SIAM J. SCI. COMPUT.
Vol. 35, No. 5, pp. A2370–A2392

THE WICK–MALLIAVIN APPROXIMATION OF ELLIPTIC PROBLEMS WITH LOG-NORMAL RANDOM COEFFICIENTS*

XIAOLIANG WAN† AND BORIS L. ROZOVSKII‡

Abstract. In this work, we discuss the approximation of elliptic problems with log-normal random coefficients using the Wick product and the Mikulevicius–Rozovskii formula. The main idea is that the multiplication between the log-normal coefficient and the gradient of the solution can be regarded as a Taylor-like expansion in terms of the Wick product and the Malliavin derivative. For the classical model, the coefficients of Wiener chaos expansion are fully coupled together in the uncertainty propagator, while the Wick–Malliavin model yields an uncertain propagator with weak coupling in the upper-triangular part for a relatively small truncation order in the Mikulevicius–Rozovskii formula. In this paper we focus on the difference between the classical model and the Wick–Malliavin model with respect to the standard deviation of the underlying Gaussian random process. Both theoretical and numerical discussions are presented.

Key words. Wiener chaos expansion, Wick product, stochastic elliptic PDE, uncertainty quantification

AMS subject classifications. 65C20, 65C30

DOI. 10.1137/130918605

1. Introduction.

Elliptic problems with random coefficients are of fundamental importance for the stochastic modeling of various physical and engineering applications (see, e.g., [19]). For instance, the permeability is often modeled by a log-normal random coefficient of the flow transport in the groundwater and oil recovery processes. With this in mind, let us consider the following equation:

$$\text{(1.1)} \qquad \text{Model I:} \quad \begin{cases} -\nabla\cdot(a(\boldsymbol{x},\omega)\nabla u(\boldsymbol{x},\omega)) = f(\boldsymbol{x}), & \boldsymbol{x}\in D, \\ u(\boldsymbol{x},\omega) = 0, & \boldsymbol{x}\in\partial D, \end{cases}$$

where $\ln a(\boldsymbol{x},\omega) = G(\boldsymbol{x},\omega)$ and $G(\boldsymbol{x},\omega)$ is a homogeneous Gaussian random process with the covariance kernel $\mathrm{cov}[G](\boldsymbol{x}_1,\boldsymbol{x}_2) = K(|\boldsymbol{x}_1-\boldsymbol{x}_2|)$. For the sake of simplicity, it is assumed in this paper that the force term is deterministic.

Mathematical analysis of (1.1) is somewhat delicate, due to the lack of uniform ellipticity. By this reason, the standard approach based on the Lax–Milgram lemma is not directly applicable. One way to deal with this problem is to consider solutions of (1.1) with appropriately weighted norms [14, 18, 7] or a weighted measure [21]. An elegant solution to the aforementioned problem based on the Fernique lower estimate for a general Gaussian kernel [5] was proposed in [3] and [4]. Another powerful approach to the analysis of (1.1), based on Wiener chaos expansion and Galerkin projection, was introduced in [1] and [6]; see also [14]. In the Wiener chaos approach, the analysis of the underlying equation is reduced to the analysis of a particular system

*Submitted to the journal's Methods and Algorithms for Scientific Computing section April 26, 2013; accepted for publication (in revised form) August 22, 2013; published electronically October 24, 2013.

http://www.siam.org/journals/sisc/35-5/91860.html

†Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803 (xlwan@math.lsu.edu). This author was supported by NSF grant DMS – 1115632.

‡Division of Applied Mathematics, Brown University, Providence, RI 02912 (rozovsky@dam.brown.edu). This author was supported by AFOSR grant FA9550-09-1-0613, ARO grant W911NF-13-1-0012, and NSF grant DMS-1148284.

of deterministic equations for the Cameron–Martin coefficients of the solution to this equation. This deterministic system is often referred to as an uncertainty propagator. Study of elliptic problems with other types of random coefficients can be found in [1, 8, 9], etc.

The numerical difficulties arising in the Wiener chaos approach are mainly due to the fact that the stiffness matrix is large and dense, which implies that an efficient preconditioner is required. To deal with this fundamental problem, some attempts were made to revise Model I (1.1). A popular strategy is to replace the flux $a\nabla u$ by $a \diamond \nabla u$,where $\diamond$ stands for the Wick product [10, 22, 25]. The Wick product can be interpreted as a stochastic convolution. In particular, the Skorokhod (anticipating) stochastic integral could be viewed as Wick product with respect to white Gaussian noise; see [15] for more details. Another Wick-type model was introduced in [26, 27], where the flux was modeled by $\left(a^{-1}\right)^{\diamond(-1)}$. When the standard deviation σ of the underlying Gaussian process $\ln a(\boldsymbol{x}, \omega)$ is less than 1, both Wick-type models provide $O(\sigma^2)$-approximations of Model I. Also, the second Wick-type model has the same solution as Model I for "spatially independent noise," i.e., the correlation length of the underlying Gaussian noise goes to infinity. The Wiener chaos expansion of both Wick-type models can be approximated efficiently because, in contrast to the fully coupled system for Model I, the uncertainty propagators of both Wick-type models correspond to a lower-triangular system.

