A mixed formulation of the Stefan problem with surface tension

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A dual formulation and finite element method is proposed and analyzed for simulating the Stefan problem with surface tension. The method uses a mixed form of the heat equation in the solid and liquid (bulk) domains, and imposes a weak formulation of the interface motion law (on the solid-liquid interface) as a constraint. The basic unknowns are the heat fluxes and temperatures in the bulk, and the velocity and temperature on the interface. The formulation, as well as its discretization, is viewed as a saddle point system. Well-posedness of the time semi-discrete and fully discrete formulations is proved in three dimensions, as well as an an a priori (stability) bound and conservation law. Simulations of interface growth (in two dimensions) are presented to illustrate the method.

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1. Introduction

1.1 Background

The Stefan problem describes the geometric evolution of a solidifying (or melting) interface. It is a classic problem in phase transitions. The model consists of time-dependent heat diffusion in the solid and liquid phases, with an interfacial condition on the solid-liquid interface known as the Gibbs-Thomson relation with kinetic undercooling [41, 42, 61]. A thermodynamic derivation of the model can be found in [29]. Applications range from modeling the freezing (or melting) of water to the solidification of crystals from a melt and dendritic growth [15, 30, 38, 51, 52, 59]. Mathematical theory for the Stefan problem with Gibbs-Thomson law is available for local and global in time solutions [13, 25, 36, 39, 45–48]. Well-posedness results are also available if the heat equation in the bulk phases is replaced by a quasi-static approximation (i.e. the Mullins–Sekerka problem) [18, 20, 24, 40, 49].

Efficient numerical schemes for simulating these models is necessary to allow for design, prediction, and optimization of these processes. Phase-field methods have been used for simulating solidification and dendrite growth [6, 35, 55]. Level set methods have also been used to handle the evolutions of the two phase interface [12, 23, 44, 54]. The method we present uses a front-tracking approach where the interface parametrization conforms to a surrounding bulk mesh. Other front-tracking methods for the Stefan problem have also been given [2, 4, 34, 35, 50–53].
Our paper presents a completely mixed formulation of the Stefan problem, including the bulk heat equations [8]. In other words, we formulate the problem in a saddle-point framework, where the heat equations are in mixed form, and the interface motion law appears as a constraint in the system of equations with a balancing Lagrange multiplier that represents the interface temperature. To the best of our knowledge, this is a new method for the Stefan problem with surface tension. Some highlights of our method are the following.

- We prove that both the time semi-discrete and fully discrete systems have a priori bounds (in time) that mimic the continuous model. This assumes the interface velocity is reasonably regular and that there are no topological changes. Moreover, we can prove that both the time semi-discrete and fully discrete systems maintain conservation of thermal energy. In [5], they only achieve this for their discrete in space, continuous in time, scheme.
- The interface is represented by a surface triangulation that conforms to the bulk mesh which deforms with the interface. Hence, occasional re-meshing is needed, which is done by the method in [63]. One advantage of this method is that all integrals in the finite element formulation can be computed exactly. In addition, we do not need to compute the intersection of meshes at adjacent time steps to transfer solution variables from one mesh to the next (e.g. for computing $L^2$ projections from one mesh to another).
- Our method can be modified to include anisotropic surface tension via [5], which is relevant to crystal growth. The well-posedness of the method remains unchanged, as well as the a priori bound and conservation law.
- Other variations of the Stefan problem (e.g. Mullins–Sekerka) can be formulated with our approach by straightforward modifications. One can even include moving contact line effects when the solid phase is attached to a rigid boundary [60, 64].

1.2 Summary

In Section 2, we describe the governing equations. Section 3 describes the fully continuous weak formulation and derives a formal a priori bound and conservation law. Section 4 explains the time-discretization and how the interface motion is handled. A variational formulation of the time semi-discrete problem is given, its well-posedness is shown, and an a priori bound and conservation law is proved. We then do the same for the fully-discrete formulation (Section 5). Section 6 concludes with numerical simulations to demonstrate the method.

2. Model for the Stefan problem with surface tension

The particular mathematical model we consider can be found in [5, 29]. In this section, we present the strong form of the Stefan problem.

2.1 Notation

Let $\Omega$ be a fixed domain in $\mathbb{R}^d$ (for $d = 2, 3$), with outer boundary $\partial \Omega$, that contains two phases, liquid and solid, denoted respectively by the open sets $\Omega_1$ and $\Omega_2$, i.e. $\Omega = \text{int}(\Omega_1 \cup \Omega_2)$ and $\Omega_1 \cap \Omega_2 = \emptyset$ (see Figure 1). Furthermore, $\partial \Omega$ partitions into two pieces: $\partial \Omega = \partial_D \Omega \cup \partial_N \Omega$ such that $\partial_D \Omega \cup \partial_N \Omega = \emptyset$ and $|\partial_D \Omega| > 0$ (set of positive measure).

The solid-liquid interface between the phases is $\Gamma = \Omega_1 \cap \Omega_2$ (a closed surface). The domains $\Omega_1$, $\Omega_2$, and $\Gamma$ are time-dependent, and we assume that $\Gamma(t) \subset \Omega$ for all $t$. Moreover, in order to
write the strong form of the Stefan problem (Section 2.2), we assume \( \Gamma(t) \) is smooth and let \( X(t) \) denote a parametrization of \( \Gamma(t) \):

\[
X(\cdot, t) : \mathbb{M} \rightarrow \mathbb{R}^d, \quad \text{where } \mathbb{M} \subset \mathbb{R}^d \text{ is a given reference manifold},
\]

i.e. \( \Gamma(t) = X(\mathbb{M}, t) \). Furthermore, we introduce fixed reference domains \( \mathring{\Omega}_1, \mathring{\Omega}_s \) for the liquid and solid domains such that \( \mathring{\Omega} = \text{int}(\mathring{\Omega}_1 \cup \mathring{\Omega}_s) \) and \( \mathbb{M} = \mathring{\Omega}_1 \cap \mathring{\Omega}_s \). We can extend \( X \) to be defined on all of \( \mathring{\Omega} \) and such that \( \mathring{\Omega}_1(t) = X(\mathring{\Omega}_1, t) \) and \( \mathring{\Omega}_s(t) = X(\mathring{\Omega}_s, t) \) (slight abuse of notation here). This is needed later when specifying the function spaces.

The surface \( \Gamma \) has a unit normal vector \( \nu \) that is assumed to point into \( \mathring{\Omega}_1 \) (see Figure 1). For quantities \( q \) in \( \mathring{\Omega}_1 (\mathring{\Omega}_s) \), we append a subscript: \( q_{l} (q_{s}) \). The symbol \( \kappa \) represents the summed curvature of the interface \( \Gamma \) (sum of the principle curvatures), and we assume the convention that \( \kappa \) is positive when \( \mathring{\Omega}_s \) is convex (contrary to [5]).

Table 1 summarizes the notation we use for the physical domain and the physical variables (e.g. temperature, etc.). The physical coefficient symbols that appear in the model, as well as their values, are given in Table 2. The non-dimensional parameters are given in Table 3.

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**Fig. 1:** Left: Domains in the Stefan problem. The entire “box” is \( \mathring{\Omega} = \text{int}(\mathring{\Omega}_1 \cup \mathring{\Omega}_s) \) (containing two phases \( \mathring{\Omega}_1, \mathring{\Omega}_s \)) with Dirichlet boundary \( \partial D \) denoted by the dashed line. A Neumann condition is applied on the remaining sides \( \partial D \). The interface between the phases is \( \Gamma = \mathring{\Omega}_1 \cap \mathring{\Omega}_s \) with unit normal vector \( \nu \) pointing into \( \mathring{\Omega}_1 \). Right: Simulation using the method developed in this paper (isotropic surface tension). Several time-lapses are shown to illustrate the evolution with initial interface having a “star” shape. See Section 6 for more simulations.
### Table 1: General notation and symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega, \Omega_l, \Omega_s$</td>
<td>Bulk Domains: Entire, Liquid, Solid</td>
<td>—</td>
</tr>
<tr>
<td>$\partial \Omega$</td>
<td>Boundary of $\Omega$</td>
<td>—</td>
</tr>
<tr>
<td>$\partial_0 \Omega, \partial_\Omega$</td>
<td>Partition of $\partial \Omega = \partial_0 \Omega \cup \partial_\Omega$</td>
<td>—</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Interface between $\Omega_l$ and $\Omega_s$ phases</td>
<td>—</td>
</tr>
<tr>
<td>$\mathbf{X}, \mathbf{V}$</td>
<td>Interface ($\Gamma$) Parametrisation and Velocity</td>
<td>m, m s(^{-1})</td>
</tr>
<tr>
<td>$u_l, u_s$</td>
<td>Temperature in $\Omega_l$ and $\Omega_s$</td>
<td>K (deg. Kelvin)</td>
</tr>
<tr>
<td>$f_l, f_s$</td>
<td>Heat sources in $\Omega_l$ and $\Omega_s$</td>
<td>J m(^{-3}) s(^{-1})</td>
</tr>
<tr>
<td>$\nabla_\Gamma, \Delta_\Gamma$</td>
<td>Surface Gradient and Laplace–Beltrami Operator</td>
<td>m(^{-1}), m(^{-2})</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Unit Normal Vector of $\Gamma$</td>
<td>—</td>
</tr>
<tr>
<td>$\nabla_\Gamma \mathbf{X} := \mathbf{I} - \nu \otimes \nu$</td>
<td>Projection onto Tangent Space of $\Gamma$</td>
<td>—</td>
</tr>
<tr>
<td>$\kappa, \kappa \nu := -\Delta_\Gamma \mathbf{X}$</td>
<td>Summed Curvature and Curvature Vector of $\Gamma$</td>
<td>m(^{-1})</td>
</tr>
</tbody>
</table>

### Table 2: Physical parameters and values

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vartheta$</td>
<td>Volumetric Heat Capacity</td>
<td>J m(^{-3}) K(^{-1})</td>
</tr>
<tr>
<td>$K_l, K_s$</td>
<td>Thermal Conductivity in $\Omega_l$ and $\Omega_s$</td>
<td>J s(^{-1}) m(^{-1}) K(^{-1})</td>
</tr>
<tr>
<td>$L$</td>
<td>Latent Heat Coefficient</td>
<td>J m(^{-3})</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Surface Tension Coefficient of $\Gamma$</td>
<td>J m(^{-2})</td>
</tr>
<tr>
<td>$S$</td>
<td>Volumetric Entropy Coefficient</td>
<td>J m(^{-3}) K(^{-1})</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Kinetic Coefficient</td>
<td>J s m(^{-4})</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Mobility Coefficient</td>
<td>—</td>
</tr>
<tr>
<td>$D$</td>
<td>Length Scale</td>
<td>m</td>
</tr>
<tr>
<td>$U_0 = T_M$</td>
<td>Temperature Scale</td>
<td>K</td>
</tr>
<tr>
<td>$t_0$</td>
<td>Time Scale</td>
<td>seconds (s)</td>
</tr>
<tr>
<td>$F_0 = \vartheta U_0 / t_0$</td>
<td>Heat Source Scale</td>
<td>J m(^{-3}) s(^{-1})</td>
</tr>
</tbody>
</table>

### Table 3: Nondimensional parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{S} = S / \vartheta$</td>
<td>entropy coefficient</td>
<td>2</td>
</tr>
<tr>
<td>$\bar{\beta}_0 = \vartheta U_0 t_0 / (\rho D)$</td>
<td>mobility coefficient</td>
<td>0.01</td>
</tr>
<tr>
<td>$\bar{\beta} = \bar{\beta}_0 \beta$</td>
<td>mobility function</td>
<td>-</td>
</tr>
<tr>
<td>$\bar{K}_l = K_l t_0 / (D^2 \vartheta)$</td>
<td>liquid conductivity</td>
<td>1</td>
</tr>
<tr>
<td>$\bar{K}_s = K_s t_0 / (D^2 \vartheta)$</td>
<td>solid conductivity</td>
<td>1</td>
</tr>
<tr>
<td>$\bar{\alpha} = \alpha / (U_0 D \vartheta)$</td>
<td>surface tension coefficient</td>
<td>0.0005</td>
</tr>
</tbody>
</table>
2.2 Strong formulation

The Stefan problem is as follows. Find $u : \Omega \times [0, T] \to \mathbb{R}$ and interface $\Gamma(t) \subset \Omega$ for all $t \in (0, T]$, such that $u|_{\Omega_t} = u_l, u|_{\Omega_s} = u_s$, and the following bulk conditions hold:

$$
\begin{align*}
\partial_t u_l - K_l \Delta u_l &= f_l, \quad \text{in } \Omega_l(t), \\
\partial_t u_s - K_s \Delta u_s &= f_s, \quad \text{in } \Omega_s(t), \\
\nu \Omega \cdot \nabla u &= 0, \quad \text{on } \partial_\Omega \Omega, \\
u_l \Omega \cdot \nabla u_l &= 0, \quad \text{on } \partial_\Omega \Omega_l, \\
u_s \Omega \cdot \nabla u_s &= 0, \quad \text{on } \partial_\Omega \Omega_s, \\
u_l \cdot \nu_s &= S, \quad \text{on } \partial_\Omega \Omega.
\end{align*}
$$

(2.2)

where $u_0$ is the initial temperature, and the following interface conditions hold:

$$
\begin{align*}
&u_l - u_s = 0, \quad \text{on } \Gamma(t), \\
&\nu \cdot (K_l \nabla u_l - K_s \nabla u_s) + L \partial_t X \cdot \nu = 0, \quad \text{on } \Gamma(t), \\
& \frac{\rho}{\beta(\nu)} \partial_t X \cdot \nu + \kappa + S u = 0, \quad \text{on } \Gamma(t), \\
&X(\cdot, 0) - X_0(\cdot) = 0, \quad \text{on } \mathbb{M}, \quad \Gamma(0) = \Gamma_0, \quad \text{in } \Omega,
\end{align*}
$$

(2.3)

where $\Gamma_0$ is the initial interface (parameterized by $X_0$) and $X(\cdot, t)$ parameterizes $\Gamma(t)$. Note that $u = T - T_M$, where $T$ is the temperature in degrees Kelvin and $T_M$ is the melting temperature at the interface $\Gamma$, and that $u$ is continuous across the interface. As noted in [5], we must have

$$
S = \frac{L}{T_M}.
$$

(2.4)

2.3 Non-dimensionalization

We non-dimensionalize the variables, but use the same variable symbols for convenience. This gives

$$
\begin{align*}
\partial_t u_l - \hat{K}_l \Delta u_l &= f_l, \quad \text{in } \hat{\Omega}_l(t), \\
\partial_t u_s - \hat{K}_s \Delta u_s &= f_s, \quad \text{in } \hat{\Omega}_s(t), \\
\nu \hat{\Omega} \cdot \nabla u &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \\
u_l \hat{\Omega} \cdot \nabla u_l &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_l, \\
u_s \hat{\Omega} \cdot \nabla u_s &= 0, \quad \text{on } \hat{\partial} \hat{\Omega}_s, \\
u_l \cdot \nu_s &= \hat{S}, \quad \text{on } \hat{\partial} \hat{\Omega}, \quad (2.5)
\end{align*}
$$

(2.5)

where $\Gamma_0$ is the initial interface (parameterized by $X_0$) and $X(\cdot, t)$ parameterizes $\Gamma(t)$. Note that $u = T - T_M$, where $T$ is the temperature in degrees Kelvin and $T_M$ is the melting temperature at the interface $\Gamma$, and that $u$ is continuous across the interface. As noted in [5], we must have

$$
S = \frac{L}{T_M}.
$$

(2.4)
Throughout the paper, we assume the non-dimensional coefficients satisfy
\[ \infty > \tilde{K}_t, \tilde{K}_s, \tilde{S} > 0, \quad \infty \geq \tilde{\beta}(\nu) \geq \tilde{\beta}_- > 0, \quad \text{where } \tilde{\beta}_- \text{ is a constant.} \]

**Remark 2.1** The case of \( \tilde{\beta} = 0 \) (i.e., \( \tilde{S}, \tilde{K}_1, \tilde{K}_s = \infty \)) corresponds to the steady-state heat equation in \( \Omega_t \) and \( \Omega_s \), and if \( \rho = 0 \) (i.e., \( \tilde{\beta}(\nu) = \infty \)) then (2.5) and (2.6) becomes the Mullins–Sekerka problem with Gibbs-Thomson law [41]. Our formulation can easily be modified to implement this model. If \( \tilde{S} = \infty \) only, then \( \partial_t \mathbf{X} \cdot \nu = 0 \), so (2.5) and (2.6) reduce to the time-dependent heat equation on a stationary domain with \( u_1 = u_s = 0 \) on \( \Gamma^p \).

