathcal{Q}(y_2)^3) \\ &\quad - \left\{ \mathcal{Q}(y_2)^2 I - \mathcal{Q}(y_1)^2 I + \nabla \delta y \nabla y_1^T + \nabla y_2 \nabla \delta y^T \right\} : \mathrm{D}^2 y_1. \end{aligned}$$

Recalling (2.10), the remaining proof is similar to the proof of Theorem 2.11 and is omitted to avoid repetition. □

Remark 2.17 For the remainder of the paper we will assume that B_1 satisfies (2.15).

3 Optimality Conditions

Using the control to state map, we can rewrite the minimization problem (1.1)-(1.2) in the following reduced form:

$$\inf\ j(u) := \mathcal{J}(\mathcal{S}(u), u) \quad \text{over } u \in U_{ad}, \tag{3.1}$$

where

$$\mathcal{J}(\mathcal{S}(u), u) = \mathcal{J}_1(\mathcal{S}(u)) + \mathcal{J}_2(u),$$

with

$$\mathcal{J}_1(\mathcal{S}(u)) = \frac{1}{2}\|\mathcal{S}(u) - y_d\|^2_{L^2(\Omega)}, \quad \mathcal{J}_2(u) = \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}.$$

We begin by introducing the notion of a minimizer for our optimal control problem.

Definition 3.1 (Optimal control) A control $\overline{u} \in U_{ad}$ is said to be *optimal* if it satisfies, together with the associated optimal state $\overline{y}(\overline{u}) := \mathcal{S}(\overline{u})$,

$$\mathcal{J}(y(u), u) \geq \mathcal{J}(\overline{y}(\overline{u}), \overline{u}) \quad \text{for all } u \in U_{ad}.$$

A control $\overline{u} \in U_{ad}$ is said to be *locally optimal* in the sense of $L^p(\Omega)$, if there exists an $\varepsilon > 0$ such that above inequality holds for all $u \in U_{ad}$ such that $\|u - \overline{u}\|_{L^p(\Omega)} \leq \varepsilon$.

The above definition clearly distinguishes between local and global solutions to our optimal control problem. Although in Theorem 3.2 below we prove the existence of a global optimal control, a local optimal control plays a central role in optimization theory and algorithms. Generally speaking, gradient based numerical schemes only guarantee convergence to a local optimal solution. Thus, we state our first order necessary optimality conditions in Theorem 3.6 in terms of a local optimal control. Uniqueness of such a local optimal control is shown in Corollary 3.12 under a second order condition (Assumption 1). In order to get to Theorem 3.6 and Corollary 3.12, we prove several new results which do not assume the local condition on the control and are central to this paper. Moreover, Proposition 3.4 and Lemmas 3.13, 3.14 hold for an arbitrary $u \in U$ (recall Definition 2.12).

3.1 Existence of an Optimal Control

Let us first show the existence (not necessarily unique) of a global optimal control.

Theorem 3.2 *There exists an optimal control $\overline{u}$ solving the reduced minimization problem* (3.1).

Proof The proof is based on a minimizing sequence argument. As $\mathcal{J}$ is bounded below, there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$, i.e.

$$\inf_{u\in U_{ad}} \mathcal{J}(S(u), u) = \lim_{n\to\infty} \mathcal{J}(S(u_n), u_n).$$

By Definition 2.12, U_{ad} is a nonempty, closed, bounded and convex subset of $L^p(\Omega)$ which is a reflexive Banach space for $n < p < \infty$, thus weakly sequentially compact. Consequently, we can extract a weakly convergent subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset L^p(\Omega)$ i.e.

$$u_{n_k} \rightharpoonup \overline{u} \text{ in } L^p(\Omega), \quad \overline{u} \in U_{ad}.$$

This $\overline{u}$ is the candidate for our optimal control.

In the sequel, we drop the index k when extracting subsequences. Using Theorem 2.11, $\mathcal{S}(u_n) = y_n$ satisfies the state equation (1.2) thus $\{y_n\} \subset \mathbb{B}$. Since $Y \subset\subset v \oplus \mathring{W}^1_\infty(\Omega)$ for $p > n$, the Rellich-Kondrachov theorem yields a strongly convergent subsequence $\{y_n\}_{n\in\mathbb{N}} \subset v \oplus \mathring{W}^1_\infty(\Omega)$, i.e.

$$y_n \to \overline{y} \text{ in } v \oplus \mathring{W}^1_\infty(\Omega).$$

Note that the limit $\overline{y}$ is the state corresponding to the control $\overline{u}$. This results from replacing $\overline{y}$ with y_n in the variational equation (2.1) taking the limit and making use of the embedding $L^p(\Omega) \subset\subset W^{-1}_\infty(\Omega)$.

Finally, using the fact that $\mathcal{J}_2(u)$ is continuous in L^2-and convex, together with the strong convergence $y_n \to \overline{y}$ in $L^\infty(\Omega)$, it follows that $\mathcal{J}$ is weakly lower semicontinous, whence

$$\inf_{u \in U_{ad}} j(u) = \liminf_{n\to\infty} (\mathcal{J}_1(\mathcal{S}(u_n)) + \mathcal{J}_2(u_n)) \geq \mathcal{J}_1(\mathcal{S}(\overline{u})) + \mathcal{J}_2(\overline{u}) = \mathcal{J}(\overline{u}).$$

This completes the proof. □

3.2 First Order Necessary Conditions

In the following, let $\overline{u}$ denote the *local* optimal control. We derive the first order necessary optimality conditions that are satisfied by $\overline{u}$ with associated state $\overline{y}$. We recall the following result from [31].

Lemma 3.3 *If $\overline{u} \in U_{ad}$ denotes a local optimal control, then the first order necessary optimality condition satisfied by $\overline{u}$ is*

$$\left\langle j'(\overline{u}), u - \overline{u}\right\rangle_{L^2(\Omega), L^2(\Omega)} \geq 0, \quad \forall u \in U_{ad}.$$

Proof Since $U_{ad} \subset L^p(\Omega)$ is nonempty and convex, and j is Fréchet differentiable (see Theorem 2.8) in an open subset of $L^p(\Omega)$ containing U_{ad}, the proof follows along the lines of [31]. □

Let

$$A[\overline{y}] = \frac{1}{Q(\overline{y})}\left(\mathcal{I} - \frac{\nabla\overline{y}\nabla\overline{y}^T}{Q(\overline{y})^2}\right). \tag{3.2}$$

Then the first and second order Fréchet derivatives of $\mathcal{S}$ satisfy the following:

