6 The Cantor middle-thirds set

In this section we construct the famous Cantor middle-thirds set $\mathcal{K}$. This set has many of the same features as $\Lambda_c$ ($c < -2$), and is in fact “topologically equivalent” (we will make the last statement precise below). We will give three distinct ways to construct $\mathcal{K}$ utilizing three different viewpoints. $\mathcal{K}$ will be our (as it was historically) first example of a “fractal,” a term that we will later make precise.

6.1 The dynamical system construction of $\mathcal{K}$

Recall the tent-looking Cantor map $T(x)$ that was defined in Section 2 as an example:

$$T(x) = \begin{cases} 
3x & \text{if } x \leq \frac{1}{2} \\
-3x + 3 & \text{if } x \geq \frac{1}{2}
\end{cases}$$

Remember that the idea behind constructing $\Lambda_c$ was to keep only those points whose orbits remained bounded, or in other words, to throw away all those points whose iterates approach infinity. The Cantor set $\mathcal{K}$ is defined as the set of all $x_0$ in which the orbit $\langle x_0 \rangle$ is bounded under iteration by $T(x)$.

Notice that if $x_0 < 0$, then the succeeding iterates of $x_0$ will go to $-\infty$. If $x_0 > 1$, then $T(x_0) < 0$ and so its iterates will go to $-\infty$. We continue this reasoning: if $T^k(x_0) > 1$ for
any $k$, then its future iterates will approach $-\infty$ and so $x_0 \notin K$. It should be clear that if $x_0 \in K$, then $T^k(x_0) \in K$ for all $k$. The graph below contains the graph of $T(x)$ and its first two iterates in the construction of $K$. The markings on the $x$-axis represent those points that have not yet been thrown away, and thus are the only points left that can possibly still belong to $K$.

In the first graph, we see that some points are already mapped outside the interval $[0,1]$, and these make up the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. What remains are the points in $\left[0, \frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]$. However, after two iterations of $T(x)$, some of these points are again mapped to values larger than 1, and such points will not belong to $K$. If we continue the iteration procedure, each of these remaining intervals will get further subdivided into thirds, and the points belonging to the “middle third” part will map to a value larger than 1. Hence those points will also be thrown away, and we just continue. Of course we might well ask which points remain, if any. Actually it is pretty clear that some do remain, namely the points 0 and 1. By inspection one can see that 0 is fixed and 1 is eventually fixed. What other points remain? Well, $\frac{1}{3}$ and $\frac{2}{3}$ also are eventually fixed, as is $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$. One might notice that we are keeping each endpoint of an interval that has appeared in the construction, for each of these points are eventually fixed (an iterate lands at 0). Does anything but these endpoints remain? It is not clear (yet).

6.2 The geometrical construction of $K$

This construction will be closely related to the previous one, but will not refer to any dynamical system. The idea is to just start with the set $K_0 := [0, 1]$, partition $K_0$ into three equally spaced parts, throw the middle part away, and call the new set $K_1$, which of course is $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Now repeat this same procedure on each of the intervals that make up $K_1$, and the result is called $K_2$. One continues this process indefinitely, each time taking one of the remaining intervals and throwing away its middle third (hence the name “middle thirds set” although perhaps “no middle thirds” would be more accurate). The “limit set” consists of all points that are never thrown away, and this is the Cantor set $K$.

A mathematically precise description of the same procedure can be given as follows:

Step 1: Define two functions

$$G_0(x) = \frac{1}{3}x \quad \text{and} \quad G_2(x) = \frac{1}{3}x + \frac{2}{3}$$

Step 2: Let $K_0 = [0, 1]$.

Step 3: Assuming that $K_k$ has been defined, define $K_{k+1}$ as the set $G_0(K_k) \cup G_2(K_k)$. That is, $x \in K_{k+1}$ if and only if there exists a point $y \in K_k$ so that either $x = G_0(y)$ or $x = G_2(y)$.

Step 4: Step 3 is repeated for all $k$. The Cantor set $K$ consists of those points belonging to every set $K_k$.

Here is the way these sets are being constructed:
A very interesting feature of the Cantor set related to this construction is that $\mathcal{K}$ is the unique closed and bounded subset of the real numbers that has the property of being “fixed” by the procedure used in Step 3 above. That is, $\mathcal{K}$ is the unique closed and bounded set in $\mathbb{R}$ so that

$$\mathcal{K} = G_0(\mathcal{K}) \cup G_2(\mathcal{K}).$$ \hspace{1cm} (1)

**Exercise 6.1.** Prove (1).

### 6.3 The analytical construction of $\mathcal{K}$

One of the reasons $\Lambda_c$ in Section 5.5 was hard to visualize is that the endpoints appearing in the construction became increasingly difficult to describe precisely - one had to find the roots of $Q^c_x(x) = x$ for each $k$, which can be done recursively but which becomes quickly a practical impossibility. The construction of $\mathcal{K}$ is easier because the endpoints are known at each iteration and can be represented analytically. We provide these details here, in which we use some properties of infinite series.

#### 6.3.1 Ternary expansions

For a number $x$ satisfying $0 \leq x \leq 1$, the *ternary* expansion of $x$ is the series

$$x = \sum_{i=1}^{\infty} \frac{s_i}{3^i} \hspace{1cm} (2)$$
where each $s_i$ is either 0, 1, or 2. For every string of $s_i$’s, the series always converges. The justification of this important fact relies on results from Math 1552. Indeed, the sequence consisting of $s_i = 2$ for all $i$ gives rise to a geometric series which converges (to 1), and if \{${s_i}$\} is any sequence consisting of 0, 1, or 2’s, then obviously $s_i \leq 2$, and so the series (2) converges by the comparison test.