### 3. Weak formulation

#### 3.1 Function spaces

Since the domain and interface deform in time, we define the function spaces using a reference domain [5]. For simplicity, we shall assume that \( \partial \Omega \cap \partial \Omega_s = \partial \Omega \) (see Figure 1); thus, \( \Omega_s \subset \Omega \). We use standard notation for denoting Sobolev spaces [1, 57], e.g., \( L^2(\Omega) \) is the space of square integrable functions on \( \Omega \). For any vector-valued function \( \eta \), if we write \( \eta \in L^2(\Omega) \), we mean each component of \( \eta \) is in \( L^2(\Omega) \). Continuing, we have \( H^1(\Omega) = \{ \eta \in L^2(\Omega) : \nabla \eta \in L^2(\Omega) \} \) and \( H(\text{div}, \Omega) = \{ \eta \in L^2(\Omega) : \nabla \cdot \eta \in L^2(\Omega) \} \). The norms on these spaces are defined in the obvious way, i.e., \( \| \eta \|_{L^2(\Omega)}^2 = \int_{\Omega} |\eta|^2 \), \( \| \nabla \eta \|_{L^2(\Omega)}^2 = \| \nabla \eta \|^2_{L^2(\Omega)} \), \( \| \eta \|^2_{H^1(\Omega)} = \| \eta \|^2_{L^2(\Omega)} + \| \nabla \eta \|^2_{L^2(\Omega)} \).

For a general function \( f : \Omega \to \mathbb{R} \), we denote its trace (or restriction) to a sub-domain \( \Sigma \subset \Omega \) (of co-dimension 1) by \( f|_{\Sigma} \). The trace of a function in \( H^1(\Omega) \) is well-defined; for a function in \( L^2(\Omega) \), the trace is not well-defined. Moreover, the trace of all functions (on \( \Sigma \subset \Omega \)) in \( H^1(\Omega) \) spans a Hilbert space, denoted \( H^{1/2}(\Sigma) \), which is a proper dense subspace of \( L^2(\Sigma) \). Referring to [7, pg. 48], the norm for \( H^{1/2}(\partial \Omega) \) is defined by

\[
\| v \|_{H^{1/2}(\partial \Omega)} := \inf_{\tilde{v} \in H^{1/2}(\Omega)} \frac{\| v \|_{H^1(\Omega)}}{\| \tilde{v} \|_{H^{1/2}(\Omega)}}, \tag{3.1}
\]

where \( \tilde{v} \) is the unique weak solution of \( -\Delta \tilde{v} + \tilde{v} = 0 \) in \( \Omega \), with \( \tilde{v} = v \) on \( \partial \Omega \). We also have \( H^{-1/2}(\partial \Omega) \), i.e., the dual space of \( H^{1/2}(\partial \Omega) \) with the dual norm,

\[
\| v \|_{H^{-1/2}(\partial \Omega)} := \sup_{\eta \in H^{1/2}(\partial \Omega)} \frac{\langle \eta, v \rangle \partial \Omega}{\| \eta \|_{H^{1/2}(\partial \Omega)}}, \tag{3.2}
\]

where \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) denotes the duality pairing between \( H^{-1/2}(\partial \Omega) \) and \( H^{1/2}(\partial \Omega) \).

It is well known [26, Theorem 1.7], [7, Lemma 2.1.1] that \( \eta \cdot \nu|_{\partial \Omega} \) is in \( H^{-1/2}(\partial \Omega) \) for all \( \eta \) in \( H(\text{div}, \Omega) \) (\( \nu_{\Omega} \) is the unit normal vector on \( \partial \Omega \)). In fact, by [26, (1.44)], we have that

\[
\| \eta \cdot \nu \|_{H^{-1/2}(\partial \Omega)} \leq \| \eta \|_{H(\text{div}, \Omega)}, \quad \text{for all } \eta \in H(\text{div}, \Omega). \tag{3.3}
\]

With this, one can show that \( \| v \|_{H^{1/2}(\partial \Omega)} \) has a dual norm realization.

**Proposition 3.1**

\[
\| v \|_{H^{1/2}(\partial \Omega)} = \sup_{\eta \in H(\text{div}, \Omega)} \frac{\langle \eta \cdot \nu, v \rangle_{\partial \Omega}}{\| \eta \|_{H(\text{div}, \Omega)}}.
\]
Enforcing boundary conditions requires the trace. To this end, let $H^1_{0,D}(\Omega) = \{u \in H^1(\Omega) : u|_{\partial_0 \Omega} = 0\}$. On the reference domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_s$, we introduce:

$$
\nabla = H(\text{div}, \Omega), \quad \nabla (g) = \{\eta \in \nabla : (\eta \cdot \nu_\Omega - g,q)_{3:\Omega} = 0, \forall q \in H^1_{0,D}(\Omega)\},
$$

$$
\nabla_1 = H(\text{div}, \tilde{\Omega}_1), \quad \nabla_1 (g) = \{\eta \in \nabla_1 : (\eta \cdot \nu_\Omega - g,q)_{3:\Omega} = 0, \forall q \in H^1_{0,D}(\Omega)\}, \quad (3.4)
$$

$$
\nabla_\varsigma = H(\text{div}, \tilde{\Omega}_s),
$$

where $g$ is in $H^{-1/2}(\partial \Omega)$ (see [7, Remark 2.1.3]). We also have the spaces

$$
Q = L^2(\Omega), \quad Q_1 = L^2(\tilde{\Omega}_1), \quad Q_\varsigma = L^2(\tilde{\Omega}_s).
$$

On the reference manifold $\mathbb{M}$, we define [1]

$$
\mathbb{M} = H^{1/2}(\mathbb{M}, \mathbb{R}), \quad \mathbb{Y} = H^1(\mathbb{M}, \mathbb{R}^d).
$$

The norm for $\mathbb{Y}$ is $\|v\|_{H^{1/2}(\mathbb{M})}^2 = \int_\Gamma |v|^2 + \int_\Gamma \left|\nabla v\right|^2$ (see Section 3.2.2 for $\nabla \Gamma$).

We use the following abuse of notation, similar to [5]. We identify functions $\eta_\Gamma$ in $\nabla_1$ with $\eta_\Gamma \circ \mathbf{X}^{-1}$ defined on $\Omega_1(t)$ (recall $\Omega_1(t) = \mathbf{X}(\tilde{\Omega}_1, t)$), and denote both functions simply as $\eta_\Gamma$; similar considerations are made for functions $\eta_\varsigma$ in $\nabla_\varsigma$. Likewise, we identify $\nabla$ in $\mathbb{Y}$ with $\nabla \circ \mathbf{X}^{-1}$ defined on $\Gamma(t)$, and denote both functions as $\nabla$; similar considerations are made for functions $\mu_\Gamma$ in $\mathbb{M}$. Along these lines, we have $\nabla_1 \simeq H(\text{div}, \Omega_1)$, $\nabla_\varsigma \simeq H(\text{div}, \Omega_s)$, $Q_1 \simeq L^2(\Omega_1)$, $Q_\varsigma \simeq L^2(\Omega_s)$, $\mathbb{M} \simeq H^{1/2}(\Gamma)$, $\mathbb{Y} \simeq H^1(\Gamma)$, provided the mapping $\mathbf{X}$ is not degenerate.

For technical reasons, we need two versions of the $H^{1/2}(\Gamma)$ norm related to $\Omega_1$ and $\Omega_s$. Define

$$
\|v\|_{H^{1/2}(\Gamma)} := \sup_{\eta_\Gamma \in \nabla_1(\mathbb{M})} \frac{\langle \eta_\Gamma \cdot \nu_\Gamma, v \rangle_{\Gamma}}{\|\eta_\Gamma\|_{H^{1/2}(\Gamma)}}, \quad \|v\|_{H^{1/2}(\Gamma)} := \sup_{\eta_\Gamma \in \nabla_\Gamma} \frac{\langle \eta_\Gamma \cdot \nu_\Gamma, v \rangle_{\Gamma}}{\|\eta_\Gamma\|_{H^{1/2}(\Gamma)}},
$$

Basically, these norms are related to the “side” of $\Gamma$ on which we take the trace. We also define the $H^{1/2}$ and $H^{-1/2}$ norm on $\Gamma$ by

$$
\|v\|_{H^{1/2}(\Gamma)} := \frac{1}{2} \left(\|v\|_{H^{1/2}(\Gamma)} + \|v\|_{H^{-1/2}(\Gamma)}\), \quad \|v\|_{H^{-1/2}(\Gamma)} := \sup_{\eta_\Gamma \in H^{1/2}(\Gamma)} \frac{\langle \eta_\Gamma \cdot \nu_\Gamma, v \rangle_{\Gamma}}{\|v\|_{H^{1/2}(\Gamma)}}.
$$

To conclude this section, we define the dual norm for $H^{-1}(\Gamma)$:

$$
\|\varphi\|_{H^{-1}(\Gamma)} := \sup_{v \in H^1(\Gamma)} \frac{\langle \varphi, v \rangle_{\Gamma}}{\|v\|_{H^1(\Gamma)}},
$$

where $\langle \varphi, v \rangle_{\Gamma}$ is understood to be the duality pairing between $H^{-1}(\Gamma)$ and $H^1(\Gamma)$.

### 3.2 Curvature

#### 3.2.1 Differential geometry

First, we review some differential geometry [17, 37]. Let $\Psi : U \to \Gamma$ be a local parameterization of $\Gamma \subset \mathbb{R}^3$ where $U \subset \mathbb{R}^2$ with local variables $(s_1, s_2)$. The first
fundamental form $g : U \to \mathbb{R}^{2 \times 2}$ is given by $g_{ij} = \partial_x \psi_i \partial_x \psi_j$ for $1 \leq i, j \leq 2$. Then the tangential gradient (or surface gradient) of $\omega : \Gamma \to \mathbb{R}$ is defined as

$$(\nabla_F \omega) \circ \psi := \sum_{i,j=1}^{2} g^{ij} \partial_{s_i} (\omega \circ \psi) \partial_{s_j} \psi,$$

where $[g^{ij}]_{i,j=1}^{2} = g^{-1}$ (matrix inverse). Given $Y : \Gamma \to \mathbb{R}^3$, we have $\nabla_F Y = (\nabla_F Y_1, \nabla_F Y_2, \nabla_F Y_3)$ (a $3 \times 3$ matrix). Moreover, we have the tangential divergence $\nabla_F \cdot Y := \text{trace}(\nabla_F Y)$.

The Laplace–Beltrami operator is defined by $\Delta_F \omega = \nabla_F \cdot \nabla_F \omega$ which expands out to

$$(\Delta_F \omega) \circ \psi := \sum_{i,j=1}^{2} g^{ij} \partial_{s_i} \left( \sum_{r,k=1}^{2} g^{rk} \partial_{s_k} (\omega \circ \psi) \partial_{s_r} \psi \right),$$

Note: When $\Gamma$ is a one-dimensional curve with oriented unit tangent vector $\tau$, we have $\nabla_F \equiv \tau \partial_s$ and $\Delta_F \equiv \partial_{s}^2$, where $\partial_s$ is the derivative with respect to arc-length.

Therefore, taking $X(.,t)$ to be a local parameterization of $\Gamma(t)$, the vector curvature $\kappa \nu$ of $\Gamma(t)$ [17, 37] is given by $-\Delta_F X = \kappa \nu$, where $\kappa$ is the sum of the principle curvatures.

3.2.2 Weak form. In the rest of the paper, we take advantage of a weak formulation of the vector curvature [3, 19]. If $\Gamma$ is a closed $C^2$ manifold, then the following integration by parts relation is true:

$$\int_\Gamma \kappa \nu \cdot Y = \int_\Gamma \nabla_F X : \nabla_F Y, \quad \text{for all } Y \text{ in } \mathcal{Y}, \quad (3.10)$$

where $\nabla_F X$ is a symmetric matrix that represents the projection operator onto the tangent space of $\Gamma$, i.e. $\nabla_F X = I - \nu \otimes \nu$. We use (3.10) to derive the weak form (3.12).

3.3 Fully continuous

We present a mixed formulation of (2.5), (2.6) that is partly related to [8] for the heat equation. Define the flux variables $\sigma_1 = -\bar{K}_1 \nabla u_1$, $\sigma_2 = -\bar{K}_2 \nabla u_2$, and take $u_3(.,t)$ in $H^4_{0,N}(\Omega) \equiv \{ u \in H^4(\Omega) : u|_{\partial_N} = 0 \}$. Then, for given input data $f_1(.,t)$, $f_2(.,t)$ in $H^4(\Omega)$, and initial data $X(.,0) = X_0$, $u_1(.,0) = u_{1,0}$, $u_2(.,0) = u_{2,0}$, find time-dependent functions $\sigma_1(.,t)$ in $\mathcal{V}_1(0)$, $\sigma_2(.,t)$ in $\mathcal{V}_2(0)$, $\sigma_3(.,t)$ in $\mathcal{Q}_1$, $X(.,t)$ in $\mathcal{Q}_2$, $u_1(.,t)$ in $\mathcal{Q}_3$, $u_2(.,t)$ in $\mathcal{Q}_4$, $\lambda(.,t)$ in $\mathcal{Q}_5$ such that

$$\int_{\Omega_1(t)} \sigma_1 \cdot \eta_1 - \int_{\Omega_1(t)} u_1 \nabla \cdot \eta_1 - \int_{\Gamma_1(t)} \lambda \eta_1 \cdot \nu = -\int_{\partial\Omega} \eta_1 \cdot \nu \eta_2, \quad \text{for all } \eta_1 \in \mathcal{V}_1(0),$$

$$-\int_{\Omega_2(t)} q_1 \nabla \cdot \sigma_1 - \int_{\Omega_2(t)} q_1 \partial_t u_1 = -\int_{\partial\Omega_2} q_1 f_1, \quad \text{for all } q_1 \in \mathcal{Q}_1,$$

$$\int_{\Omega_1(t)} \sigma_2 \cdot \eta_2 - \int_{\Omega_1(t)} u_2 \nabla \cdot \eta_2 + \int_{\Gamma_1(t)} \lambda \eta_2 \cdot \nu = 0, \quad \text{for all } \eta_2 \in \mathcal{V}_2,$$

$$-\int_{\Omega_2(t)} q_2 \nabla \cdot \sigma_2 - \int_{\Omega_2(t)} q_2 \partial_t u_2 = -\int_{\partial\Omega_2} q_2 f_2, \quad \text{for all } q_2 \in \mathcal{Q}_2,$$

(3.11)
Integration by parts shows that
\[
\int_{\Gamma(t)} \frac{1}{\beta(\nu)} (\partial_t X \cdot \nu)(Y \cdot \nu) + \hat{\alpha} \int_{\Gamma(t)} \nabla_{\Gamma} X : \nabla_{\Gamma} Y + \hat{S} \int_{\Gamma(t)} \lambda (Y \cdot \nu) = 0, \quad \text{for all } Y \in Y,
\]
where we have accounted for the orientation of the normal vector
\[
\hat{S} \int_{\Gamma(t)} \mu \partial_t X \cdot \nu - \int_{\Gamma(t)} \mu \sigma_t \cdot \nu + \int_{\Gamma(t)} \mu \sigma_s \cdot \nu = 0, \quad \text{for all } \mu \in M,
\]
(3.12)
where we have dropped the differential measure symbols \(d\mathbf{x}, dS(\mathbf{x})\), etc., for brevity. Note: Integration by parts shows that \(\lambda = u_1 = u_s\) on \(\Gamma(t)\).

### 3.4 Formal estimates

Well-posedness of the fully continuous problem (3.11), (3.12) is challenging. One must handle the parameterized deforming domain appropriately and be able to obtain a priori estimates of the interface velocity, curvature, and improved regularity estimates of the variables [14, 31]. However, one may formally derive a priori bounds by assuming existence and uniqueness of a solution as well as sufficient regularity to allow for choosing test functions.

#### 3.4.1 A priori bound

For simplicity, take \(u_{10} = 0\). In (3.11) and (3.12), choose \(\eta_1 = \sigma_t, \eta_s = \sigma_s\), \(Y = \partial_t X, q_1 = -u_t, q_s = -u_s, \mu = -\lambda\), and add the equations together to get:

\[
\frac{1}{K_1} \int_{\Omega(t)} |\sigma_t|^2 + \frac{1}{K_2} \int_{\Omega(t)} |\sigma_s|^2 + \int_{\Gamma(t)} \frac{1}{\beta(\nu)} |\partial_t X \cdot \nu|^2 + \hat{\alpha} \int_{\Gamma(t)} \nabla_{\Gamma}(\partial_t X) : \nabla_{\Gamma} X
\]

\[
+ \int_{\Omega(t)} u_t \partial_t u_t + \int_{\Omega(t)} u_s \partial_t u_s = \int_{\Omega(t)} u_{1t} + \int_{\Omega(t)} u_{st} f_s.
\]

(3.13)

Next, we make some preliminary calculations for some of the terms in (3.13). By standard shape differentiation [16, 32, 56], we have

\[
\frac{d}{dt} \left( \int_{\Omega(t)} u_t^2 \right) = \int_{\Omega(t)} \partial_t (u_t^2) - \int_{\Gamma(t)} u_t^2 (\partial_t X) \cdot \nu,
\]

(3.14)

where we have accounted for the orientation of the normal vector \(\nu\) of \(\Gamma(t)\). Thus,

\[
\int_{\Omega(t)} u_t \partial_t u_t + \int_{\Omega(t)} u_s \partial_t u_s = \frac{1}{2} \int_{\Omega(t)} \partial_t (u_t^2) + \int_{\Omega(t)} \partial_t (u_s^2)
\]

\[
= \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega(t)} u_t^2 + \int_{\Omega(t)} u_s^2 \right) + \frac{1}{2} \int_{\Gamma(t)} (u_t^2 - u_s^2) (\partial_t X) \cdot \nu
\]

(3.15)

where the last term is dropped because (formally) \(u_1 = u_s\) on \(\Gamma(t)\).