Proposition 3.4 *For every $u \in U$ and every $h_1, h_2 \in L^p(\Omega)$ the first and second order Fréchet derivatives $\mathcal{S}'(u)h_1 \in W^2_p(\Omega)$ and $\mathcal{S}''(u)h_1h_2 \in W^2_p(\Omega)$ at $\mathcal{S}(u) \in Y$ satisfy*

$$-\operatorname{div}(A[\mathcal{S}(u)]\nabla\mathcal{S}'(u)h_1) = h_1 \quad \text{in } \Omega, \tag{3.3}$$

$$-\operatorname{div}(A[\mathcal{S}(u)]\nabla\mathcal{S}''(u)h_1h_2) = \operatorname{div}(D_u A[\mathcal{S}(u)]\langle h_2\rangle \nabla\mathcal{S}'(u)h_1) \quad \text{in } \Omega, \tag{3.4}$$

with $\mathcal{S}'(u)h_1 = 0$, $\mathcal{S}''(u)h_1h_2 = 0$ on $\partial\Omega$ and $A[\cdot]$ given in (3.2). Moreover,

$$\|\mathcal{S}'(u)h_1\|_{W^1_2(\Omega)} \leq C(n,p,\Omega)\|h_1\|_{W^{-1}_2(\Omega)}, \quad \|\mathcal{S}'(u)h_1\|_{W^2_p(\Omega)} \leq C(n,p,\Omega)\|h_1\|_{L^p(\Omega)} \tag{3.5}$$

$$\|\mathcal{S}''(u)h_1h_2\|_{W^1_2(\Omega)} \leq C(n,p,B_1,\Omega)\|h_1\|_{W^{-1}_2(\Omega)}\|h_2\|_{L^p(\Omega)}. \tag{3.6}$$

Proof In terms of the control to state map, (1.2) can be written as $-\operatorname{div}\frac{\nabla\mathcal{S}(u)}{Q(\mathcal{S}(u))} = u$. Since the control to state map is twice Fréchet differentiable, then differentiating with respect to u in the directions h_1 and h_2 leads to (3.3) and (3.4). The first inequality in (3.5) is due to the characterization of $W_2^{-1}(\Omega)$ functions [17, P. 283, Theorem 1] and the second inequality is due to Theorem 2.14. Using both of these results, in conjunction with the Sobolev embedding $W_p^2(\Omega) \subset\subset W_\infty^1(\Omega)$ for $p > n$, gives (3.6). □

Following [31, Sect. 4.6], we introduce the adjoint state as follows.

Definition 3.5 (Adjoint state) The adjoint state $\overline{\varphi} \in \mathring{W}_2^1(\Omega)$ is the unique weak solution to the *adjoint equation*

$$-\operatorname{div}(A[\overline{y}]\nabla\overline{\varphi}) = \overline{y} - y_d \quad \text{in } \Omega, \qquad \overline{\varphi} = 0 \quad \text{on } \partial\Omega. \tag{3.7}$$

The adjoint state enables us to rewrite the first order necessary conditions presented in Lemma 3.3 as follows:

Theorem 3.6 *Every locally optimal control $\overline{u} \in U_{ad}$ for problem (1.1)–(1.2) satisfies, together with the associated adjoint state $\overline{\varphi} \in \mathring{W}_2^1(\Omega)$ defined by (3.7), the variational inequality*

$$\langle \overline{\varphi} + \alpha\overline{u}, u - \overline{u} \rangle_{L^2(\Omega),L^2(\Omega)} \geq 0, \quad \forall u \in U_{ad}. \tag{3.8}$$

Proof Using Theorem 2.8 we can infer that j is Fréchet differentiable, and the Fréchet derivative of j at $\overline{u}$ in a direction h is

$$\left\langle j'(\overline{u}), h \right\rangle_{L^{p'}(\Omega),L^p(\Omega)} = \left\langle \mathcal{J}_1'(S(\overline{u})), \mathcal{S}'(\overline{u})h \right\rangle_{L^2(\Omega),L^2(\Omega)} + \left\langle \mathcal{J}_2'(\overline{u}), h \right\rangle_{L^2(\Omega),L^2(\Omega)},$$

whence

$$\left\langle j'(\overline{u}), h \right\rangle_{L^{p'}(\Omega),L^p(\Omega)} = \left\langle \overline{y} - y_d, \mathcal{S}'(\overline{u})h \right\rangle_{L^2(\Omega),L^2(\Omega)} + \alpha \left\langle \overline{u}, h \right\rangle_{L^2(\Omega),L^2(\Omega)},$$

where $\mathcal{S}'(\overline{u})h$ is the weak solution to (3.3). Setting $\mathcal{S}'(\overline{u})h$ as a test function in the weak formulation of (3.7), and integrating by parts, yields

$$\left\langle \overline{y} - y_d, \mathcal{S}'(\overline{u})h \right\rangle_{L^2(\Omega),L^2(\Omega)} = \int_\Omega A[\overline{y}]\nabla\overline{\varphi} \cdot \nabla\mathcal{S}'(\overline{u})h = \langle \overline{\varphi}, h \rangle_{L^2(\Omega),L^2(\Omega)}$$

where the final equality is a consequence of using $\overline{\varphi}$ as a test function in the weak form of (3.3). Collecting all the estimates and using Lemma 3.3 leads to asserted inequality. We remark that the pairing $\left\langle j'(\overline{u}), h \right\rangle_{L^{p'}(\Omega),L^p(\Omega)}$ can be simply treated as the L^2-pairing. □

Remark 3.7 In general, $j'(u) = \varphi(y) + \alpha u$ for an arbitrary u in U_{ad}, where y solves (1.2) with u as right-hand-side, and $\varphi(y)$ solves (3.7) with right-hand-side given by $y - y_d$.

We next study the regularity of the adjoint $\overline{\varphi}$ and the control $\overline{u}$.

Proposition 3.8 (Regularity of the adjoint) *For every local optimal control $\overline{u}$, there exists a unique $\overline{\varphi} \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)$. If in addition $y_d \in L^p(\Omega)$, $p > n$, then $\overline{\varphi} \in W_p^2(\Omega)$.*

Proof Rewriting (3.7) in the non-divergence form, we get

$$-A[\overline{y}] : \mathrm{D}^2\overline{\varphi} - \mathrm{div}\,(A[\overline{y}]) \cdot \nabla\overline{\varphi} = \overline{y} - y_d \quad \text{in } \Omega, \qquad \overline{\varphi} = 0 \quad \text{on } \partial\Omega.$$

Since $\overline{y} \in W^2_p(\Omega)$, $p > n$, therefore $A[\overline{y}] \in W^1_p(\Omega)$, and $\mathrm{div}\,(A[\overline{y}]) \in L^p(\Omega)$, then invoking Theorem 2.14, with $q = p$, we obtain the desired result. □

Proposition 3.9 (Regularity of the optimal control) *Let $\overline{u}$ denote a local optimal control. Then $\overline{u} \in W^2_2(\Omega)$ and further if $y_d \in L^p(\Omega)$, $p > n$, then $\overline{u} \in W^2_p(\Omega)$.*

Proof In view of (3.8) we have $\overline{u} = \mathbb{P}_{U_{ad}}(-\frac{\overline{\varphi}}{\alpha})$ where $\mathbb{P}_{U_{ad}}(z) = z$ when z is strictly inside U_{ad} and $\mathbb{P}_{U_{ad}}(z) = \theta z/\|z\|_{L^p(\Omega)}$ otherwise. Then invoking Proposition 3.8 we obtain the assertion. □

3.3 Second Order Sufficient Conditions

We investigate the second order behavior of the cost functional $\mathcal{J}$. Starting from Assumption 1, we build up several intermediate results that allow us to prove Corollary 3.12 which is a quadratic growth condition on j near the optimal solution $\overline{u}$. In order to carefully handle the L^2-L^p norm discrepancy, we prove a Lipschitz continuity type result for j'' in Lemma 3.15. This requires several intermediate results which are shown in Proposition 3.4 and Lemmas 3.13 and 3.14.