### 6.3.2 A visualization

Here is one way to think about the representation (2): Suppose $0 \leq x < 1$, then $x$ lies in exactly one of the intervals $I_0 := [0, \frac{1}{3}]$, $I_1 := [\frac{1}{3}, \frac{2}{3}]$, or $I_2 := [\frac{2}{3}, 1]$. Let $s_1$ equal an index $i$ so that $x \in I_i$. If $s_1 = 1$, then $x$ lies in one of the intervals $I_{0,0} := [0, \frac{1}{9}]$, $I_{0,1} := (\frac{1}{9}, \frac{2}{9})$, or $I_{0,2} := [\frac{2}{9}, \frac{1}{3}]$. Let $s_2$ be the index $i$ so that $x \in I_{0,i}$. If $s_1 = 1$, then one divides $I_1$ into three parts and determines $s_2$ from these, and similarly if $s_1 = 2$. Inductively, suppose $s_1, \ldots, s_{k-1}$ are chosen associated to $x$; then $s_k$ is chosen as 0 (respectively 1, 2), if $x$ belongs to the interval

\[
I_{s_1, \ldots, s_{k-1}, 0} := \left[ \frac{1}{3} \sum_{i=1}^{k-1} \frac{s_i}{3^i}, \frac{1}{3} \sum_{i=1}^{k-1} \frac{s_i}{3^i} + \frac{1}{3^{k+1}} \right],
\]

respectively

\[
I_{s_1, \ldots, s_{k-1}, 1} := \left[ \frac{1}{3} \sum_{i=1}^{k-1} \frac{s_i}{3^i} + \frac{2}{3^{k+1}}, \frac{1}{3} \sum_{i=1}^{k-1} \frac{s_i}{3^i} + \frac{1}{3^{k+1}} \right],
\]

\[
I_{s_1, \ldots, s_{k-1}, 2} := \left[ \frac{2}{3} \sum_{i=1}^{k-1} \frac{s_i}{3^i} + \frac{3}{3^{k+1}}, \frac{2}{3} \sum_{i=1}^{k-1} \frac{s_i}{3^i} + \frac{3}{3^{k+1}} \right],
\]

One can think of $x$ as represented by the string

\[x \prec .s_1 s_2 s_3 s_4 \ldots\]  

(3)

in a manner similar to a decimal expansion, except that the $s_i$’s are 0, 1, or 2 rather than a digit from 0 to 9.

### 6.3.3 Nonuniqueness

Now the ternary expansion (2) is not unique, since for example

\[\frac{1}{3} = \sum_{i=1}^{\infty} \frac{s_i}{3^i} = \sum_{i=1}^{\infty} \frac{t_i}{3^i}\]

where

\[s_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases} \quad \text{and} \quad t_i = \begin{cases} 0 & \text{if } i = 1 \\ 2 & \text{if } i \neq 1 \end{cases}\]

But it turns out that the only $x \in [0,1]$ without a unique expansion is of the form $\frac{p}{3^k}$ for some positive integer $k$ and some integer $p$, $0 \leq p < 3^k$. In fact, $\frac{p}{3^k}$ is one of the following two types: it either has two ternary expansions of the form

\[
\frac{p}{3^k} \prec .s_1 s_2 \ldots s_n 1 0 0 \ldots
\]

or

\[
\prec .s_1 s_2 \ldots s_n 0 2 2 \ldots,
\]
or two of the form

\[ \frac{p}{3^k} \sim .s_1s_2 \ldots s_n200 \ldots \sim .s_1s_2 \ldots s_n122 \ldots . \]

This is because

\[ \frac{1}{3^{n+1}} = \sum_{i=n+1}^{\infty} \frac{2}{3^i} \quad \text{and} \quad \frac{2}{3^{n+1}} = \frac{1}{3^{n+1}} + \sum_{i=n+2}^{\infty} \frac{2}{3^i}. \]

If we agree to always use the form of the ternary expansion that uses no 1’s if possible, then the expansion is unique.

### 6.3.4 Another (equivalent) definition of \( K \)

Now we define the Cantor set \( K \) as the set of all \( x \in [0,1] \) whose ternary expansion (3) has the property that \( s_i = 0 \) or 2 for all \( i \).

Here is why the previous description coincides with the previous two definitions given earlier. Consider the ternary expansion (3) for a fixed \( x \in [0,1] \). Notice that \( s_1 = 1 \) means precisely that \( x \in \left( \frac{1}{3}, \frac{2}{3} \right) \) (recall the convention of how elements with different expansions are treated). Similarly, \( s_2 = 1 \) means that \( x \) belongs to either to \( \left( \frac{1}{9}, \frac{2}{9} \right) \), \( x \in \left( \frac{4}{9}, \frac{5}{9} \right) \), or \( x \in \left( \frac{7}{9}, \frac{8}{9} \right) \). Then it is clear that if \( s_1 \neq 1 \) and \( s_2 \neq 1 \) we must have \( x \) belonging to

\[ \left[ 0, \frac{1}{9} \right] \cup \left[ \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right]. \]

These are the four intervals indicated in third Graph above that contains all the points that belong to \( K \). Going further along in the ternary expansion and not using the coordinate 1 has the effect of breaking each of the previous intervals into equal thirds and throwing the middle third away.

