

Analysis of the Finite Element Method for the Laplace–Beltrami Equation on Surfaces with Regions of High Curvature Using Graded Meshes

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Abstract We derive error estimates for the piecewise linear finite element approximation of the Laplace–Beltrami operator on a bounded, orientable, C^3 , surface without boundary on general shape regular meshes. As an application, we consider a problem where the domain is split into two regions: one which has relatively high curvature and one that has low curvature. Using a graded mesh we prove error estimates that do not depend on the curvature on the high curvature region. Numerical experiments are provided.

Keywords Finite elements · Laplace–Beltrami · Curvature

1 Introduction

Since the publication of the seminal paper [16], there has been a growing interest in the discretization of surface partial differential equations (PDEs) using finite element methods

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(FEMs). Such interest is motivated by important applications related to physical and biological phenomena, and also by the potential use of numerical methods to answer theoretical questions in geometry [16, 17, 33].

In this paper, we focus on linear finite element methods for the Poisson problem with the Laplace–Beltrami operator on $\Gamma \subset \mathbb{R}^3$, a C^3 two-dimensional compact orientable surface without boundary. That is, we consider

$$-\Delta_{\Gamma} u = f \quad \text{on } \Gamma.$$

In order to motivate the results in our paper, we start by giving a short overview of previous results. A piecewise linear finite element method is proposed and analyzed in [16, 17]. The basic idea is to consider a piecewise linear approximation of the surface, and pose a finite element method over the discretized surface. Discretizing the surface, of course, creates a *geometric error*, however, the advantage is that for a given discretization a surface parametrization is not necessary.

In [12] a generalization of the piecewise linear FEM is considered, based on higher order polynomials that approximate both the geometry and the PDE; the same paper proposes a variant of the method which employs parametric elements, and the method is posed on the surface, originating thus no geometric error. Discontinuous Galerkin schemes are considered in [1, 11], and HDG and mixed versions are considered in [8]. Adaptive schemes are presented in [3, 7, 10, 13, 14]. An alternative approach, where a discretization of an outer domain induces the finite element spaces is proposed in [26, 27]. See also [4, 6, 14, 28, 29]. In [2, 7, 9, 30], the PDE itself is extended to a neighborhood of the surface before discretization.

Other problems and methods were considered as well, as a multiscale FEM for PDEs posed on rough surface [18], and stabilized methods [4, 21, 22, 29]. In [23] the finite element exterior calculus framework was considered. Finally, transient and nonlinear PDEs were also subject of consideration, as reviewed in [17].

A common ground between all aforementioned papers is that the a-priori choices of the surface discretization do not consider how to locally refine the mesh following some optimality criterion. It is however reasonable to expect that some geometrical traits, as the curvatures, have a local influence on the solution, and thus the mesh refinement could account for that locally. Not surprisingly, numerical tests using adaptive schemes confirm that high curvature regions require refined meshes [3, 6, 10, 13]. This is no different from problems in nonconvex *flat* domains, where corner singularities arise, and meshes are used to tame the singularity at an optimal cost [31, 32].

As far as we can tell, the development and analysis of a-priori strategies to deal with high curvature regions have not been an object of investigation, so far. In this paper we consider a simple setting: We suppose that the domain $\Gamma = \Gamma_1 \cup \Gamma_2$, and assume that the maximum of the principal curvatures in Γ_1 is much larger than those in Γ_2 . We then seek a graded mesh that gives us optimal error estimate. Of course, in the region Γ_1 the triangles will be much smaller than the mesh size in regions far from Γ_1 . We consider the method originally proposed by Dziuk [16].

To carry out the analysis, we first need to track the geometric constants carefully. This, as far as we can tell, has not explicitly appeared in literature, although it is not a difficult task. We do this by following [16, 17] although in some cases we give different arguments while trying to be as precise as possible. The estimate we obtain is found below in (41). If u_h^ℓ is the finite element solution approximation to u then the result reads (see sections below for precise notation):

$$\begin{aligned} \left\| \nabla_{\Gamma} \left(u - u_h^{\ell} \right) \right\|_{L^2(\Gamma)} &\leq C c_p \left[(\Lambda_h + \Psi_h) \|f\|_{L^2(\Gamma)} \right. \\ &\quad \left. + \left\| f - f_h^{\ell} \right\|_{L^2(\Gamma)} \right] + C \left(\sum_{T \in \mathcal{T}_h} h_T^2 \left\| \nabla_{\Gamma}^2 u \right\|_{L^2(T^{\ell})}^2 \right)^{1/2}. \end{aligned}$$

Here f_h^{ℓ} is an approximation to f , c_p is the Poincaré’s constant, and the numbers Λ_h, Ψ_h are geometric quantities. For instance, $\Psi_h = \max_T \kappa_T^2 h_T^2$ where h_T is the diameter of the triangle T and κ_T measures the maximum principal curvatures on T^{ℓ} (T^{ℓ} is the surface triangle corresponding to T ; see sections below). The important point here is that $\Lambda_h + \Psi_h$ can be controlled locally. That is, if one wants to reach a certain tolerance, one needs to make h_T small enough only depending on the geometry in T^{ℓ} . We point out that $\max_T \kappa_T$ large is a necessary but not sufficient condition for c_p to be large, this is related to a *dumbbell* shape of the surface. Surface with bumps for instance can allow large $\max_T \kappa_T$ and $c_p = O(1)$.

On the other hand, $\|\nabla_{\Gamma}^2 u\|_{L^2(T^{\ell})}$ does not depend only on the local geometry. In order to deal with this term, in the case of two sub-regions, we prove local H^2 regularity results. Combining the local regularity and the a-priori error estimate (41) we are able to define a mesh grading strategy and prove Theorem 16. The error estimate contained in Theorem 16 is independent of the curvature in region Γ_1 , and in some sense is the best error estimate one can hope for given the available information.

The paper is organized as follows. In Sect. 2 we set the notation and derive several fundamental estimates highlighting the influence of the curvatures. Section 3 regards the finite element and interpolation approximations. Finally, we present in Sect. 4 a local H^2 estimate and a mesh grading scheme culminating in an error estimate that is independent of the *bad* curvature. The paper ends with numerical results in Sect. 5.

2 Preliminaries

As mentioned above, we assume that Γ is bounded, orientable, C^3 surface without boundary. Furthermore, we assume that there is a high curvature region $\Gamma_1 \subsetneq \Gamma$, and define $\Gamma_2 = \Gamma \setminus \overline{\Gamma_1}$. For $f \in L^2(\Gamma)$ with $\int_{\Gamma} f \, dA = 0$, let $u \in \dot{H}^1(\Gamma)$ be such that

$$\int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v \, dA = \int_{\Gamma} f v \, dA \quad \text{for all } v \in \dot{H}^1(\Gamma), \tag{1}$$

where $\dot{H}^1(\Gamma) = \{v \in H^1(\Gamma) : \int_{\Gamma} v \, dA = 0\}$. We denote by ∇_{Γ} the tangential gradient [33], and (1) is nothing but the weak formulation of the Poisson problem for the Laplace–Beltrami operator. Existence and uniqueness of solution follows easily from the Poincaré’s inequality (Lemma 8) and the Lax–Milgram theorem. Details on the definition of $\nabla_{\Gamma} v$ are given below. Consider now a triangulation Γ_h of the surface Γ . By that we mean that Γ_h is a two-dimensional compact orientable polyhedral C^0 surface, and denoting by \mathcal{T}_h the set of closed nonempty triangles such that $\cup_{T \in \mathcal{T}_h} T = \Gamma_h$, we assume that all vertices belong to Γ , and that any two triangles have as intersection either the empty set, a vertex or an edge. Let $h_T = \text{diam}(T)$ and $h = \max\{h_T : T \in \mathcal{T}_h\}$. For all $T \in \mathcal{T}_h$ assume that there is a three-dimensional neighborhood \mathcal{N}_T of T where for every $\mathbf{x} \in \mathcal{N}_T$ there is a unique closest point $\mathbf{a}(\mathbf{x}) \in \Gamma$ (see Fig. 1) such that

$$\mathbf{x} = \mathbf{a}(\mathbf{x}) + d(\mathbf{x})\mathbf{v}(\mathbf{x}), \tag{2}$$

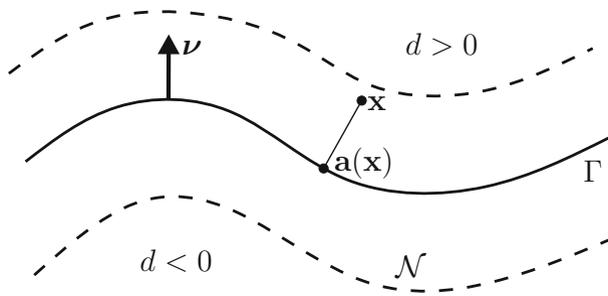


Fig. 1 Diagram of closest point map (2). The exact surface is denoted by Γ which is contained in the “tubular” neighborhood \mathcal{N} . Since d is the signed distance function to Γ , the zero level set of d coincides with Γ

$d(\mathbf{x})$ is the signed distance function from \mathbf{x} to Γ and define $\mathbf{v}(\mathbf{x}) = (\nabla d(\mathbf{x}))^t$ and note that $\mathbf{v}(\mathbf{a}(\mathbf{x})) = \mathbf{v}(\mathbf{x})$ and it is the unit normal to Γ at $\mathbf{a}(\mathbf{x})$. Note that by using local tubular neighborhoods, we avoid unnecessary restrictions on the mesh size.

Here we would like to explain some notational conventions that we use. From now on, the gradient of a scalar function will be a row vector. The normal vector \mathbf{v} is a column vector (as well as \mathbf{v}_h which is defined below).

We can now define, for every $T \in \mathcal{T}_h$, the surface triangle $T^\ell = \{\mathbf{a}(\mathbf{x}) : \mathbf{x} \in T\}$. Then $\Gamma = \cup_{T \in \mathcal{T}_h} T^\ell$. For $\mathbf{x} \in \mathcal{N}_T$, define the self-adjoint extended Weingarten operator [15] $H(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $H(\mathbf{x}) = \nabla^2 d(\mathbf{x})$ and let $\kappa_i(\mathbf{x})$, $i = 1, 2$, and 0 be the eigenvalues of $H(\mathbf{x})$. We note that $\kappa_1(\mathbf{a}(\mathbf{x}))$ and $\kappa_2(\mathbf{a}(\mathbf{x}))$ are the principal curvatures. We note that

$$\kappa_i(\mathbf{x}) = \frac{\kappa_i(\mathbf{a}(\mathbf{x}))}{1 + d(\mathbf{x})\kappa_i(\mathbf{a}(\mathbf{x}))} \tag{3}$$

for curvature of parallel surfaces [20, Lemma 14.17] (also see [13]).

We assume the natural geometrical condition (see also [12] for a similar condition).

$$|d(\mathbf{x})| \max\{|\kappa_1(\mathbf{a}(\mathbf{x}))|, |\kappa_2(\mathbf{a}(\mathbf{x}))|\} \leq c_0 < 1 \quad \text{for all } \mathbf{x} \in \mathcal{N}_T, \tag{4}$$

where c_0 is sufficiently small.