We introduce a set of admissible directions.

Definition 3.10 (Admissible set) Given $u \in U_{ad}$, the convex set $\mathcal{C}(u)$ comprises all directions $h \in L^p(\Omega)$ such that $u + th \in U_{ad}$ for some $t > 0$, i.e.

$$\mathcal{C}(u) := \left\{h \in L^p(\Omega) : u + th \in U_{ad}, \;\; \text{for some} \;\; t > 0\right\}.$$

Assumption 1 *We make the following standard assumption about the second order behavior of the cost functional:*

$$j''(\overline{u})(u - \overline{u})^2 \geq \delta\|u - \overline{u}\|^2_{L^2(\Omega)}, \quad \forall u - \overline{u} \in \mathcal{C}(\overline{u}), \quad \textit{for some fixed}\ \delta > 0. \tag{3.9}$$

Remark 3.11 (Admissible set vs critical cone) In the context of second order sufficient conditions, generally, one uses the cone of critical directions [9,12,31]

$$\mathcal{K}(\overline{u}) = \left\{\overline{\mathcal{C}(\overline{u})} : j'(\bar{u})h = 0\right\}, \tag{3.10}$$

and an assumption

$$j''(\overline{u})h^2 > 0, \quad \forall h \in \mathcal{K}(\overline{u}) \setminus \{0\} \tag{3.11}$$

to prove the so-called quadratic growth condition (3.13). We refer to [9, Theorem 4.1] for a proof of the quadratic growth condition where the state equation is a semilinear elliptic partial differential equation. In addition, given u_a, u_b in $L^\infty(\Omega)$ with $u_a \leq u_b$, the control u in [9] fulfills

$$u_a(x) \leq u(x) \leq u_b(x) \quad x \text{ a.e. in } \Omega.$$

We notice that the control constraints in our case are of integral type and the state equation is a degenerate quasilinear partial differential equation. Thus the result of [9] do not apply directly. However, it is plausible to work under the assumption (3.11) but we believe that this is a subject requiring investigation, even in case of extremely well-studied semilinear problems [9,12,31]. Instead we will prove the quadratic growth condition under the coercivity condition (3.9). Recently, a similar condition was used in [4,5], and a rigorous proof was provided.

Our next goal is to prove the following crucial result:

Corollary 3.12 (Quadratic growth near a local optimal control) *Let the control $\overline{u} \in U_{ad}$ satisfy the first order necessary optimality condition* (3.8) *and assume that* (3.9) *holds. Then there exists an $\epsilon > 0$ such that, for all $u \in U_{ad}$ with $\|u - \overline{u}\|_{L^p(\Omega)} \leq \epsilon$, we have*

$$\left\langle j'(u) - j'(\overline{u}), u - \overline{u} \right\rangle_{L^2(\Omega) \times L^2(\Omega)} \geq \frac{\delta}{2} \|u - \overline{u}\|^2_{L^2(\Omega)}, \tag{3.12}$$

and

$$j(u) \geq j(\overline{u}) + \frac{\delta}{4} \|u - \overline{u}\|^2_{L^2(\Omega)}. \tag{3.13}$$

In particular, $\overline{u}$ is a unique local optimal control (see Definition 3.1).

The proof requires a non-trivial estimate which we will prove in Lemma 3.15. Such an estimate is needed to deal with the so-called *2-norm discrepancy*, we refer to [11] for further reading on the subject. We will conclude this section with a proof of Corollary 3.12, first we need several intermediate results.

Lemma 3.13 (A is Lipschitz) *If $u_1, u_2 \in U$, with $u_1 \neq u_2$, the map $A : Y \to W^1_p(\Omega)$ in* (3.7) *satisfies*

$$\|A[\mathcal{S}(u_1)] - A[\mathcal{S}(u_2)]\|_{L^\infty(\Omega)} \leq C(n, p, B_1, \Omega) \|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{W^1_\infty(\Omega)}, \tag{3.14}$$

$$\|A[\mathcal{S}(u_1)] - A[\mathcal{S}(u_2)]\|_{L^2(\Omega)} \leq C(n, p, B_1, \Omega) \|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{W^1_2(\Omega)}, \tag{3.15}$$

and for $h_1 \in L^p(\Omega)$, $\mathcal{S}' : U \to \mathcal{L}(L^p(\Omega), Y)$:

$$\|D_u(A[\mathcal{S}(u_1)] - A[\mathcal{S}(u_2)]) \langle h_1 \rangle \|_{L^2(\Omega)} \leq C(n, p, B_1, \Omega) \|(\mathcal{S}'(u_1) - \mathcal{S}'(u_2)) h_1\|_{W^1_2(\Omega)}. \tag{3.16}$$

Proof In order to ease the notation we set $y_1 = \mathcal{S}(u_1)$ and $y_2 = \mathcal{S}(u_2)$. It is enough to show (3.14), the same proof works for (3.15) and (3.16). Now

$$\|A[y_1] - A[y_2]\|_{L^\infty(\Omega)} \leq \left\| \frac{1}{\mathcal{Q}(y_1)} - \frac{1}{\mathcal{Q}(y_2)} \right\|_{L^\infty(\Omega)} + \left\| \frac{\nabla y_1 \nabla y_1^T}{\mathcal{Q}(y_1)^3} - \frac{\nabla y_2 \nabla y_2^T}{\mathcal{Q}(y_2)^3} \right\|_{L^\infty(\Omega)}.$$

We consider each term on the right hand side separately. For the first term, we recall (2.5). Setting $\delta y = y_1 - y_2$ and invoking the triangle inequality on the second term leads to

$$\begin{aligned}\left\|\frac{\nabla y_1 \nabla y_1^T}{\mathcal{Q}(y_1)^3} - \frac{\nabla y_2 \nabla y_2^T}{\mathcal{Q}(y_2)^3}\right\|_{L^\infty(\Omega)} &\leq \left\|\frac{\nabla y_1 \nabla y_1^T \mathcal{Q}(y_2)^3 - \nabla y_2 \nabla y_2^T \mathcal{Q}(y_1)^3}{\mathcal{Q}(y_1)^3 \mathcal{Q}(y_2)^3}\right\|_{L^\infty(\Omega)} \\ &\leq \left\|\frac{\nabla \delta y \nabla y_1^T \mathcal{Q}(y_2)^3}{\mathcal{Q}(y_1)^3 \mathcal{Q}(y_2)^3}\right\|_{L^\infty(\Omega)} + \left\|\frac{\nabla y_2 \nabla \delta y \mathcal{Q}(y_2)^3}{\mathcal{Q}(y_1)^3 \mathcal{Q}(y_2)^3}\right\|_{L^\infty(\Omega)} \\ &\quad + \left\|\frac{\nabla y_2 \nabla y_2^T (\mathcal{Q}(y_2)^3 - \mathcal{Q}(y_1)^3)}{\mathcal{Q}(y_1)^3 \mathcal{Q}(y_2)^3}\right\|_{L^\infty(\Omega)} \\ &\leq C |\delta y|_{W^1_\infty(\Omega)},\end{aligned}$$