Now note that shape differentiation also tells us that [16, 32, 56]

\[
\int_{\Gamma(t)} \nabla_{\Gamma}(\partial_t X) : \nabla_{\Gamma} X = \frac{d}{dt} |\Gamma(t)|. \tag{3.16}
\]
Therefore, we arrive at an identity
\[
\int_{\Omega(t)} \frac{1}{\beta(\nu)} [\langle \partial_t \mathbf{X}, \cdot \rangle \cdot \nu]^2 + \frac{1}{K_1} \| \sigma_t \|_{L^2(\Omega(t))}^2 + \frac{1}{K_2} \| \sigma_s \|_{L^2(\Omega(t))}^2 + \frac{\tilde{\alpha}}{d} |\Gamma(t)| \\
+ \frac{1}{d} \left( \int_{\Omega(t)} u_t^2 + \int_{\Omega(t)} u_s^2 \right) = \int_{\Omega(t)} u_1 f_1 + \int_{\Omega(t)} u_s f_s,
\] (3.17)
which is a variation of a result in [5, eqn. (2.13)].

To continue, we recall a variant of a Grönwall type inequality in [43, Lemma 3.1].

**Lemma 3.2** Let \( g, h, r, w : [0, T] \to \mathbb{R} \) be measurable and positive functions such that
\[
r^2(t) + g(t) + \int_0^t w(\tau) \, d\tau \leq r^2(0) + g(0) + \int_0^t r(\tau) h(\tau) \, d\tau.
\] (3.18)

Then,
\[
r^2(t) + g(t) + \int_0^t w(\tau) \, d\tau \leq 2(r^2(0) + g(0)) + t \int_0^t h^2(\tau) \, d\tau.
\] (3.19)

Now make the following identifications with the functions in Lemma 3.2:
\[
h(t) = \sqrt{2} \left( \| f_1(t) \|_{L^2(\Omega(t))}^2 + \| f_s(t) \|_{L^2(\Omega(t))}^2 \right)^{1/2},
\]
\[
r(t) = \frac{1}{\sqrt{2}} \left( \| u(t) \|_{L^2(\Omega(t))}^2 + \| u_s(t) \|_{L^2(\Omega(t))}^2 \right)^{1/2},
\]
\[
g(t) = \tilde{\alpha}|\Gamma(t)|,
\]
\[
w(t) = \| \tilde{\beta}^{-1/2}(\nu)(\partial_t \mathbf{X}) \cdot \nu \|_{L^2(\Gamma(t))}^2 + \frac{1}{K_1} \| \sigma_t(t) \|_{L^2(\Omega(t))}^2 + \frac{1}{K_2} \| \sigma_s(t) \|_{L^2(\Omega(t))}^2.
\]

Using Cauchy–Schwarz twice on the right-hand-side of (3.17) and integrating, we get
\[
\int_0^t w(\tau) \, d\tau + r^2(t) + g(t) \leq r^2(0) + g(0) + \int_0^t r(\tau) h(\tau) \, d\tau.
\]
Applying Lemma 3.2 delivers the a priori estimate:
\[
\frac{1}{2} \left( \| u(t) \|_{L^2(\Omega(t))}^2 + \| u_s(t) \|_{L^2(\Omega(t))}^2 \right) + \tilde{\alpha}|\Gamma(t)|
\]
\[
+ \int_0^t \left( \| \tilde{\beta}^{-1/2}(\nu)(\partial_t \mathbf{X}) \cdot \nu \|_{L^2(\Gamma(t))}^2 + \frac{1}{K_1} \| \sigma_t(t) \|_{L^2(\Omega(t))}^2 + \frac{1}{K_2} \| \sigma_s(t) \|_{L^2(\Omega(t))}^2 \right) \, d\tau
\]
\[
\leq \left( \| u(t) \|_{L^2(\Omega(t))}^2 + \| u_s(t) \|_{L^2(\Omega(t))}^2 \right) + 2\tilde{\alpha}|\Gamma(0)|
\]
\[
+ 2t \int_0^t \left( \| f_1(\tau) \|_{L^2(\Omega(t))}^2 + \| f_s(\tau) \|_{L^2(\Omega(t))}^2 \right) \, d\tau.
\] (3.20)
See (4.27) for the semi-discrete version of (3.20).
3.4.2 Conservation law. We also have a conservation law for the system which is simply a thermal energy balance. Choosing \( q_i = 1, q_s = 1 \) in (3.11), and \( \mu = 1 \) in (3.12) gives

\[
- \int_{\partial_i \Omega} \sigma_1 \cdot \nu \Omega + \int_{\Gamma(t)} \sigma_1 \cdot \nu = \int_{\Omega_i(t)} \partial_t u_i - \int_{\Omega_s(t)} f_i,
\]

\[
- \int_{\Gamma(t)} \sigma_s \cdot \nu = \int_{\Omega_s(t)} \partial_t u_s - \int_{\Omega_i(t)} f_s,
\]

\[
\bar{S} \int_{\Gamma(t)} (\partial_t \mathbf{X}) \cdot \nu = \int_{\Gamma(t)} \sigma_1 \cdot \nu - \int_{\Gamma(t)} \sigma_s \cdot \nu.
\]

Adding them together gives the balance law:

\[
\int_{\Omega_i(t)} f_i + \int_{\Omega_s(t)} f_s - \int_{\partial_i \Omega} \sigma_1 \cdot \nu \Omega = \int_{\Omega_i(t)} \partial_t u_i + \int_{\Omega_s(t)} \partial_t u_s - \bar{S} \int_{\Gamma(t)} (\partial_t \mathbf{X}) \cdot \nu,
\]

where the left side is the thermal (power) input and the right side is the rate of change in the stored thermal energy of the system. Note that energy is stored in the phase change associated with the velocity \( \partial_t \mathbf{X} \) of \( \Gamma(t) \). See (4.34) for the semi-discrete version of (3.21).

4. Time semi-discrete formulation

We now partition the time interval \((0, T)\) into subintervals of size \( \Delta t \). We use a superscript \( i \) to denote a time dependent quantity at time \( t_i \). Furthermore, let \( (\cdot, \cdot)_\Sigma \) denote the \( L^2 \) inner product on the generic domain \( \Sigma \). For a general domain \( \Sigma \), let \( (\cdot, \cdot)_\Sigma \) denote the duality pairing on \( \Sigma \) between \( H^{-1/2}(\Sigma) \) and \( H^{1/2}(\Sigma) \) or between \( H^{-1}(\Sigma) \) and \( H^1(\Sigma) \) (the context will make it clear).

4.1 Domain velocity

4.1.1 Map \( \Gamma^i \) to \( \Gamma^{i+1} \). We introduce the interface velocity \( \mathbf{V} := \partial_t \mathbf{X} \) as a new variable. Thus, we approximate the interface position at time \( t_{i+1} \) by a backward Euler scheme:

\[
\mathbf{X}^{i+1} = \mathbf{X}^i + \Delta t \mathbf{V}^{i+1}, \quad \text{where } \mathbf{V}^{i+1} : \Gamma^i \rightarrow \mathbb{R}^3.
\]

(4.1)

Thus, knowing \( \mathbf{V}^{i+1} \) and \( \mathbf{X}^i \) we can update the parametrization of the interface and obtain the interface \( \Gamma^{i+1} \) at \( t_{i+1} \). Note that \( \mathbf{X}^i(\cdot) = \text{id}_{\Gamma^i}(\cdot) \) (the identity map) on \( \Gamma^i \).

REMARK 4.1 We shall assume throughout this paper that \( \mathbf{V}^{i+1} \) (for all \( i \)) is at least in \( W^{1,\infty}(\Gamma^i) \) in order for the update (4.1) to make sense.

4.1.2 Map \( \Omega^i, \Omega^{i+1} \) to \( \Omega^{i+1} \). Clearly, the bulk domains \( \Omega_i, \Omega^{i+1} \) follow the interface \( \Gamma^i \). Given \( \mathbf{V}^{i+1} \) on \( \Gamma^i \), it can be extended to the entire domain \( \Omega \) by a harmonic extension [22, 65], i.e. if \( \mathbf{V}_E^{i+1} \) denotes the extension, then

\[
\mathbf{V}_E^{i+1} \in H^1(\Omega) : \quad -\Delta \mathbf{V}_E^{i+1} = 0, \quad \text{in } \Omega_i \cup \Omega^{i+1}, \quad \mathbf{V}_E^{i+1} = \mathbf{V}^{i+1}, \quad \text{on } \Gamma^i, \quad \mathbf{V}_E^{i+1} = 0, \quad \text{on } \partial \Omega.
\]

(4.2)

In the following, we drop the \( E \) subscript and use \( \mathbf{V}^{i+1} \) to denote the extension. This induces a map \( \Phi_{i+1} : \Omega^i \rightarrow \Omega^{i+1} \) for “updating” the domain:

\[
\Phi_{i+1}(x) = \text{id}_{\Omega^i}(x) + \Delta t \mathbf{V}^{i+1}(x), \quad \text{for all } x \in \Omega^i.
\]

(4.3)
See [27, 28] for similar constructions in an ALE (Arbitrary-Lagrangian-Eulerian) context.

Note that \( \Phi_{i+1} \) is defined over both \( \Omega'_{i} \) and \( \Omega'_{i} \), and \( \Omega'_{i+1} := \Phi_{i+1}(\Omega'_{i}) \), \( \Omega'_{i+1} := \Phi_{i+1}(\Omega'_{i}) \) conform to \( \Gamma_{i+1}' \). Similarly as for (4.1), we assume \( \mathbf{v}^{i+1} \) (on \( \Omega'_{i} \)) is at least in \( W^{1,\infty}(\Omega'_{i}) \). Furthermore, we assume \( \Phi_{i+1} \) is a bijective map and \( \text{det}([\nabla_{x}\Phi_{i+1}(x)]) > 0 \). We note the following properties satisfied by \( \Phi_{i+1} \) [33, 58].

- If \( y = \Phi_{i+1}(x) \), then \( (\nabla_{x}\Phi_{i+1}^{-1}(\Phi_{i+1}(x)) = [\nabla_{x}\Phi_{i+1}(x)]^{-1} \).
- If \( f : \Omega'_{i+1} \rightarrow \mathbb{R} \), then \( \int_{\Omega'_{i+1}} f(y) \, dy = \int_{\Omega'_{i+1}} f(\Phi_{i+1}(x)) \det([\nabla_{x}\Phi_{i+1}(x)]) \, dx \).

We use the map \( \Phi_{i+1} \) to transform the functions \( u_{i+1}^{j}, u_{i+1}^{s} \) on \( \Omega'_{i} \) to new functions on \( \Omega'_{i+1} \) in order to advance the solution to the next time step (see (4.7)).

**Remark 4.2 (Time Step Restriction)** In order for (4.3) to remain bijective, \( \Delta t \) cannot be too large. In fact, it depends on \( \|\nabla\mathbf{v}^{i+1}\|_{L_{\infty}(\Omega)} \) because \( \text{det}(\nabla_{x}\Phi_{i+1}) \) depends on \( \nabla\mathbf{v}^{i+1} \). There is also a similar restriction on the time step in Theorem 4.14 (a priori bound).

### 4.1.3 Time derivative: Eulerian vs. Lagrangian

Similar to (4.1), we use a backward Euler method to discretize the temperature time derivatives at each time step:

\[
(\partial_{t}u_{i+1}^{j})^{i+1} \approx \frac{u_{i+1}^{j} - u_{i}^{j}}{\Delta t}, \quad (\partial_{t}u_{i+1}^{s})^{i+1} \approx \frac{u_{i+1}^{s} - u_{i}^{s}}{\Delta t}.
\]

But, because the domain is changing, \( u_{i+1}^{j}, u_{i+1}^{s} \) (\( j = 1, s \)) are defined on different domains (\( \Omega'_{i}, \Omega'_{i-1} \), respectively; see next section). This means \( u_{i}^{j} \) must be transferred to the new domain in order to compute the (discrete) Eulerian time derivative. The transference can be accomplished by an \( L^{2} \) projection, for instance, but is not so convenient for a numerical method.

Therefore, we make use of the material derivative [58]. Using the standard formula \( \dot{u}_{j} = \partial_{t}u_{j} + \mathbf{v} \cdot \nabla u_{j} \), and introducing the flux variables, we have \( \dot{u}_{j} = \partial_{t}u_{j} - K_{1}^{-1}\mathbf{v} \cdot \sigma_{j} \) for \( j = 1, s \). Thus, we adopt the following discretization of \( \partial_{t}u_{1} \) and \( \partial_{t}u_{s} \):

\[
(\partial_{t}u_{j})^{i+1} \approx \frac{u_{i+1}^{j} - u_{i}^{j} \circ \Phi_{i}^{-1}}{\Delta t} + \frac{1}{K_{j}} (\sigma_{j}^{i} \cdot \mathbf{v}^{i}) \circ \Phi_{i}^{-1}, \quad \text{for } j = 1, s.
\]

Note that we have treated the convective term explicitly, and (formally) taking \( \Delta t \rightarrow 0 \) recovers the standard material derivative formula. The advantage here is that computing \( u_{i}^{j} \circ \Phi_{i}^{-1} \) and \( (\sigma_{j}^{i} \cdot \mathbf{v}^{i}) \circ \Phi_{i}^{-1} \) (\( j = 1, s \)), in the fully discrete method, is straightforward (see (5.3) and Remark 5.8).

### 4.2 Weak formulation

We now present the semi-discrete formulation of equations (3.11) and (3.12). The main idea is to write all integrals over the current domain \( \Omega'_{i} \), \( \Gamma'_{i} \) but set all of the solution variables at the next time step \( t_{i+1} \) (i.e. a semi-implicit method). Moreover, we apply (4.1) and (4.4) and set \( u_{i+1}^{1} = u_{D}(\cdot, t_{i+1}), \quad f_{i+1}^{1} = f_{i}(\cdot, t_{i+1}), \) and \( f_{i+1}^{s} = f_{i}(\cdot, t_{i+1}) \). Thus, we arrive at the following weak formulation. At time \( t_{i} \), find \( \sigma_{i}^{1} \in V_{i}^{1}(0), \sigma_{i}^{s} \in V_{s}^{1} \) in \( V_{i}^{1} \), \( \mathbf{v}^{i+1} \) in \( Y_{i}^{s} \), \( u_{i}^{1} \in Q_{i}^{1}, \sigma_{i}^{s} \in Q_{s}^{1} \), and \( \lambda_{i+1}^{1} \)
in $\mathbb{M}^i$ such that
\[
\frac{1}{K_l}(\sigma_l^{i+1}, \eta_l^{i+1})_{\Omega_l^i} - (u_l^{i+1}, \nabla \cdot \eta_l^{i+1})_{\Omega_l^i} - (\eta_l^{i+1}, \nu_l^{i+1})_{\Omega_l^i} = -(\eta_l^{i+1}, \nu_l^{i+1})_{\Omega_l^i},
\]
for all $\eta_l \in \mathcal{V}_l(0)$.
\[
-(\nabla \cdot \sigma_l^{i+1}, q_l)_{\Omega_l^i} - \frac{1}{\Delta t}(u_l^{i+1}, q_l)_{\Omega_l^i} + \frac{1}{\Delta t}(u_l^{i+1}, q_l)_{\Omega_l^i} = -(f_l^{i+1}, q_l)_{\Omega_l^i}, \quad \text{for all } q_l \in \mathcal{Q}_l^i, \quad (4.5)
\]
\[
\frac{1}{K_s}(\sigma_s^{i+1}, \eta_s^{i+1})_{\Omega_s^i} - (u_s^{i+1}, \nabla \cdot \eta_s^{i+1})_{\Omega_s^i} + (\eta_s^{i+1}, \nu_s^{i+1})_{\Omega_s^i} = 0, \quad \text{for all } \eta_s \in \mathcal{V}_s^i,
\]
\[
-(\nabla \cdot \sigma_s^{i+1}, q_s)_{\Omega_s^i} - \frac{1}{\Delta t}(u_s^{i+1}, q_s)_{\Omega_s^i} + \frac{1}{\Delta t}(u_s^{i+1}, q_s)_{\Omega_s^i} = -(f_s^{i+1}, q_s)_{\Omega_s^i}, \quad \text{for all } q_s \in \mathcal{Q}_s^i,
\]
\[
(\beta^{-1}(\nu_l^{i+1}) \nu_l^{i+1}, \nu_l^{i+1})_{\Omega_l^i} + \Delta t \tilde{g}(\nabla P_l^{i+1}, \nabla P_l^{i+1}, \nabla Y_l^{i+1})_{\Omega_l^i}
\]
\[
+ \tilde{S}(Y_l^{i+1}, \nu_l^{i+1})_{\Omega_l^i} = -\tilde{g}(\nabla P_l^{i+1}, \nabla P_l^{i+1}, Y_l^{i+1})_{\Omega_l^i},
\]
\[
\tilde{S}(V_l^{i+1}, \nu_l^{i+1}, \mu_l^{i+1}, \nu_l^{i+1})_{\Omega_l^i} - (\sigma_l^{i+1}, \nu_l^{i+1})_{\Omega_l^i} + (\sigma_s^{i+1}, \nu_s^{i+1})_{\Omega_s^i} = 0, \quad \text{for all } \mu_l \in \mathcal{M}_l^i,
\]
where the function spaces are defined over the current (known) domain $\Omega_l^i, I_l^i$. Then we use (4.1) to obtain the new interface position, which induces a map $\Phi_l^{i+1} : \Omega_l^i \rightarrow \Omega_l^{i+1}$. Because of (4.4), the temperature from the previous time index, $u_l^{i+1} : \Omega_l^{i+1} \rightarrow \mathbb{R}$, is mapped onto $\Omega_l^i$ by
\[
\bar{u}_l^i := u_l^{i+1} \circ \Phi_l^{-1} - \Delta t \frac{1}{K_l}(\sigma_l^i, \nu_l^i) \circ \Phi_l^{-1}, \quad \text{for } j = 1, s.
\]
Iterating this procedure gives a time semi-discrete approximation of the fully continuous problem (3.11), (3.12).