### 6.3.5 Finding ternary expansions

Given \( x \in [0,1] \), how can one find the ternary expansion \( x \sim .s_1s_2s_3 \ldots \)? We first illustrate with an example. Consider \( x = \frac{3}{4} \). Clearly \( \frac{2}{3} < \frac{3}{4} \), so \( s_1 = 2 \). To find \( s_2 \), we need to look at \( \frac{3}{4} - \frac{2}{3} = \frac{1}{12} \) and determine which interval among \( \left[ 0, \frac{1}{9} \right], \left[ \frac{1}{9}, \frac{2}{9} \right] \), and \( \left[ \frac{2}{9}, 1 \right] \) that it belongs. It is clearly the first interval, and so \( s_2 = 0 \). The next digit \( s_3 \) is determined by which interval among \( \left[ 0, \frac{1}{27} \right], \left[ \frac{1}{27}, \frac{2}{27} \right] \), and \( \left[ \frac{2}{27}, 1 \right] \) that \( \frac{1}{12} \) belongs. Since \( \frac{1}{12} = \frac{2}{27} > \frac{2}{27} \), we see that \( s_3 = 2 \). Now for \( s_4 \), etc. This is getting tedious.

Here is a simpler way based on the tripling map, which recall is given by \( F(x) = 3x \) (mod 1). We first note as above that \( s_1 = 2 \); this part was easy. We found \( s_2 \) by considering which of the three intervals of length \( \frac{1}{9} \) that \( \frac{3}{4} - \frac{2}{3} = \frac{1}{12} \) belonged. The key observation is to note that the determination of this interval coincides if we just expand everything by a factor of 3. In other words, among the intervals \( \left[ 0, \frac{1}{9} \right], \left[ \frac{1}{9}, \frac{2}{9} \right] \), and \( \left[ \frac{2}{9}, 1 \right] \), one does \( 3 \cdot \frac{1}{12} = \frac{1}{4} \) belong? This arithmetic is much easier to do, and we see immediately that \( s_2 = 0 \). What did we do? In effect, we multiplied \( x - \frac{a_1}{3} \) by 3, threw away the “integer” part, and determined to which of the one-third-length intervals the remainder belonged. But
now note that “throwing away the integer part” is precisely the same as using the tripling map!

To summarize, as in Section 6.3.2, let $I_0 := \left[0, \frac{1}{3}\right]$, $I_1 := \left[\frac{1}{3}, \frac{2}{3}\right]$, or $I_2 := \left[\frac{2}{3}, 1\right]$.

Now suppose we are given $x \in [0, 1]$. With $x_1 = x$, define $x_k$ for $k = 2, 3, \ldots$ by setting $x_k = 3^k x_{k-1} \pmod{1}$. This is the same as recursively defining each $x_k$ by $x_{k+1} = 3 x_k \pmod{1}$, $k = 1, 2, \ldots$. Then one determines $s_k$ as the index for which $x_k \in I_{s_k}$. That’s it.

For the example $x = \frac{3}{4}$ we started with above, one has

\begin{align*}
x_1 = \frac{3}{4} &\quad \implies s_1 = 2, \\
x_2 = \frac{9}{4} \pmod{1} = \frac{1}{4} &\quad \implies s_2 = 0, \\
x_3 = \frac{3}{4} &\quad \implies s_3 = 2 \\
x_4 = \frac{9}{4} \pmod{1} = \frac{1}{4} &\quad \implies s_4 = 0
\end{align*}

Now wait a second, this is starting to look familiar, no?

**Exercise 6.2.** Can you determine the entire ternary expansion of $\frac{3}{4}$? Why?

A ternary expansion $\ldots s_1 s_2 s_3 \ldots$ is said to repeat if there exists $k$ so that $s_i = s_{i+k}$ for all $i = 1, 2, 3, \ldots$. It is said to eventually repeat if there exists $k_0$ and $k$ so that $s_i = s_{i+k}$ for all $i = k_0, k_0 + 1, k_0 + 2, \ldots$.

**Exercise 6.3.** Show that every rational number $q \in \mathbb{Q}$, $q \in [0, 1]$ has an eventually repeating ternary expansion. Recall that $q$ is a rational number provided it can be written as $q = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Conversely, show that every number with a repeating ternary expansion is a member of $\mathbb{Q}$.

### 6.4 The connections among the viewpoints

We explore in further detail how the three viewpoints produce the same set $\mathcal{K}$.

#### 6.4.1 Between ternary expansions and the geometrical construction

Recall the maps

$$G_0(x) = \frac{1}{3} x \quad \text{and} \quad G_2(x) = \frac{1}{3} x + \frac{2}{3}$$

Suppose $x \in \mathcal{K}$, and has a ternary expansion $x = \ldots s_1 s_2 s_3 \ldots$ where each $s_k$ is 0 or 2. Now, $s_1 = 0$ if and only if $x \in [0, \frac{1}{3}]$, in which case $x \in G_0([0, 1])$. On the other hand, if $s_1 = 2$, then $x \in G_2([0, 1])$. We can write both cases as $x \in G_{s_1}([0, 1])$. Similar reasoning
leads to the observation that if \( s_2 = 0 \), then \( x \in G_{s_1} \left( G_0([0, 1]) \right) \), whereas if \( s_2 = 2 \), then \( x \in G_{s_1} \left( G_2([0, 1]) \right) \). Again, both cases can be written as \( x \in G_{s_1} \left( G_{s_2}([0, 1]) \right) \). One can continue and see that similar reasoning can be applied at each stage of the construction, and that in fact for all \( k \), one has

\[
x \in G_{s_1} \left( G_{s_2} \left( \ldots \left( G_{s_k}([0, 1]) \right) \ldots \right) \right).
\]

Thus the ternary expansion is in effect a “record” of the order in which the maps \( G_0 \) and \( G_2 \) were applied in the geometric construction in order to have \( x \in K_i \) for all \( i \).