where $C > 0$ is a generic uniform constant depending on n, p, Ω and B_1. □

Lemma 3.14 ($\mathcal{S}'$ is Lipschitz) *Let* $u, u_1, u_2 \in U$, *and* $h_1 \in L^p(\Omega)$. *Then* $\mathcal{S}' : U \to \mathcal{L}(L^p(\Omega), Y)$ *satisfies*

$$\|(\mathcal{S}'(u_1) - \mathcal{S}'(u_2))h_1\|_{W^1_2(\Omega)} \leq C(n, p, B_1, \Omega)\|u_1 - u_2\|_{L^p(\Omega)}\|h_1\|_{L^2(\Omega)}. \tag{3.17}$$

Proof Consider the system satisfied by $\mathcal{S}'(u_1)h_1$ and $\mathcal{S}'(u_2)h_1$ from Proposition 3.4:

$$\begin{aligned} -\operatorname{div}(A[\mathcal{S}(u_1)]\nabla \mathcal{S}'(u_1)h_1) &= h_1 \quad \text{in } \Omega, \qquad \mathcal{S}'(u_1)h_1 = 0 \quad \text{on } \partial\Omega \\ -\operatorname{div}(A[\mathcal{S}(u_2)]\nabla \mathcal{S}'(u_2)h_1) &= h_1 \quad \text{in } \Omega, \qquad \mathcal{S}'(u_2)h_1 = 0 \quad \text{on } \partial\Omega \end{aligned}$$

On subtracting and rearranging

$$\begin{aligned} -\operatorname{div}(A[\mathcal{S}(u_1)]\nabla(\mathcal{S}'(u_1) - \mathcal{S}'(u_2))h_1) &= \operatorname{div}(A[\mathcal{S}(u_1)] - A[\mathcal{S}(u_2)]\nabla \mathcal{S}'(u_2)h_1) \text{ in } \Omega \\ (\mathcal{S}'(u_1) - \mathcal{S}'(u_2))h_1 &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Using the characterization of $W_2^{-1}(\Omega)$ functions [17, P. 283, Theorem 1] we deduce

$$\|(\mathcal{S}'(u_1) - \mathcal{S}'(u_2))h_1\|_{W^1_2(\Omega)} \leq C(\Omega)\|A[\mathcal{S}(u_1)] - A[\mathcal{S}(u_2)]\|_{L^\infty(\Omega)}\|\mathcal{S}'(u_2)h_1\|_{W^1_2(\Omega)}.$$

Using (3.14) and (3.5), we obtain

$$\|(\mathcal{S}'(u_1) - \mathcal{S}'(u_2))h_1\|_{W^1_2(\Omega)} \leq C(n, p, B_1, \Omega)\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{W^1_\infty(\Omega)}\|h_1\|_{W_2^{-1}(\Omega)}.$$

Using (2.17) and $W_2^{-1}(\Omega) \hookrightarrow L^2(\Omega)$ we get (3.17). □

The treatment of the $L^2 - L^p$ norm discrepancy requires a technical result. This result makes use of the previous estimates in this section.

Lemma 3.15 (Auxiliary result for the $L^2 - L^p$ norm discrepancy) *Let* $u \in U$ *and* $y_d, h, h_1, h_2 \in L^p(\Omega)$. *Then there exists a constant* $L(n, p, B_1, \Omega) > 0$ *such that*

$$
\begin{aligned}
&|\, j''(u+h)\,\langle h_1, h_2\rangle - j''(u)\,\langle h_1, h_2\rangle\,| \\
&\quad \le L(\|h\|_{L^2(\Omega)}\|h_2\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}\|h_2\|_{L^2(\Omega)})\|h_1\|_{L^2(\Omega)}.
\end{aligned}
\tag{3.18}
$$

Proof Using the reduced cost functional (3.1), a simple calculation gives

$$
\begin{aligned}
j''(u+h)\,\langle h_1, h_2\rangle - j''(u)\,\langle h_1, h_2\rangle &= \int_\Omega (\mathcal{S}'(u+h)^2 - \mathcal{S}'(u)^2)h_1h_2 \\
&\quad + \int_\Omega \big[(\mathcal{S}(u+h) - y_d)\mathcal{S}''(u+h) - (\mathcal{S}(u) - y_d)\mathcal{S}''(u)\big]\, h_1h_2 \\
&= \int_\Omega (\mathcal{S}'(u+h) - \mathcal{S}'(u))h_1(\mathcal{S}'(u+h) + \mathcal{S}'(u))h_2 \\
&\quad + \int_\Omega \big[(\mathcal{S}(u+h) - \mathcal{S}(u))\mathcal{S}''(u+h) + (\mathcal{S}(u) - y_d)(\mathcal{S}''(u+h) - \mathcal{S}''(u))\big]\, h_1h_2.
\end{aligned}
$$

Using the triangle inequality in combination with Cauchy-Schwarz, we arrive at

$$
\begin{aligned}
&|\, j''(u+h)\,\langle h_1, h_2\rangle - j''(u)\,\langle h_1, h_2\rangle\,| \\
&\le \|(\mathcal{S}'(u+h) - \mathcal{S}'(u))h_1\|_{L^2(\Omega)}\|(\mathcal{S}'(u+h) + \mathcal{S}'(u))h_2\|_{L^2(\Omega)} \\
&\quad + \|\mathcal{S}(u+h) - \mathcal{S}(u)\|_{L^2(\Omega)}\|\mathcal{S}''(u+h)h_1h_2\|_{L^2(\Omega)} \\
&\quad + \Big|\int_\Omega (\mathcal{S}(u) - y_d)(\mathcal{S}''(u+h) - \mathcal{S}''(u))h_1h_2\Big| \\
&= \mathsf{I} + \mathsf{II} + \mathsf{III}.
\end{aligned}
$$

We will estimate each term $\mathsf{I} - \mathsf{III}$ individually. In view of (3.17), (3.5)

$$
\mathsf{I} \le C(n, p, B_1, \Omega)\|h\|_{L^p(\Omega)}\|h_1\|_{L^2(\Omega)}\|h_2\|_{L^2(\Omega)}
$$

and using (2.16) and (3.6)

$$
\mathsf{II} \le C(n, p, B_1, \Omega)\|h\|_{L^2(\Omega)}\|h_1\|_{L^2(\Omega)}\|h_2\|_{L^p(\Omega)}.
$$