**Remark 4.3 (How To Start The Method)** From (4.7), it is clear we need $V^0$ to compute $\bar{u}_l^0$, $\bar{u}_s^0$. However, we start solving (4.5), (4.6) at $i = 0$, which only gives $V^1$. Hence, we must do one of the following. (i) specify $V^0$, $\sigma_l^0$, $\sigma_s^0$; (ii) set $V^0 = \sigma_l^0 = \sigma_s^0 = 0$ (i.e. choose $\bar{u}_l^0$, $\bar{u}_s^0$ directly); (iii) or apply (ii) with a small time step to obtain an approximation of $V^0$, $\sigma_l^0$, $\sigma_s^0$. Either way, the error in approximating $\bar{u}_l^0$, $\bar{u}_s^0$ is only $O(\Delta t)$.

### 4.3 Abstract formulation

In order to simplify notation, we shall drop the time index notation and remember that we are solving for all variables on the current known domain $\Omega = \Omega_l^i$, $I = I_l^i$ with the current known normal vector $\nu \equiv \nu_l$. In particular, we take
\[
\sigma_l^{i+1} = \sigma_l, \quad \sigma_s^{i+1} = \sigma_s, \quad \nabla l^{i+1} = \nabla l, \quad u_l^{i+1} = u_l, \quad u_s^{i+1} = u_s, \quad \lambda_l^{i+1} = \lambda_l, \quad f_l^{i+1} = f_l, \quad f_s^{i+1} = f_s, \quad u_D^{i+1} = u_D, \quad \bar{u}_l^i = \bar{u}_l, \quad \bar{u}_s^i = \bar{u}_s, \quad X_l^i = X_l, \quad \nabla P_l = \nabla P_l.
4.3.1 \textit{Bilinear and linear forms.} For notational convenience, we introduce the following bilinear forms. The primal form is
\[ a((\eta, \eta), (\sigma, \sigma)) = \frac{1}{K_1}(\eta, \eta)_\Omega + \frac{1}{K_2}(\sigma, \sigma)_\Omega \]
where
\[ \beta^{-1}(\nu, \nu)_\Gamma + \Delta t \alpha(\nabla \eta, \nabla \zeta)_\Gamma. \]
the constraint form is
\[ b((\eta, \eta), (q, q, \mu)) = -(\nabla \cdot \eta)_{\Omega} - (\nabla \cdot q, \zeta)_{\Omega} \]
\[ - (\eta \cdot \nu, \zeta)_\Gamma + (\eta \cdot \nu, \mu)_\Gamma + \bar{S}(\nu, \mu)_\Gamma. \]
and the lower diagonal form is
\[ c((q, q, \mu), (u, u, \lambda)) = \frac{1}{\Delta t}(q, q)_{\Omega} + \frac{1}{\Delta t}(q, u)_{\Omega}. \]

4.3.2 \textit{Saddle-point formulation.} Define the primal space by
\[ Z = V_1(0) \times V_s \times Y, \]
and the multiplier space by
\[ T = Q_1 \times Q_s \times M. \]

4.4 \textit{Norms}
4.4.1 \textit{Non-degenerate interface.} The purpose of the following assumption is to avoid a case where \( \Gamma \) is closed and very flat (e.g. the surface of a pancake). It is necessary to ensure the equivalence of the norms in Proposition 4.6.

\textbf{Assumption 4.5} Assume that \( \Gamma \) is a Lipschitz or polyhedral manifold. In addition, for any non-zero constant vector \( a \in \mathbb{R}^3 \), assume there exists an open neighborhood \( \mathcal{N} \subset \Gamma \) such that \( |\mathcal{N}| \geq c_0 > 0 \) and
\[ a \cdot \nu(x) > 0, \quad \forall x \in \mathcal{N}, \quad \text{or} \quad a \cdot \nu(x) < 0, \quad \forall x \in \mathcal{N}. \]
4.4.2 Primal norm. Clearly, \( \| (\eta, \eta, Y) \|_2^2 := \| \eta \|_{H^1(\Omega)}^2 + \| \eta \|_{H^1(\Omega)}^2 + \| Y \|_{H^1(\Gamma)}^2 \) is a norm on \( \mathbb{Z} \). But because of the form of the equations, we shall use a different norm. First, we note an equivalent norm to the standard \( H^1 \) norm on \( \Gamma \) (recall that \( \| Y \|_{H^1(\Gamma)}^2 = \| Y \|_{L^2(\Gamma)}^2 + \| \nabla Y \|_{L^2(\Gamma)}^2 \)).

**Proposition 4.6** Let \( \Gamma \) be a Lipschitz or polyhedral manifold. Define:

\[
||| Y |||^2 = || Y \cdot \nu ||_{H^{-1/2}(\Gamma)}^2 + \| \nabla Y \|_{L^2(\Gamma)}^2.
\]

Then, \( ||| Y \parallel \approx \| Y \|_{H^1(\Gamma)} \), with constants that only depend on the domain.

**Proof.** First, verify that \( ||| Y ||| \) is a norm on \( H^1(\Gamma) \). We just need to check that \( ||| Y ||| = 0 \Leftrightarrow Y = 0 \) since the other norm properties are trivial to verify. If \( ||| Y ||| = 0 \), then \( \| \nabla Y \|_{L^2(\Gamma)} = 0 \), so \( Y = 0 \). If \( a \neq 0 \), then by Assumption 4.5, \( a \cdot \nu > 0 \) (or < 0) on a set of positive measure. Thus, \( || Y \cdot \nu ||_{H^{-1/2}(\Gamma)} \neq 0 \), but this is a contradiction, so then \( a = 0 \). Since \( \| \cdot \| \) is a norm on \( H^1(\Gamma) \), the equivalence with \( \| \cdot \|_{H^1(\Gamma)} \) follows by a classical compactness argument [1, 21].

In light of the above, we define the following primal norm:

\[
\|(\eta, \eta, Y)\|_2^2 = \frac{1}{K_1} \| \eta \|_{H^1(\Omega)}^2 + \frac{1}{K_2} \| \eta \|_{H^1(\Omega)}^2 + \| Y \|_{H^1(\Gamma)}^2
\]

\[
+ \| Y \cdot \nu \|_{H^{-1/2}(\Gamma)} + \Delta t \hat{\alpha} \| \nabla Y \|_{L^2(\Gamma)}^2. \quad (4.15)
\]

The choice of \( H^{-1/2}(\Gamma) \) is the most convenient for our formulation.

4.4.3 Multiplier norm. The obvious multiplier norm is \( \|(q, q_\nu, \mu)\|_{2,0} := \| q \|_{L^2(\Omega)}^2 + \| q_\nu \|_{L^2(\Omega)}^2 + \| \mu \|_{H^{1/2}(\Gamma)}^2 \). However, because of the form of the bilinear form \( b \) (4.9), it is more advantageous to use the following equivalent norm:

\[
\|(q, q_\nu, \mu)\|_T^2 = \| \hat{q}_i \|_{L^2(\Omega)}^2 + \| \hat{q}_\nu \|_{L^2(\Omega)}^2 + \| \mu - \hat{q}_\nu \|_{H^{1/2}(\Gamma)}^2
\]

\[
+ \| \mu - \hat{q}_\nu \|_{H^{1/2}(\Gamma)}^2 + \hat{S} \| \mu \nu \|_{H^{-1/2}(\Gamma)}^2. \quad (4.16)
\]

where we introduced the mean value: \( \hat{q}_i := \frac{1}{|\Omega_i|} \int_{\Omega_i} q_i \), and \( \hat{q}_\nu := q_i - \hat{q}_i \) (for \( i = 1, s \)). We also define the mean value on \( \Gamma \): \( \hat{\mu} := \frac{1}{|\Gamma|} \int_{\Gamma} \mu \), and \( \hat{\mu} := \mu - \hat{\mu} \).

**Proposition 4.7** (Equivalence of Multiplier Norms) Let \( \Gamma \) be a Lipschitz or polyhedral manifold. Then, \( \|(q, q_\nu, \mu)\|_{\pi,0} \approx ||(q, q_\nu, \mu)||_{\pi} \), with constants that only depend on the domain and \( \hat{S} \).

**Proof.** Again, use a compactness argument.

4.5 Well-posedness

This section verifies the conditions needed for well-posedness of (4.14) [9, 11].
4.5.1 Main conditions.

**Lemma 4.8 (Continuity of Forms)** For all \((\eta, \eta, Y), (\sigma, \sigma, V)\) in \(\mathbb{Z}\) and \((q, q_c, \mu), (u, u, \lambda)\) in \(\mathbb{T}\),

\[
\begin{align*}
|a((\eta, \eta, Y), (\sigma, \sigma, V))| &\leq C_a \|(\eta, \eta, Y)\|_{\mathbb{Z}} \|(\sigma, \sigma, V)\|_{\mathbb{Z}}, \\
|b((\eta, \eta, Y), (q, q_c, \mu))| &\leq C_b \|(\eta, \eta, Y)\|_{\mathbb{Z}} \|(q, q_c, \mu)\|_{\mathbb{Z}}, \\
|c((q, q_c, \mu), (u, u, \lambda))| &\leq \Delta t^{-1} \left(\|q\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|q_c\|_{L^2(\Omega)} \|u_c\|_{L^2(\Omega)}\right), \\
|\varphi(\eta, \eta, Y)| &\leq C_\varphi \|(\eta, \eta, Y)\|_{\mathbb{Z}}, \\
|\psi(q, q_c, \mu)| &\leq C_\psi \|(q, q_c, \mu)\|_{\mathbb{T}},
\end{align*}
\]

where \(C_a, C_b, C_\varphi, C_\psi > 0\) are constants that depend on physical parameters and domain geometry. In addition, \(C_\varphi\) depends on \(u_{D}, \Delta t^{-1/2}\), and \(C_\psi\) depends on \(f, f, \pi, \pi, \Delta t^{-1}\).

**Proof.** The first result comes from two uses of the Schwarz inequality. The second estimate follows by noting

\[
-(\nabla \cdot Y) q \Omega - (\eta \cdot \nu) t \Omega \leq \left(\|q\|_{L^2(\Omega)} + \|\mu\|_{H^{1/2}(\Gamma)}\right)\|\eta\|_{H(\mathrm{div}, \Omega)},
\]

\[
-(\nabla \cdot Y) q_c \Omega + (\eta \cdot \nu) t \Omega \leq \left(\|q_c\|_{L^2(\Omega)} + \|\mu\|_{H^{1/2}(\Gamma)}\right)\|\eta\|_{H(\mathrm{div}, \Omega)},
\]

where we used Cauchy–Schwarz and (3.7). In addition, by (3.8), we have

\[
\tilde{S} \int_\Gamma \mu (Y \cdot \nu) = \tilde{S} (Y \cdot \nu) \mu \leq \tilde{S} \|Y \cdot \nu\|_{H^{-1/2}(\Gamma)} \|\mu\|_{H^{1/2}(\Gamma)},
\]

The bound on \(b\) then follows by combining these results and using Proposition 4.7. The bound on \(c\) is obvious. Next, we have

\[
\varphi(\eta, \eta, Y) \leq \|u_D\|_{H^{1/2}(\partial \Omega)} \|\eta \cdot \nu\|_{H^{-1/2}(\partial \Omega)} + C_1 \tilde{a} \|\nabla_Y Y\|_{L^2(\Gamma)} \leq C \|(\eta, \eta, Y)\|_{\mathbb{Z}},
\]

where \(C\) depends on \(\Delta t^{-1/2}\) and the data \(u_D\). The last inequality follows from (4.11) where the constant depends on \(\Delta t^{-1}\) and the problem data. \(\square\)

**Lemma 4.9 (Coercivity)** Let \((\eta, \eta, Y) \in \mathbb{Z}\) with \(b((\eta, \eta, Y), (q, q_c, \mu)) = 0\) for all \((q, q_c, \mu)\) in \(\mathbb{T}\). Then,

\[
|a((\eta, \eta, Y), (\eta, \eta, Y))| \geq C \|(\eta, \eta, Y)\|_{\mathbb{Z}}^2,
\]

where \(C > 0\) is a constant depending on \(\tilde{S}, \tilde{K}, \tilde{K}_c\), and the domain. This is true even if \(\tilde{b} \to \infty\).

**Proof.** From (4.8), we get

\[
a((\eta, \eta, Y), (\eta, \eta, Y)) \geq \frac{1}{\tilde{K}_1} \|\eta\|_{L^2(\Omega)}^2 + \frac{1}{\tilde{K}_1} \|\eta\|_{H(\mathrm{div}, \Omega)}^2 + \tilde{b}^{-1/2} \|\nabla_Y Y\|_{L^2(\Gamma)}^2 + \Delta t \tilde{a} \|\nabla_Y Y\|_{L^2(\Gamma)}^2,
\]

\[
= \frac{1}{\tilde{K}_1} \|\eta\|_{H(\mathrm{div}, \Omega)}^2 + \frac{1}{\tilde{K}_c} \|\eta\|_{H(\mathrm{div}, \Omega)}^2 + \tilde{b}^{-1/2} \|\nabla_Y Y\|_{L^2(\Gamma)}^2 + \Delta t \tilde{a} \|\nabla_Y Y\|_{L^2(\Gamma)}^2,
\]
where the last step follows from the hypothesis $\nabla \cdot \eta = \nabla \cdot \eta_0 = 0$. Also by hypothesis, we have
\[
\tilde{S}(\mathbf{Y} \cdot \mathbf{\nu}, \mu) = (\eta \cdot \mathbf{\nu}, \mu) = (\eta \cdot \mathbf{\nu}, \mu) - (\eta \cdot \mathbf{\nu}, \mu).
\]
for all $\mu \in H^{1/2}(\Gamma)$.

Hence, using (3.7) and (3.8), we have
\[
\tilde{S}\|\mathbf{Y} \cdot \mathbf{\nu}\|_{H^{-1/2}(\Gamma)} = \sup_{\mu \in H^{1/2}(\Gamma)} \frac{\tilde{S}(\mathbf{Y} \cdot \mathbf{\nu}, \mu)}{\|\mu\|_{H^{1/2}(\Gamma)}} \leq 2 \left( \|\eta \|_{H(\text{div}, \Omega)} + \|\eta_0 \|_{H(\text{div}, \Omega_0)} \right).
\]

Combining these inequalities yields the assertion.

\section*{Lemma 4.10 (Inf-Sup)}

For all $(q_i, q_\nu, \mu) \in \mathcal{T}$, the following “inf-sup” condition holds
\[
\sup_{(\eta, \mathbf{\nu}, \mathbf{Y}) \in \mathcal{Z}} \frac{b((\eta, \mathbf{\nu}, \mathbf{Y}), (q_i, q_\nu, \mu))}{\|\eta \|_{L^\infty(\Omega)}} \geq C \|(q_i, q_\nu, \mu)\|_{\mathcal{T}},
\]
where $C > 0$ depends on the domain and $\tilde{S}$. If $\|\eta \|_{L^\infty(\Omega)}$ is replaced by $\|\eta \|_{L^\infty(\Omega)}$ in the denominator, then the inf-sup still holds, except $C$ also depends on $K_1, K_\nu, \alpha$, and $\beta$. Furthermore, $C$ does not depend on the time step $\Delta t$, as long as $\Delta t \leq 1$.