At the estimates that follow in this paper we denote by C a generic constant that might not assume the same value at all occurrences, but that does not depend on h_T , u , f or on Γ . It might however depend for instance on the shape regularity of $T \in \mathcal{T}_h$.

Given $T \in \mathcal{T}_h$, let

$$\kappa_T = \|H\|_{L^\infty(T)} := \max_{ij} \|H_{ij}\|_{L^\infty(T)}, \tag{5}$$

and $\mathbf{v}_h \in \mathbb{R}^3$ be unit-normal vector to T such that $\mathbf{v}_h \cdot \mathbf{v} > 0$. We note that since H is symmetric, κ_T is also equivalent to the L^∞ norm of the spectral radius of $H(\mathbf{a}(\cdot))$, or $\max\{|\kappa_1(\mathbf{a}(\cdot))|_{L^\infty(T)}, |\kappa_2(\mathbf{a}(\cdot))|_{L^\infty(T)}\}$, since from (4) we have

$$\frac{1}{1 + c_0} |\kappa_i(\mathbf{a}(\mathbf{x}))| \leq |\kappa_i(\mathbf{x})| \leq \frac{1}{1 - c_0} |\kappa_i(\mathbf{a}(\mathbf{x}))|.$$

Assumption Throughout the paper we will assume

$$h_T^2 \kappa_T^2 \leq c_1 < 1 \quad \text{for all } T \in \mathcal{T}_h, \tag{6}$$

where c_1 is sufficiently small.

It is easy to see that

$$\|d\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T, \tag{7}$$

$$\|\mathbf{v} - \mathbf{v}_h\|_{L^\infty(T)} \leq Ch_T \kappa_T. \tag{8}$$

To show (7), assume without loss of generality that $T \subset \mathbb{R}^2 \times \{0\}$ and that $v_3, v_{3,h} > 0$. Let $I_h d$ be the Lagrange linear interpolant of d in T . Since d vanishes at the vertices, $I_h d \equiv 0$. By [19] we have

$$\begin{aligned} \|d\|_{L^\infty(T)} + h_T \|\partial d / \partial x_i\|_{L^\infty(T)} &= \|d - I_h d\|_{L^\infty(T)} + h_T \|\partial(d - I_h d) / \partial x_i\|_{L^\infty(T)} \\ &\leq Ch_T^2 \|\nabla^2 d\|_{L^\infty(T)}, \end{aligned} \tag{9}$$

for $i = 1, 2$. Next, to prove (8), start by noting that $\mathbf{v}'_h = (0, 0, 1)$, and then the estimate for the first two components $\|v_i\|_{L^\infty(T)} = \|\partial d / \partial x_i\|_{L^\infty(T)}$ of \mathbf{v} follow from (9) and (5). The third component estimate follows from

$$\begin{aligned} \|v_3 - 1\|_{L^\infty(T)} &\leq \|(v_3 - 1)(v_3 + 1)\|_{L^\infty(T)} = \|v_3^2 - 1\|_{L^\infty(T)} \leq \|v_1\|_{L^\infty(T)}^2 + \|v_2\|_{L^\infty(T)}^2 \\ &\leq Ch_T^2 \kappa_T^2 \leq Ch_T \kappa_T. \end{aligned}$$

Here we used (6).

From (7) and (5) we see that

$$\|dH\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2. \tag{10}$$

Therefore, making c_1 sufficiently small in (6) so that the eigenvalues of $d(\mathbf{x})H(\mathbf{x})$ are smaller or equal to $1/2$ for every $\mathbf{x} \in \Gamma_h$, then we will have

$$\|(I - dH)^{-1}\|_{L^\infty(T)} \leq C. \tag{11}$$

We define tangential projections onto Γ and Γ_h , respectively, as $P = I - \mathbf{v} \otimes \mathbf{v}$ and $P_h = I - \mathbf{v}_h \otimes \mathbf{v}_h$, where $\mathbf{q} \otimes \mathbf{r} = \mathbf{q}\mathbf{r}^t$ for two column vectors \mathbf{q} and \mathbf{r} . We recall that the tangential derivatives for a functions defined on a neighborhood of Γ (or Γ_h) are given by

$$\nabla_\Gamma v = (\nabla v)P, \quad \nabla_{\Gamma_h} v = (\nabla v)P_h. \tag{12}$$

By using that $\mathbf{v} \cdot \mathbf{v} = 1$, we have

$$0 = \frac{1}{2} \nabla(\mathbf{v} \cdot \mathbf{v}) = (\nabla \mathbf{v})\mathbf{v} = H(\mathbf{x})\mathbf{v}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in T, \tag{13}$$

Hence, we, of course, have

$$PH = H = HP, \tag{14}$$

which we use repeatedly. Also, we can show that

$$P(I - dH)^{-1}\mathbf{v} = 0. \tag{15}$$

Indeed, $\mathbf{v} = (I - dH)\mathbf{v}$ by (13) and so $P(I - dH)^{-1}\mathbf{v} = P(I - dH)^{-1}(I - dH)\mathbf{v} = P\mathbf{v} = 0$.

2.1 Local Parametrization

Let $\hat{T} = \{(\theta_1, \theta_2): 0 \leq \theta_1, \theta_2 \leq 1, 0 \leq \theta_1 + \theta_2 \leq 1\}$ be the reference triangle. Fix $T \in \mathcal{T}_h$, let \mathbf{x}_0 be one of the vertices, and let \mathbf{x}_1 and \mathbf{x}_2 be vectors in \mathbb{R}^3 representing two edges of T (i.e. $T = \{\mathbf{x}_0 + \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2: 0 \leq \theta_1, \theta_2 \leq 1, 0 \leq \theta_1 + \theta_2 \leq 1\}$). Let $\mathbf{X} : \hat{T} \rightarrow T$ be given by

$\mathbf{X}(\theta_1, \theta_2) = \mathbf{x}_0 + \theta_1 \mathbf{x}_1 + \theta_2 \mathbf{x}_2$. We also define $Y : \hat{T} \rightarrow T^\ell$ by $Y(\theta_1, \theta_2) = \mathbf{a}(\mathbf{X}(\theta_1, \theta_2))$. Since $\nabla \mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2]$ we have $(\nabla \mathbf{X})^t \mathbf{v}_h = 0$. From the definition of \mathbf{a} we have

$$\nabla \mathbf{a}(\mathbf{x}) = P(\mathbf{x}) - d(\mathbf{x})H(\mathbf{x}),$$

and, hence

$$\nabla Y = (P - dH)\nabla X. \tag{16}$$

Therefore, using that P and H are symmetric and (13) we have $(\nabla Y)^t \mathbf{v} = 0$. Collecting the two results we have

$$(\nabla \mathbf{X})^t \mathbf{v}_h = 0, \quad (\nabla Y)^t \mathbf{v} = 0. \tag{17}$$

Given a function $\eta \in L^1(T^\ell)$ we define the pullback lift $\eta_\ell \in L^1(T)$ as

$$\eta_\ell(\mathbf{x}) = \eta(\mathbf{a}(\mathbf{x})),$$

and for $\eta \in L^1(T)$ we define the push-forward lift $\eta^\ell \in L^1(T^\ell)$ as

$$\eta^\ell(\mathbf{a}(\mathbf{x})) = \eta(\mathbf{x}), \tag{18}$$

and associate $\hat{\eta} : \hat{T} \rightarrow \mathbb{R}$ defined by

$$\hat{\eta}(\theta_1, \theta_2) = \eta(\mathbf{X}(\theta_1, \theta_2)) = \eta^\ell(Y(\theta_1, \theta_2)).$$

Note that $(\eta_\ell)^\ell = \eta$ for $\eta \in L^1(T^\ell)$ and $(\eta^\ell)_\ell = \eta$ for $\eta \in L^1(T)$.

Consider also the metric tensors

$$G_X(\theta_1, \theta_2) = (\nabla \mathbf{X}(\theta_1, \theta_2))^t \nabla \mathbf{X}(\theta_1, \theta_2), \quad G_Y(\theta_1, \theta_2) = (\nabla Y(\theta_1, \theta_2))^t \nabla Y(\theta_1, \theta_2).$$

From the definition of tangential derivative it is possible to show [33, Section 4.2.1] (see also (2.2) in [17]) that for a function $\eta : \Gamma_h \rightarrow \mathbb{R}$,

$$\nabla_{\Gamma_h} \eta(X) = \nabla \hat{\eta} G_X^{-1} \nabla X^t, \quad \nabla_\Gamma \eta^\ell(Y) = \nabla \hat{\eta} G_Y^{-1} \nabla Y^t,$$

and multiplying by ∇X and ∇Y we gather that

$$\nabla \hat{\eta} = \nabla_{\Gamma_h} \eta(X) \nabla X, \quad \nabla \hat{\eta} = \nabla_\Gamma \eta^\ell(Y) \nabla Y. \tag{19}$$

Hence,

$$\nabla_{\Gamma_h} \eta(X) = \nabla_\Gamma \eta^\ell(Y) \nabla Y G_X^{-1} \nabla X^t, \tag{20}$$

$$\nabla_\Gamma \eta^\ell(Y) = \nabla_{\Gamma_h} \eta(X) \nabla X G_Y^{-1} \nabla Y^t. \tag{21}$$

Note that we can also write

$$P = \nabla Y G_Y^{-1} \nabla Y^t, \quad P_h = \nabla X G_X^{-1} \nabla X^t. \tag{22}$$

To see that this is the case, note first from (17) that $\nabla Y G_Y^{-1} \nabla Y^t \mathbf{v} = 0$. Next, consider for $\epsilon > 0$ an arbitrary differentiable curve $s : (-\epsilon, \epsilon) \rightarrow \hat{T}$ and $\alpha(t) = Y(s(t))$. Then

$$\nabla Y G_Y^{-1} \nabla Y^t \alpha' = \nabla Y G_Y^{-1} \nabla Y^t \nabla Y s' = \alpha' = P \alpha',$$

since α' is tangent to Γ . The same arguments hold for the identity regarding P_h .

The following identities have appeared in the literature under different forms; see for example [12, 13]. Again, we give a proof for completeness and to show the independence of C with respect to Γ .

Lemma 1 Let $\eta : \Gamma_h \rightarrow \mathbb{R}$ be differentiable, and define its forward lift η^ℓ as in (18). It then holds that

$$\nabla_{\Gamma_h} \eta = \left(\nabla_{\Gamma} \eta^\ell \circ \mathbf{a} \right) Q \quad \text{on } \Gamma_h, \tag{23}$$

where $Q = (I - dH)P_h$, and

$$\nabla_{\Gamma} \eta^\ell \circ \mathbf{a} = (\nabla_{\Gamma_h} \eta)R \quad \text{on } \Gamma_h, \tag{24}$$

where $R = \left[I - \frac{(\mathbf{v}_h - \mathbf{v}) \otimes (\mathbf{v} - \mathbf{v}_h)}{\mathbf{v}_h \cdot \mathbf{v}} \right] (I - dH)^{-1} P$. Moreover, there exists a constant C such that

$$\|Q\|_{L^\infty(T)} + \|R\|_{L^\infty(T)} \leq C. \tag{25}$$

Proof Using (20) and (16) we get

$$\nabla_{\Gamma_h} \eta(\mathbf{X}) = \nabla_{\Gamma} \eta^\ell(\mathbf{Y}) [I - d(\mathbf{X})H(\mathbf{X})] \nabla X G_X^{-1} \nabla X^t$$

where we used that $\nabla_{\Gamma} \eta^\ell \mathbf{v} \otimes \mathbf{v} = 0$. Then (23) follows from (22).