The estimate for the term III is more involved. Recall (3.4), namely the system satisfied by $\mathcal{S}''(u+h)h_1h_2$ and $\mathcal{S}''(u)h_1h_2$:

$$
\begin{aligned}
-\operatorname{div}(A[\mathcal{S}(u+h)]\nabla\mathcal{S}''(u+h)h_1h_2) &= \operatorname{div}(D_uA[\mathcal{S}(u+h)]\,\langle h_2\rangle\,\nabla\mathcal{S}'(u+h)h_1) \;\text{ in } \Omega, \\
-\operatorname{div}(A[\mathcal{S}(u)]\nabla\mathcal{S}''(u)h_1h_2) &= \operatorname{div}(D_uA[\mathcal{S}(u)]\,\langle h_2\rangle\,\nabla\mathcal{S}'(u)h_1) \;\text{ in } \Omega,
\end{aligned}
$$

with $\mathcal{S}''(u+h)h_1h_2 = 0$ and $\mathcal{S}''(u)h_1h_2 = 0$ on $\partial\Omega$. On subtracting and rearranging, we obtain

$$\begin{aligned} &-\operatorname{div}\left(A[\mathcal{S}(u)]\nabla(\mathcal{S}''(u) - \mathcal{S}''(u+h))h_1h_2\right) \\ &= \operatorname{div}\left((A[\mathcal{S}(u)] - A[\mathcal{S}(u+h)])\nabla\mathcal{S}''(u+h)h_1h_2\right) \\ &+ \operatorname{div}\left(D_uA[\mathcal{S}(u)]\langle h_2\rangle\nabla\mathcal{S}'(u)h_1 - D_uA[\mathcal{S}(u+h)]\langle h_2\rangle\nabla\mathcal{S}'(u+h)h_1\right). \end{aligned}$$

For $u \in U$, we denote the variable satisfying (3.7) by φ, with right hand side $\mathcal{S}(u) - y_d$. We further deduce

$$\begin{aligned} \mathsf{III} &= \left|\int_\Omega \nabla\varphi \cdot \Big\{((A[\mathcal{S}(u)] - A[\mathcal{S}(u+h)])\nabla\mathcal{S}''(u+h)h_1h_2) \right. \\ &\quad \left. + (D_uA[\mathcal{S}(u)]\langle h_2\rangle\nabla\mathcal{S}'(u)h_1 - D_uA[\mathcal{S}(u+h)]\langle h_2\rangle\nabla\mathcal{S}'(u+h)h_1)\Big\}\right| \\ &\le \|\varphi\|_{W^1_\infty(\Omega)}\|A[\mathcal{S}(u)] - A[\mathcal{S}(u+h)]\|_{L^2(\Omega)}\|\mathcal{S}''(u+h)h_1h_2\|_{W^1_2(\Omega)} \\ &\quad + \|\varphi\|_{W^1_\infty(\Omega)}\|D_uA[\mathcal{S}(u)]\langle h_2\rangle\|_{L^2(\Omega)}\|(\mathcal{S}'(u) - \mathcal{S}'(u+h))h_1\|_{W^1_2(\Omega)} \\ &\quad + \|\varphi\|_{W^1_\infty(\Omega)}\|D_u(A[\mathcal{S}(u)] - A[\mathcal{S}(u+h)])\langle h_2\rangle\|_{L^2(\Omega)}\|\mathcal{S}'(u+h)h_1\|_{W^1_2(\Omega)}. \end{aligned}$$

Using (3.15), (2.16), (3.6), (3.16), (3.17) and (3.5), we obtain

$$\mathsf{III} \le C(n, p, B_1, \Omega)\|\varphi\|_{W^1_\infty(\Omega)}(\|h\|_{L^2(\Omega)}\|h_2\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}\|h_2\|_{L^2(\Omega)})\|h_1\|_{L^2(\Omega)}.$$

This completes the proof. □

Lemma 3.16 (Second order behavior in a neighborhood) *If $\overline{u}$ satisfies (3.9) then*

$$j''(u)(\overline{u} - u)^2 \ge \frac{\delta}{2}\|\overline{u} - u\|^2_{L^2(\Omega)}, \tag{3.19}$$

for all $u \in U_{ad}$ with $\|u - \overline{u}\|_{L^p(\Omega)} < \frac{\delta}{4L}$. Note: the argument of j'' is different from that in (3.9).

Proof We begin by rewriting $j''(u)(\overline{u} - u)^2$:

$$\begin{aligned} j''(u)(\overline{u} - u)^2 &= j''(\overline{u})(\overline{u} - u)^2 + (j''(u)(\overline{u} - u)^2 - j''(\overline{u})(\overline{u} - u)^2) \\ &\ge j''(\overline{u})(\overline{u} - u)^2 - |(j''(u)(\overline{u} - u)^2 - j''(\overline{u})(\overline{u} - u)^2)| = \mathsf{I} - \mathsf{II} \end{aligned}$$

Using (3.9), we obtain $\mathsf{I} \ge \delta\|\overline{u} - u\|^2_{L^2(\Omega)}$. And invoking (3.18) yields

$$\mathsf{II} \le L(\|u - \overline{u}\|_{L^2(\Omega)}\|u - \overline{u}\|_{L^p(\Omega)} + \|u - \overline{u}\|_{L^p(\Omega)}\|u - \overline{u}\|_{L^2(\Omega)})\|u - \overline{u}\|_{L^2(\Omega)}.$$

Finally, combining the estimates for I and II gives

$$j''(u)(u - \overline{u})^2 \ge \delta\|\overline{u} - u\|^2_{L^2(\Omega)} - 2L\|\overline{u} - u\|_{L^p(\Omega)}\|\overline{u} - u\|^2_{L^2(\Omega)}.$$

For $\|u-\overline{u}\|_{L^p(\Omega)} < \frac{\delta}{4L}$, we obtain (3.19). □

We now arrive at the main result of this section.

Proof **[Proof of Corollary** 3.12] We proceed in two steps:
$\boxed{1}$ Let $u \in U_{ad}$ and $\|u-\overline{u}\|_{L^p(\Omega)} < \frac{\delta}{8L}$. By Taylor's theorem, there is a $t \in (0,1)$ such that

$$\begin{aligned} j(u) &= \mathcal{J}(\overline{u}) + \langle j'(\overline{u}), u-\overline{u}\rangle + \frac{1}{2} j''(tu+(1-t)\overline{u})(u-\overline{u})^2 \\ &= j(\overline{u}) + \langle j'(\overline{u}), u-\overline{u}\rangle + \frac{1}{2} j''(\overline{u})(u-\overline{u})^2 + \frac{1}{2}(j''(tu+(1-t)\overline{u}) - j''(\overline{u}))(u-\overline{u})^2 \\ &\geq j(\overline{u}) + \langle j'(\overline{u}), u-\overline{u}\rangle + \frac{\delta}{2}\|u-\overline{u}\|^2_{L^2(\Omega)} - \left|\frac{1}{2}\big(j''(tu+(1-t)\overline{u}) - j''(\overline{u})\big)(u-\overline{u})^2\right| \end{aligned}$$

where the last inequality is due to (3.9). Next, (3.18) gives

$$j(u) \geq j(\overline{u}) + \langle j'(\overline{u}), u-\overline{u}\rangle + \frac{\delta}{2}\|u-\overline{u}\|^2_{L^2(\Omega)} - 2L\|u-\overline{u}\|_{L^p(\Omega)}\|u-\overline{u}\|^2_{L^2(\Omega)},$$

which implies

$$j(u) \geq j(\overline{u}) + \langle j'(\overline{u}), u-\overline{u}\rangle + \frac{\delta}{4}\|u-\overline{u}\|^2_{L^2(\Omega)}. \tag{3.20}$$