In this work, we consider a strategy for sequential improvement and generalization of Wick-type approximations based on the Mikulevicius–Rozovskii (M-R) formula [16]. Providing proper stochastic smoothness, the M-R formula shows that the product of two functions of a Gaussian vector, say, X and Y, has the following Taylor-like expansion:

$$XY = X \diamond Y + \sum_{n=1}^{\infty} \frac{\mathcal{D}^n X \diamond \mathcal{D}^n Y}{n!}, \tag{1.2}$$

where $\mathcal{D}$ stands for Malliavin derivative [17]. In particular, formula (1.2) indicates that $X \diamond Y$ is the lowest (zeroth-order) approximation of the standard product XY. We will apply the M-R formula to approximate the nonlinear term $a\nabla u$ in (1.1) by adding high-order terms $(n!)^{-1}\mathcal{D}^n a \diamond \nabla \mathcal{D}^n u$ to the Wick-type model. By doing so, we will (sequentially) upgrade the corresponding uncertainty propagator (at a fraction of the computational cost) and reduce the approximation errors. The inclusion of high-order terms will introduce nonzero components into the upper-triangular part of the uncertainty propagator but the coupling in the upper-triangular part will be weak if the truncation order in the M-R formula is relatively small. For instance, for the one-dimensional case a kth-order truncation adds only k nonzero upper-diagonal lines. The focus of this paper is the comparison of the new Wick–Malliavin model and the original model (1.1).

The paper is organized as follows. In section 2 we describe our problem. The new Wick–Malliavin model based on the M-R formula is introduced in section 3, where, for simplicity and clarity, we focus on the first-order approximation as $a\nabla u \approx a \diamond \nabla u + \mathcal{D}^1 a \diamond \nabla \mathcal{D}^1 u$. In section 4 we generalize the the setting of section 3 to high-order Wick–Malliavin approximation of Model I. Numerical results are given in section 5, followed by a summary.

2. Problem description. Let $G(\boldsymbol{x}, \omega)$ be a homogeneous Gaussian random process on the complete probability space $\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P})$ with the covariance kernel $K \in \mathcal{C}^{0,1}(\mathbb{R}, \mathbb{R})$ and let $\ln a(\boldsymbol{x}, \omega) = G(\boldsymbol{x}, \omega)$.

Then the trajectories of a are almost surely (a.s.) continuous on the compact domain $\bar{D}$, which implies that one can define the following random variables:

$$a_{\min} = \min_{\boldsymbol{x}\in\bar{D}} a(\boldsymbol{x},\omega), \quad a_{\max} = \max_{\boldsymbol{x}\in\bar{D}} a(\boldsymbol{x},\omega).$$

Since $0 < a_{\min}(\omega) < a(\boldsymbol{x},\omega) < a_{\max}(\omega) < \infty$ holds a.s., we have that equation

$$\int_D a(\boldsymbol{x},\omega)\nabla u\cdot\nabla v d\boldsymbol{x} = \int_D f v d\boldsymbol{x} \quad \forall v\in H_0^1(D) \tag{2.1}$$

admits a unique solution. Furthermore, the following estimate holds a.s.:

$$\|u(\boldsymbol{x},\omega)\|_{H_0^1(D)} \le C_D \frac{\|f\|_{H^{-1}(D)}}{a_{\min}(\omega)}, \tag{2.2}$$

where C_D is the Poincaré constant. It follows now (from the Fernique theorem [5, 20]) that $a_{\min}^{-1}(\omega)\in L_p(\mathbb{F})$, $p>0$. Therefore $u(\boldsymbol{x},\omega)\in L_p(\mathbb{F};H_0^1(D))$, $p>0$, where

$$\|u\|_{L_p(\mathbb{F};H_0^1(D))} = \mathbb{E}\left[\|u\|_{H_0^1(D)}^p\right]^{1/p}. \tag{2.3}$$

We will limit our considerations to the setting with $p=2$. More information along these lines can be found in [3].

For numerical computations, the Gaussian process $G(\boldsymbol{x},\omega)$ is usually approximated by a truncated Karhunen–Loeve expansion

$$G(\boldsymbol{x},\omega) \approx G_N(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i=1}^N \sigma\sqrt{\lambda_i}\phi_i(\boldsymbol{x})\xi_i, \tag{2.4}$$

where the random vector $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_N)\in\mathbb{R}^N$ consists of independent Gaussian random variables with zero mean and unit variance, $(\lambda_i,\phi_i(\boldsymbol{x}))$ are the eigen-pairs of the covariance kernel $\mathrm{cov}[G](\boldsymbol{x}_1,\boldsymbol{x}_2)$, and σ characterizes the standard deviation of $G(\boldsymbol{x},\omega)$.