### 6.4.2 Between the dynamical system and ternary expansions

Let \( K_D \) be the Cantor set as defined by dynamical system viewpoint, and \( K_A \) be that defined analytically:

\[
K_D = \left\{ x_0 : x_k \not\to -\infty \text{ as } k \to \infty \right\}
\]

\[
K_A = \left\{ x_0 : \exists \text{ ternary expansion } x_0 \sim s_1 s_2 s_3 \ldots , \text{ with } s_k \in \{0, 2\} \forall k \right\},
\]

where \( x_k = T^k(x) \) and \( T(x) \) is the Cantor map. Of course we have claimed that \( K_D = K_A \). Let’s see why.

Suppose \( x_0 \in [0, 1] \) has the ternary expansion \( x_0 \sim s_1 s_2 s_3 \ldots \), in which if possible there are no ones in the expansion. Thus \( s_1 = 1 \) if and only if \( \frac{1}{3} < x_0 < \frac{2}{3} \), which implies

\[
K_A \cap \left( \frac{1}{3}, \frac{2}{3} \right) = \emptyset.
\]

Also, note that \( T(x_0) > 1 \) if and only if \( \frac{1}{3} < x_0 < \frac{2}{3} \), and since \( x_k \to -\infty \) in this case, we have

\[
K_D \cap \left( \frac{1}{3}, \frac{2}{3} \right) = \emptyset.
\]

Now if \( x_0 \in [0, 1] \) but \( x_0 \notin \left( \frac{1}{3}, \frac{2}{3} \right) \), then \( T(x) \in [0, 1] \). We need to have a better understanding of how \( T \) affects a ternary expansion. That is, suppose

\[
x \sim .s_1 s_2 s_3 \ldots \tag{4}
\]

is a ternary expansion with \( s_1 \neq 1 \). We have just observed that \( T(x) \in [0, 1] \), and so it has a ternary expansion

\[
T(x) \sim .t_1 t_2 t_3 \ldots \tag{5}
\]

Knowing the \( s_i \)'s, is there a simple way to determine the \( t_i \)'s?

We first analyze how \( T \) maps the interval

\[
\left[ 0, \frac{1}{3} \right] = \left[ 0, \frac{1}{9} \right] \cup \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left[ \frac{2}{9}, \frac{1}{3} \right].
\]
Obviously one has
\[
T \left( \left[ 0, \frac{1}{9} \right] \right) = \left[ 0, \frac{1}{3} \right], \\
T \left( \left[ \frac{1}{9}, \frac{2}{9} \right] \right) = \left( \frac{1}{3}, \frac{2}{3} \right), \\
T \left( \left[ \frac{2}{9}, \frac{1}{3} \right] \right) = \left[ \frac{2}{3}, 1 \right].
\]

Thus with \( x \) as in (4) with \( s_1 = 0 \), we see that the ternary expansion for \( T(x) \) as in (5) satisfies
\[
t_1 = \begin{cases} 
0 & \text{if } s_2 = 0 \\
1 & \text{if } s_2 = 1 \\
2 & \text{if } s_2 = 2
\end{cases}
\]

In fact, each of the subintervals are similarly stretched by 3, and it follows that for each \( k = 1, 2, 3, \ldots \),
\[
t_k = \begin{cases} 
0 & \text{if } s_{k+1} = 0 \\
1 & \text{if } s_{k+1} = 1 \\
2 & \text{if } s_{k+1} = 2
\end{cases}
\]

We next analyze how \( T \) maps the interval
\[
\left[ \frac{2}{3}, 1 \right] = \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left( \frac{7}{9}, \frac{8}{9} \right) \cup \left[ \frac{8}{9}, 1 \right].
\]

For these \( x \in \left[ \frac{2}{3}, 1 \right] \), one has \( T(x) = -3x + 3 \), and so
\[
T \left( \left[ \frac{2}{3}, \frac{7}{9} \right] \right) = \left[ \frac{2}{3}, 1 \right], \\
T \left( \left( \frac{7}{9}, \frac{8}{9} \right) \right) = \left( \frac{1}{3}, \frac{2}{3} \right), \\
T \left( \left[ \frac{8}{9}, 1 \right] \right) = \left[ 0, \frac{1}{3} \right].
\]

Thus with \( x \) as in (4) with \( s_1 = 2 \), we see that the ternary expansion for \( T(x) \) as in (5) satisfies
\[
t_1 = \begin{cases} 
2 & \text{if } s_2 = 0 \\
1 & \text{if } s_2 = 1 \\
0 & \text{if } s_2 = 2
\end{cases}
\]

But note now that these interval are “flipped” in the sense that \( T \left( \frac{2}{7} \right) = 1 \) and \( T \left( \frac{7}{9} \right) = \frac{2}{3} \), etc. In fact all of the subintervals are “flipped” as well, which means for all \( k = 1, 2, 3, \ldots \),
\[
t_k = \begin{cases} 
2 & \text{if } s_{k+1} = 0 \\
1 & \text{if } s_{k+1} = 1 \\
0 & \text{if } s_{k+1} = 2
\end{cases}
\]
A consequence of this analysis is that if all of the $s_i$’s are either 0 or 2, then so is each of the $t_i$’s. Therefore $K_A \subseteq K_D$.