Using Lemma 3.3, we obtain (3.13).
$\boxed{2}$ Since $\|u-\overline{u}\|_{L^p(\Omega)} < \frac{\delta}{8L}$ (i.e. u satisfies (3.19)), we can repeat all the steps in $\boxed{1}$ with u replaced by $\overline{u}$ and vice-versa to get

$$j(\overline{u}) \geq j(u) + \langle j'(u), \overline{u}-u\rangle + \frac{\delta}{4}\|\overline{u}-u\|^2_{L^2(\Omega)}. \tag{3.21}$$

Adding (3.20) and (3.21) and setting $\epsilon = \frac{\delta}{8L}$ proves the corollary. □

4 Discrete Control Problem

Let $\mathcal{T}$ denote a geometrically conforming, quasiuniform triangulation of the domain Ω such that $\overline{\Omega} = \cup_{K\in\mathcal{T}} K$ with K closed and h the meshsize of $\mathcal{T}$. Consider the following finite dimensional spaces

$$\begin{aligned} Y^h &= \left\{ y_h \in C^0(\overline{\Omega}) : y_h|_K \in \mathbb{P}_1(K),\, K \in \mathcal{T} \right\}, \\ \mathring{Y}^h &= Y^h \cap \mathring{W}^1_\infty(\Omega), \\ U^h_{ad} &= Y^h \cap U_{ad}. \end{aligned} \tag{4.1}$$

The spaces U^h_{ad}, Y^h will be used to approximate the continuous solution of (1.1) and (1.2). The spaces are based on the finite dimensional space $\mathbb{P}_1$ which are the linear polynomials on the domain K, where K is a triangle. This discretization is classical and can be found in any standard finite element book, for instance [8,14]. We remark that in our numerical implementation the L^p constraints in U^h_{ad} are enforced by scaling the functions with their L^p-norm, we refer to §s:computations for more details. Using $I_h : W^1_r(\Omega) \to Y^h$ we denote the interpolation operator, i.e. if $r > n$ then I_h is the standard Lagrange interpolation operator, otherwise it indicates the so-called Scott-Zhang interpolation operator [30].

The discrete version of the continuous optimal control problem (1.1) is

$$\inf \mathcal{J}_h(y_h, u_h) := \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_h\|^2_{L^2(\Omega)} \quad \text{over } y_h - I_h v \in \mathring{Y}^h, u_h \in U^h_{ad}, \tag{4.2}$$

subject to $y_h - I_h v \in \mathring{Y}_h$ solving the discrete state equation

$$\int_\Omega \frac{\nabla y_h}{\mathcal{Q}(y_h)} \cdot \nabla z_h = \int_\Omega u_h z_h, \quad \text{for all } z_h \in \mathring{Y}^h. \tag{4.3}$$

We remark that in (4.2), for simplicity, we have not discretized y_d.

The discrete optimality conditions amount to the state (4.3); the adjoint, find $\overline{\varphi}_h \in \mathring{Y}^h$ such that

$$\int_\Omega \nabla z_h^T A[\overline{y}_h] \nabla \overline{\varphi}_h = \int_\Omega (\overline{y}_h - y_d) z_h \quad \text{for all } z_h \in \mathring{Y}^h, \tag{4.4}$$

where $A[\overline{y}_h] = \frac{1}{\mathcal{Q}(\overline{y}_h)}\left(\mathcal{I} - \frac{\nabla \overline{y}_h \nabla \overline{y}_h^T}{\mathcal{Q}(\overline{y}_h)^2}\right)$, and the discrete variational inequality for the optimal control

$$\left\langle \overline{\varphi}_h + \alpha \overline{u}_h, u_h - \overline{u}_h \right\rangle_{L^2(\Omega), L^2(\Omega)} \geq 0, \quad \text{for all } u_h \in U^h_{ad}. \tag{4.5}$$

Remark 4.1 Similar to Remark 3.7, the discrete functional derivative is given by $j_h'(u_h) = \varphi_h(y_h) + \alpha u_h$ for an arbitrary u_h in U^h_{ad}, where y_h solves (4.3) with u_h as right-hand-side, and $\varphi_h(y_h)$ solves (4.4) with right-hand-side given by $y_h - y_d$.

5 Numerical Examples

5.1 Setup

We present numerical examples for the discrete optimal control problem in Sect. 4. We solve the optimization problem using MATLAB's optimization toolbox with an SQP method, where we provide the gradient information.

The gradient of the cost functional (4.2), at each iteration of the optimization algorithm, is computed by first solving the state equation (4.3) for y_h with the control u_h taken from the previous iteration. Then, the adjoint problem (4.4) is solved for φ_h using the discrete solution y_h. We then define the linear form (see Remark 4.1)

$$\left\langle j_h'(u_h), v_h \right\rangle_{L^2(\Omega), L^2(\Omega)} = \int_\Omega (\varphi_h + \alpha u_h) v_h, \quad \text{for all } v_h \in Y^h,$$

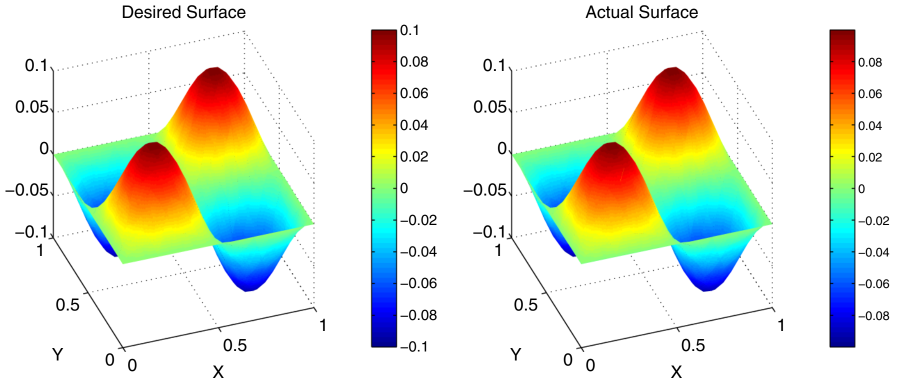

Fig. 2 *Left* Desired surface height $y_d = \sin(2\pi x)\sin(2\pi y)$. *Right* Actual surface height $\overline{y}$ (after the optimization method converges). Boundary data is $v = 0$

and pass the discrete gradient vector (and cost value) to MATLAB's optimization algorithm at the current iteration. The constraint on the control U_{ad}^h is handled by MATLAB's optimization algorithm by specifying an inequality constraint on u_h.