To prove (24), we use (16) and (14) to get $\nabla \mathbf{Y} = P[I - d(\mathbf{X})H(\mathbf{X})] \nabla X$. Hence, we have

$$(\nabla \mathbf{Y})^t = (\nabla X)^t [I - d(\mathbf{X})H(\mathbf{X})] P. \tag{26}$$

Using (14) we get

$$\begin{aligned} & (I - dH)P(I - dH)^{-1} \left(I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \right) \\ &= P(I - dH)(I - dH)^{-1} \left(I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \right) = P \left(I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \right). \end{aligned}$$

However,

$$P \left(I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \right) = I - \mathbf{v} \otimes \mathbf{v} - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} + \frac{(\mathbf{v} \otimes \mathbf{v})\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} = I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}}.$$

This gives

$$(I - dH)P(I - dH)^{-1} \left(I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \right) = I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \tag{27}$$

So, from (27), (26) and (17) we have

$$(\nabla \mathbf{Y})^t (I - dH)^{-1} \left(I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \right) = (\nabla X)^t \left(I - \frac{\mathbf{v}_h \otimes \mathbf{v}}{\mathbf{v}_h \cdot \mathbf{v}} \right) = (\nabla X)^t.$$

Thus, using (21) and the above identity we gather that

$$\begin{aligned} \nabla_{\Gamma} \eta^\ell(\mathbf{Y}) &= \nabla_{\Gamma_h} \eta(\mathbf{X}) \left(I - \frac{\mathbf{v} \otimes \mathbf{v}_h}{\mathbf{v}_h \cdot \mathbf{v}} \right) (I - dH)^{-1} \nabla Y G_Y^{-1} \nabla Y^t \\ &= \nabla_{\Gamma_h} \eta(\mathbf{X}) \left(I - \frac{\mathbf{v} \otimes \mathbf{v}_h}{\mathbf{v}_h \cdot \mathbf{v}} \right) (I - dH)^{-1} P, \end{aligned}$$

from (22). Clearly we have $(\nabla_{\Gamma_h} \eta) \mathbf{v}_h \otimes \mathbf{v}_h = 0 = (\nabla_{\Gamma_h} \eta) \mathbf{v}_h \otimes \mathbf{v}$, and $P(I - dH)^{-1} \mathbf{v} \otimes \mathbf{v} = 0$ follows from (15). So we get

$$\nabla_{\Gamma} \eta^\ell(\mathbf{Y}) = \nabla_{\Gamma_h} \eta \left[I - \frac{(\mathbf{v}_h - \mathbf{v}) \otimes (\mathbf{v} - \mathbf{v}_h)}{\mathbf{v}_h \cdot \mathbf{v}} \right] (I - dH)^{-1} P.$$

Here we used that $(\mathbf{v}_h - \mathbf{v}) \otimes (\mathbf{v} - \mathbf{v}_h) = -\mathbf{v}_h \otimes \mathbf{v}_h - \mathbf{v} \otimes \mathbf{v} + \mathbf{v}_h \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{v}_h$. This proves (24). Finally, (25) follows from (8), (10), (11) and (6). \square

Next, we write an integration identity.

Lemma 2 *Let $\eta \in L^1(T)$. Then, if dA is the surface measure in T^ℓ and dA_h is the surface measure in T it follows that*

$$\int_{T^\ell} \eta^\ell dA = \int_T \eta \delta_T dA_h, \tag{28}$$

where

$$\delta_T = \sqrt{\det(G_Y G_X^{-1})}. \tag{29}$$

Proof The result follows from the change of variables formulas [15]

$$\int_{T^\ell} \eta^\ell dA = \int_{\hat{T}} \hat{\eta} \sqrt{\det G_Y} d\theta_1 d\theta_2, \quad \int_{\hat{T}} \hat{\eta} \sqrt{\det G_X^{-1}} d\theta_1 d\theta_2 = \int_T \eta dA_h.$$

□

Combining (28), (23) and (24), we have that

$$\int_T \nabla_{\Gamma_h} \eta \cdot \nabla_{\Gamma_h} \psi dA_h = \int_{T^\ell} \nabla_{\Gamma} \eta^\ell Q^\ell \cdot \nabla_{\Gamma} \psi^\ell Q^\ell \frac{1}{\delta_T^\ell} dA, \tag{30}$$

$$\int_{T^\ell} \nabla_{\Gamma} \eta^\ell \cdot \nabla_{\Gamma} \psi^\ell dA = \int_T \nabla_{\Gamma_h} \eta R \cdot \nabla_{\Gamma_h} \psi R \delta_T dA_h. \tag{31}$$

Next, we prove some bounds for δ_T .

Lemma 3 *Assuming that (6) holds and defining δ_T by (29) we have that*

$$\|\delta_T - 1\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2, \tag{32}$$

$$\left\| \frac{1}{\delta_T} - 1 \right\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2. \tag{33}$$

Proof From (16) and (17) we have

$$\begin{aligned} G_Y &= \nabla X^t (I - dH - \mathbf{v} \otimes \mathbf{v}) \nabla Y \\ &= \nabla X^t (I - dH) \nabla Y \\ &= \nabla X^t (I - dH)^2 \nabla X - \nabla X^t (I - dH) \mathbf{v} \otimes \mathbf{v} \nabla X. \end{aligned}$$

Using (13) we get

$$(\nabla X)^t (I - dH) \mathbf{v} \otimes \mathbf{v} \nabla X = \nabla X^t \mathbf{v} \otimes \mathbf{v} \nabla X.$$

By (17),

$$(\nabla X)^t \mathbf{v}_h \otimes \mathbf{v}_h \nabla X = (\nabla X)^t \mathbf{v}_h \otimes \mathbf{v} \nabla X = (\nabla X)^t \mathbf{v} \otimes \mathbf{v}_h \nabla X = 0.$$

Hence,

$$(\nabla X)^t \mathbf{v} \otimes \mathbf{v} \nabla X = (\nabla X)^t (\mathbf{v} - \mathbf{v}_h) \otimes (\mathbf{v} - \mathbf{v}_h) \nabla X.$$

Therefore, we get

$$G_Y = (\nabla X)^t [(I - dH)^2 - (\mathbf{v} - \mathbf{v}_h) \otimes (\mathbf{v} - \mathbf{v}_h)] \nabla X,$$

or $G_Y = (\nabla X)^t (I + B) \nabla X$ where $B = -2dH + d^2 H^2 - (\mathbf{v} - \mathbf{v}_h) \otimes (\mathbf{v} - \mathbf{v}_h)$. Therefore,

$$G_Y G_X^{-1} = I + M \quad \text{where } M = \nabla X^t B \nabla X G_X^{-1}.$$

It is clear that $\|\nabla X\|_{L^\infty(T)} \leq Ch_T$. Also, not difficult to see that $\|G_X^{-1}\|_{L^\infty(T)} \leq Ch_T^{-2}$. Moreover, using (8) and (10) we gather that $\|B\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2$. Hence, $\|M\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2$. Since M is symmetric, consider the spectral decomposition $M = V \Lambda V^{-1}$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2)$ and V is orthogonal. Denoting the i th column of V by v_i , we have that $\lambda_i = v_i^t M v_i$ and then $\|\lambda_i\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2$ (for $i = 1, 2$). We also note that

$$G_Y G_X^{-1} = I + V \Lambda V^{-1} = V(I + \Lambda)V^{-1}.$$

Therefore, we obtain

$$\delta_T^2 = \det(G_Y G_X^{-1}) = \det(V) \det(I + \Lambda) \det(V^{-1}) = (1 + \lambda_1)(1 + \lambda_2),$$

which yields

$$\|\delta_T^2 - 1\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2.$$

Using that $\delta_T - 1 = (\delta_T^2 - 1)/(\delta_T + 1)$, we obtain (32). The inequality (33) follows from the previous inequality and the fact $(\delta_T^{-1} - 1) = \delta_T^{-1}(1 - \delta_T)$. □

Remark 4 An alternative way to prove Lemma 3 is to use [13, Proposition 2.1] given by

$$\delta_T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{v}_h (1 - d(\mathbf{x})\kappa_1(\mathbf{x}))(1 - d(\mathbf{x})\kappa_2(\mathbf{x})),$$

the identity $1 - \mathbf{v} \cdot \mathbf{v}_h = \frac{1}{2}|\mathbf{v} - \mathbf{v}_h|^2$ and (3), (4), (7) and (8).

We can now state the following result which follows from Lemmas 1 and 3, and Eqs. (30), (31).

Lemma 5 *Assuming the hypotheses of Lemmas 1 and 3, we have that*

$$\|\nabla_\Gamma \eta^\ell\|_{L^2(T^\ell)} \leq C \|\nabla_{\Gamma_h} \eta\|_{L^2(T)} \leq C \|\nabla_\Gamma \eta^\ell\|_{L^2(T^\ell)}.$$

In the following, we use the notation $\underline{D}_i u = (\nabla_\Gamma u)_i$, and write $\nabla_\Gamma^2 u$ as the 3×3 matrix with entries $\underline{D}_i \underline{D}_j u$ (also denoted by $\underline{D}_{ij} u$).

Lemma 6 *Assuming that (6) holds, we have*

$$\|\nabla_{\Gamma_h}^2 \eta\|_{L^2(T)} \leq C \|\nabla_\Gamma^2 \eta^\ell\|_{L^2(T^\ell)} + C (h_T \kappa_T^2 + h_T^2 \kappa_T \gamma_T) \|\nabla_\Gamma \eta^\ell\|_{L^2(T^\ell)},$$

where $\gamma_T = \max_{ij} \|\nabla H_{ij}\|_{L^\infty(T)}$.

Proof Let $\mathbf{w} = \nabla_{\Gamma_h} \eta$. Using (23) we see that for $\mathbf{x} \in T$,

$$w_i(\mathbf{x}) = (P_h)_{ik} (I - d(\mathbf{x})H(\mathbf{x}))_{kj} \underline{D}_j \eta^\ell(\mathbf{a}(\mathbf{x})),$$

where we use Einstein summation convention.