Then problem (1.1) can be approximated by

$$\begin{cases} -\nabla\cdot(\hat{a}_N(\boldsymbol{x},\boldsymbol{\xi})\nabla u_N(\boldsymbol{x},\boldsymbol{\xi})) = f(\boldsymbol{x}), & \boldsymbol{x}\in D,\\ u_N(\boldsymbol{x},\boldsymbol{\xi}) = 0, & \boldsymbol{x}\in\partial D, \end{cases} \tag{2.5}$$

where $\ln\hat{a}_N(\boldsymbol{x},\boldsymbol{\xi}) = G_N(\boldsymbol{x},\boldsymbol{\xi})$. The strong and the weak convergence of u_N to u as $N\to\infty$ were studied in [3, 4]. The weak form of (2.5) is given by

$$\int_D \mathbb{E}\left[\hat{a}_N(\boldsymbol{x},\boldsymbol{\xi})\nabla u\cdot\nabla v\right]d\boldsymbol{x} = \int_D f v d\boldsymbol{x} \tag{2.6}$$

for an appropriate set of test functions v. Since $-\nabla\cdot(\hat{a}_N(\boldsymbol{x},\boldsymbol{\xi})\nabla u_N)$ is not a strongly elliptic operator, the choice of the spaces where u_N converges to u is an important and delicate problem. In [18], the existence and uniqueness of the solution was discussed from the white noise analysis (see, e.g., [11]) point of view and established in a weighted version of $L_2(\mathbb{F};H_0^1(D))$. Motivated by the estimate (2.2), this idea was generalized in [21], where a regular $L_2(\mathbb{F};H_0^1(D))$ space was chosen for the solution and a weighted probabilistic space $L_2(\Omega,\mathcal{F},a_{\min}^{-1}(\omega)\mathbb{P}(d\omega))\otimes H_0^1(D)$ for the test functions.

The difficulties for numerical approximation of problem (2.5) are twofold. First, if one employs the analytical results from [18, 21], then $L_2(\mathbb{F}; H_0^1(D))$ is not an appropriate space for test functions. Second, if one takes $L_2(\mathbb{F}; H_0^1(D))$ as the test space and the Wiener chaos basis, then, although no divergence with respect to the $L_2(\mathbb{F}; H_0^1(D))$ norm has been numerically observed, the stiffness matrix is full and dense, which would make numerical computations very challenging. To make this point more transparent, let us consider the truncated Wiener chaos expansion $u = \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{N,p}} u_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}$ (see Theorem 3.1) and implement the Galerkin projection in the probability space, which results in the following semidiscrete form:

$$-\sum_{\boldsymbol{\alpha}\in\mathcal{J}_{N,p}} \nabla\cdot\left(\mathbb{E}\left[\hat{a}_N(\boldsymbol{x},\boldsymbol{\xi})\mathrm{H}_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\gamma}}\right]\nabla u_{\boldsymbol{\alpha}}\right) = f(\boldsymbol{x})1_{\boldsymbol{\gamma}=0} \quad \forall\boldsymbol{\gamma}\in\mathcal{J}_{N,p}. \tag{2.7}$$

Here $\mathcal{J}_{N,p} = \{\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_N)||\boldsymbol{\alpha}| \le p,\ \boldsymbol{\alpha}\in\mathbb{N}_0^N\}$ is a set of multi-indices. It is easy to see that the coefficients $u_{\boldsymbol{\alpha}}$ are fully coupled and for any $\boldsymbol{x}\in D$, $\mathbb{E}\left[\hat{a}_N(\boldsymbol{x},\boldsymbol{\xi})\mathrm{H}_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\gamma}}\right]$ is a full and dense symmetric matrix with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$.

3. M-R formula.

3.1. Preliminaries. Consider a spatial white noise

$$\dot{W} = \sum_{k\ge1}\dot{W}(\mathfrak{u}_k)\mathfrak{u}_k \tag{3.1}$$

on the probability space $\mathbb{F}$ taking values in Hilbert space $\mathcal{U} = L_2(D)$, where $\{\mathfrak{u}_k\}_{k\ge1}$ is a complete orthonormal basis of $\mathcal{U}$. For $h(\boldsymbol{x})\in\mathcal{U}$, $\dot{W}(h)$ defines a zero-mean Gaussian random variable satisfying

$$\mathbb{E}\left[\dot{W}(h_1),\dot{W}(h_2)\right] = (h_1,h_2)_{\mathcal{U}}, \quad h_1,h_2\in\mathcal{U}, \tag{3.2}$$

where $(\cdot,\cdot)_{\mathcal{U}}$ is the inner product defined on $\mathcal{U}$. Thus $\dot{W}$ can be rewritten as $\dot{W} = \sum_{k\ge1}\mathfrak{u}_k\xi_k$, where ξ_k are independent Gaussian random variables with zero mean and unit variance. Denote by $\mathcal{J}$ the collection of multi-indices $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\ldots)$ such that each $\alpha_k\in\mathbb{N}_0$ and $|\boldsymbol{\alpha}| := \sum_{k\ge1}\alpha_k < \infty$. Define a collection of random variables $\Xi = \{\mathrm{h}_{\boldsymbol{\alpha}}, \boldsymbol{\alpha}\in\mathcal{J}\}$ as follows:

$$\mathrm{h}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{k\ge1}\frac{1}{\sqrt{\alpha_k!}}\mathrm{H}_{\alpha_k}(\xi_k),$$

where $\boldsymbol{\xi} = (\xi_1,\xi_2,\ldots)$ and $\mathrm{H}_{\alpha_k}(\xi_k)$ are one-dimensional Hermite polynomials of order α_k. Recall the following results.