The non-linear state equation is solved with Newton's method and a direct solver (backslash); we also use a direct solver for the adjoint problem. This was all implemented in MATLAB using the FELICITY toolbox [33]. The following sections show some examples of our computational method. In all cases, we set $\alpha = 10^{-6}$ and $p = 2.5$. For most examples, we set $\theta = 20$ in the definition of U_{ad}^h, except in Sect. 5.2.2 where $\theta = 2$. The first two examples are posed on a unit square domain, which technically does not satisfy the $C^{1,1}$ domain assumption. The last example is posed on a C^∞ domain in the shape of a four-leaf clover.

5.2 Sine on a Square

5.2.1 $\theta = 20$

We take y_d to be a product of sine functions and set the boundary data to $v = 0$. The domain Ω is the unit square. See Figs. 2 and 3 for plots of y_d, $\overline{y}$, $\overline{u}$, and the optimization history. This example shows that we can recover the desired surface almost exactly when the boundary condition v matches y_d on $\partial\Omega$. Note: for this optimal control, we have $\|\overline{u}\|_{L^p(\Omega)} \approx 3.75$.

5.2.2 $\theta = 2$

We run the same example as in Sect. 5.2.1, except we choose a smaller value of θ to see the impact on the quality of the optimal control; all other parameters are identical. See Figs. 4 and 5 for plots of y_d, $\overline{y}$, $\overline{u}$, and the optimization history. The value of $\|\overline{u}\|_{L^p(\Omega)}$ in the previous example was ≈ 3.75. Here, $\|\overline{u}\|_{L^p(\Omega)}$ is constrained to be ≤ 2 (in fact, it is equal to 2).

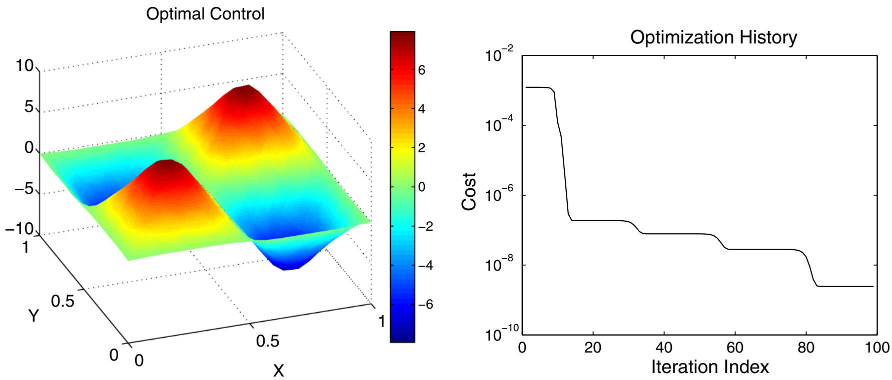

Fig. 3 *Left* Optimal control function $\overline{u}$ for y_d in Fig. 2. *Right* Decrease of cost functional $\mathcal{J}$

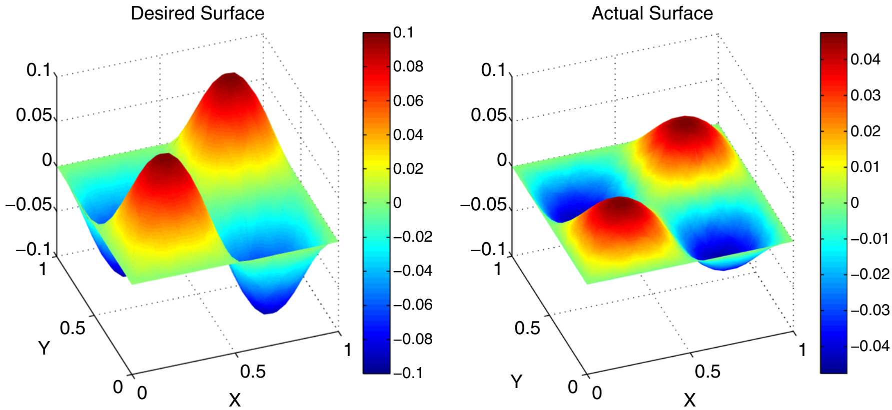

Fig. 4 *Left* desired surface height $y_d = \sin(2\pi x)\sin(2\pi y)$. *Right* actual surface height $\overline{y}$ (after the optimization method converges). Boundary data is $v = 0$. Note: $\theta = 2$ here

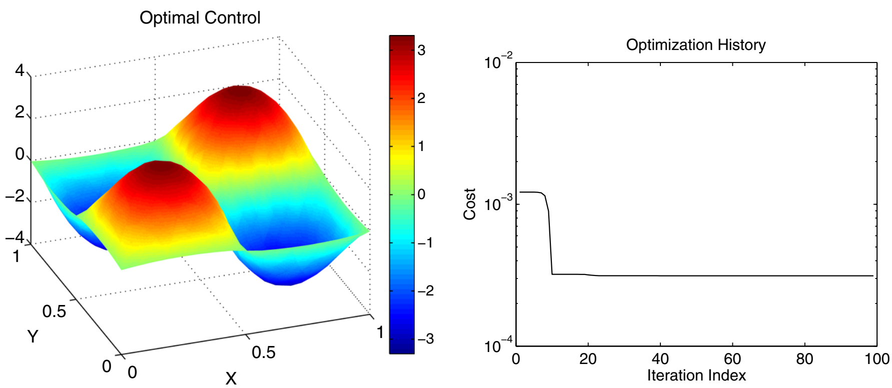

Fig. 5 *Left* optimal control function $\overline{u}$ for y_d in Fig. 4. *Right* decrease of cost functional $\mathcal{J}$

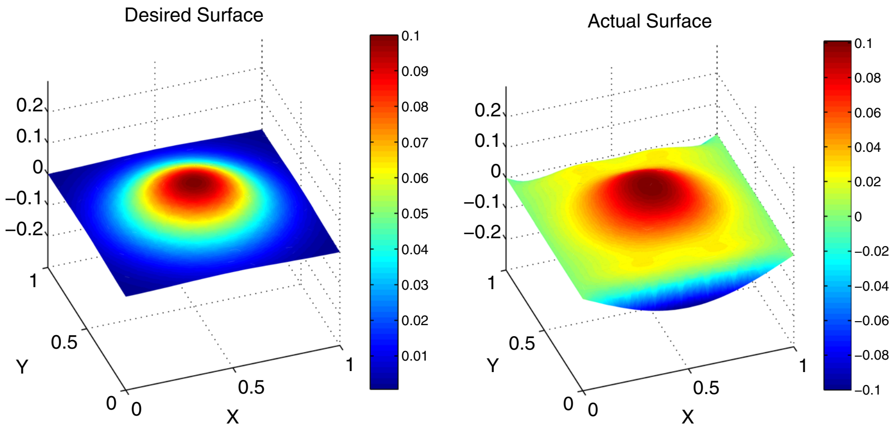

Fig. 6 *Left* Desired surface height $y_d = 0.1\exp\left(-((x-0.5)^2+(y-0.5)^2)/0.1\right)$. *Right* Actual surface height $\overline{y}$ (after the optimization method converges). Boundary data is $v = -0.1\sin(\pi x)\cos(2\pi y)$

It is clear from Fig. 5 that the height of the optimal control is less than in Fig. 3 (note the different scale in the plot). Moreover, $\overline{u}$ is not as "peaked" as before (more rounded), but is qualitatively the same. This, in turn, affects the obtained surface height $\overline{y}$ in Fig. 4, i.e. it appears to be uniformly scaled with respect to the result in Fig. 2. In other words, the main effect that θ has is to *scale down* the optimal control, which shrinks the obtained surface height. But the qualitative shape of $\overline{u}$ and $\overline{y}$ is essentially the same as before.