\textbf{Proof.} Setting $\eta_\nu = 0$ on $\partial \Omega$, and accounting for the orientation of the normal vector and using the divergence theorem, we have
\[
b((\eta, \mathbf{\nu}, \mathbf{Y}), (q_i, q_\nu, \mu))
= (\nabla \cdot \eta, q_i)_{\Omega_1} - (\nabla \cdot \eta_\nu, q_\nu)_{\Omega_1} - (\eta \cdot \mathbf{\nu}, \mu) - (\eta_\nu \cdot \mathbf{\nu}, \mu) + \tilde{S}(\mathbf{Y} \cdot \mathbf{\nu}, \mu)
= (\nabla \cdot \eta_\nu, q_i)_{\Omega_1} - (\nabla \cdot \eta_\nu, q_\nu)_{\Omega_1} - (\eta \cdot \mathbf{\nu}, \mu) - (\eta_\nu \cdot \mathbf{\nu}, \mu) + \tilde{S}(\mathbf{Y} \cdot \mathbf{\nu}, \mu).
\]

By definition of the $H^{1/2}(\Gamma)$ norm (3.7), there exists a $\xi \in V_1(0)$ such that $-\xi \cdot \mathbf{\nu}, \mu - q_i, \mu = 0$ and $\|\xi \|_{H(\text{div}, \Omega_1)} = 1$. With this, we construct the vector field $\eta_\nu \in H(\text{div}, \Omega_1)$. Let $\phi_1, \phi_2 \in H^1(\Omega_1)$ (with zero mean value) be weak solutions of the following elliptic problems,
\[
-\Delta \phi_1 = -\frac{q_i}{\tilde{q}_i} \in \Omega_1, \quad \mathbf{\nu} \cdot \nabla \phi_1 = 0, \quad \mathbf{\nu} \cdot \nabla \phi_2 = \mathbf{\nu} \cdot \mathbf{\nu}, \quad \mathbf{\nu} \cdot \nabla \phi_2 = 0, \quad \mathbf{\nu} \cdot \nabla \phi_2 = 0.
\]
and define $\eta_\nu = \nabla \phi_1 + \nabla \phi_2$ (note that $\nabla \phi_1$ and $\nabla \phi_2$ are in $V_1(0)$). This gives
\[
\nabla \cdot (\eta_\nu q_\nu)_{\Omega_1} \cdot (\eta_\nu \cdot \mathbf{\nu}, \mu - q_i) = (\nabla \phi_1, 0)_{\Omega_1} + (\nabla \phi_2, 0)_{\Omega_1} - (\nabla \phi_2, 0)_{\Omega_1}.
\]

Now bound $\|\eta \|_{H(\text{div}, \Omega_1)}$. Since (3.2) and (3.3) hold with $\Omega$ replaced by $\Omega_1$, we get $\|\Delta \phi_2\|_{L^2(\Omega_1)} \leq C_0(\|\xi \cdot \mathbf{\nu}, \mu - \xi\|_{H^{-1/2}(\Omega_1)} \leq C_1(\|\xi \|_{H(\text{div}, \Omega_1)} \leq C_1(\xi \cdot \mathbf{\nu}, \mu - \xi_1) \leq C_2(\|\xi \|_{H(\text{div}, \Omega_1)} \leq C_2(\xi \cdot \mathbf{\nu}, \mu - \xi_1)) \leq C_3(\xi \cdot \mathbf{\nu}, \mu - \xi_1)).$ Similarly, we deduce that $\|\Delta \phi_1\|_{L^2(\Omega_1)} \leq 1$ and $\|\phi_1\|_{H^2(\Omega_1)} \leq C_3$. Hence, we arrive at the following result
\[
\|\eta \|_{H(\text{div}, \Omega_1)} \leq \|\Delta \phi_1\|_{L^2(\Omega_1)} + \|\Delta \phi_2\|_{L^2(\Omega_1)} + \|\phi_1\|_{H^2(\Omega_1)} + \|\phi_2\|_{H^1(\Omega_1)} \leq C_4.
\]
where $C_4 > 0$ depends on $\Omega_t$ and $\Gamma$. Analogous results show there exists an $\eta_h$ in $V_h$ such that

$$-\langle \nabla \cdot \eta_h, \hat{q}_t \rangle_{\Omega_t} + \langle \eta_h \cdot \nu, \mu - \hat{q}_t \rangle_{\Gamma} = \| \hat{q}_t \|_{L^2(\Omega_t)} + \| \mu - \hat{q}_t \|_{H^{1/2}(\Gamma)},$$

where $C_5 > 0$ depends on $\Omega_t$ and $\Gamma$.

By the definition of the $H^{-1}(\Gamma)$ norm (3.9), there exists a $Y$ in $H^1(\Gamma)$ such that

$$(Y \cdot \nu, \mu)_{\Gamma} = (Y, \mu \nu)_{\Gamma} = \| \mu \nu \|_{H^{-1}(\Gamma)}, \quad \| Y \|_{H^1(\Gamma)} = 1.$$
Proposition 4.12 Let $A$ be a constant $3 \times 3$ matrix and define $G(\gamma) = 1 + \gamma A$ for all $\gamma$ in $\mathbb{R}$. Then

$$\det G(\gamma) = 1 + \gamma \text{trace}(A) + \frac{1}{2} \gamma^2 \left[ \left( \text{trace}(A) \right)^2 - \text{trace}(A^2) \right] + \frac{1}{6} \gamma^3 \Xi_2(A).$$

where $\Xi_1(A)$ and $\Xi_2(A)$ are functions of $A$ that satisfy $|\Xi_1(A)| \leq C |A|^2$, $|\Xi_2(A)| \leq C |A|^3$, where $|\cdot|$ is any matrix norm and $C > 0$ is a constant that only depends on the norm.

Lemma 4.13 (Discrete Grönwall Inequality) Let $c \geq 0$ and suppose $\{r_i\}_{i \geq 0}$ and $\{g_i\}_{i \geq 0}$ are non-negative sequences. Then the following is true:

$$r_n \leq c + \sum_{k=0}^{n-1} g_k r_k, \quad \text{for all } n \geq 0, \quad \Rightarrow \quad r_n \leq c \exp \left( \sum_{k=0}^{n-1} g_k \right), \quad \text{for all } n \geq 0,$$

where the sum is zero when $n = 0$.

4.6.1 A priori bound. We begin as we did in Section 3.4.1. Again, take $u^{i+1}_D = 0$ for $i \geq 0$. In (4.5) and (4.6), choose $\eta_l = \sigma^{i+1}_l$, $\eta_s = \sigma^{i+1}_s$, $Y = \nu^{i+1}$, $q_l = -u^{i+1}_l$, $q_s = -u^{i+1}_s$, $\mu = -\lambda^{i+1}$, and add the equations together to get

$$\frac{1}{K_1} \left\| \sigma^{i+1}_l \right\|^2_{L^2(\Omega_t^l)} + \frac{1}{K_s} \left\| \sigma^{i+1}_s \right\|^2_{L^2(\Omega_t^s)} + \frac{\beta^{i+1/2}}{\Delta t} \left( \nu^{i+1}_l \nu^{i+1}_s \right)_{\Omega_t^l} + \frac{1}{\Delta t} \left( (u^{i+1}_l, (u^{i+1}_l - \overline{u}_l))_{\Omega_t^l} + \frac{1}{\Delta t} (u^{i+1}_s, (u^{i+1}_s - \overline{u}_s))_{\Omega_t^s} \right)$$

$$= (u^{i+1}_l, (u^{i+1}_l - \overline{u}_l))_{\Omega_t^l} + (u^{i+1}_s, (u^{i+1}_s - \overline{u}_s))_{\Omega_t^s}. \quad (4.17)$$

Next, focus on the discrete time derivative terms. Using $2a(a-b) = a^2 - b^2 + (a-b)^2$, we obtain

$$(u^{i+1}_l, (u^{i+1}_l - \overline{u}_l))_{\Omega_t^l} = \frac{1}{2} \left( \int_{\Omega_t^l} (u^{i+1}_l)^2 - \int_{\Omega_t^l} (\overline{u}_l)^2 + \int_{\Omega_t^l} (u^{i+1}_l - \overline{u}_l)^2 \right). \quad (4.18)$$

Now use (4.7) and a change of variables to show

$$\int_{\Omega_t^l} (\overline{u}_l)^2 = \int_{\Omega_t^{i-1}} (\overline{u}_l^i \circ \Phi_l(x))^2 |\det (\nabla \Phi_l(x))| \, dx$$

$$= \int_{\Omega_t^{i-1}} \left( u^{i-1}_l - \Delta t \tilde{K}_l^{-1} \sigma_l \cdot \nabla \right)^2 |\det (\nabla \Phi_l)|. \quad (4.19)$$
Combining (4.3) and Proposition 4.12 with (4.19), and expanding, gives

\[
\int_{\Omega^i} (\mathfrak{M}^i)^2 \leq \int_{\Omega^i} \left( (u_i^i)^2 + 2\Delta t \tilde{K}_i^{-1} |u_i^i| |\sigma_i^i| |V_i^i| + \Delta t^2 \tilde{K}_i^{-2} |\sigma_i^i|^2 |V_i^i|^2 \right) \cdot (1 + \Delta t |\nabla \cdot V_i^i| + (\Delta t^2 / 2) |\nabla \nabla V_i^i| + (\Delta t^3 / 6) |\nabla^2 \nabla V_i^i|) \, dx 
\]

\[
\leq \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + M_{i,i} \Delta t \left( \frac{2}{K_i} \|u_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} \|\sigma_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} + \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} \right) 
\quad + (M_{i,i}^2 \Delta t^2 + M_{i,i}^3 \Delta t^3) \left( \frac{1}{K_i} \|\sigma_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + \frac{2C}{K_i} \|u_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} \|\sigma_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} + C \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} \right) 
\quad + CM_{i,i}^4 \Delta t^4 \left( \frac{1}{K_i} \|\sigma_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + \frac{2}{K_i} \|u_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} \|\sigma_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} \right) 
\quad + CM_{i,i}^5 \Delta t^5 \frac{1}{K_i} \|\sigma_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})}
\]

\[
\leq \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + 2M_{i,i} \frac{\Delta t}{K_i} \|u_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} \|\sigma_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} + \Delta t M_{i,i} \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + CM_{i,i}^2 \Delta t^2 + CM_{i,i}^3 \Delta t^3 \]

\[
+ \Delta t M_{i,i} \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + CM_{i,i}^2 \Delta t^2 + CM_{i,i}^3 \Delta t^3 
\]  

(4.20)

where \(M_{i,i} = \|\nabla V_i^i\|_{W^{1,\infty}(\Omega^i;\mathbb{G}^{-1})}\) and \(C > 0\) is an independent constant. Next, choose \(\Delta t\) such that

\[
M_{i,i} \Delta t, \quad CM_{i,i} \Delta t, \quad CM_{i,i}^2 \Delta t^2, \quad CM_{i,i}^3 \Delta t^3 \leq 1/3, \quad \Delta t M_{i,i}^2 / \tilde{K}_i \leq 1/4, 
\]  

(4.21)

and note the following weighted Young’s inequality:

\[
\frac{\Delta t}{K_i} M_{i,i}^2 \|u_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} \|\sigma_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})} \leq \frac{\Delta t}{K_i} \left( 4M_{i,i}^2 \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + \frac{1}{4} \|\sigma_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} \right).
\]

Hence, (4.20) implies

\[
\int_{\Omega^i} (\mathfrak{M}^i)^2 \leq \|u_i^i\|_{L^2(\Omega^i;\mathbb{G}^{-1})}^2 + \Delta t \left( \frac{8M_{i,i}^2}{K_i} + 2M_{i,i} \right) \|u_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})} + \left( \frac{\Delta t}{K_i} \right) \|\sigma_i^i\|^2_{L^2(\Omega^i;\mathbb{G}^{-1})},
\]

for all \(i \geq 0\).  

(4.22)

A similar result holds for \(u_i^i\), with constant \(\tilde{M}_{i,i} = (8M_{i,i}^2 / \tilde{K}_i) + 2M_{i,i}\) where \(M_{i,i} = \|\nabla V_i^i\|_{W^{1,\infty}(\Omega^i;\mathbb{G}^{-1})}\).
Next, we note a result from [3, Lemma 1] which says that

$$\int_{F_i} \nabla \mathbf{r} \cdot \mathbf{X}^i + \nabla \mathbf{r} (\mathbf{X}^i - \mathbf{X}^i) \geq |\mathbf{X}^i (F_i^i)| - |F_i^i| = |F_i^i + 1| - |F_i^i|,$$

where $F_i^i + 1 := \mathbf{X}^i (F_i^i)$. Hence,

$$(\nabla \mathbf{r} (\Delta t \mathbf{X}^i + \mathbf{X}^i), \nabla \mathbf{r} \cdot \mathbf{X}^i)_{F_i} + (\nabla \mathbf{r} \cdot \mathbf{X}^i, \nabla \mathbf{r} \mathbf{X}^i)_{F_i} = \Delta t (\nabla \mathbf{r} \mathbf{X}^i + \nabla \mathbf{r} (\mathbf{X}^i - \mathbf{X}^i))_{F_i} \geq \frac{|F_i^i + 1| - |F_i^i|}{\Delta t}.$$  

(4.23)

Now combine (4.22) with (4.18) and plug into (4.17) (and do the same for $u_i^1$). Using (4.23) then yields

$$\frac{1}{K_1} \|\sigma_i^i + 1\|_{L^2(\Omega_i)}^2 - \frac{1}{2} \frac{1}{K_1} \|\sigma_i^i\|_{L^2(\Omega_i)}^2 + \frac{1}{K_2} \|\sigma_i^i + 1\|_{L^2(\Omega_i)}^2 - \frac{1}{2} \frac{1}{K_2} \|\sigma_i^i\|_{L^2(\Omega_i)}^2$$

$$+ \frac{1}{\Delta t} \bigg( \frac{\beta^{-1/2}(\nu') \mathbf{X}^i + \nu'}{L^2(\Omega_i)} \bigg) \geq \Delta t \bigg( \frac{1}{\Delta t} \bigg( \frac{\beta^{-1/2}(\nu') \mathbf{X}^i + \nu'}{L^2(\Omega_i)} \bigg) \bigg)$$

$$\leq (u_i^i, \mathbf{X}^i)_{\Omega_i} + \frac{1}{2} \widetilde{M}_{\nu} N \|u_i^i\|_{L^2(\Omega_i)}^2 + (u_i^i, \mathbf{X}^i)_{\Omega_i} + \frac{1}{2} \widetilde{M}_{\nu} N \|u_i^i\|_{L^2(\Omega_i)}^2.$$  

(4.24)

Applying a weighted Young’s inequality to the right-hand-side, multiplying by $\Delta t$, summing over $i$, and cancelling similar terms, we get

$$\frac{\Delta t}{2} \sum_{i=0}^{N-1} \left[ \frac{1}{K_1} \|\sigma_i^i + 1\|_{L^2(\Omega_i)}^2 + \frac{1}{K_2} \|\sigma_i^i\|_{L^2(\Omega_i)}^2 \right] + \Delta t \sum_{i=0}^{N-1} \left[ \frac{\beta^{-1/2}(\nu') \mathbf{X}^i + \nu'}{L^2(\Omega_i)} \right]$$

$$+ \Delta t \bigg( \frac{1}{\Delta t} \bigg( \frac{\beta^{-1/2}(\nu') \mathbf{X}^i + \nu'}{L^2(\Omega_i)} \bigg) \bigg)$$

$$\leq \Delta t \left[ (u_i^i, \mathbf{X}^i)_{\Omega_i} + \frac{1}{2} \|u_i^i\|_{L^2(\Omega_i)}^2 \right] + \Delta t \left[ \left( \frac{1}{\Delta t} \bigg( \frac{\beta^{-1/2}(\nu') \mathbf{X}^i + \nu'}{L^2(\Omega_i)} \bigg) \right) \right]$$

$$+ \Delta t \sum_{i=0}^{N-1} \left[ \frac{1}{2} \|\sigma_i^i + 1\|_{L^2(\Omega_i)}^2 + \|\sigma_i^i\|_{L^2(\Omega_i)}^2 \right]$$

$$+ \Delta t \sum_{i=0}^{N-1} \left[ \frac{1}{2} \|\sigma_i^i + 1\|_{L^2(\Omega_i)}^2 + \|\sigma_i^i\|_{L^2(\Omega_i)}^2 \right] + \frac{1}{2} \|u_i^i + 1\|_{L^2(\Omega_i)}^2 + \frac{1}{2} \|u_i^i\|_{L^2(\Omega_i)}^2.$$  

where $N$ is the last time index to compute. Making further simplifications, and assuming $\Delta t \leq 1$,
we arrive at

\[
\frac{\Delta t}{2} \sum_{i=0}^{N-1} \left[ \frac{1}{K_1} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2 + \frac{1}{K_s} \| \sigma^{i+1}_s \|_{L^2(\Omega^i)}^2 \right] + \Delta t \sum_{i=0}^{N-1} \frac{1}{4} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2 = \frac{\Delta t}{2} \sum_{i=0}^{N-1} \left[ \frac{1}{K_1} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2 + \frac{1}{K_s} \| \sigma^{i+1}_s \|_{L^2(\Omega^i)}^2 \right] + \Delta t \sum_{i=0}^{N-1} \frac{1}{4} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2
\]

for some independent constant \( B_0 > 0 \), which we have now proved.