THEOREM 3.1 (Cameron and Martin [2]). *The set* Ξ *is an orthonormal basis in* $L_2(\mathbb{F})$*: if* $\eta\in L_2(\mathbb{F})$*, then* $\eta = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\eta_{\boldsymbol{\alpha}}\mathrm{h}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\eta_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})/\sqrt{\boldsymbol{\alpha}!}$, $\eta_{\boldsymbol{\alpha}} = \mathbb{E}[\eta\mathrm{h}_{\boldsymbol{\alpha}}]$*, and* $\mathbb{E}[\eta^2] = \sum_{\boldsymbol{\alpha}\in\mathcal{J}}\eta_{\boldsymbol{\alpha}}^2$.

The Cameron–Martin expansion is often called Wiener chaos expansion or Fourier–Hermite expansion. For convenience, in this paper we will mostly consider Wiener chaos expansion in terms of $\{\mathrm{H}_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\mathcal{J}}$. Denote by $\mathcal{D}$ the Malliavin derivative on $L_2(\mathbb{F})$ [17]. For $\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, $\mathcal{D}$ is defined as

$$\mathcal{D}\mathrm{H}_{\boldsymbol{\alpha}} = \sum_{k\ge1}\alpha_k\mathrm{H}_{\boldsymbol{\alpha}-\boldsymbol{\varepsilon}_k}\mathfrak{u}_k = \sum_{\boldsymbol{\beta}\in\mathcal{J}}\sum_{\boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\varepsilon}_k}\alpha_k\mathfrak{u}_k\mathrm{H}_{\boldsymbol{\beta}}\in L_2(\mathbb{F};\mathcal{U}), \tag{3.3}$$

where $\boldsymbol{\varepsilon}_k$ is a multi-index such that only the kth component is nonzero and equal to 1. It is seen that the Malliavin derivative of $\mathrm{H}_{\boldsymbol{\alpha}}$ is a random element on $\mathcal{U}$. By induction, one can show that the nth-order Malliavin derivative is given by

$$\mathcal{D}^n \mathrm{H}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} \left(\sum_{|\boldsymbol{\gamma}|=n} 1_{\boldsymbol{\gamma}+\boldsymbol{\beta}=\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!} \mathfrak{u}^{(\boldsymbol{\gamma})} \right) \mathrm{H}_{\boldsymbol{\beta}}, \tag{3.4}$$

where $\mathfrak{u}^{(\boldsymbol{\gamma})} = \sum_{k_1,k_2,\dots,k_n} \sum_{\boldsymbol{\varepsilon}_{k_1}+\cdots+\boldsymbol{\varepsilon}_{k_n}=\boldsymbol{\gamma}} \mathfrak{u}_{k_1} \otimes \cdots \otimes \mathfrak{u}_{k_n} \in \mathcal{U}^{\otimes n}$.

The Wick product can be defined as follows:

$$\mathrm{h}_{\boldsymbol{\alpha}} \diamond \mathrm{h}_{\boldsymbol{\beta}} := \sqrt{\frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!\boldsymbol{\beta}!}} \mathrm{h}_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \quad \text{or} \quad \mathrm{H}_{\boldsymbol{\alpha}} \diamond \mathrm{H}_{\boldsymbol{\beta}} = \mathrm{H}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \tag{3.5}$$

For random elements $f = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}$ and $g = \sum_{\boldsymbol{\alpha}} g_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}$ with $f_{\boldsymbol{\alpha}}, g_{\boldsymbol{\alpha}} \in \mathcal{U}$, the Wick product is defined as

$$f \diamond g = \sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta} \le \boldsymbol{\alpha}} (g_{\boldsymbol{\beta}}, f_{\boldsymbol{\alpha}-\boldsymbol{\beta}})_{\mathcal{U}} \mathrm{H}_{\boldsymbol{\alpha}}, \tag{3.6}$$

which is a random variable on $\mathbb{R}$.

Next we shall recall two versions of the M-R formula presented in [16].

LEMMA 3.2. *Let* $\mathrm{h}_{\boldsymbol{\alpha}}$ *and* $\mathrm{h}_{\boldsymbol{\beta}}$ *be elements of the Cameron–Martin basis. Then with probability* 1

$$\mathrm{h}_{\boldsymbol{\alpha}} \mathrm{h}_{\boldsymbol{\beta}} = \sum_{n=0}^{\infty} \frac{\mathcal{D}^n \mathrm{h}_{\boldsymbol{\alpha}} \diamond \mathcal{D}^n \mathrm{h}_{\boldsymbol{\beta}}}{n!} \text{ and } \mathrm{H}_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\beta}} = \sum_{n=0}^{\infty} \frac{\mathcal{D}^n \mathrm{H}_{\boldsymbol{\alpha}} \diamond \mathcal{D}^n \mathrm{H}_{\boldsymbol{\beta}}}{n!}. \tag{3.7}$$

If $|\boldsymbol{\alpha}| \le Q$ *or* $|\boldsymbol{\beta}| \le Q$, *then*

$$\mathrm{h}_{\boldsymbol{\alpha}} \mathrm{h}_{\boldsymbol{\beta}} = \sum_{n=0}^{Q} \frac{\mathcal{D}^n \mathrm{h}_{\boldsymbol{\alpha}} \diamond \mathcal{D}^n \mathrm{h}_{\boldsymbol{\beta}}}{n!} \text{ and } \mathrm{H}_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\beta}} = \sum_{n=0}^{Q} \frac{\mathcal{D}^n \mathrm{H}_{\boldsymbol{\alpha}} \diamond \mathcal{D}^n \mathrm{H}_{\boldsymbol{\beta}}}{n!}.$$

Proof. This result follows from the definition of the Malliavin derivative and the M-R formula (1.2). ∎

Remark 3.3. It is readily seen that if $X, Y \in L_2(\mathbb{F})$ and the Wiener chaos expansion for at least one of the elements of the product converges sufficiently fast, then we have

$$XY = \sum_{n=0}^{\infty} \frac{\mathcal{D}^n X \diamond \mathcal{D}^n Y}{n!}, \tag{3.8}$$

where the convergence is in L_1. In other words, the regular product of two general random elements can also have the M-R expansion provided some smoothness in the probability space.