5.3 Gaussian on a Square (Nonzero Boundary Condition)

We take y_d to be a Gaussian bump and set the boundary data to $v = -0.1\sin(\pi x)\cos(2\pi y)$. The domain Ω is the unit square. See Figs. 6 and 7 for plots of y_d, $\overline{y}$, $\overline{u}$, and the optimization history. In this case, we impose a mismatch between the imposed boundary condition v and the desired surface y_d. The results show that the optimization does the "best it can" by trying to match y_d in the interior of Ω. Note the large value of the control $\overline{u}$ at the boundary of Ω in Fig. 7.

5.4 Cosine on a Clover

We take y_d to be a product of cosine functions and set the boundary data to $v = 0$. The domain Ω is a four-leaf clover (smooth domain). See Figs. 8 and 9 for plots of y_d, $\overline{y}$, $\overline{u}$, and the optimization history. This example also has a mismatch between the imposed boundary condition v and y_d. Again, the optimal surface $\overline{y}$ matches y_d well in the interior of Ω, but not at the boundary. Moreover, in Fig. 9, it is evident from the convergence history of the optimization algorithm that the path to the optimal control is non-trivial.

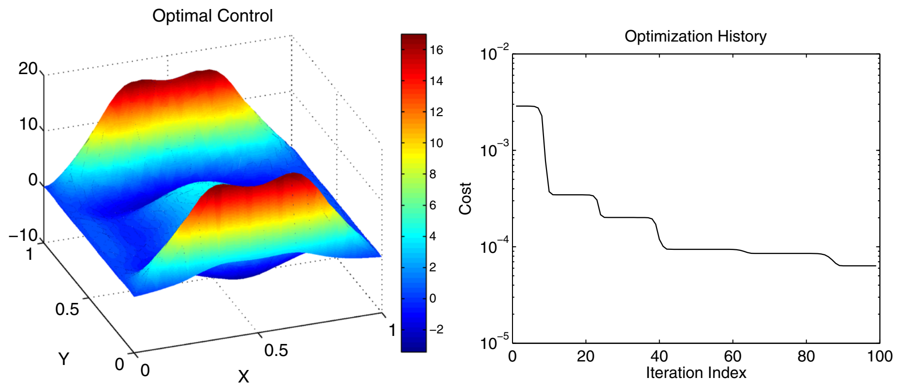

Fig. 7 *Left* Optimal control function $\overline{u}$ for y_d in Fig. 6. *Right* Decrease of cost functional $\mathcal{J}$

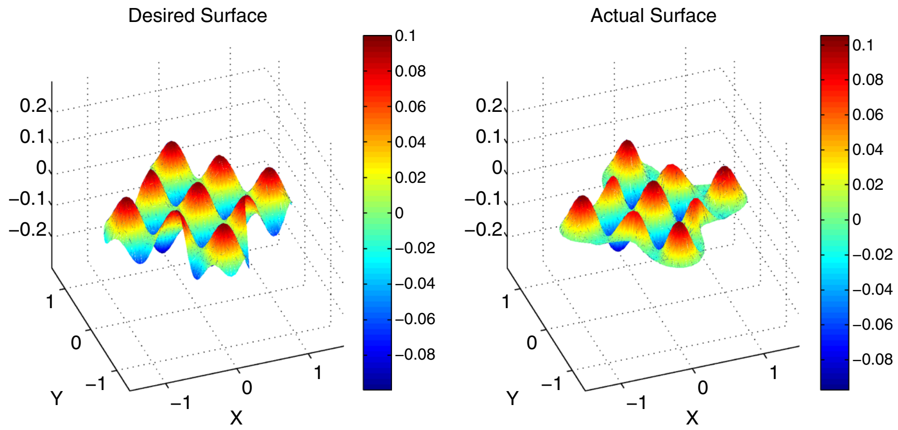

Fig. 8 *Left* Desired surface height $y_d = 0.1\cos(2\pi x)\cos(2\pi y)$. *Right* Actual surface height $\overline{y}$ (after the optimization method converges). Boundary data is $v = 0$

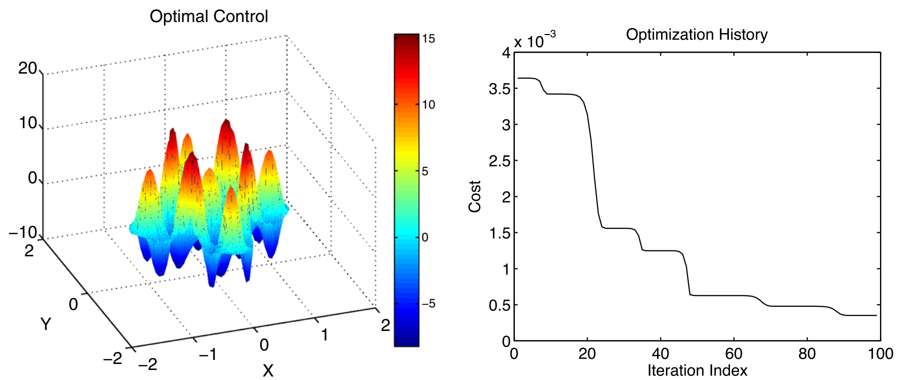

Fig. 9 *Left* Optimal control function $\overline{u}$ for y_d in Fig. 8. *Right* Decrease of cost functional $\mathcal{J}$

6 Conclusion and Future Work

The mean curvature operator is only locally-coercive, which leads to several difficulties in proving the existence of solution to the PDE. Using two approaches, (i) the implicit function theorem (see Theorem 2.8) and (ii) a fixed point theorem (see Theorem 2.11), we provide a complete second order analysis to this PDE. This requires a smallness condition on the boundary data v and right-hand-side u. We handle (i) by proving various Fréchet differentiability results, where as for (ii) we prove a new result for second order elliptic PDEs in non-divergence form, where the lower order coefficients need not be bounded (for the bounded coefficient case, see [20, Theorem 9.15]).

By using the regularity results for the PDE, we rigorously justify the first and second order sufficient optimality conditions and further tackle the *2-norm discrepancy* in the $L^p - L^2$ pair. The discretization of the PDE uses a finite element method.

There are some possible extensions of this work. The first could be boundary control. The second is where the surface tension coefficient $K \in \mathbb{R}^{n\times n}$ in the operator

$$- \operatorname{div} K \frac{\nabla y}{\mathcal{Q}(y)}$$

acts as an optimal control, and the right-hand-side u acts as a driving force. This would be especially applicable to material science, where the presence of colloidal particles on a surface, or interface, can modulate surface tension.

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