Conversely, if $x_0 \in [0, 1]$ and $x_0 \notin K_A$, then there exists a least $k \geq 0$ so that $T^k(x_0) > 1$. Our analysis above has showed that $s_{k+1} = 1$, and so $x_0 \notin K_A$.

Putting everything together implies that $K_D = K_A$.

What we have really showed is that for $x \in K$ with ternary expansion $x \sim s_1s_2s_3\ldots$, then $T(x) \in K$ and has a ternary expansion $T(x) \sim t_1t_2t_3\ldots$, where for each $k = 1, 2, 3, \ldots$,

$$t_k = \begin{cases} s_{k+1} & \text{if } s_1 = 0 \\ 2 - s_{k+1} & \text{if } s_1 = 2. \end{cases}$$

(6)

Exercise 6.4. Using (6) of how $T$ affects ternary expansions, verify that $\frac{3}{4}$ is a fixed point of $T(x)$. That is, compare the ternary expansion of $\frac{3}{4}$ with the one of $T(x)$ as described in (6).

We can deduce (6) also directly by manipulating ternary expansions, as follows. Suppose $x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$. If $s_1 = 0$, then

$$T(x) = 3 \cdot x = 3 \cdot \sum_{i=1}^{\infty} \frac{s_i}{3^i}$$

$$= 3 \cdot \sum_{i=2}^{\infty} \frac{s_i}{3^i} \quad \text{(since } s_1 = 0)$$

$$= \sum_{i=2}^{\infty} \frac{s_i}{3^{i-1}} \quad \text{(by multiplication)}$$

$$= \sum_{i=1}^{\infty} \frac{s_i+1}{3^i} \quad \text{(by reindexing)}.$$
6.5 Distance between points of $K$

We use the notation of Subsection 6.3. Suppose $x$ and $y$ are two points with ternary expansions

$$x \sim .s_1 s_2 s_3, \ldots \quad \text{and} \quad y \sim .t_1 t_2 t_3, \ldots . \quad (7)$$

If $s_1 = t_1 = 0$, then $x$ and $y$ both belong to $I_0$, and if $s_1 = t_1 = 1$, then $x$ and $y$ both belong to $I_2$. In either case we note that $|x - y| \leq \frac{1}{3}$. This observation can be extended to any string of first entries in the ternary expansion: If $s_1 = t_1$, $s_2 = t_2$, \ldots, $s_{k_0} = t_{k_0}$, then $x$ and $y$ both belong to the interval $I_{s_1 \ldots s_{k_0}} = I_{t_1 \ldots t_{k_0}}$ which has total length $\frac{1}{3^{k_0}}$, and thus $|x - y| \leq \frac{1}{3^{k_0}}$.

Another way to see the same result is to note that if $x$ and $y$ are as in (7), then

$$|x - y| \leq \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{3^k}.$$ 

In particular, if $s_k = t_k$ for $k = 1, 2, \ldots k_0$, then

$$|x - y| \leq \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{3^k} = \sum_{k=k_0+1}^{\infty} \frac{|s_k - t_k|}{3^k}$$

$$\leq \sum_{k=k_0+1}^{\infty} \frac{2}{3^k} = \frac{2}{3^{k_0+1}} \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{2}{3^{k_0+1}} \frac{1}{1 - (1/3)}$$

$$= \frac{1}{3^{k_0}}.$$

The same argument actually shows that if $s_{k_0} \neq t_{k_0}$ for some $k_0$, then

$$|x - y| \geq \frac{1}{3^{k_0}}. \quad (8)$$

The important point here is that an upper bound on the distance between two points of $K$ can be obtained directly from the ternary expansions (7). We shall use this as a starting point when we discuss symbolic dynamics below.

6.6 $K$ is uncountable

We begin to explore in more detail the nature of the elements in $K$.

**Exercise 6.5.** Show that the endpoints of an interval used in the construction $K$ belongs to $K$. That is, $K$ contains $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \ldots$. Give an explanation of why this so using each of the various viewpoints.

A natural question is whether $K$ contains more elements besides those endpoints? Thinking in terms of ternary expansions, one might think there is, and we shall see in this section that in fact there are many, many more points of $K$ that are not one of these endpoints.
6.6.1 Bijections

How can we tell whether one set has more elements than another? Of course, if the sets are finite, then we just count the elements, and whichever has the larger number is the larger set. This is more complicated if the sets are infinite. We need the following concepts.

**Definition 6.1.** Suppose $X$ and $Y$ are two sets, and $f : X \to Y$ is map between them.

1. The map $f$ is $1-1$ (other names: one-to-one, injection) provided that for any $x_1, x_2$ belong to $X$, then
   \[ f(x_1) = f(x_2) \implies x_1 = x_2. \]

2. The map $f$ is onto (also called a surjection) provided for all $y \in Y$, there exists $x \in X$ so that $f(x) = y$.

3. The map $f$ is a one-to-one correspondent (also called a bijection) if it is $1-1$ and onto.

4. The map $f$ has an inverse $g : Y \to X$ provided $g \circ f = \text{Id}_X$ and $f \circ g = \text{Id}_Y$, where $\text{Id}_X$ and $\text{Id}_Y$ are the identity maps on $X$ and $Y$, respectively.