Using the product rule and the fact that P_h is constant we have

$$\nabla_{\Gamma_h} w_i(\mathbf{x}) = \mathbf{J}_1(\mathbf{x}) + \mathbf{J}_2(\mathbf{x}) + \mathbf{J}_3(\mathbf{x}),$$

where

$$\begin{aligned} \mathbf{J}_1 &= -(P_h)_{ik} H_{kj} \left(\underline{D}_j \eta^\ell \circ \mathbf{a} \right) \nabla_{\Gamma_h} d, \\ \mathbf{J}_2 &= -(P_h)_{ik} d \left(\underline{D}_j \eta^\ell \circ \mathbf{a} \right) \nabla_{\Gamma_h} H_{kj}, \\ \mathbf{J}_3 &= (P_h)_{ik} (I - dH)_{kj} \nabla_{\Gamma_h} \left(\underline{D}_j \eta^\ell \circ \mathbf{a} \right). \end{aligned}$$

We start with J_3 . Using (23) we have

$$\nabla_{\Gamma_h} \left(\underline{D}_j \eta^\ell \circ \mathbf{a} \right) (x) = \nabla_{\Gamma} \underline{D}_j \eta^\ell (\mathbf{a}(x)) Q$$

Hence, using (25), (28), (32) and (6) we get

$$\left\| \nabla_{\Gamma_h} \left(\underline{D}_j \eta^\ell \circ \mathbf{a} \right) \right\|_{L^2(T)} \leq C \left\| \nabla_{\Gamma} \eta^\ell \right\|_{L^2(T^\ell)}.$$

If we combine this inequality with (10) and (6), we have

$$\|J_3\|_{L^2(T)} \leq C \left\| \nabla_{\Gamma} \eta^\ell \right\|_{L^2(T^\ell)}.$$

Next, using (12) and the fact $\mathbf{v}_h^t P_h = 0$, we obtain

$$\nabla_{\Gamma_h} d = (\nabla d) P_h = \mathbf{v}^t P_h = (\mathbf{v} - \mathbf{v}_h)^t P_h.$$

Hence, using (8), (5), (28), (32) and (6) yields

$$\|J_1\|_{L^2(T)} \leq Ch_T \kappa_T^2 \left\| \nabla_{\Gamma} \eta^\ell \right\|_{L^2(T^\ell)}.$$

Similarly, (7), (28), (32) and (6) yield

$$\|J_2\|_{L^2(T)} \leq Ch_T^2 \kappa_T \gamma_T \left\| \nabla_{\Gamma} \eta^\ell \right\|_{L^2(T^\ell)}.$$

Combining the above estimates gives the desired result. □

We note that a similar result, with the γ_T dependence, can also be found in [5, Lemma 2.1]

Remark 7 Note that the C^3 regularity imposed on Γ is needed in Lemma 6. That seems unavoidable, at least under the choice of the local mapping (2) used in the analysis. The same regularity is used in [13].

3 Finite Element Spaces and Local Approximations

We introduce the following finite dimensional approximation of (1). The finite element space is given by

$$S_h = \left\{ v_h \in C^0(\Gamma_h) : \int_{\Gamma_h} v_h dA_h = 0, v_h|_T \text{ is linear for all } T \in \mathcal{T}_h \right\},$$

$$S_h^\ell = \left\{ v_h^\ell : v_h \in S_h \right\}.$$

For $f_h \in L^2(\Gamma_h)$ with $\int_{\Gamma_h} f_h dA_h = 0$, let $u_h \in S_h$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h dA_h = \int_{\Gamma_h} f_h v_h dA_h \quad \text{for all } v_h \in S_h. \tag{34}$$

Existence and uniqueness of the finite-dimensional problem (34) follows from noting that if u_h is a solution with $f_h = 0$, then u_h must be constant with zero average. Thus $u_h = 0$.

We will need a Poincaré’s inequality, as follows [17].

Lemma 8 Assuming $\Gamma \subset \mathbb{R}^3$ a C^3 two-dimensional compact orientable surface without boundary, there exists a constant c_p such that

$$\|\phi\|_{L^2(\Gamma)} \leq c_p \|\nabla_\Gamma \phi\|_{L^2(\Gamma)} \quad \text{for all } \phi \in \dot{H}^1(\Gamma). \tag{35}$$

Then we can state a simple energy estimate.

Lemma 9 Let u solve (1), then

$$\|\nabla_\Gamma u\|_{L^2(\Gamma)} \leq c_p \|f\|_{L^2(\Gamma)}. \tag{36}$$

Before proving an a-priori estimate for $u_h^\ell - u$ we will need to prove an important lemma that measures the inconsistency in going from Γ to Γ_h . First, we need to develop notation to use in the next proof. Since $\delta_T dA_h = dA$ with respect to a given triangle T and its lifting T^ℓ , let us define

$$\delta_h(\mathbf{x}) = \delta_T, \quad \text{if } \mathbf{x} \in T.$$

Lemma 10 Let v_h and z_h belong to S_h . Then the following holds

$$\left| \int_{\Gamma_h} \nabla_{\Gamma_h} v_h \cdot \nabla_{\Gamma_h} z_h dA_h - \int_\Gamma \nabla_\Gamma v_h^\ell \cdot \nabla_\Gamma z_h^\ell dA \right| \leq C \Psi_h \left\| \nabla_\Gamma v_h^\ell \right\|_{L^2(\Gamma)} \left\| \nabla_\Gamma z_h^\ell \right\|_{L^2(\Gamma)} \tag{37}$$

$$\left| \int_{\Gamma_h} v_h z_h dA_h - \int_\Gamma v_h^\ell z_h^\ell dA \right| \leq C \Psi_h \left\| v_h^\ell \right\|_{L^2(\Gamma)} \left\| z_h^\ell \right\|_{L^2(\Gamma)}, \tag{38}$$

where $\Psi_h = \max_{T \in \mathcal{T}_h} \kappa_T^2 h_T^2$.

Proof Using (28) we get

$$\left| \int_{\Gamma_h} v_h z_h dA_h - \int_\Gamma v_h^\ell z_h^\ell dA \right| = \left| \int_\Gamma v_h^\ell z_h^\ell \left(\frac{1}{\delta_h^\ell} - 1 \right) dA \right|$$

Hence, (38) follows from (33).

To prove (37) we use (30) to get

$$\int_{\Gamma_h} \nabla_{\Gamma_h} v_h \cdot \nabla_{\Gamma_h} z_h dA_h = \int_\Gamma \left(\nabla_\Gamma v_h^\ell Q^\ell \right) \cdot \left(\nabla_\Gamma z_h^\ell Q^\ell \right) \frac{1}{\delta_h^\ell} dA.$$

Using that $\nabla_\Gamma(\cdot)P = \nabla_\Gamma(\cdot)$ we have

$$\int_{\Gamma_h} \nabla_{\Gamma_h} v_h \cdot \nabla_{\Gamma_h} z_h dA_h = \int_\Gamma \nabla_\Gamma v_h^\ell M \cdot \nabla_\Gamma z_h^\ell dA,$$

where

$$M = \frac{1}{\delta_h^\ell} (PQ^\ell)(PQ^\ell)^t.$$

On the other hand, again using that $\nabla_\Gamma(\cdot)P = \nabla_\Gamma(\cdot)$ we get

$$\int_\Gamma \nabla_\Gamma v_h^\ell \cdot \nabla_\Gamma z_h^\ell dA = \int_\Gamma \nabla_\Gamma v_h^\ell P \cdot \nabla_\Gamma z_h^\ell dA.$$

Hence,

$$\left| \int_{\Gamma_h} \nabla_{\Gamma_h} v_h \cdot \nabla_{\Gamma_h} z_h dA_h - \int_\Gamma \nabla_\Gamma v_h^\ell \cdot \nabla_\Gamma z_h^\ell dA \right| = \left| \int_\Gamma \nabla_\Gamma v_h^\ell (M - P) \cdot \nabla_\Gamma z_h^\ell dA \right|. \tag{39}$$

We now proceed to bound $(M - P)$. We first use (14), to get on Γ_h

$$PQ^\ell \circ \mathbf{a} = PQ = P(I - dH)P_h = (I - dH)PP_h.$$

Hence, we get

$$M \circ \mathbf{a} = \frac{1}{\delta_h}(I - dH)S(I - dH),$$

where $S = PP_hP$, and then the triangle inequality yields

$$\begin{aligned} \|M - P\|_{L^\infty(T^\ell)} &= \left\| \delta_h^{-1}(I - dH)S(I - dH) - P \right\|_{L^\infty(T)} \\ &\leq \left\| \delta_h^{-1}(I - dH)S(I - dH) - S \right\|_{L^\infty(T)} + \|S - P\|_{L^\infty(T)}. \end{aligned}$$

Now using (10), (6) and (33) and the fact that S is bounded we obtain

$$\left\| \delta_h^{-1}(I - dH)S(I - dH) - S \right\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2.$$

Since $P\mathbf{v}_h \otimes \mathbf{v}P = P\mathbf{v} \otimes \mathbf{v}_hP = P\mathbf{v} \otimes \mathbf{v}P = 0$ we have that on Γ_h ,

$$S = P(I - \mathbf{v}_h \otimes \mathbf{v}_h)P = P(I - (\mathbf{v}_h - \mathbf{v}) \otimes (\mathbf{v}_h - \mathbf{v}))P.$$

Finally using that $P^2 = P$ we have

$$S - P = -P(\mathbf{v}_h - \mathbf{v}) \otimes (\mathbf{v}_h - \mathbf{v})P.$$

Therefore, it follows from (8) that $\|S - P\|_{L^\infty(T)} \leq Ch_T^2 \kappa_T^2$. Using the previous inequalities we obtain

$$\|M - P\|_{L^\infty(T^\ell)} \leq Ch_T^2 \kappa_T^2,$$

and hence

$$\|M - P\|_{L^\infty(\Gamma)} \leq C\Psi_h,$$

and thus (37) follows from this inequality and (39). □

Theorem 11 *Let $u \in \hat{H}^1(\Gamma)$ be the solution of (1) and let $u_h \in S_h$ that solves (34). Assume that f_h^ℓ satisfies $\|f_h^\ell\|_{L^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}$. Then there exists a constant C such that*

$$\begin{aligned} \left\| \nabla_\Gamma (u - u_h^\ell) \right\|_{L^2(\Gamma)} &\leq C \min_{\phi_h \in S_h} \left\| \nabla_\Gamma (u - \phi_h^\ell) \right\|_{L^2(\Gamma)} \\ &\quad + Cc_p \left(\Psi_h \|f\|_{L^2(\Gamma)} + \|f - f_h^\ell\|_{L^2(\Gamma)} \right), \end{aligned}$$

where $\Psi_h = \max_{T \in \mathcal{T}_h} \kappa_T^2 h_T^2$ is as in Lemma 10.