Remark 3.4. It is known that the Wick product is related to the Skorokhod stochastic integral. (The Skorokhod integral is a generalization of Itô integral to the integrands not adapted to the filtration provided by the driving Brownian motion.) Thus, it is more appropriate to think of the Wick product as a convolution in the probability space. The M-R formula implies that the regular product can be approximated by the Wick products using a Taylor-like expansion in terms of Malliavin derivative.

3.2. Wick–Malliavin approximation of Model I. A popular modification of Model I (1.1) is represented by the following Wick-type stochastic model:

$$-\nabla \cdot (a(\boldsymbol{x}, \omega) \diamond \nabla u) = f(\boldsymbol{x}), \tag{3.9}$$

where the regular product in the nonlinear term is replaced by the Wick product. The motivation for the introduction of this model is related to the fact that (3.9) is an "unbiased" model. That is, $\bar{u} := \mathbb{E}[u]$ solves the deterministic equation

$$-\nabla \cdot (\mathbb{E}\left[a(\boldsymbol{x}, \omega)\right] \diamond \nabla \bar{u}) = f(\boldsymbol{x}).$$

Certainly, the absence of the statistical bias is very desirable in stochastic modeling. Another consideration is that since the Wick product is a stochastic convolution rather than a product, solving this model by Wiener chaos expansion is easier.

Originally, model (3.9) was investigated analytically and numerically in [23, 22]. The approach taken in these papers is based on Wiener chaos expansion of the solution. It leads to the reduction of the stochastic problem to the system of deterministic equations for the Wiener chaos expansion coefficients. This system is often referred to as the *propagator*. It was shown that the solution of (3.9) belongs to the scale of Kondratiev spaces $\mathcal{S}_{\rho,q}(H_0^1(D))$ defined by the norm

$$\|u\|_{\rho,q}^2 := \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \|u_{\boldsymbol{\alpha}}\|_{H_0^1(D)}^2 (\boldsymbol{\alpha}!)^{1+\rho} (2\mathbb{N})^{q\boldsymbol{\alpha}}, \tag{3.10}$$

where $u_{\boldsymbol{\alpha}}$ is given by the Wiener chaos expansion $u = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} u_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}$. The existence and uniqueness of the solution of (3.9) follow from the Lax–Milgram lemma on the Hilbert space $\mathcal{S}_{-1,q}(H_0^1(D))$. Stochastic finite element method was was also applied in this setting by several authors [23, 22].

The most important property of the Wick-type modeling is that it reduces the computations to solving a *lower-triangular system* of deterministic equations for the Wiener chaos expansion coefficients. More specifically, for the truncated Wiener chaos expansion $u = \sum_{\boldsymbol{\alpha} \in \mathcal{J}_{N,p}} u_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}$, where $\mathcal{J}_{N,p}$ is defined as in (2.7), the Galerkin projection in the probability space yields

$$-\sum_{\boldsymbol{\alpha} \le \boldsymbol{\gamma}} \nabla \cdot (a_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}(\boldsymbol{x}) \nabla u_{\boldsymbol{\alpha}}(\boldsymbol{x})) = f(\boldsymbol{x}) 1_{\boldsymbol{\gamma}=0} \quad \forall \boldsymbol{\gamma} \in \mathcal{J}_{N,p}, \tag{3.11}$$

where $a_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}(\boldsymbol{x})$ are the WCE coefficients of $\hat{a}_N(\boldsymbol{x}, \boldsymbol{\omega})$. It is seen that in (3.11), $u_{\boldsymbol{\gamma}}$ depends only on $u_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} < \boldsymbol{\gamma}$. In other words, $u_{\boldsymbol{\gamma}}$ can be solved sequentially (one by one). In contrast, the uncertainty propagator of Model I leads to a system of fully coupled deterministic PDEs. Based on the solution of the uncertainty propagator, an appropriate weighted Wiener chaos space $\mathcal{R}L_2(\mathbb{F}; H_0^1(D))$ [13] can be defined with respect to the norm

$$\|u\|_{\mathcal{R}L_2(H_0^1(D))}^2 = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \|u_{\boldsymbol{\alpha}}\|_{H_0^1(D)}^2 \boldsymbol{\alpha}! r_{\boldsymbol{\alpha}}^2, \tag{3.12}$$

where $\{r_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathcal{J}}$ is a set of uniformly bounded positive numbers. Obviously, by the choice of $r_{\boldsymbol{\alpha}}$, the weighted Wiener chaos space and the Kondratiev space can be made consistent with each other. The numerical analysis of the Wick-type model (3.9) using the weighted Wiener chaos space can be found in [14, 25].