**Exercise 6.6.** Suppose $X$ and $Y$ are two sets, and $f : X \to Y$ is map between them.

(i) Show that $f$ has an inverse if and only if $f$ is a bijection.

(ii) Show that if an inverse exists, then it is unique. (One usually then writes $f^{-1}$ for this unique map).

(iii) Give an example where $f$ is $1-1$ but not onto, and another where it is onto but not $1-1$.

**Definition 6.2.**

1. The sets $X$ and $Y$ have the same cardinality provided there exists a bijection $f : X \to Y$.

2. A set $X$ has finite cardinality if there exists a natural number $n$ so that $X$ has the same cardinality as the set $\{1, 2, 3, \ldots \}$.

3. A set $X$ is said to be countable provided it has the same cardinality as the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots \}$.

**Exercise 6.7.**

1. Show that the natural numbers $\mathbb{N}$, the integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$, and the even numbers $\mathbb{E} = \{0, 2, 4, 6, \ldots \}$ all have the same cardinality. In particular, find bijections between $\mathbb{N}$ and $\mathbb{Z}$, and between $\mathbb{N}$ and $\mathbb{E}$.

2. Show that if $X$ is countable and $Y \subseteq X$, then $Y$ is countable.

3. Show that if $f : \mathbb{N} \to X$ is onto, then $X$ is countable (i.e. $f$ may not be one-to-one, but there is a map $g : \mathbb{N} \to X$ that is one-to-one and onto).

4. Suppose both $X$ and $Y$ are countable. Show that $X \times Y := \{(x, y) : x \in X, y \in Y\}$ is also countable.
In fact, show that if $X_1, X_2, X_3, \ldots$ is a (countable) collection of countable sets, then \[ \bigcup_{i=1}^{\infty} X_i \] is also a countable set.

6.6.2 Equivalence relations

Definition 6.3. Suppose $X$ is any set. A relation defined on $X$ is any subset $R$ of the product space $X \times X := \{(x, y) : x \in X, y \in X\}$. A relation $R$ is called an equivalence relation provided

(i) $(x, x) \in R \quad \forall x \in X \quad \text{(reflexivity)}$

(ii) $(x, y) \in R \Rightarrow (y, x) \in R \quad \forall x, y \in X \quad \text{(symmetry)}$

(iii) $(x, y), (y, z) \in R \rightarrow (x, z) \in R \quad \forall x, y, z \in X \quad \text{(transitivity)}$

One often writes $x \sim y$ (or if there is some ambiguity as to which relation, $x \sim_R y$ is used) if and only if $(x, y) \in R$. Note the (small, subtle) difference in the notation “$\sim$” that we have been using in associating a number with its ternary expansion. Yes, the notation is very similar (no pun intended), but this should cause no confusion.

Exercise 6.8. Consider the class $\mathcal{U}$ of all sets (sometimes called the universal class), and suppose we define the relation $\sim$ on $\mathcal{U}$ by saying that $X \sim Y$ if and only if $X$ and $Y$ have the same cardinality. Show that $\sim$ is an equivalence relation.

Exercise 6.9. Consider the integers $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and suppose $p \in \mathbb{N}$. For $z_1, z_2 \in \mathbb{Z}$, we say that $z_1$ is equivalent to $z_2$ mod $p$ and write $z_1 \sim_p z_2$ (also written as $z_1 = z_2 \pmod{p}$) provided there exists $k \in \mathbb{Z}$ so that $z_1 - z_2 = k \cdot p$. Show that $\sim_p$ is an equivalence relation.

6.6.3 Binary expansions

Completely analogous to a ternary expansion, each number $x \in [0, 1]$ has a binary expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i},$$

where each $a_i$ is either 0 or 1. It is well-known that the set of numbers in the interval $[0, 1]$ is uncountable, that is, there is no one-to-one correspondence between $[0, 1]$ and the natural numbers $\mathbb{N} := \{1, 2, \ldots\}$. What is perhaps surprising is that the Cantor set $\mathcal{K}$ is also uncountable. A consequence is that $\mathcal{K}$ certainly must contain points (and a “lot” of them) other than the endpoints of the intervals used in the geometric construction, since those endpoints are countable.

We give two proofs of the important fact that $\mathcal{K}$ is uncountable.
6.6.4 \( K \) is in 1-to-1 correspondence with \([0,1]\)

The first proof that \( K \) is uncountable is the one given in Devaney, and goes like this. Define \( H : [0,1] \to K \) in the following manner: let \( x \in [0,1] \), and let \( \{a_i\} \) be the coefficients in the binary expansion of \( x \) as in (9). The idea is to use these same coefficients in a ternary expansion. For each \( i \), let \( s_i = 2a_i \), and note that \( s_i \) is either 0 or 2. Then \( H(x) \) is defined as the number associated to the ternary expansion \( .s_1 s_2 s_3 s_4 \ldots \), or saying the same thing,

\[
H(x) = \sum_{i=1}^{\infty} \frac{s_i}{3^i}.
\]

Then \( H \) is one-to-one, which means that if \( H(x) = H(y) \), then \( x = y \). This is because \( H(x) = H(y) \) implies that both \( H(x) \) and \( H(y) \) have the same ternary expansion, and consequently the coefficients used in the binary expansion for \( x \) and \( y \) must also be the same. It is also the case that every element in \( K \) is mapped onto by some element of \([0,1]\). Specifically, this is true because if \( z \in K \) has ternary expansion \( .s_1 s_2 s_3 s_4 \ldots \), then \( z = H(x) \) where \( x \) is given by

\[
x = \sum_{i=1}^{\infty} \frac{a_i}{2^i}
\]

where \( a_i := \frac{1}{2}s_i \). Thus we have seen that \([0,1]\) and \( K \) have the same \textit{cardinality}, which, as defined above, means they can be put into one-to-one correspondence with each other. Since we “know” that \([0,1]\) is uncountable, so is \( K \). Actually, though, this is cheating. \textit{How do we know \([0,1]\) is uncountable?} The argument to show that is the same that will be used to show \( K \) is uncountable.