Proof For an arbitrary $\phi_h \in S_h$, set $\xi_h = u_h - \phi_h$ and $\xi_h^\ell = u_h^\ell - \phi_h^\ell$. By Lemma 5 we have

$$\left\| \nabla_\Gamma \xi_h^\ell \right\|_{L^2(\Gamma)} \leq C \left\| \nabla_{\Gamma_h} (u_h - \phi_h) \right\|_{L^2(\Gamma_h)}.$$

Then, we write for an arbitrary constant c

$$\begin{aligned} \|\nabla_{\Gamma_h}(u_h - \phi_h)\|_{L^2(\Gamma_h)}^2 &= \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} \xi_h dA_h - \int_{\Gamma_h} \nabla_{\Gamma_h} \phi_h \cdot \nabla_{\Gamma_h} \xi_h dA_h \\ &= \int_{\Gamma_h} f_h \xi_h dA_h - \int_{\Gamma_h} \nabla_{\Gamma_h} \phi_h \cdot \nabla_{\Gamma_h} \xi_h dA_h \quad \text{by (34)} \\ &= \int_{\Gamma_h} f_h \xi_h dA_h - \int_{\Gamma} f \cdot \xi_h^\ell dA \\ &\quad + \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \xi_h^\ell dA - \int_{\Gamma_h} \nabla_{\Gamma_h} \phi_h \cdot \nabla_{\Gamma_h} \xi_h dA_h, \quad \text{by (1)} \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned}$$

where after using that $\int_{\Gamma} f dA = 0 = \int_{\Gamma_h} f_h dA_h$,

$$\begin{aligned} J_1 &= \int_{\Gamma_h} f_h (\xi_h - c) dA_h - \int_{\Gamma} f_h^\ell (\xi_h^\ell - c) dA, \quad J_2 = \int_{\Gamma} (f_h^\ell - f) (\xi_h^\ell - c) dA, \\ J_3 &= \int_{\Gamma} \nabla_{\Gamma} (u - \phi_h^\ell) \cdot \nabla_{\Gamma} \xi_h^\ell dA, \quad J_4 = \int_{\Gamma} \nabla_{\Gamma} \phi_h^\ell \cdot \nabla_{\Gamma} \xi_h^\ell dA - \int_{\Gamma_h} \nabla_{\Gamma_h} \phi_h \cdot \nabla_{\Gamma_h} \xi_h dA_h. \end{aligned}$$

By applying (38) and the Poincaré’s inequality (35) we get

$$J_1 \leq C c_p \Psi_h \left\| f_h^\ell \right\|_{L^2(\Gamma)} \left\| \nabla_{\Gamma} \xi_h^\ell \right\|_{L^2(\Gamma)},$$

where we chose $c = \frac{1}{|\Gamma|} \int_{\Gamma} \xi_h^\ell dA$. Using the Cauchy–Schwarz inequality and the Poincaré’s inequality (35) we have

$$J_2 \leq c_p \left\| f_h^\ell - f \right\|_{L^2(\Gamma)} \left\| \nabla_{\Gamma} \xi_h^\ell \right\|_{L^2(\Gamma)}.$$

Using the Cauchy–Schwarz inequality we get

$$J_3 \leq \left\| \nabla_{\Gamma} (u - \phi_h^\ell) \right\|_{L^2(\Gamma)} \left\| \nabla_{\Gamma} \xi_h^\ell \right\|_{L^2(\Gamma)}.$$

Using (37) gives

$$J_4 \leq C \Psi_h \left\| \nabla_{\Gamma} \phi_h^\ell \right\|_{L^2(\Gamma)} \left\| \nabla_{\Gamma} \xi_h^\ell \right\|_{L^2(\Gamma)}$$

Hence, combining the above results we get

$$\begin{aligned} \left\| \nabla_{\Gamma} \xi_h^\ell \right\|_{L^2(\Gamma)} &\leq C c_p \left(\Psi_h \|f\|_{L^2(\Gamma)} + \left\| f_h^\ell - f \right\|_{L^2(\Gamma)} \right) \\ &\quad + C \left(\left\| \nabla_{\Gamma} (u - \phi_h^\ell) \right\|_{L^2(\Gamma)} + \Psi_h \left\| \nabla_{\Gamma} \phi_h^\ell \right\|_{L^2(\Gamma)} \right), \end{aligned}$$

where we used that $\|f_h^\ell\|_{L^2(\Gamma)} \leq \|f\|_{L^2(\Gamma)}$. If we use the triangle inequality and (36) we obtain

$$\left\| \nabla_{\Gamma} \phi_h^\ell \right\|_{L^2(\Gamma)} \leq \left\| \nabla_{\Gamma} (u - \phi_h^\ell) \right\|_{L^2(\Gamma)} + c_p \|f\|_{L^2(\Gamma)}.$$

The result now follows after taking the minimum over $\phi_h \in S_h$ and use the fact that $\Psi_h \leq 1$ which follows from (6). □

Let $I_{h,\ell}$ be the standard Lagrange interpolant in Γ_h onto S_h and define $I_h \eta^\ell = (I_{h,\ell} \eta)^\ell \in S_h^\ell$. We then have the following estimate.

Lemma 12 *Let u be the solution of (1). Then,*

$$\|\nabla_{\Gamma}(u - I_h u)\|_{L^2(\Gamma)} \leq C c_p (\Lambda_h + \Psi_h) \|f\|_{L^2(\Gamma)} + \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla_{\Gamma}^2 u\|_{L^2(T^\ell)}^2 \right)^{1/2},$$

where $\Lambda_h = \max_{T \in \mathcal{T}_h} h_T^3 \gamma_T \kappa_T$ and γ_T was defined in Lemma 6.

Proof Recall that u_ℓ is the pullback lift of u . Using Lemma 5, and approximation properties of the Lagrange interpolant, we have

$$\|\nabla_{\Gamma}(u - I_h u)\|_{L^2(T^\ell)} \leq C \|\nabla_{\Gamma_h}(u_\ell - I_{h,\ell} u_\ell)\|_{L^2(T)} \leq C h_T \|\nabla_{\Gamma_h}^2 u_\ell\|_{L^2(T)}.$$

We get from Lemma 6 that

$$\|\nabla_{\Gamma}(u - I_h u)\|_{L^2(T^\ell)} \leq C \left(h_T \|\nabla_{\Gamma}^2 u\|_{L^2(T^\ell)} + (h_T^2 \kappa_T^2 + h_T^3 \kappa_T \gamma_T) \|\nabla_{\Gamma} u\|_{L^2(T^\ell)} \right). \tag{40}$$

The result follows easily by summing over T and applying (36). □

We can combine Theorem 11 and Lemma 12 to get the following theorem.

Theorem 13 *Assume that the hypothesis of Theorem 11 holds. Then,*

$$\begin{aligned} \|\nabla_{\Gamma}(u - u_h^\ell)\|_{L^2(\Gamma)} &\leq C c_p \left((\Lambda_h + \Psi_h) \|f\|_{L^2(\Gamma)} + \|f - f_h^\ell\|_{L^2(\Gamma)} \right) \\ &+ C \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla_{\Gamma}^2 u\|_{L^2(T^\ell)}^2 \right)^{1/2}. \end{aligned} \tag{41}$$

4 Graded Meshes for Two Subdomains

In this section we consider the case where there is a high curvature region $\Gamma_1 \subsetneq \Gamma$, and $\Gamma_2 = \Gamma \setminus \overline{\Gamma_1}$. We let $\kappa^{(i)} = \|H\|_{L^\infty(\Gamma_i)}$ and assume that $\kappa^{(1)} \gg \kappa^{(2)}$. We also let $\kappa = \|H\|_{L^\infty(\Gamma)} = \kappa^{(1)}$. We use our above results from the previous sections and local regularity results in order to grade a mesh so that the error is balanced.

We start by just stating the global regularity result which is found in [17]. We note that this result does not fit our graded meshing strategy; instead, we establish Lemma 15.

Lemma 14 *Assume that Γ is a C^3 orientable compact surface without boundary, and that $u \in \dot{H}^1(\Gamma)$ solves (1). Then $u \in H^2(\Gamma)$, and there exists a constant C that is independent of the curvatures of Γ such that*

$$\|\nabla_{\Gamma}^2 u\|_{L^2(\Gamma)} \leq (1 + C c_p \kappa) \|f\|_{L^2(\Gamma)}.$$

Applying Theorem 13 and Lemma 14 it easily follows that

$$\|\nabla_{\Gamma}(u - u_h^\ell)\|_{L^2(\Gamma)} \leq C c_p \|f - f_h^\ell\|_{L^2(\Gamma)} + C(h + c_p(\Lambda_h + \Psi_h + \kappa h)) \|f\|_{L^2(\Gamma)}. \tag{42}$$

Then, requiring $\Lambda_h + \Psi_h \leq h(1 + \kappa)$ we get

$$\|\nabla_{\Gamma}(u - u_h^\ell)\|_{L^2(\Gamma)} \leq C c_p \|f - f_h\|_{L^2(\Gamma)} + Ch(1 + c_p(1 + \kappa)) \|f\|_{L^2(\Gamma)}. \tag{43}$$

Of course, this is not a very good estimate in the case $\kappa = \kappa^{(1)} \gg \kappa^{(2)}$ since we would have the mesh size far away from Γ_1 still depending on $\kappa^{(1)}$. Instead, we would like to have the mesh size to only depend on $\kappa^{(2)}$ a distance order one away from Γ_1 . In order to do this we will need a local regularity result, as follows.

Lemma 15 (Weighted- H^2 regularity) *Let $u \in H^2(\Gamma)$ be the solution of (1), and consider the subset $\Gamma_2 \subseteq \Gamma$. Let $\rho \in W^{1,\infty}(\Gamma)$ be such that $\text{supp } \rho \subseteq \Gamma_2$. Then there is an universal constant C such that*

$$\|\rho \nabla_{\Gamma}^2 u\|_{L^2(\Gamma_2)} \leq C \|\rho\|_{W^{1,\infty}(\Gamma)} (1 + c_p(1 + \kappa^{(2)})) \|f\|_{L^2(\Gamma)}. \tag{44}$$

Proof Note that $\underline{D}_i(\rho^2 \underline{D}_j u) = 2\rho \underline{D}_i \rho \underline{D}_j u + \rho^2 \underline{D}_{ij} u$, and

$$\begin{aligned} \int_{\Gamma_2} \rho^2 \underline{D}_{ij} u \underline{D}_{ij} u \, dA &= \int_{\Gamma_2} \underline{D}_{ij} u \underline{D}_i (\rho^2 \underline{D}_j u) \, dA - 2 \int_{\Gamma_2} \rho \underline{D}_{ij} u \underline{D}_i \rho \underline{D}_j u \, dA \\ &\leq \int_{\Gamma_2} \underline{D}_{ij} u \underline{D}_i (\rho^2 \underline{D}_j u) \, dA + 2 \|\nabla_{\Gamma} \rho\|_{L^{\infty}(\Gamma)} \|\rho \underline{D}_{ij} u\|_{L^2(\Gamma_2)} \|\underline{D}_j u\|_{L^2(\Gamma_2)} \\ &\leq \int_{\Gamma_2} \underline{D}_{ij} u \underline{D}_i (\rho^2 \underline{D}_j u) \, dA + 2 \|\nabla_{\Gamma} \rho\|_{L^{\infty}(\Gamma)}^2 \|\underline{D}_j u\|_{L^2(\Gamma_2)}^2 + \frac{1}{2} \|\rho \underline{D}_{ij} u\|_{L^2(\Gamma_2)}^2. \end{aligned}$$