**Theorem 4.14** Suppose (4.5), (4.6), (4.7) is solved on \( \Omega^i \) at time index \( i \), with \( u^{i+1}_i = 0 \), and assume \( \nu^{i+1} \) is in \( W^{1,\infty}(\Omega^i) \) and that \( \Phi^{i+1} \) is a bijective map in \( W^{1,\infty}(\Omega^i) \) with bounded inverse. Suppose this holds for \( i = 0, ..., N-1 \). If \( \Delta t \leq 1 \) also satisfies

\[
\Delta t \leq \frac{B_0}{\max_{0 \leq i \leq N} \| \nu^{i+1} \|_{W^{1,\infty}(\Omega^i)}} \quad \text{and} \quad \Delta t \leq \frac{B_0}{\max_{0 \leq i \leq N} \| \nu^{i+1} \|_{W^{1,\infty}(\Omega^i)}}
\]

for some independent constant \( B_0 > 0 \), then

\[
\frac{\Delta t}{2} \sum_{i=0}^{N-1} \left[ \frac{1}{K_1} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2 + \frac{1}{K_s} \| \sigma^{i+1}_s \|_{L^2(\Omega^i)}^2 \right] + \Delta t \sum_{i=0}^{N-1} \frac{1}{4} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2 = \frac{\Delta t}{2} \sum_{i=0}^{N-1} \left[ \frac{1}{K_1} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2 + \frac{1}{K_s} \| \sigma^{i+1}_s \|_{L^2(\Omega^i)}^2 \right] + \Delta t \sum_{i=0}^{N-1} \frac{1}{4} \| \sigma^{i+1}_i \|_{L^2(\Omega^i)}^2
\]
where
\[ A_0 = \tilde{\alpha}[f^0] + \frac{1}{2} \left( \frac{\Delta t}{K_1} \| \sigma^0 \|_{L^2(\Omega_1^{-1})}^2 + \frac{\Delta t}{K_s} \| \sigma^0_{r,} \|_{L^2(\Omega_1^{-1})}^2 + \| u^0_l \|_{L^2(\Omega_1^{-1})}^2 + \| u^0_{r,} \|_{L^2(\Omega_1^{-1})}^2 \right) \]
\[ + \Delta t \sum_{i=0}^{N-1} \left[ \| f_s^{i+1} \|_{L^2(\Omega_i)}^2 + \| f_s^{i+1} \|_{L^2(\Omega_i)}^2 \right]. \]

where \( \mathcal{U}_i > 0 \) is a constant that depends on \( \| \mathbf{V} \|_{W^{1,\infty}(\Omega)} \). Note: the final time for the semi-discrete evolution is \( T = \Delta t N \).

**Remark 4.15** Using an \( L^2 \) projection for the temperatures from one time step to the next would give a better estimate (i.e. more in-line with the fully continuous result (3.20)). The approach taken here is more complicated because we introduced the material derivative with an explicit treatment of the convective term (recall (4.4) and (4.7)); see Remark 5.8 for the reason. Theorem 4.14 can be easily modified to allow \( u_{l,r}^{i+1} \neq 0 \).

4.6.2 **Conservation law.** Analogous to Section 3.4.2, choose \( q_l = 1, q_r = 1 \) in (4.5), and \( \mu = 1 \) in (4.6) to get
\[
- \int_{\partial_0 \Omega} \sigma^{i+1}_l \cdot \nu_2 + \int_{\partial_0 r} \sigma^{i+1}_r \cdot \nu^l = \frac{1}{\Delta t} \left( \int_{\Omega_i} u^{i+1}_l - \int_{\Omega_i} \mathbf{w}_i \right) - \int_{\Omega_i} f_s^{i+1},
\]
\[ - \int_{\partial_0 r} \sigma^{i+1}_r \cdot \nu^l = \frac{1}{\Delta t} \left( \int_{\Omega_i} u^{i+1}_s - \int_{\Omega_i} \mathbf{w}_s \right) - \int_{\Omega_i} f_s^{i+1}, \quad (4.28) \]
\[
\tilde{S} \int_{\partial_0 r} \mathbf{v}^{i+1} \cdot \nu^l = \int_{\partial_0 r} \sigma^{i+1}_l \cdot \nu^l - \int_{\partial_0 r} \sigma^{i+1}_s \cdot \nu^l.
\]

Just as in (4.19), we have
\[
\int_{\Omega_i} \mathbf{w}_i = \int_{\partial_0^{-1} \Omega_i} \mathbf{w}_i \circ \Phi_i(\mathbf{x}) \det([\nabla \Phi_i(\mathbf{x})]) \ dx = \int_{\partial_0^{-1} \Omega_i} \left( u^{i}_l - \Delta t \tilde{K}^{-1}_l \sigma^{i}_l \cdot \mathbf{v}^l \right) \det([\nabla \Phi_i]).
\]

Combining (4.3) and Proposition 4.12 with (4.29), and expanding, gives
\[
\int_{\Omega_i} \mathbf{w}_i = \int_{\partial_0^{-1} \Omega_i} \left( u^{i}_l - \Delta t \tilde{K}^{-1}_l \sigma^{i}_l \cdot \mathbf{v}^l \right)
\]
\[ \cdot \left( 1 + \Delta t \nabla \cdot \mathbf{v}^l + (\Delta t^2/2) \mathcal{E}_1(\nabla \mathbf{v}^l) + (\Delta t^3/6) \mathcal{E}_2(\nabla \mathbf{v}^l) \right)
\]
\[ = \int_{\partial_0^{-1} \Omega_i} u^{i}_l + \Delta t \int_{\partial_0^{-1} \Omega_i} \left( u^{i}_l \nabla \cdot \mathbf{v}^l - \tilde{K}^{-1}_l \sigma^{i}_l \cdot \mathbf{v}^l \right) + \Delta t^2 I_1,
\]

where \( I_1 \) contains the higher order terms (c.f. (4.20)):
\[
I_1 = \int_{\partial_0^{-1} \Omega_i} \left\{ (u^{i}_l/2) \mathcal{E}_1(\nabla \mathbf{v}^l) - \tilde{K}^{-1}_l \sigma^{i}_l \cdot \mathbf{v}^l (\nabla \cdot \mathbf{v}^l) \right.
\]
\[ + \Delta t \left[ (u^{i}_l/6) \mathcal{E}_2(\nabla \mathbf{v}^l) - (1/2) \tilde{K}^{-1}_l \sigma^{i}_l \cdot \mathbf{v}^l \mathcal{E}_1(\nabla \mathbf{v}^l) \right] - \Delta t^2 (1/6) \tilde{K}^{-1}_l \sigma^{i}_l \cdot \mathbf{v}^l \mathcal{E}_2(\nabla \mathbf{v}^l).\]
Next, by making judicious use of the weak formulation \((4.5)\), we can simplify the \(O(\Delta t)\) term in \((4.30)\). Choose \(\eta^i = \nabla |_{\Omega_i} \in H^1(\Omega_i)\) in \((4.5)\) on \(\Omega_i^{t-1}\) for any \(i \geq 1\), and note \((4.2)\):

\[
\int_{\Omega_i^{t-1}} \bar{K}_i^{-1} \sigma_i^t \cdot \nabla i - \int_{\Omega_i^{t-1}} u_i^t \cdot \nabla i = \int_{\Gamma_i^{t-1}} \lambda_i \nabla i \cdot \nu_i^{t-1}. \tag{4.31}
\]

Thus, we arrive at

\[
\int_{\Omega_i^t} \bar{m}_i^t = \int_{\Omega_i^{t-1}} u_i^t - \Delta t \int_{\Gamma_i^{t-1}} \bar{K}_i \nabla i \cdot \nu_i^{t-1} + \Delta t^2 I_i, \quad \text{for all } i \geq 1,
\]

where \(M_i = \|\nabla i\|_{W^{1,\infty}(\Omega_i)}\) and \(C_0 > 0\) is an independent constant. Note that setting the initial velocity \(\nabla i^0 = 0\) gives \(\int_{\Omega_i^0} \bar{m}_i^0 = \int_{\Omega_i^{-1}} u_i^0\). Similar results hold for \(u_i^t\):

\[
\int_{\Omega_i^t} \bar{u}_i^t = \int_{\Omega_i^{t-1}} u_i^t + \Delta t \int_{\Gamma_i^{t-1}} \lambda_i \nabla i \cdot \nu_i^{t-1} + \Delta t^2 I_i, \quad \text{for all } i \geq 1,
\]

where \(M_i = \|\nabla i\|_{W^{1,\infty}(\Omega_i)}\).

Therefore, adding the equations in \((4.28)\), and using \((4.32)\) and \((4.33)\), gives a time-discrete thermal power balance for each \(i = 0, \ldots, N - 1\):

\[
\int_{\Omega_i^{t+1}} f_i^t + \int_{\Omega_i^t} f_i^t = \int_{\Omega_i^{t-1}} \sigma_i^t \cdot \nu_i + \frac{1}{\Delta t} \left( \int_{\Omega_i^t} u_i^{t+1} - \int_{\Omega_i^{t-1}} u_i^t \right)
+ \frac{1}{\Delta t} \left( \int_{\Omega_i^t} u_i^{t+1} - \int_{\Omega_i^{t-1}} u_i^t \right)
- \Delta t (I_i + I_s).
\]

Finally, summing \((4.34)\) over the time steps, and bounding \(I_i\) and \(I_s\), yields the following theorem.

THEOREM 4.16 Assume the hypothesis of Theorem 4.14 and suppose \(\nabla i^0 = 0\) on \(\Omega_i\). Then,

\[
\left| \Delta t \sum_{i=0}^{N-1} \left( \int_{\Omega_i^t} f_i^t + \int_{\Omega_i^t} f_i^t - \int_{\Omega_i^{t-1}} \sigma_i^t \cdot \nu_i + \frac{1}{\Delta t} \left( \int_{\Omega_i^t} u_i^{t+1} - \int_{\Omega_i^{t-1}} u_i^t \right)
+ \frac{1}{\Delta t} \left( \int_{\Omega_i^t} u_i^{t+1} - \int_{\Omega_i^{t-1}} u_i^t \right)
- \Delta t (I_i + I_s) \right) \right| \leq \Delta t B_1,
\]

where

\[
B_1 = \Delta t \sum_{i=0}^{N-1} u_i \left[ \frac{1}{K_1} \|\sigma_i^t\|_{L^1(\Omega_i)} + \frac{1}{K_2} \|\sigma_i^t\|_{L^1(\Omega_i)} + \|u_i^t\|_{L^1(\Omega_i)} + \|u_i^t\|_{L^1(\Omega_i)} \right].
\]
and \( U \) is a constant that depends on \( \| \mathbf{V}^i \|_{W^{1,\infty}(\Omega)} \). Note that \( B_1 \) is uniformly bounded (with respect to \( \Delta t \)) by Theorem 4.14.

**Remark 4.17** Theorem 4.16 is a discrete time integral version of (3.21), except with an \( O(\Delta t) \) error. The conservation property in [5, Remark 3.5] is exact but only applies to a time-continuous version of their numerical scheme. We note that Theorem 4.16 can be modified to allow \( u_{i+1} \neq 0 \) and \( \mathbf{V} \neq 0 \).

5. Fully discrete formulation

5.1 Discretization

5.1.1 Non-degenerate interface. Let \( \nu_h \) denote the unit normal vector on \( \Gamma_h \) and \( \partial \Omega_h \). The following assumption is the space discrete version of Assumption 4.5 in Section 4.4.1. It is necessary to ensure the equivalence of the norms in the space discrete version of Proposition 4.6 when \( \| \cdot \|_{H^{1/2}} \) is replaced by a discrete norm \( \| \cdot \|_{H^{1/2}} \).

**Assumption 5.1** Assume that \( \Gamma_h \) is a polyhedral manifold (i.e. a surface triangulation). For any vertex \( v \), let \( \text{Star}(v) \) be the set of triangle faces in \( \Gamma_h \) that contain \( v \) as a vertex. For any non-zero constant vector \( a \in \mathbb{R}^3 \), assume there exists a vertex \( v \) in \( \Gamma_h \) such that \( |\text{Star}(v)| \geq c_0 > 0 \) and

\[
\mathbf{a} \cdot \nu_h(x) > 0, \quad \forall x \in \text{Star}(v), \quad \text{or} \quad \mathbf{a} \cdot \nu_h(x) < 0, \quad \forall x \in \text{Star}(v).
\]

5.1.2 Formulation. We begin by approximating the domains \( \Omega^i \), \( \Omega^i \) by three dimensional triangulations \( \Omega^i_{s,h} \), \( \Omega^i_{s,h} \) such that \( \Gamma^i = \Omega^i_{s,h} \cap \overline{\Omega^i_{s,h}} \) is an embedded polyhedral surface contained in the faces of the mesh. A standard Galerkin approximation of equations (4.5), (4.6) takes the form:

\[
\frac{1}{K_s}(\sigma_{s,h}^i, \eta)_{\Omega^i_{s,h}} - (u^i_{s,h}, \nabla \cdot \eta)_{\Gamma^i_{s,h}} - (\eta \cdot \nu^i_{s,h}, \lambda^i_{s,h})_{\Omega^i_{s,h}} = -(\eta \cdot \nu^i_{s,h}, u^i_{s,h})_{\Omega^i_{s,h}},
\]

\[
-(\nabla \cdot \sigma^i_{s,h}, q)_{\Omega^i_{s,h}} = -\frac{1}{\Delta t}(u^i_{s,h}, q)_{\Omega^i_{s,h}} + \frac{1}{\Delta t}(u_{s,h}, q)_{\Gamma^i_{s,h}} = -(f^i_{s,h}, q)_{\Omega^i_{s,h}}.
\]

\[
\frac{1}{K_s}(\sigma_{s,h}^i, \eta)_{\Omega^i_{s,h}} - (u^i_{s,h}, \nabla \cdot \eta)_{\Gamma^i_{s,h}} + (\eta \cdot \nu^i_{s,h}, \lambda^i_{s,h})_{\Omega^i_{s,h}} = 0,
\]

\[
-(\nabla \cdot \sigma^i_{s,h}, q)_{\Omega^i_{s,h}} = -\frac{1}{\Delta t}(u^i_{s,h}, q)_{\Omega^i_{s,h}} + \frac{1}{\Delta t}(u_{s,h}, q)_{\Gamma^i_{s,h}} = -(f^i_{s,h}, q)_{\Omega^i_{s,h}},
\]

\[
(\mathbf{\beta}^i \nu^i_{s,h}, \lambda^i_{s,h})_{\Omega^i_{s,h}} + \Delta t \tilde{\sigma}((\nabla_r \mathbf{Y}_{s,h}^{i+1}, \nabla_r \mathbf{Y})_{\Gamma^i_{s,h}}
\]

\[
\tilde{S}(\mathbf{Y}_{s,h}^{i+1}, \mathbf{Y}, \nu^i_{s,h})_{\Gamma^i_{s,h}} = -\tilde{\sigma}((\nabla_r \mathbf{X}, \nabla_r \mathbf{Y})_{\Gamma^i_{s,h}}
\]

where the discrete spaces are defined over the current (known) domain \( \Omega^i_{s,h} \), \( \Gamma^i_{s,h} \). We then use the space discrete version of (4.1) to compute the new interface \( \Gamma^{i+1} \), followed by the space discrete version of (4.2), (4.3) to compute the map \( \Phi^{i+1} : \Omega^i_{s,h} \rightarrow \Omega^{i+1}_{s,h} \).
Remark 5.2 (Finite Element Space For Domain Velocity) The extension (4.2) of \( V^j_{h} \) to all of \( \Omega^j_{h} \) is computed by solving a discrete Laplace equation using a finite element space \( L^j_{h} \) on \( \Omega^j_{h} \) whose restriction to \( \Gamma^j_{h} \) contains \( Y^j_{h} \). Because of (4.1), (4.3), the shape of the tetrahedral elements \( T \) in \( \Omega^j_{h} \) must be representable by functions in \( L^j_{h} \), i.e. the parametrization of \( T \) must be expressed as a linear combination of basis functions in the local finite element space of \( L^j_{h} \). For example, this is achieved when \( L^j_{h} \) is piecewise linear and \( \Omega^j_{h} \) consists of affine tetrahedra.