6.6.5 Cantor’s diagonal argument

We next give Cantor’s original proof that \( K \) is uncountable. It is based on contradiction, and so assume on the contrary that \( K \) is countable. This means that \( K \) can be written in one-to-one correspondence with \( \mathbb{N} \), and that we can label the elements as

\[
K = \left\{ S_1, S_2, S_3, \ldots \right\}.
\]

Now each of the \( S_j \) is an element of \( K \), and so has a ternary expansion that we label as

\[
S_j \sim .s_{j,1} s_{j,2} s_{j,3} s_{j,4} \ldots .
\]

Let us write us now write this in the form of an infinite array:

\[
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix} s_{1,1} \\
s_{2,1} \\
s_{3,1} \\
s_{4,1} \
\end{bmatrix} & s_{1,2} & s_{1,3} & s_{1,4} & \ldots \\
\begin{bmatrix} s_{2,2} \\
s_{3,2} \\
s_{4,2} 
\end{bmatrix} & s_{2,3} & s_{2,4} & \ldots \\
\begin{bmatrix} s_{3,3} \\
s_{4,3} 
\end{bmatrix} & s_{3,4} & \ldots \\
\vdots & \vdots & \vdots & \vdots 
\end{bmatrix}
\tag{10}
\]
Now remember that we are assuming every element of $K$ belongs to the list $\{S_1, S_2, \ldots\}$. But now we can construct some $\bar{S}$ that belongs to $K$ but NOT to the list. The idea is to make sure the ternary expansion of $\bar{S} \sim \bar{s}_1 \bar{s}_2 \bar{s}_3 \ldots$ cannot be the ternary expansion of any of the $S_i$’s, and of course we have listed the ternary expansions of all the $S_i$’s in (10). The boxed diagonal entries in (10) are the coefficients that determine how $\bar{s}_i$ is chosen. Define

$$\bar{s}_i = \begin{cases} 0 & \text{if } s_{i,i} = 2 \\ 2 & \text{if } s_{i,i} = 0. \end{cases}$$

Then the $\bar{s}_i$’s are all either 0 or 2, and so they determine a ternary expansion for some element of $K$ which we call $\bar{S}$. It is also clear that $\bar{S} \neq S_i$ for all $i$ because the ternary expansion of $\bar{S}$ is different than the ternary expansion of every $S_i$. In fact, by (8) one has $|\bar{S} - S_i| \geq \frac{1}{3^i}$ for every $i$. The conclusion is that our original assumption that $K$ could be put into a one-to-one correspondence with $\mathbb{N}$ cannot be correct, and therefore $K$ must be uncountable.

**Exercise 6.10.** Let $X$ be the set consisting of all sequences of the form $s_1 s_2 s_3 \ldots$ where each $s_i = 0$ or 1. Show that $X$ is uncountable.

**Exercise 6.11.** Suppose $X$ is any set, and $\mathcal{P}(X)$ is the collection of all nonempty subsets of $X$.

1. Show there is a one-to-one correspondence between $\mathcal{P}(X)$ and the set denoted (and then defined) by

$$2^X := \left\{ g : g \text{ is a function from } X \text{ into } \{0, 1\} \right\}$$

This explains why $\mathcal{P}(X)$ is usually called the *power set* of $X$. (Hint: Define $f : \mathcal{P}(X) \to 2^X$ by letting $f(A)$ be the function $g : X \to \{0, 1\}$ defined as

$$g(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$

Show that $f$ is a bijection.)

2. Suppose $X$ is countable. Show that $\mathcal{P}(X)$ is not countable. (Hint: Mimic Cantor’s diagonal argument.)

### 6.7 What is the dimension of $K$?

Why ask this question? Well, it is because $K$ sort of falls “in between” a set of 0 dimension (which consists of some isolated points) and a set of dimension 1 (which would be a segment of a curve or a line). Indeed, there seems to be a lot of stuff in $K$ since it is uncountable. On the other hand, a lot was thrown away in the construction: a set of length $\frac{1}{3}$ at the first step, and a set of size $\frac{2^{i-1}}{3^i}$ at each succeeding step. This means a set contained in $[0, 1]$ with “measure”

$$\sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} \leq \frac{1}{3} \sum_{i=0}^{\infty} \left( \frac{2}{3} \right)^i = \frac{1}{3} \frac{1}{(1 - 2/3)} = 1$$

14
is thrown out, and all that is left is $K$. Hence $K$ has in effect no length! ($K$ is what is left belonging to $[0,1]$ after a set of length 1 was thrown away). One surely would not want such a thing to have dimension 1 since it has no length. We shall see later in our discussion of Iterated Function Systems (IFS) that a number can be assigned as the dimension in such a way that it agrees with our usual notions but allows for objects to have fractional dimension (hence the name “fractal” although it is a misnomer since these dimensions are rarely rational). It will turn out that $K$ has dimension $\frac{\ln 2}{\ln 3}$ with this definition.