Then

$$\int_{\Gamma_2} \rho^2 \underline{D}_{ij} u \underline{D}_{ij} u \, dA \leq 2 \int_{\Gamma_2} \underline{D}_{ij} u \underline{D}_i (\rho^2 \underline{D}_j u) \, dA + 4 \|\nabla_{\Gamma} \rho\|_{L^{\infty}(\Gamma)}^2 \|\underline{D}_j u\|_{L^2(\Gamma_2)}^2.$$

From Lemma 19,

$$\begin{aligned} &\sum_{i,j=1}^3 \int_{\Gamma_2} \underline{D}_{ij} u \underline{D}_i (\rho^2 \underline{D}_j u) \, dA \\ &= \sum_{j=1}^3 \int_{\Gamma_2} \Delta_{\Gamma} u \underline{D}_j (\rho^2 \underline{D}_j u) \, dA - \int_{\Gamma_2} \rho^2 (\text{tr}(H)H - 2H^2) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} u \, dA \\ &\leq \int_{\Gamma_2} \Delta_{\Gamma} u (2\rho \nabla_{\Gamma} \rho \cdot \nabla_{\Gamma} u + \rho^2 \Delta_{\Gamma} u) \, dA + C(\kappa^{(2)})^2 \int_{\Gamma_2} \rho^2 |\nabla_{\Gamma} u|^2 \, dA \\ &\leq C \left(\|\nabla_{\Gamma} \rho\|_{L^{\infty}(\Gamma)} \|\rho \Delta_{\Gamma} u\|_{L^2(\Gamma_2)} \|\nabla_{\Gamma} u\|_{L^2(\Gamma_2)} + \|\rho \Delta_{\Gamma} u\|_{L^2(\Gamma_2)}^2 \right. \\ &\quad \left. + (\kappa^{(2)})^2 \|\rho\|_{L^{\infty}(\Gamma)}^2 \|\nabla_{\Gamma} u\|_{L^2(\Gamma_2)}^2 \right). \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{i,j=1}^3 \int_{\Gamma_2} (\rho \underline{D}_{ij} u)^2 \, dA \\ &\leq C \left(\|\rho \Delta_{\Gamma} u\|_{L^2(\Gamma_2)}^2 + \left(\|\nabla_{\Gamma} \rho\|_{L^{\infty}(\Gamma)} + \kappa^{(2)} \|\rho\|_{L^{\infty}(\Gamma)} \right)^2 \|\nabla_{\Gamma} u\|_{L^2(\Gamma_2)}^2 \right). \end{aligned}$$

The result now follows after using the energy estimate (36). □

Lemma 15 holds with a generic function ρ , but we will apply the result with $\rho(\mathbf{x}) = \text{dist}(\mathbf{x}, \Gamma_1)$ which is a 1-Lipschitz function.

4.1 The Graded Mesh

We start by defining $d_1 = (1 + c_p(1 + \kappa^{(2)}))/(1 + c_p\kappa^{(1)})$. We then define the region $\mathcal{D}_1 = \{\mathbf{x} \in \Gamma: \text{dist}(\Gamma_1, \mathbf{x}) \leq d_1\}$. We also let $\tilde{\mathcal{T}}_h^1 = \{T \in \mathcal{T}_h: T^\ell \cap \mathcal{D}_1 \neq \emptyset\}$ and define $\tilde{\mathcal{T}}_h^2 = \{T \in \mathcal{T}_h: T \notin \tilde{\mathcal{T}}_h^1\}$. We set $h_1 = hd_1$. Finally, we let $\rho(\mathbf{x}) = \text{dist}(\mathbf{x}, \Gamma_1)$ and also set $\rho_T = \text{dist}(T^\ell, \Gamma_1)$ for all $T \in \mathcal{T}_h$, where $\rho(\cdot)$ is defined using the geodesic distance [25] and $\|\nabla\rho\|_{L^\infty(\Gamma_2)} \leq 1$.

Our graded mesh will then satisfy:

- (M1) $\Lambda_h + \Psi_h \leq h(1 + \kappa^{(2)})$
- (M2) $h_T \leq h_1$ for every $T \in \tilde{\mathcal{T}}_h^1$
- (M3) $h_T \leq \min\{\rho_T, 1\}h$ for every $T \in \tilde{\mathcal{T}}_h^2$

We recall that Ψ_h and Λ_h were respectively defined in Theorem 11 and Lemma 12.

A few comments are in order. First, note that condition (M1) is completely local, and in the case $O(\kappa^{(1)}) = O(\kappa^{(2)})$, condition (M1) would be necessary to get an estimate of the form (43), as the argument above (43) shows.

Note that if $O(\kappa^{(1)}) = O(\kappa^{(2)})$ then $O(h_1) = O(h)$. The mesh size for triangles that are unit distance from Γ_1 (i.e. $\rho_T = O(1)$) can be chosen so that $O(h_T) = O(h)$. In the intermediate region a grading giving by (M3) needs to be satisfied. Finally, note that there is a smooth transition for triangles in the border of \mathcal{D}_1 . Indeed, if $\rho_T = d_1$ then by (M3), $h_T \leq d_1h = h_1$

We now state and prove our main result.

Theorem 16 *Suppose that $u \in \dot{H}^1(\Gamma)$ solves (1) and $u_h \in S_h$ solves (34). Assume that the mesh satisfies (M1–M3). Then we have*

$$\left\| \nabla_\Gamma \left(u - u_h^\ell \right) \right\|_{L^2(\Gamma)} \leq C(1 + c_p(1 + \kappa^{(2)}))h \|f\|_{L^2(\Gamma)} + Cc_p \left\| f - f_h^\ell \right\|_{L^2(\Gamma)}. \tag{45}$$

Before proving the result let us state a few comments. Note that the right hand side of (45) looks like the right hand side of (43) with $\kappa^{(2)}$ instead of κ . Therefore, with the available information, (45) is essentially the best result we can hope for. So, we found a graded mesh where one has a fine mesh in the region where the curvature is high to get the best error estimate.

Proof (of Theorem 16) By (41) and our assumption (M1) we have

$$\begin{aligned} \left\| \nabla_\Gamma \left(u - u_h^\ell \right) \right\|_{L^2(\Gamma)} &\leq Cc_p \left((1 + \kappa^{(2)})h \|f\|_{L^2(\Gamma)} + \left\| f - f_h^\ell \right\|_{L^2(\Gamma)} \right) \\ &+ C \left(\sum_{T \in \mathcal{T}_h} h_T^2 \left\| \nabla_\Gamma^2 u \right\|_{L^2(T^\ell)}^2 \right)^{1/2}. \end{aligned}$$

Next, we estimate

$$\sum_{T \in \mathcal{T}_h} h_T^2 \left\| \nabla_\Gamma^2 u \right\|_{L^2(T^\ell)}^2 = \sum_{T \in \tilde{\mathcal{T}}_h^1} h_T^2 \left\| \nabla_\Gamma^2 u \right\|_{L^2(T^\ell)}^2 + \sum_{T \in \tilde{\mathcal{T}}_h^2} h_T^2 \left\| \nabla_\Gamma^2 u \right\|_{L^2(T^\ell)}^2.$$

By our (M2) we get

$$\sum_{T \in \tilde{\mathcal{T}}_h^1} h_T^2 \left\| \nabla_\Gamma^2 u \right\|_{L^2(T^\ell)}^2 \leq Ch_1^2 \left\| \nabla_\Gamma^2 u \right\|_{L^2(\Gamma)}^2,$$

and we gather from Lemma 14 that

$$\sum_{T \in \tilde{\mathcal{T}}_h^1} h_T^2 \|\nabla_{\Gamma}^2 u\|_{L^2(T^\ell)}^2 \leq Ch_1^2 \left(1 + c_p \kappa^{(1)}\right)^2 \|f\|_{L^2(\Gamma)}^2 = Ch^2 (1 + c_p (1 + \kappa^{(2)}))^2 \|f\|_{L^2(\Gamma)}^2.$$

The other term we bound in the following way by using (M3):

$$\sum_{T \in \tilde{\mathcal{T}}_h^2} h_T^2 \|\nabla_{\Gamma}^2 u\|_{L^2(T^\ell)}^2 \leq \sum_{T \in \tilde{\mathcal{T}}_h^2} \frac{h_T^2}{\rho_T^2} \|\rho \nabla_{\Gamma}^2 u\|_{L^2(T^\ell)}^2 \leq h^2 \|\rho \nabla_{\Gamma}^2 u\|_{L^2(\Gamma_2)}^2.$$

Now using (44), we have

$$\sum_{T \in \tilde{\mathcal{T}}_h^2} h_T^2 \|\nabla_{\Gamma}^2 u\|_{L^2(T^\ell)}^2 \leq Ch^2 (1 + c_p (1 + \kappa^{(2)}))^2 \|f\|_{L^2(\Gamma)}^2.$$

The result now follows. □

4.2 An $L^2(\Gamma)$ Estimate

We now derive an error estimate in the $L^2(\Gamma)$ norm, based on the usual duality argument. We note that the conditions (M1–M3) are no longer enough to guarantee a h^2 convergence that is independent of $\kappa^{(1)}$. Actually, (M1) is reinforced by imposing that

$$(M4) \quad \Psi_h \leq h^2 (1 + \kappa^{(2)})$$

Theorem 17 *Suppose that $u \in \dot{H}^1(\Gamma)$ solves (1) and $u_h \in S_h$ solves (34). Assume that f_h^ℓ satisfies $\|f_h^\ell\|_{L^2(\Gamma)} \leq C \|f\|_{L^2(\Gamma)}$, and that the mesh satisfies (M1–M4). Then we have*

$$\begin{aligned} \|u - u_h^\ell\|_{L^2(\Gamma)} &\leq Ch^2 \Upsilon (\Upsilon + c_p + h(1 + \kappa^{(2)})(c_p + h\Upsilon)) \|f\|_{L^2(\Gamma)} \\ &\quad + Cc_p(c_p + h\Upsilon) \|f - f_h^\ell\|_{L^2(\Gamma)}. \end{aligned}$$

Here $\Upsilon = 1 + c_p(1 + \kappa^{(2)})$.