The space discrete version of the temperature update formula (4.7) is then given by

\[
\hat{u}_{j,h} := \left[ u_{j,h} - \Delta t \frac{1}{K_h} \Pi_{\varphi_{j,h}} \left( \sigma_{j,h} \cdot \varphi_{j,h} V_h^j \right) \right] \circ \Phi_{i,h}^{-1}, \quad j = 1, s, \tag{5.3}
\]

where \( \Pi_{\varphi_{j,h}} : L^2(\Omega^j_{h}) \to Q^j_{h} \) is the standard \( L^2 \) projection onto \( Q^j_{h} \) and \( \varphi_{j,h} : H^1_0(\Omega^j_{h}) \to \mathcal{V}_h^j \) is a suitable interpolant; see Section 5.2 for a description of these operators and Section 5.4 for the reasons we need them. Iterating this procedure gives the fully discrete approximation of (3.11), (3.12).

Just as in Section 4.3, we drop the time index notation when considering (5.1), (5.2) at a single time step. This leads to a fully discrete version of (4.14).

Variational Formulation 5.3 Find \( (\sigma_{1,h}, \sigma_{s,h}, V_h) \) in \( \mathcal{Z}_h \) and \( (u_{1,h}, u_{s,h}, \lambda_h) \) in \( \mathcal{T}_h \) such that

\[
a_h((\eta_h, \eta_h, Y), (\sigma_{1,h}, \sigma_{s,h}, V_h)) + b_h((\eta_h, \eta_h, Y), (u_{1,h}, u_{s,h}, \lambda_h)) = j_h(\eta_h, \eta_h, Y),
\]

\[
+ b_h((\sigma_{1,h}, \sigma_{s,h}, V_h), (q_1, q_s, \mu), (u_{1,h}, u_{s,h}, \lambda_h)) = \psi_h(q_1, q_s, \mu),
\]

for all \( (\eta_h, \eta_h, Y) \) in \( \mathcal{Z}_h \), and \( (q_1, q_s, \mu) \) in \( \mathcal{T}_h \).

The discrete version of the forms in Section 4.3.1 are defined in the obvious way. The discrete product spaces are defined similar to (4.12), (4.13); \( \mathcal{Z}_h = \mathcal{V}_{1,h}(0) \times \mathcal{V}_{s,h} \times \mathcal{Y}_h \), \( \mathcal{T}_h = \mathcal{Q}_{1,h} \times \mathcal{Q}_{s,h} \times \mathcal{M}_h \).

5.1.3 Discrete norms. The discrete multiplier norm is slightly different. We first introduce a discrete version of the \( H^{1/2}(\Gamma_h) \) norm. For any \( \mu \in H^{1/2}(\Gamma_h) \), define the discrete version of (3.7):

\[
\| \mu \|_{H^{1/2}(\Gamma_h)} := \sup_{\eta_h \in \mathcal{V}_{1,h}(0)} \frac{\langle \eta_h \cdot \nu_h, \mu \rangle_{\Gamma_h}}{\| \eta_h \|_{H^{1/2}(\mathcal{D}_{\Omega,h})}}, \quad \| \mu \|_{H^{1/2}(\Gamma_h)} := \sup_{\eta_h \in \mathcal{V}_{1,h}(0)} \frac{\langle \eta_h \cdot \nu_h, \mu \rangle_{\Gamma_h}}{\| \eta_h \|_{H^{1/2}(\mathcal{D}_{\Omega,h})}}. \tag{5.5}
\]

Clearly, for \( j = 1, s, \| \mu \|_{H^{1/2}(\Gamma_h)} \leq \| \mu \|_{H^{1/2}(\Gamma_h)} \) and \( \| \eta_h \cdot \nu_h, \mu \|_{\Gamma_h} \leq \| \eta_h \|_{H^{1/2}(\mathcal{D}_{\Omega,h})} \| \mu \|_{H^{1/2}(\Gamma_h)} \) (discrete Schwarz inequality). We shall also use a discrete version of the \( H^{-1}(\Gamma_h) \) norm to control the mean value of \( \mu \in \mathcal{M}_h \). For all \( v \) in \( H^{-1}(\Gamma_h) \), define

\[
\| v \|_{H^{-1}(\Gamma_h)} := \sup_{\nu_h \in \mathcal{V}_{1,h}} \frac{\langle v, \nu_h \rangle_{\Gamma_h}}{\| v \|_{H^{1/2}(\Gamma_h)}}, \tag{5.6}
\]

which also satisfies \( \| v \|_{H^{-1}(\Gamma_h)} \leq \| v \|_{H^{-1}(\Gamma_h)} \) and \( \langle v, \nu_h \rangle_{\Gamma_h} \leq \| v \|_{H^{-1}(\Gamma_h)} \| \nu_h \|_{H^{1/2}(\Gamma_h)} \) (discrete Schwarz inequality). Then the discrete version of \( \| (q_1, q_s, \mu) \|_{\mathcal{V}_{1,h}(0)}^2 \) is \( \| (q_1, q_s, \mu) \|_{\mathcal{V}_{1,h}(0)}^2 = \| q_1 \|_{L^2(\Omega_{1,h})}^2 + \| q_s \|_{L^2(\Omega_{s,h})}^2 + 1/2 \left( \| \mu \|_{H^{1/2}(\Gamma_h)}^2 + \| \mu \|_{H^{1/2}(\Gamma_h)}^2 \right) \).

\[
\| \mu \|_{H^{1/2}(\Gamma_h)} := \frac{1}{2} \left( \| \mu \|_{H^{1/2}(\Gamma_h)}^2 + \| \mu \|_{H^{1/2}(\Gamma_h)}^2 \right). \tag{5.7}
\]
and the discrete version of (4.16) is
\[
\|\langle q, q_h, \mu \rangle \|^2_{T_h} = \|q_h\|^2_{L^2(\Omega_h, \mathcal{H})} + \|q_h\|^2_{L^2(\Omega_h, \mathcal{S})} + \|\mu - \hat{q}_h\|^2_{H^{1/2}(\Gamma_h)} + \|q_h\|^2_{H^{-1/2}(\Gamma_h)}.
\]

(5.8)

A discrete version of Proposition 4.7 also holds, i.e. \(\|\langle q, q_h, \mu \rangle \|_{\mathcal{T}^2_h} \approx \|\langle q, q_h, \mu \rangle \|_{\mathcal{T}_h}\).

The discrete version of the primal norm (4.15) is also slightly different. It requires a discrete version of the \(H^{-1/2}(\Gamma_h)\) norm to control the mean value of \(Y \cdot \nu_h\) for \(Y \in \mathcal{V}_h\). For any \(Y \cdot \nu_h \in H^{-1/2}(\Gamma_h)\), define
\[
\|Y \cdot \nu_h\|_{H^{-1/2}(\Gamma_h)} := \sup_{\mu_h \in \mathcal{V}_h} \frac{\langle Y \cdot \nu_h, \mu_h \rangle_{\Gamma_h}}{\|\mu_h\|_{H^{1/2}(\Gamma_h)}}.
\]

(5.9)

Clearly, \(\langle Y \cdot \nu_h, \mu_h \rangle_{\Gamma_h} \lesssim \|Y \cdot \nu_h\|_{H^{-1/2}(\Gamma)}\|\mu_h\|_{H^{1/2}(\Gamma_h)}\) (discrete Schwarz inequality). Then the discrete version of \(\|\langle \eta, \eta, Y \rangle \|_{\mathcal{T}^2_h}^2\) is obtained by replacing \(\|Y \cdot \nu\|_{H^{-1/2}(\Gamma)}\) with \(\|Y \cdot \nu_h\|_{H^{-1/2}(\Gamma_h)}\).

A discrete version of Proposition 4.6 also holds.

### 5.2 Space assumptions

To prove well-posedness of the discrete system, we must prove the discrete version of Lemmas 4.8, 4.9, and 4.10. In addition, we want to obtain discrete versions of Theorems 4.14 and 4.16. To facilitate this, we make the following general assumptions on the choice of finite dimensional subspaces (see Section 5.5 for the specific spaces used).

Let \(\mathcal{V}_h\) be a conforming finite dimensional subspace, i.e. \(\mathcal{V}_h \subset \mathcal{V} = H(\text{div}, \Omega)\), and define
\[
\mathcal{V}_h := \{\eta \in \mathcal{V}_h : \eta \cdot \nu_h = 0, \text{ on } \partial \Omega_h \} \subset \{\eta \in \mathcal{V} : \langle \eta \cdot \nu_h, q \rangle_{\partial \Omega_h} = 0, \forall q \in H^1(\Omega)\}.
\]

Furthermore, assume that for any \(\eta\) in \(\mathcal{V}_h\), we have \(\eta|_{\partial \Omega_h} \in \mathcal{V}_h(0)\) and \(\eta|_{\partial \Omega_h} \in \mathcal{V}_{s,h}\).

Next, take \(\mathcal{V}_{s,h} = \{\eta \in \mathcal{V}_h : \eta \cdot \nu_h = 0, \text{ on } \partial \Omega_h\} \cup \mathcal{Q}_{i,h} \cup \mathcal{Q}_{s,h} = \{q \in \mathcal{Q}_{i,h} : \int_{\Omega_h} q \, dx = 0\},\) and assume that \(\mathcal{V} \cdot \mathcal{V}_{s,h} = \mathcal{Q}_{i,h}, \mathcal{V} \cdot \mathcal{V}_{s,h} = \mathcal{Q}_{s,h}, \) and \(\mathcal{V}_{s,h}\) contains continuous piecewise linear functions on \(\Gamma_h\). Analogous definitions are made for \(\mathcal{V}_{s,h}\) and \(\mathcal{Q}_{i,s,h}\). Moreover, assume \(\langle \mathcal{V}_{i,h}, \mathcal{Q}_{i,h} \rangle\) and \(\langle \mathcal{V}_{s,h}, \mathcal{Q}_{s,h} \rangle\) satisfy
\[
\sup_{\eta \in \mathcal{V}_{s,h}} \frac{-(\mathcal{V} \cdot \eta, q)_{\Omega_h}}{\|\eta\|_{H(\text{div}, \Omega_h)}} \leq c \|q\|_{L^2(\Omega_h)}, \quad \sup_{\eta \in \mathcal{V}_{s,h}} \frac{-(\mathcal{V} \cdot \eta, q)_{\Omega_h}}{\|\eta\|_{H(\text{div}, \Omega_h)}} \geq c \|q\|_{L^2(\Omega_h)}.
\]

(5.10)

for all \(q \in \mathcal{Q}_{i,h}, q \in \mathcal{Q}_{s,h}\), with \(c\) independent of \(h\) and that an analogous condition is satisfied for \(\langle \mathcal{V}_{i,h}, \mathcal{Q}_{i,h} \rangle\) and \(\langle \mathcal{V}_{s,h}, \mathcal{Q}_{s,h} \rangle\). This implies that we can solve the discrete mixed form of Laplace’s equation. As for \(\mathcal{V}_h\) and \(\mathcal{V}_{s,h}\), assume they are spaces of continuous functions.

Regarding (5.3), we have a “Fortin interpolant” \([9, 11]\) \(\mathcal{Q}_h : H^1_0(\Omega_h) \to \mathcal{V}_h\) that satisfies for any \(\mathcal{V} \in H^1_0(\Omega_h)\):
\[
\|\mathcal{Q}_h \mathcal{V}\|_{L^2(\Omega_h)} \leq C \|\mathcal{V}\|_{H^1(\Omega_h)}, \quad \text{and} \quad (q, \mathcal{V} - \mathcal{Q}_h \mathcal{V})_{\Omega_j,h} = 0, \forall q \in \mathcal{Q}_{j,h}, \text{ for } j = 1, s.
\]

(5.11)

And the \(L^2\) projections \(\Pi_{\mathcal{Q}_{j,h}} : L^2(\Omega_{j,h}) \to \mathcal{Q}_{j,h}\) for \(j = 1, s\) satisfy for any \(v \in L^2(\Omega_{j,h})\):
\[
\|\Pi_{\mathcal{Q}_{j,h}} v\|_{L^2(\Omega_{j,h})} \leq \|v\|_{L^2(\Omega_{j,h})}, \quad \text{and} \quad (q - \Pi_{\mathcal{Q}_{j,h}} q)_{\Omega_j,h} = 0, \forall q \in \mathcal{Q}_{j,h}, \text{ for } j = 1, s.
\]

(5.12)
5.3 Well-posedness
We follow a similar outline as Section 4.5.

5.3.1 Main conditions.

Lemma 5.4 (Continuity of Forms)

\[ |a_h((\eta, \eta, Y), (\sigma, \sigma, V))| \leq C_{ah} \| (\eta, \eta, Y) \|_{Z_h} \| (\sigma, \sigma, V) \|_{Z_h}, \forall (\eta, \eta, Y), (\sigma, \sigma, V) \in Z_h, \]

\[ |b_h((\eta, \eta, Y), (q_l, q_l, \mu))| \leq C_{bh} \| (\eta, \eta, Y) \|_{Z_h} \| (q_l, q_l, \mu) \|_{T_h}, \forall (\eta, \eta, Y) \in Z_h, (q_l, q_l, \mu) \in T_h, \]

where \( C_{ah}, C_{bh}, C_{\chi h} > 0 \) are constants that depend on physical parameters and domain geometry. In addition, \( C_{\chi h} \) depends on \( u_D, \Delta t^{-1/2} \), and \( C_{\phi h} \) depends on \( f, f, \overline{h}, \overline{\nu}, T \) and \( \Delta t^{-1} \).

Proof. The proof is analogous to the proof of Lemma 4.8. Minor modifications are: one must use the discrete Schwarz inequalities associated with the discrete \( H_{1/2}^h, H_{s,1/2}^h \), and \( H_{1/2}^{-1} \) norms, and use the discrete versions of Propositions 4.6 and 4.7.

Lemma 5.5 (Coercivity) Let \( (\eta, \eta, Y) \in Z_h \) with \( b_h((\eta, \eta, Y), (q_l, q_l, \mu)) = 0 \) for all \( (q_l, q_l, \mu) \in T_h \). Then,

\[ |a_h((\eta, \eta, Y), (\eta, \eta, Y))| \geq C \| (\eta, \eta, Y) \|_{Z_h}^2, \]

where \( C > 0 \) is a constant depending on \( \hat{S}, \hat{K}, \hat{K}, \) and the domain. This is true even if \( \hat{\beta} \rightarrow \infty \).

Proof. Follows the same argument as in Lemma 4.9, except the discrete \( H_{1/2}^{-1} \) norm (5.9) is used.

Lemma 5.6 (Inf-Sup) For all \( (q_l, q_l, \mu) \in T_h \), the following "inf-sup" condition holds

\[ \sup_{(\eta, \eta, Y) \in Z_h} \frac{b_h((\eta, \eta, Y), (q_l, q_l, \mu))}{\| (\eta, \eta, Y) \|_{Z_h}^2} \geq C \| (q_l, q_l, \mu) \|_{T_h}, \]

where \( C > 0 \) depends on the domain and \( \hat{S} \). If \( \| (\eta, \eta, Y) \|_{Z_h}^2 \) is replaced by \( \| (\eta, \eta, Y) \|_{Z_h} \) in the denominator, then the inf-sup still holds, except \( C \) also depends on \( \hat{K}, \hat{K}, \hat{\alpha}, \) and \( \hat{\beta} \). Furthermore, \( C \) does not depend on the time step \( \Delta t \), as long as \( \Delta t \leq 1 \).