6.8 Other Cantor-like constructions

There is really nothing that special about the number three being used in defining the Cantor set, except perhaps that it is the simplest to work with. We can do similar constructions with other numbers. For example, instead of dividing the interval into three equal pieces, we divide it into five, and throw the middle fifth away. Of the remaining two intervals we repeat the process, etc. Or we could throw away the middle three fifths, and repeat, etc.

Exercise 6.12. What is the “length” of the set described above that is left by throwing away the middle fifths? What is its fractal dimension? Is this dimension larger or smaller than the dimension of $K$? Why should this be?

Exercise 6.13. Just like binary, ternary and decimal expansions, all points $x \in [0,1]$ have a fifth order expansion described as follows: associate with $x \in [0,1]$ a string $s_1s_2\ldots$ of digits where $s_i \in \{0,1,2,3,4\}$, and satisfy $x = \sum_{i=1}^{\infty} \frac{s_i}{5^i}$. Suppose we construct the set of points $x \in [0,1]$ who have a fifth order expansion that contains no 2’s. how does this set differ from the set constructed in the previous exercise? What is its “length”?

Although $K$ has no “length,” we can make a similar type of construction so that the amount thrown away does not add up to 1. Let $0 < \alpha < 1$, and if the middle $\alpha$ part is thrown out at each stage. If an interval $I$ has its length denoted by $|I|$, then throwing out the middle $\alpha$ part means throwing out an interval of length $\alpha|I|$ and leaves two intervals of length $|I| - \alpha|I| = |I|(1 - \alpha)$. The total amount thrown out is

$$\text{length removed} = \alpha + \frac{2}{\text{# intervals}} (1 - \alpha) \frac{\alpha}{2} + \frac{2^2}{\text{# intervals}} (1 - \alpha)^2 \frac{\alpha}{2^2} + \ldots$$

$$= \alpha \sum_{k=0}^{\infty} (1 - \alpha)^k = \alpha \frac{1}{1 - (1 - \alpha)} = 1.$$
though this set contains no interval. Notice now that if \( \alpha > K \), then \( x \rightarrow \infty \) for all \( x \) if \( c < 6.9 \) with the set of \( x \) that satisfy \( |x| > p_{+}(c) \), then \( x \rightarrow \infty \). Consider the case \( c < -2 \), and recall (see Section ??) that the set of \( x \) for which \( x_1 < -p_{+}(c) \) consists of the points where \( |x_0| < z_0(c) \), where \( z_0(c) \) was defined in (??). Hence in this case, \( x \rightarrow \infty \) and so \( x_0 \notin \Lambda_c \).

Let

\[ J_0 = [-p_{+}(c), -z_0(c)] \quad \text{and} \quad J_1 = [z_0(c), p_{+}(c)] \]

The disjoint intervals \( J_0 \) and \( J_1 \) play the same role for \( \Lambda_c \) that \([0, \frac{1}{3}] \) and \([\frac{2}{3}, 1] \) played for \( \mathcal{K} \). This is because \( x_0 \in \Lambda_c \) if and only if \( x_k \in J_0 \cup J_1 \) for all \( k = 1, 2, \ldots \). Why is that? Well, it is obvious that if \( x_k \in J_0 \cup J_1 \) for all \( k \), then \( x_k \rightarrow \infty \) because \( J_0 \) and \( J_1 \) are bounded intervals, and so \( x_0 \in \Lambda_c \). Conversely, if \( x_k \notin J_0 \cup J_1 \), then at least \( x_{k+2} > p_{+}(c) \) and the orbit will go to \( \infty \). Therefore, we state again that

\[ x_0 \in \Lambda_c \quad \iff \quad x_k \in J_0 \cup J_1 \quad \text{for all} \quad k = 0, 1, 2, \ldots \]  \( (11) \)

Now we can use (11) to represent elements in \( \Lambda_c \) in much the same way that ternary expansions were used in the analytical description of elements in \( \mathcal{K} \). We associate to every \( x_0 \in \Lambda_c \) a sequence \( x_0 \sim s_1s_2s_3 \ldots \) where each \( s_i = 0 \) or 1, and here is how:

\[ s_k = \begin{cases} 0 & \text{if } x_{k-1} \in J_0 \\ 1 & \text{if } x_{k-1} \in J_1. \end{cases} \]

We have just proved the following.

**Proposition 6.1.** For \( c < -2 \) the set \( \Lambda_c \) has the same cardinality as the sequence space that was introduced in Exercise 6.10. In particular, \( \Lambda_c \) is uncountable.
Exercise 6.14. Show \( \Lambda_c \) (for \( c < -2 \)) and the Cantor set \( K \) have the same cardinality; that is, show there exists a bijection between them.

Research Project: For a fixed value of \( c < -2 \), determine the “length” of the set that is thrown away in the construction of \( \Lambda_c \).

6.10 Summary of properties

We list some of the properties of the Cantor set \( K \).

1. \( K \) is closed: i.e. \( K \) is the complement of the union of open intervals. An interval \( I \subseteq \mathbb{R} \) is said to be open provided for every \( x \in I \), there exists \( \delta > 0 \) so that \( \{ y : |y - x| < \delta \} \subseteq I \).

2. \( K \) is of measure zero.

3. \( K \) has positive fractal dimension, and in fact has dimension \( \frac{\ln 2}{\ln 3} \).

4. \( K \) is uncountable.

5. \( K \) is totally disconnected.