Proof First, let $v \in \dot{H}^1(\Gamma)$ be the weak solution of

$$-\Delta_{\Gamma} v = u - \tilde{u}_h^\ell \quad \text{on } \Gamma$$

where

$$\tilde{u}_h^\ell = u_h^\ell - \frac{1}{|\Gamma|} \int_{\Gamma} u_h^\ell \, dA.$$

Then

$$\begin{aligned} \|u - \tilde{u}_h^\ell\|_{L^2(\Gamma)}^2 &= \int_{\Gamma} \nabla_{\Gamma} v \cdot \nabla_{\Gamma} (u - \tilde{u}_h^\ell) \, dA \\ &= \int_{\Gamma} \nabla_{\Gamma} (v - I_h v) \cdot \nabla_{\Gamma} (u - \tilde{u}_h^\ell) \, dA \\ &\quad - \int_{\Gamma} \nabla_{\Gamma} I_h v \cdot \nabla_{\Gamma} \tilde{u}_h^\ell \, dA + \int_{\Gamma} f I_h v \, dA, \end{aligned}$$

where we used (1). Then, using (34), the fact that derivatives of constants are zero, and that $\int_{\Gamma} f \, dA = 0 = \int_{\Gamma_h} f_h \, dA_h$, we can show that

$$\|u - \tilde{u}_h^\ell\|_{L^2(\Gamma)}^2 = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned} J_1 &= \int_{\Gamma} f_h^\ell (I_h v - c) \, dA - \int_{\Gamma_h} f_h (I_h v - c)_\ell \, dA_h \\ J_2 &= \int_{\Gamma} (f - f_h^\ell) (I_h v - c) \, dA \\ J_3 &= \int_{\Gamma} \nabla_{\Gamma} (v - I_h v) \cdot \nabla_{\Gamma} (u - u_h^\ell) \, dA, \\ J_4 &= \int_{\Gamma_h} \nabla_{\Gamma_h} ((I_h v)_\ell) \cdot \nabla_{\Gamma_h} u_h \, dA_h - \int_{\Gamma} \nabla_{\Gamma} I_h v \cdot \nabla_{\Gamma} u_h^\ell \, dA. \end{aligned}$$

Here we choose $c = \frac{1}{|\Gamma|} \int_{\Gamma} I_h v \, dA$. Using (38), the Poincaré’s inequality (35), and that $\|f_h^\ell\|_{L^2(\Gamma)} \leq C \|f\|_{L^2(\Gamma)}$,

$$J_1 \leq C c_p \Psi_h \|f\|_{L^2(\Gamma)} \|\nabla_{\Gamma} (I_h v)\|_{L^2(\Gamma)}.$$

Using the Poincaré’s inequality (35) we get

$$J_2 \leq C c_p \|f - f_h^\ell\|_{L^2(\Gamma)} \|\nabla_{\Gamma} (I_h v)\|_{L^2(\Gamma)}.$$

Using (37) we get

$$J_4 \leq C \Psi_h \|\nabla_{\Gamma} u_h^\ell\|_{L^2(\Gamma)} \|\nabla_{\Gamma} (I_h v)\|_{L^2(\Gamma)}.$$

Using the triangle inequality and the energy estimate (36) we have

$$J_4 \leq C \Psi_h \left(\|\nabla_{\Gamma} (u_h^\ell - u)\|_{L^2(\Gamma)} + c_p \|f\|_{L^2(\Gamma)} \right) \|\nabla_{\Gamma} (I_h v)\|_{L^2(\Gamma)}.$$

To bound J_3 we use the Cauchy–Schwarz inequality

$$J_3 \leq \|\nabla_{\Gamma} (u_h^\ell - u)\|_{L^2(\Gamma)} \|\nabla_{\Gamma} (I_h v - v)\|_{L^2(\Gamma)}.$$

Using Lemma 12 we get the estimate

$$\|\nabla_{\Gamma} (v - I_h v)\|_{L^2(\Gamma)}^2 \leq C c_p^2 (\Lambda_h + \Psi_h)^2 \|u - \tilde{u}_h^\ell\|_{L^2(\Gamma)}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla_{\Gamma}^2 v\|_{L^2(T^\ell)}^2.$$

As we did in the proof of Theorem 16 [using (M1–M3)] we can show that

$$\sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla_{\Gamma}^2 v\|_{L^2(T^\ell)}^2 \leq C h^2 \Upsilon^2 \|u - \tilde{u}_h^\ell\|_{L^2(\Gamma)}^2.$$

Hence, using (M1) we have

$$\|\nabla_{\Gamma} (v - I_h v)\|_{L^2(\Gamma)} \leq C h \Upsilon \|u - \tilde{u}_h^\ell\|_{L^2(\Gamma)}. \tag{46}$$

Therefore, we get the bound

$$J_3 \leq Ch\Upsilon \left\| \nabla_\Gamma \left(u_h^\ell - u \right) \right\|_{L^2(\Gamma)} \left\| u - \tilde{u}_h^\ell \right\|_{L^2(\Gamma)}.$$

Using (46), the triangle inequality and a energy estimate we have

$$\left\| \nabla_\Gamma (I_h v) \right\|_{L^2(\Gamma)} \leq C(c_p + h\Upsilon) \left\| u - \tilde{u}_h^\ell \right\|_{L^2(\Gamma)}.$$

It follows then that

$$\begin{aligned} J_1 + J_2 + J_4 &\leq C(c_p + h\Upsilon) \\ &\left(c_p \Psi_h \|f\|_{L^2(\Gamma)} + c_p \left\| f - f_h^\ell \right\|_{L^2(\Gamma)} + \Psi_h \left\| \nabla_\Gamma \left(u_h^\ell - u \right) \right\|_{L^2(\Gamma)} \right) \\ &\times \left\| u - \tilde{u}_h^\ell \right\|_{L^2(\Gamma)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| u - \tilde{u}_h^\ell \right\|_{L^2(\Gamma)} &\leq C(c_p + h\Upsilon) \left(c_p \Psi_h \|f\|_{L^2(\Gamma)} + c_p \left\| f - f_h^\ell \right\|_{L^2(\Gamma)} \right) \\ &+ C((c_p + h\Upsilon)\Psi_h + h\Upsilon) \left\| \nabla_\Gamma \left(u_h^\ell - u \right) \right\|_{L^2(\Gamma)}. \end{aligned}$$

We gather from (M4) and (45) that

$$\begin{aligned} \left\| u - \tilde{u}_h^\ell \right\|_{L^2(\Gamma)} &\leq C(c_p + h\Upsilon)h^2\Upsilon \|f\|_{L^2(\Gamma)} + Cc_p(c_p + h\Upsilon) \left\| f - f_h^\ell \right\|_{L^2(\Gamma)} \\ &+ C((c_p + h\Upsilon)(1 + \kappa^{(2)})h^2 + h\Upsilon)h\Upsilon \|f\|_{L^2(\Gamma)}. \end{aligned} \tag{47}$$

The triangle inequality yields

$$\left\| u - u_h^\ell \right\|_{L^2(\Gamma)} \leq \left\| u - \tilde{u}_h^\ell \right\|_{L^2(\Gamma)} + \frac{1}{|\Gamma|^{1/2}} \left| \int_\Gamma u_h^\ell dA \right|. \tag{48}$$

We now use that $\int_{\Gamma_h} u_h dA_h = 0$ and (38) to get

$$\frac{1}{|\Gamma|^{1/2}} \left| \int_\Gamma u_h^\ell dA \right| = \frac{1}{|\Gamma|^{1/2}} \left| \int_\Gamma u_h^\ell dA - \int_{\Gamma_h} u_h dA_h \right| \leq C\Psi_h \left\| u_h^\ell \right\|_{L^2(\Gamma)}.$$

Using the triangle inequality, (35) and (36) we gather that

$$\frac{1}{|\Gamma|^{1/2}} \left| \int_\Gamma u_h^\ell dA \right| \leq C \left(\Psi_h \left\| u_h^\ell - u \right\|_{L^2(\Gamma)} + \Psi_h c_p^2 \|f\|_{L^2(\Gamma)} \right).$$

Finally, from (48), (6), and (M4) we have

$$\left\| u - u_h^\ell \right\|_{L^2(\Gamma)} \leq C \left\| u - \tilde{u}_h^\ell \right\|_{L^2(\Gamma)} + Cc_p h^2 \Upsilon \|f\|_{L^2(\Gamma)}.$$

The result follows from this inequality and (47). □

Remark 18 Although we only proved results for domains without boundaries, we anticipate that our analysis will carry over to surfaces with boundary. In fact, in the next section we will provide numerical experiments for a surface Γ with a boundary.

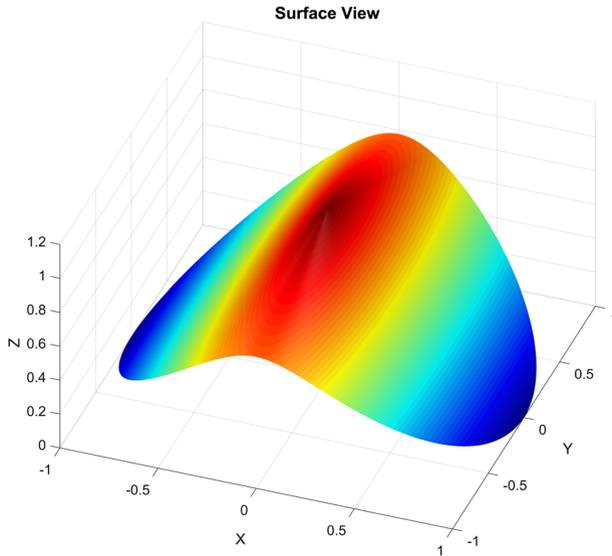


Fig. 2 Visualization of the surface Γ

5 Numerical Experiments

We consider a simple example of a surface with a high curvature “ridge,” and show our adapted mesh as well as properties of the solution.

5.1 Surface Parameterization

Let Γ be the surface parameterized by

$$X(x, y) = \left(x, y, 1 - \sqrt{x^2 + 5 \cdot 10^{-2}y^2 + 2.5 \cdot 10^{-5}} \right), \quad \text{for } (x, y) \in U, \quad (49)$$

where $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ (see Fig. 2). The curvature regions Γ_1, Γ_2 are defined by

$$U_1 = \left\{ (x, y) \in \mathbb{R}^2 : \left(\frac{x}{0.05} \right)^2 + \left(\frac{y}{0.5} \right)^2 \leq 1 \right\}, \quad \Gamma_1 = X(U_1), \quad \Gamma_2 = \Gamma \setminus \Gamma_1. \quad (50)$$

This leads to the following maximum curvatures on Γ_1, Γ_2 :

$$\kappa^{(1)} = \max_{\Gamma_1} \kappa = 199.970, \quad \kappa^{(2)} = \max_{\Gamma_2} \kappa = 8.701, \quad \frac{\kappa^{(1)}}{\kappa^{(2)}} = 22.984.$$

5.2 “Exact” Solution

We use zero boundary conditions on $\partial\Gamma$ and choose the right-hand-side $f : \Gamma \rightarrow \mathbb{R}$ to be

$$f(x, y, z) = \begin{cases} 50.0 \exp\left(\frac{1}{(x-0.2)^2 + y^2 - 0.2}\right), & \text{if } (x - 0.2)^2 + y^2 < 0.2, \\ 0, & \text{else.} \end{cases} \quad (51)$$

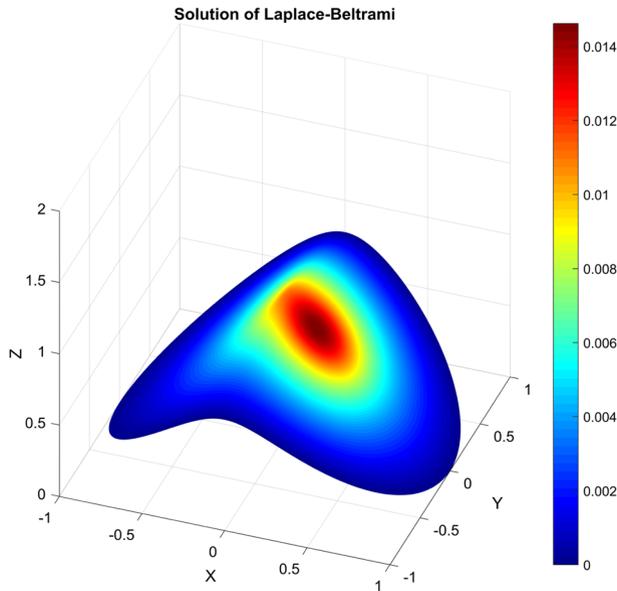


Fig. 3 Plot of the “exact” solution u on Γ

Note that f is similar to a “bump” function [24] and is $C^\infty(\Gamma)$ and has compact support on Γ .