Although the Wick-type model (3.9) can be approximated more efficiently, it is a different elliptic model. In fact, it is actually a second-order approximation of Model I with respect to the standard deviation σ [26, 27]. Another Wick-type model

$$-\nabla\cdot\left((a^{-1})^{\diamond(-1)}\diamond\nabla u\right)=f(\boldsymbol{x}) \tag{3.13}$$

was given in [26, 27], which is also a second-order approximation of Model I with respect to σ. However, it provides a much better approximation than (3.9), since the multiplier of σ^2 is smaller and goes to zero as the correlation length increases.

In this work, we focus on how to approximate Model I sequentially using the Wick product based on the M-R formula. Applying the M-R formula to approximate the flux, Model I can be rewritten as

$$-\nabla\cdot\left(\sum_{n=0}^{\infty}\frac{\mathcal{D}^n a(\boldsymbol{x},\omega)\diamond\nabla\mathcal{D}^n u}{n!}\right)=f(\boldsymbol{x}). \tag{3.14}$$

When $n=0$, (3.14) recovers the Wick-type model (3.9). Note that if we consider the truncated Wiener chaos expansion

$$a(\boldsymbol{x},\omega)\nabla u\approx\widehat{a\nabla u}=\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}_{N,p}}a_{\boldsymbol{\alpha}}\nabla u_{\boldsymbol{\beta}}\mathrm{H}_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\beta}},$$

then $\widehat{a\nabla u}$ can be fully recovered (at least as long as $\mathcal{J}_{N,p}<\infty$.). From now on, we will refer to (3.14) as the Wick–Malliavin approximation of Model I. For clarity, we first focus on the following first-order approximation:

$$\text{Model II:}\quad\begin{cases}-\nabla\cdot(a\diamond\nabla u+\mathcal{D}a\diamond\nabla\mathcal{D}u)=f(\boldsymbol{x}), & \boldsymbol{x}\in D,\\ \qquad\qquad\qquad\qquad u(\boldsymbol{x},\omega)=0, & \boldsymbol{x}\in\partial D,\end{cases} \tag{3.15}$$

where only the first-order Malliavin derivative is included. Later on in section 4, we consider a more general case with up to Qth-order Malliavin derivatives included. The application of the M-R formula to other types of PDEs can be found in [24]. From now on, we use u^{I} and u^{II} to indicate the solutions of Models I and II, respectively.

Remark 3.5. It is seen that the Malliavin derivative is defined with respect to the white noise $\dot{W}=\sum_{k\geq1}\mathrm{u}_k\xi_k$ while the underlying Gaussian random process $\ln a(\boldsymbol{x},\omega)$ is colored noise. Since the colored Gaussian random process can be regarded as a smoothed white noise [10], the M-R formula is well defined for our problem.

Remark 3.6. To verify that the Wick–Malliavin model (3.14) converges to Model I, the regularity of the solution needs to be examined, which is beyond the scope of this paper.

3.3. Uncertainty propagator of Model II. We now derive the uncertainty propagator of Model II. For $\mathrm{H}_{\boldsymbol{\alpha}}$, we have

$$\mathcal{D}\mathrm{H}_{\boldsymbol{\alpha}}=\sum_{\boldsymbol{\theta}}\sum_{k\geq1}1_{\boldsymbol{\varepsilon}_k+\boldsymbol{\theta}=\boldsymbol{\alpha}}\alpha_k\mathrm{u}_k\mathrm{H}_{\boldsymbol{\theta}}.$$

Thus,

$$\begin{aligned}\mathcal{D}\mathrm{H}_{\boldsymbol{\alpha}} \diamond \mathcal{D}\mathrm{H}_{\boldsymbol{\beta}} &= \sum_{\boldsymbol{\theta},\boldsymbol{\kappa}} \sum_{k\geq 1} 1_{\boldsymbol{\varepsilon}_k+\boldsymbol{\theta}=\boldsymbol{\alpha}} 1_{\boldsymbol{\varepsilon}_k+\boldsymbol{\kappa}=\boldsymbol{\beta}} \alpha_k \beta_k \mathrm{H}_{\boldsymbol{\theta}+\boldsymbol{\kappa}} \\ &= \sum_{\boldsymbol{\theta}} \sum_{\boldsymbol{\kappa}\leq\boldsymbol{\theta}} \sum_{k\geq 1} 1_{\boldsymbol{\varepsilon}_k+\boldsymbol{\theta}-\boldsymbol{\kappa}=\boldsymbol{\alpha}} 1_{\boldsymbol{\varepsilon}_k+\boldsymbol{\kappa}=\boldsymbol{\beta}} \alpha_k \beta_k \mathrm{H}_{\boldsymbol{\theta}} \\ &= \sum_{\boldsymbol{\theta}} c_{\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\beta}} \mathrm{H}_{\boldsymbol{\theta}},\end{aligned}$$

where the coefficient $c_{\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\beta}}$ depends on $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\theta}$. Note that $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| = |\boldsymbol{\theta}| + 2$. Therefore, with respect to the Wiener chaos expansions of a and u^{II}

$$\begin{aligned}\mathcal{D}a \diamond \nabla\mathcal{D}u^{\mathsf{II}} &= \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}} a_{\boldsymbol{\alpha}} \nabla u^{\mathsf{II}}_{\boldsymbol{\beta}} \mathcal{D}\mathrm{H}_{\boldsymbol{\alpha}} \diamond \mathcal{D}\mathrm{H}_{\boldsymbol{\beta}} \\ &= \sum_{\boldsymbol{\theta}} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}} c_{\boldsymbol{\theta},\boldsymbol{\alpha},\boldsymbol{\beta}} a_{\boldsymbol{\alpha}} \nabla u^{\mathsf{II}}_{\boldsymbol{\beta}} \mathrm{H}_{\boldsymbol{\theta}}.\end{aligned}$$