Proof. Starting as we did in the proof of Lemma 4.10, we have

\[ b_h((\eta, \eta, Y), (q_l, q_l, \mu)) = - (\nabla \cdot \eta \cdot \hat{\eta} - (\nabla \cdot \eta \cdot \hat{\eta}) \hat{\eta} - (\eta \cdot \nu_h, \mu - \hat{\eta}) \hat{r}_h + (\eta \cdot \nu_h, \mu - \hat{\eta}) \hat{r}_h, \]

\[ + \hat{S} (\nabla \cdot \nu_h, \mu - \hat{\eta}) \hat{r}_h, \]

\[ + \hat{S} (\nabla \cdot \nu_h, \mu - \hat{\eta}) \hat{r}_h. \]
Next, let us focus on \(- (\nabla \cdot \eta_h, \tilde{q}_h)_{\Omega_h} + (\eta_h \cdot \nu_h, \mu - \hat{q}_h)_{\Gamma_h}\) only. By (5.10), there exists a unique \((w, o)\) in \((\tilde{V}_{s,h}, \tilde{Q}_{s,h})\) such that

\[
(w, v)_{\Omega_h} - (o, \nabla \cdot v)_{\Omega_h} = 0, \quad \forall v \in \tilde{V}_{s,h},
\]

\[
-(\nabla \cdot w, r)_{\Omega_h} = (\hat{q}_s, r)_{\Omega_h}, \quad \forall r \in \tilde{Q}_{s,h},
\]

and \(\|w\|_{H^1(\Omega_h)} \leq C_0\|\tilde{q}_s\|_{L^2(\Omega_h)}\). By (5.5), there exists \(\xi \in V_s\) such that

\[
(\xi \cdot \nu_h, \mu - \hat{q}_s)_{\Gamma_h} = \|\mu - \hat{q}_s\|_{H^{1/2}(\Gamma_h)}, \quad \|\xi\|_{H^1(\Omega_h)} = \|\mu - \hat{q}_s\|_{H^{1/2}(\Gamma_h)}.
\]

Similar to (5.13), there exists a \(z\) in \(\tilde{V}_{s,h}\) such that

\[
-\nabla \cdot z = \nabla \cdot \xi - \frac{1}{|\Omega_{s,h}|} \left( \int_{\Gamma_h} \xi \cdot \nu_h \right), \quad \text{on } \Omega_{s,h}, \quad \|z\|_{H^1(\Omega_h)} \leq C_1 \|\xi\|_{H^1(\Omega_h)}.
\] 

(5.14)

Now let \(d = z + \xi\). Then,

\[
\nabla \cdot d = \frac{1}{|\Omega_{s,h}|} \left( \int_{\Gamma_h} \xi \cdot \nu_h \right), \quad \text{on } \Omega_{s,h}, \quad d \cdot \nu_h = \xi \cdot \nu_h, \quad \text{on } \Gamma_h,
\]

where \(\|d\|_{H^1(\Omega_h)} \leq (1 + C_1) \|\xi\|_{H^1(\Omega_h)} = (1 + C_1) \|\mu - \hat{q}_s\|_{H^{1/2}(\Gamma_h)}\).

Next, define \(y := w + d \in \tilde{V}_{s,h}\) and note \(\|y\|_{H^1(\Omega_h)} \leq C_0 \|\hat{q}_s\|_{L^2(\Omega_h)} + (1 + C_1) \|\mu - \hat{q}_s\|_{H^{1/2}(\Gamma_h)}\). Thus, setting \(\eta_h := y/\|y\|_{H^1(\Omega_h)}\) gives

\[
-(\nabla \cdot \eta_h, \tilde{q}_h)_{\Omega_h} + (\eta_h \cdot \nu_h, \mu - \hat{q}_h)_{\Gamma_h} = \frac{1}{\|y\|_{H^1(\Omega_h)}} \left( \|\tilde{q}_s\|_{L^2(\Omega_h)}^2 + \|d \cdot \nu_h, \mu - \hat{q}_h\|_{\Gamma_h} \right)
\]

\[
\geq C_2 \left( \|\tilde{q}_s\|_{L^2(\Omega_h)} + \|\mu - \hat{q}_h\|_{H^{1/2}(\Gamma_h)} \right),
\]

with \(\|\eta_h\|_{H^1(\Omega_h)} = 1\). Similarly, there exists \(\eta \in V_{s,h}(0)\) such that

\[
-(\nabla \cdot \eta, \tilde{q}_h)_{\Omega_h} + (\eta \cdot \nu_h, \mu - \hat{q}_h)_{\Gamma_h} \geq C_3 \left( \|\tilde{q}_s\|_{L^2(\Omega_h)} + \|\mu - \hat{q}_h\|_{H^{1/2}(\Gamma_h)} \right),
\]

with \(\|\eta\|_{H^1(\Omega_h)} = 1\).

By the definition of the discrete \(H^{-1}(\Gamma_h)\) norm (5.6), there exists a \(Y\) in \(\mathbb{V}_h\) such that

\[
(Y \cdot \nu_h, \mu)_{\Gamma_h} = (Y, \mu \nu_h)_{\Gamma_h} = \| \mu \nu_h \|_{H^{-1}(\Gamma_h)}, \quad \|Y\|_{H^1(\Gamma_h)} = 1.
\]

Combining the above results gives the assertion.

5.3.2 Summary: A discussion analogous to the one in Section 4.5.2 applies to the fully discrete problem also. Hence, the discrete problem is well-posed, but one must modify the norm \(\|\cdot\|_{H^{1/2}}\) to include an extra factor of \(\Delta t\) multiplying \(\|\nabla \cdot \eta_h\|_{L^2(\Omega_h)}, \|\nabla \cdot \eta_h\|_{L^2(\Omega_h)}, \text{and } \|Y \cdot \nu_h\|_{H^{-1/2}(\Gamma_h)}\).
5.4 Discrete estimates

Applying the same arguments in Section 4.6.1 to the fully discrete problem (5.1), (5.2), (5.3), and using the stability properties in (5.11), (5.12), we get the fully discrete version of Theorem 4.14.

For the conservation law, the argument in Section 4.6.2 changes slightly. Recalling (5.3), the discrete counterpart of (4.30) is

\[
\int_{\Omega_{i,h}^{l+1}} u_{i,h}^l = \int_{\Omega_{i,h}^{l}} u_{i,h}^l + \Delta t \int_{\Omega_{i,h}^{l}} \left( u_{i,h}^l \nabla \cdot V_h^i - \tilde{K}_i^{-1} \Pi_{Q_{i,h}} \left( \sigma_{i,h}^l \cdot \Psi_{i,h} V_h^i \right) \right) + O(\Delta t^2).
\]

(5.15)

Using the properties of \( \Psi_{i,h} \) and \( \Pi_{Q_{i,h}} \), we see that

\[
R_1 = \int_{\Omega_{i,h}^{l}} u_{i,h}^l \nabla \cdot \Psi_{i,h} V_h^i = \int_{\Omega_{i,h}^{l}} \tilde{K}_i^{-1} \sigma_{i,h}^l \cdot \Psi_{i,h} V_h^i = -\int_{\Omega_{i,h}^{l}} \tilde{K}_i^{-1} \sigma_{i,h}^l \cdot \Psi_{i,h} V_h^i \cdot \nu_{i,h}^l.
\]

where the last equality follows by choosing \( \eta = \Psi_{i,h} V_h^i |_{\Omega_{i,h}^{l}} \in V_{i,h}^{l-1} \) in (5.1) on \( \Omega_{i,h}^{l-1} \) for any \( i \geq 1 \). Thus, we arrive at

\[
\int_{\Omega_{i,h}^{l}} u_{i,h}^l = \int_{\Omega_{i,h}^{l}} u_{i,h}^l - \Delta t \int_{\Omega_{i,h}^{l-1}} \tilde{K}_i^{-1} \sigma_{i,h}^l \cdot \Psi_{i,h} V_h^i \cdot \nu_{i,h}^l + O(\Delta t^2), \quad \text{for all } i \geq 1.
\]

A similar relation holds for \( u_{i,h}^l \), except with \( + \Delta t \). The rest of the derivation in Section 4.6.2 remains the same (note that \( \Psi_{i,h} V_h^i \) is continuous across \( \Gamma_h^{l-1} \)), which delivers the fully discrete version of Theorem 4.16.

We summarize these results in the following theorem.

THEOREM 5.7 (A Priori Bound and Conservation Law) Suppose (5.1), (5.2), (5.3) is solved on \( \Omega_{i,h}^{l} \) at time index \( i \), with \( u_{i,0}^0 \equiv 0 \), and that \( \Phi_{i+1,h} \) is a bijective map with bounded inverse. Suppose this holds for \( i = 0, \ldots, N - 1 \). If \( \Delta t \leq 1 \) also satisfies

\[
\Delta t \leq \frac{B_0}{\max_{0 \leq i \leq N} \| V_h^i \|_{W^{1,\infty}(\Omega)}} \quad \text{and} \quad \Delta t \leq \frac{\max(\tilde{K}_1, \tilde{K}_2)}{\max_{0 \leq i \leq N} \| V_h^i \|_{W^{1,\infty}(\Omega)}}
\]

for some independent constant \( B_0 > 0 \), then the fully discrete version of the a priori bound (4.27) is true, i.e. replace all pertinent variables in (4.27) by their discrete counterparts. Moreover, if \( V_h^0 \equiv 0 \) on \( \Omega_h \), then the fully discrete version of the conservation law (4.35) is also true.

We emphasize that the time step \( \Delta t \) does not depend on the mesh size \( h \) to guarantee stability or the conservation law; it only depends on \( \| V_h^i \|_{W^{1,\infty}(\Omega)} \) (see Remark 4.2).

REMARK 5.8 (Reason For The Lagrangian Update) Using a Lagrangian approach to update the temperatures (5.3) avoids having to compute the intersection of the mesh from one time step to the next (i.e. the \( L^2 \) projections (5.12) are only computed on the previous domains \( \Omega_{i,h}^{l-1}, \Omega_{i+1,h}^{l-1} \)). The alternative would have been to compute the \( L^2 \) projection (for \( j = 1, s \)) of \( u_{j,h}^l \) from \( \Omega_{j,h}^{l-1} \) to \( \Omega_{j,h}^l \), which would require computing the intersection of the meshes representing \( \Omega_{j,h}^{l-1} \) and \( \Omega_{j,h}^l \).
5.5 Specific realization

The particulars of our implementation are as follows. Let $\mathcal{T}_h$ denote a quasi-uniform, shape regular triangulation of $\Omega_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{s,h}$ consisting of affine tetrahedra $T$ of maximum size $h_T = hT$ [10]. We choose the finite element spaces in the bulk to be $V_{l,h}^{1} = \text{BDM}_1 \subset H(\text{div}, \Omega_{l,h})$, $V_{s,h}^{1} = \text{BDM}_1 \subset H(\text{div}, \Omega_{s,h})$, i.e. the lowest order Brezzi-Douglas-Marini space of piecewise linear vector functions $[7, 26]$, and $Q_{l,h}$, $Q_{s,h}$ to be the set of piecewise constants.

Next, assume that $\Gamma_h$ is represented by a conforming set of faces $\mathcal{T}_h$ in the triangulation $\mathcal{T}_h$, i.e. $\mathcal{T}_h$ is the surface triangulation obtained by restricting $\mathcal{T}_h$ to $\Gamma_h$. Then choose $M_h$ to be the space of continuous piecewise linear functions over $\mathcal{T}_h$ and each of the three components of the space $\mathcal{V}_h$ to be continuous piecewise linear functions over $\mathcal{T}_h$. Recalling Remark 5.2, we choose $L_h$ to be the space of continuous piecewise linear functions over $\Omega_h$.

Remark 5.9 (Choice Of Finite Element Spaces) It is well-known that these spaces satisfy the assumptions in Section 5.2. Indeed, it is possible to enforce zero boundary values point-wise with $\text{BDM}_1$. If different spaces were chosen that did not allow this, then one needs a reasonable compatibility condition between $\mathcal{V}_{l,h}$, $\mathcal{V}_{s,h}$ and $M_h$ in order to prove Lemma 5.6.

Moreover, we take $\mathcal{I}_{\mathcal{V}_{l,h}}$ in (5.11) to be the classic $\text{BDM}_1$ interpolant $[7, 11]$; the $L^2$ projections $\Pi_{Q_{l,h}}$, $\Pi_{Q_{s,h}}$ are standard $[10]$. Note that this allows (5.3) to be computed locally (i.e. element-by-element).

6. Numerical results

We present two dimensional simulations to illustrate our method (2-D for simplicity). All simulations were implemented in the package FELICITY $[62]$. The linear systems are solved by MATLAB’s “backslash” command. Alternatively, one can use an iterative procedure such as Uzawa’s algorithm; see [22, Section 7] for an example in a related problem.

For all simulations, the Dirichlet boundary is the entire outer boundary, i.e. $\partial D \Omega \equiv \partial \Omega$ with $\mathcal{D} = -0.5$. The initial temperature is $\mathcal{U}_0 := 0$ in $\Omega$, and $\mathcal{U}_0$ is a smooth function between 0 and $-0.5$ in $\Omega_t$. For updating the temperatures, we initialized $V_0 = 0$. We verified the conservation law by computing the left-hand-side of (4.35). The error was less than $10^{-3}$, which is consistent with the $O(\Delta t)$ error estimate in Theorem 4.16. During the course of a simulation, the mesh topology was regenerated between three and five times which did not impact the computational time.

Error estimates for the spatial discretization will be discussed in a future publication.

6.1 Isotropic surface energy

The model in Section 2 assumes the surface tension coefficient $\tilde{\alpha}$ is constant (isotropic). In Figure 2, we show a simulation of our method with a non-trivial initial shape. Also see Figure 1 for another example with a different initial shape.
6.2 Anisotropic surface energy

The model can be generalized to have an anisotropic surface tension coefficient, i.e. $\tilde{\alpha} = \tilde{\alpha}(\nu)$. In particular, we consider anisotropies of the form:

$$\tilde{\alpha} = \tilde{\alpha}(\nu) := \tilde{\alpha}_0 \sum_{j=1}^{K} (\nu^T G_j \nu)^{1/2},$$  \hspace{1cm} (6.1)

where $\tilde{\alpha}_0 = 0.0005$ is a material constant, $K$ is the number of anisotropies, and $G_j$ is a symmetric positive definite matrix in $\mathbb{R}^{d \times d}$. We consider a class of matrices that have the structure $G_j = R_j^T D_j R_j$, where $R_j$ is a rotation matrix that determines the “directions” of the anisotropy, and $D_j$ is a diagonal matrix consisting of ones and small numbers, which controls the strength of the anisotropy. For our simulations, we set $\tilde{\beta} = \tilde{\beta}_0 \tilde{\alpha}(\nu)$, although this is not required. Note that isotropic surface tension is modeled by this as well with $K = 1$ and $G_1 = I_{2 \times 2}$ so that $\tilde{\alpha}(\nu) = \tilde{\alpha}_0$. 

Fig. 2: Simulation with isotropic surface tension. Several time-lapses are shown to illustrate the evolution with initial interface having a “clover” shape.
Fig. 3: Simulation with anisotropic surface tension. Several time-lapses are shown to illustrate the evolution with initial interface shape being a circle. A one-fold anisotropy is used which breaks the initial radial symmetry.

With the above, we can derive the modified form of (4.6) by standard shape differentiation [16, 32, 56]. Indeed,

$$\frac{d}{dt} \int_{\Gamma(t)} \alpha(\nu) = \int_{\Gamma(t)} \alpha(\nu) \nabla_{\Gamma} \mathbf{X} : \nabla_{\Gamma} \mathbf{V} - \int_{\Gamma(t)} \nu [\alpha'(\nu)]^T : \nabla_{\Gamma} \mathbf{V},$$

(6.2)

where \(\mathbf{V}\) is the velocity of \(\Gamma\), and for \(\mathbf{p} \in \mathbb{R}^d\), \(\alpha'(\mathbf{p})\) is the gradient of \(\alpha\) with respect to \(\mathbf{p}\). We now obtain a semi-discrete formulation for the anisotropic case by combining (4.5), (4.6), and (6.2):

$$\begin{align*}
(\tilde{\beta}^{-1}(\nu^i) \mathbf{V}^{i+1} \cdot \nu^i, \mathbf{Y} \cdot \nu^i)_{\Gamma^i} & + \Delta t (\tilde{\alpha}(\nu^i) \nabla_{\Gamma^i} \mathbf{V}^{i+1}, \nabla_{\Gamma^i} \mathbf{Y})_{\Gamma^i} + \tilde{S}(\mathbf{Y} \cdot \nu^i, \lambda^{i+1})_{\Gamma^i} \\
& = - (\tilde{\alpha}(\nu^i) \nabla_{\Gamma^i} \mathbf{X}^i, \nabla_{\Gamma^i} \mathbf{Y})_{\Gamma^i} + (\nu^i [\tilde{\alpha}'(\nu^i)]^T, \nabla_{\Gamma^i} \mathbf{Y})_{\Gamma^i} \quad \text{for all } \mathbf{Y} \in \mathbb{H}^i.
\end{align*}$$

(6.3)
The fully discrete formulation follows straightforwardly. This type of anisotropy is studied in [5] where they handle the anisotropic surface energy by defining the local finite element basis functions to capture the anisotropic energy. Their approach allows for obtaining an energy law, which can also be combined with our method. But (6.3) is easier to implement. In fact, it allows us to consider more general coefficients $\bar{a}(\nu)$ other than (6.1). The main drawback of (6.3) is it makes the numerical scheme slightly explicit, which puts a constraint on the time step. From our experience, we need $\Delta t \leq Ch$ for some uniform constant $C$. Using the anisotropic approach in [5] would circumvent this.

In Figure 3, we present a simulation using (6.1) with $K = 1$ (i.e. a one-fold anisotropy). Figure 4 shows a simulation with $K = 3$ (i.e. a three-fold anisotropy).

Fig. 4: Simulation with anisotropic surface tension. Several time-lapses are shown to illustrate the evolution with initial interface shape being a circle. A three-fold anisotropy is used which breaks the initial radial symmetry.
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