In lieu of an exact analytic solution, we compute a reference “exact” solution (denoted u) on a mesh consisting of 3,679,489 vertices and 7,356,928 triangles obtained from refining an initial coarse mesh. The number of free degrees-of-freedom of the reference solution is 3,677,441 (after eliminating boundary degrees-of-freedom). See Fig. 3 for a plot of u .

We also plot an approximation of $|\nabla_\Gamma^2 u|$ to illustrate how the hessian is influenced by the high curvature of the domain, which is concentrated at the high curvature ridge of the surface (see Fig. 4).

5.3 Adapted Mesh and Solution

Our adapted mesh is generated by first starting with a coarse mesh that satisfies (10), (6), (11). We then iteratively check the criteria in (M1), (M2), (M3) in Sect. 4.1. At each iteration, if any triangle does not satisfy the criteria, then it is marked for refinement. We then refine all marked triangles using standard longest-edge bisection. Figure 5 shows a plot of our final adapted mesh.

Figure 6 shows the “pointwise” error $|\nabla_\Gamma(u - u_h)|$, where u_h is the numerical solution on the adapted mesh. Note that the graded mesh, essentially, eliminates the error in the high curvature region. However, the grading strategy does not specifically account for f , so the error is larger where f is large.

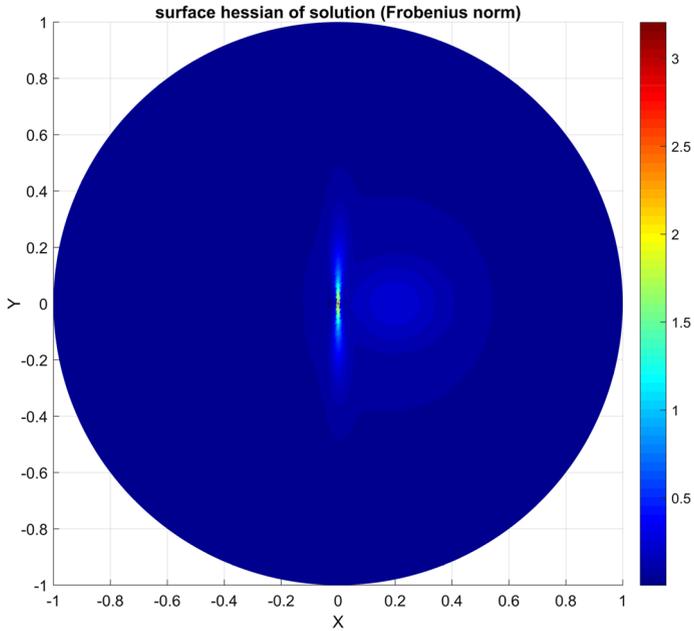


Fig. 4 Plot of the hessian of u (viewed from the top). Note that it peaks in the high curvature region

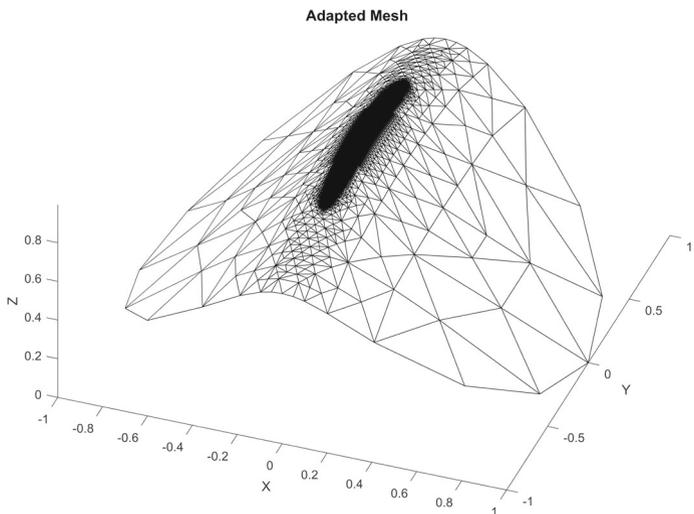


Fig. 5 Visualization of the adapted mesh of Γ using our grading criteria

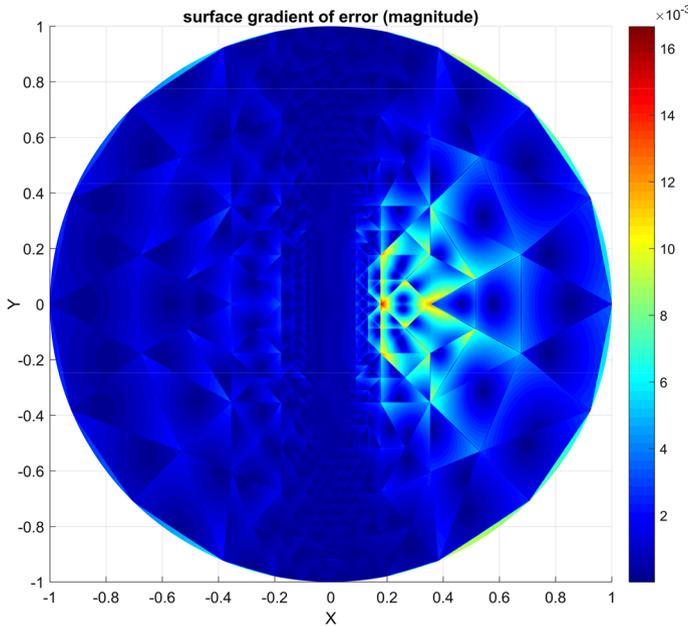


Fig. 6 Plot of $|\nabla_{\Gamma}(u - u_h)|$ (viewed from the top). The adapted mesh essentially eliminates the error in the high curvature region

Appendix A: One Technical Result

Lemma 19 *Assume that Γ is a C^3 two-dimensional compact orientable surface without boundary, and that $u \in H^2(\Gamma)$. Then*

$$\begin{aligned} & \sum_{i,j=1}^3 \int_{\Gamma_2} \underline{D}_{ij}u \underline{D}_i (\rho^2 \underline{D}_j u) \, dA \\ &= \sum_{j=1}^3 \int_{\Gamma_2} \Delta_{\Gamma} u \underline{D}_j (\rho^2 \underline{D}_j u) \, dA - \int_{\Gamma_2} \rho^2 (\text{tr}(H)H - 2H^2) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} u \, dA. \end{aligned}$$

Proof In what follows, we use the two identities [17]

$$\int_{\Gamma} \underline{D}_i u v \, dA = - \int_{\Gamma} u \underline{D}_i v \, dA + \int_{\Gamma} u v \text{tr}(H) v_i \, dA, \tag{52}$$

and

$$\underline{D}_{ij}u = \underline{D}_{ji}u + (H \nabla_{\Gamma} u)_j v_i - (H \nabla_{\Gamma} u)_i v_j, \tag{53}$$

for all $C^3(\Gamma)$ functions u, v . Also we will use that of course

$$\nabla_{\Gamma} u \cdot \mathbf{v} = 0. \tag{54}$$

We assume for the proof that $u \in C^3(\Gamma)$, and the general result follows from density arguments. Following [17, Lemma 3.2], and using the Einstein summation convention,

$$\begin{aligned}
 & \int_{\Gamma} \underline{D}_{ij}u \underline{D}_i \left(\rho^2 \underline{D}_j u \right) dA \\
 &= - \int_{\Gamma} \underline{D}_{iij}u \left(\rho^2 \underline{D}_j u \right) dA + \int_{\Gamma} \underline{D}_{ij}u \rho^2 \underline{D}_j u \operatorname{tr}(H)v_i dA && \text{by (52)} \\
 &= - \int_{\Gamma} \underline{D}_{iij}u \left(\rho^2 \underline{D}_j u \right) dA && \text{by (54)} \\
 &= - \int_{\Gamma} \underline{D}_i \left[\underline{D}_{ji}u + (H\nabla_{\Gamma}u)_j v_i - (H\nabla_{\Gamma}u)_i v_j \right] \rho^2 \underline{D}_j u dA && \text{by (53)} \\
 &= - \int_{\Gamma} \underline{D}_i \left[\underline{D}_{ji}u + H_{jk} \underline{D}_k u v_i - H_{ik} \underline{D}_k u v_j \right] \rho^2 \underline{D}_j u dA \\
 &= - \int_{\Gamma} \left(\underline{D}_{ijj}u + H_{jk} H_{ii} \underline{D}_k u - H_{ik} H_{ij} \underline{D}_k u \right) \rho^2 \underline{D}_j u dA && \text{by (54)} \\
 &= - \int_{\Gamma} \left(\underline{D}_{ijj}u \rho^2 \underline{D}_j u + \rho^2 (\operatorname{tr}(H)H - H^2) \nabla_{\Gamma}u \cdot \nabla_{\Gamma}u \right) dA.
 \end{aligned}$$

To handle the first term on the right hand side, we use (53) and the fact that $\underline{D}_{ii}u = \Delta_{\Gamma}u$ to write:

$$\begin{aligned}
 - \int_{\Gamma} \underline{D}_{iij}u \rho^2 \underline{D}_j u dA &= - \int_{\Gamma} \left[\underline{D}_j \Delta_{\Gamma}u + (H\nabla_{\Gamma} \underline{D}_i u)_j v_i - (H\nabla_{\Gamma} \underline{D}_i u)_i v_j \right] \rho^2 \underline{D}_j u dA \\
 &= - \int_{\Gamma} \underline{D}_j \Delta_{\Gamma}u \rho^2 \underline{D}_j u + H_{jk} \underline{D}_{ki}u v_i \rho^2 \underline{D}_j u dA,
 \end{aligned}$$

where we used (54) in the last equation. But $\underline{D}_{ki}u v_i = \underline{D}_k(\underline{D}_i u v_i) - \underline{D}_i u H_{ki} = -\underline{D}_i u H_{ki}$, and then

$$\begin{aligned}
 - \int_{\Gamma} \underline{D}_{iij}u \rho^2 \underline{D}_j u dA &= \int_{\Gamma} -\underline{D}_j \Delta_{\Gamma}u \rho^2 \underline{D}_j u + H_{jk} \underline{D}_i u H_{ki} \rho^2 \underline{D}_j u dA \\
 &= \int_{\Gamma} \Delta_{\Gamma}u \underline{D}_j \left(\rho^2 \underline{D}_j u \right) + \rho^2 H^2 \nabla_{\Gamma}u \cdot \nabla_{\Gamma}u dA.
 \end{aligned}$$

In the last equation we used (52) and (54). This completes the proof. □

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