Hence, the uncertainty propagator of Model II takes the form

$$-\sum_{\boldsymbol{\alpha}\leq\boldsymbol{\gamma}} \nabla\cdot\left(a_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\alpha}}\right) - \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}} c_{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}} \nabla\cdot\left(a_{\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\beta}}\right) = f(\boldsymbol{x})1_{\boldsymbol{\gamma}=0} \;\; \forall\boldsymbol{\gamma}\in\mathcal{J}. \tag{3.16}$$

Due to the appearance of the first-order Malliavin derivative, the system (3.16) for the chaos coefficients $u^{\mathsf{II}}_{\boldsymbol{\alpha}}$, is not lower-triangular any more. The coupling is due to the coefficient

$$c_{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\boldsymbol{\kappa}\leq\boldsymbol{\gamma}} \sum_{k\geq 1} 1_{\boldsymbol{\varepsilon}_k+\boldsymbol{\gamma}-\boldsymbol{\kappa}=\boldsymbol{\alpha}} 1_{\boldsymbol{\varepsilon}_k+\boldsymbol{\kappa}=\boldsymbol{\beta}} \alpha_k \beta_k.$$

The coefficient $u_{\boldsymbol{\gamma}}$ is coupled with N higher-order terms $u_{\boldsymbol{\varepsilon}_k+\boldsymbol{\gamma}}$, which are located in the upper-triangular part of the system. Considering that there exist $\frac{(N+|\boldsymbol{\gamma}|)!}{(N-1)!(|\boldsymbol{\gamma}|+1)!}$ terms of polynomial order $|\boldsymbol{\gamma}|+1$, the upper-triangular part of the system is sparse.

The mean of the solution of (3.9) solves a simple elliptic PDE

$$-\nabla\cdot(\mathbb{E}[a]\nabla\mathbb{E}[u]) = f(\boldsymbol{x})$$

in contrast to the equation for $\mathbb{E}[u^{\mathsf{II}}]$,

$$-\nabla\cdot\left(\mathbb{E}[a]\nabla\mathbb{E}[u^{\mathsf{II}}] + \mathbb{E}[\mathcal{D}a]\nabla\mathbb{E}[\mathcal{D}u^{\mathsf{II}}]\right) = f(\boldsymbol{x}),$$

which is not closed.

3.4. Comparison of Models I and II via the variance of $\ln a(\boldsymbol{x},\omega)$. We now look at the difference between Models I and II with respect to the standard deviation σ of $G(\boldsymbol{x},\omega)$. We choose the random coefficient

$$a(\boldsymbol{x},\omega) = e^{\diamond G(\boldsymbol{x},\omega)} = e^{G(\boldsymbol{x},\omega)-\frac{1}{2}\sigma^2}. \tag{3.17}$$

Then, the random coefficient $a(\boldsymbol{x},\boldsymbol{\xi})$ has the following Wiener chaos expansion:

$$a(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{\sigma^{|\boldsymbol{\alpha}|}\Phi^{\boldsymbol{\alpha}}(\boldsymbol{x})}{\boldsymbol{\alpha}!}\mathrm{H}_{\boldsymbol{\alpha}}, \tag{3.18}$$

where $\Phi(\boldsymbol{x}) = (\sqrt{\lambda_1}\phi_1(\boldsymbol{x}), \ldots)$ with $(\lambda_i, \phi_i(\boldsymbol{x}))$ being the eigen-pairs of the covariance kernel of $G(\boldsymbol{x}, \omega)$. We assume that $\lambda_1 \geq \lambda_2 \geq \cdots > 0$. In our numerical approximation, we keep only the first N Gaussian random variables and define

$$a_N(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathcal{J}_N} \frac{\sigma^{|\boldsymbol{\alpha}|}\Phi^{\boldsymbol{\alpha}}(\boldsymbol{x})}{\boldsymbol{\alpha}!} \mathrm{H}_{\boldsymbol{\alpha}}, \tag{3.19}$$

where $\mathcal{J}_N = \{\boldsymbol{\alpha} | \boldsymbol{\alpha} \in \mathbb{N}_0^N, |\boldsymbol{\alpha}| < \infty\}$. Note that $a_N(\boldsymbol{x}, \boldsymbol{\xi})$ is slightly different from $e^{G_N(\boldsymbol{x}, \boldsymbol{\xi}) - \frac{1}{2}\sigma^2}$ with G_N being the truncated Karhunen–Loeve expansion of $G(\boldsymbol{x}, \omega)$. Actually

$$a_N(\boldsymbol{x}, \boldsymbol{\xi}) = e^{G_N(\boldsymbol{x}, \boldsymbol{\xi}) - \frac{1}{2}\sigma^2 \sum_{i=1}^N \lambda_i \phi_i^2(\boldsymbol{x})}. \tag{3.20}$$

Note that

$$\mathrm{cov}[G](\boldsymbol{x}, \boldsymbol{x}) = K(0) = \sum_{i=1}^{\infty} \lambda_i \phi_i^2(\boldsymbol{x}) = 1. \tag{3.21}$$

When N is large enough, $a_N(\boldsymbol{x}, \boldsymbol{\xi})$ is also a good approximation of $a(\boldsymbol{x}, \omega)$ defined in (3.17). In fact, we choose the approximation $a_N(\boldsymbol{x}, \boldsymbol{\xi})$ only for simplicity; see Remark 3.11 for more comments.

Define $\mathcal{L}^{\mathsf{I}} u^{\mathsf{I}} = a_N \nabla u^{\mathsf{I}}$ and $\mathcal{L}^{\mathsf{II}} u^{\mathsf{II}} = a_N \diamond u^{\mathsf{II}} + \mathcal{D} a_N \diamond \mathcal{D} \nabla u^{\mathsf{II}}$, where the operators $\mathcal{L}^{\mathsf{I}}$ and $\mathcal{L}^{\mathsf{II}}$ characterize the fluxes of Models I and II, respectively. Then, $u^{\mathsf{I}} - u^{\mathsf{II}}$ solves the equation

$$-\nabla \cdot \left(\mathcal{L}^{\mathsf{I}}(u^{\mathsf{I}} - u^{\mathsf{II}})\right) = \nabla \cdot \left(\left(\mathcal{L}^{\mathsf{I}} - \mathcal{L}^{\mathsf{II}}\right) u^{\mathsf{II}}\right). \tag{3.22}$$

It is readily seen that the difference between u^{I} and u^{II} is determined by the difference between $\mathcal{L}^{\mathsf{I}}$ and $\mathcal{L}^{\mathsf{II}}$ with respect to u^{II}.

We now look at the right-hand side of (3.22), where the solution u^{II} needs to be understood in more detail. For clarity, we first consider the one-dimensional case, i.e., $N = 1$. Then, $a_N(\boldsymbol{x}, \boldsymbol{\xi})$ is defined as follows:

$$a_{N=1}(\boldsymbol{x}, \xi_1) = \sum_{n=0}^{\infty} a_n \mathrm{H}_n(\xi_1) = \sum_{n=0}^{\infty} \frac{\sigma^n \Phi_1^n(\boldsymbol{x})}{n!} \mathrm{H}_n(\xi_1). \tag{3.23}$$

Assume that u_n^{II} can be expanded in a power series with respect to σ when σ is small enough. The scaling of the chaos coefficients of u^{II}, with respect to σ, is given by the following result.

PROPOSITION 3.7. *For Model* II*, the Wiener chaos expansion* $u^{\mathsf{II}} = \sum_{n=0}^{p} u_n^{\mathsf{II}} \mathrm{H}_n(\xi_1)$ *with one-dimensional random coefficient given by* (3.23) *admits the following asymptotics:*

$$u_n^{\mathsf{II}} \sim \sum_{k=0}^{\infty} O(\sigma^{n+2k}) \sim O(\sigma^n), \quad n = 0, \ldots, p, \tag{3.24}$$

where $p \geq 1$.

Proof. The uncertainty propagator for u^{II} is given by

$$-\sum_{i \leq n} \nabla \cdot (a_{n-i} \nabla u_i^{\mathsf{II}}) - \sum_{i \geq 1, j \geq 1} c_{n,i,j} \nabla \cdot (a_i \nabla u_j^{\mathsf{II}}) = f(\boldsymbol{x}) 1_{n=0}, \tag{3.25}$$

where $n = 0, \ldots, p$, $j = 0, \ldots, p$, and $i + j = n + 2$. Let

$$u_n^{\text{II}} = \sum_{m=0}^{\infty} u_{n,m}^{\text{II}} \sigma^m, \quad a_i = \hat{a}_i \sigma^i,$$

where we use the fact that $a_i = O(\sigma^i)$. Then,

$$-\sum_{m=0}^{\infty} \left(\sum_{i \le n} \nabla \cdot (\hat{a}_{n-i} \nabla u_{i,m}^{\text{II}}) \sigma^{n-i+m} - \sum_{i \ge 1, j \ge 1} c_{n,i,j} \nabla \cdot (\hat{a}_i \nabla u_{j,m}^{\text{II}}) \sigma^{m+i} \right) = f(\boldsymbol{x}) 1_{n=0}. \tag{3.26}$$

Next, we shall compare the coefficients of σ^i on both sides of the above equation. The coefficient $u_{n,0}^{\text{II}}$, corresponding to σ^0, is defined by the equation

$$-\nabla \cdot (\hat{a}_0 \nabla u_{n,0}^{\text{II}}) = f(\boldsymbol{x}) 1_{n=0}. \tag{3.27}$$

Since $\hat{a}_0 = a_0 = \mathbb{E}[a] > 0$, then

$$u_{0,0}^{\text{II}} \neq 0, \; u_{n,0}^{\text{II}} = 0, \quad n = 1, \ldots, p.$$

For $u_{n,1}^{\text{II}}$ corresponding to σ^1, the following equation holds:

$$-\nabla \cdot (\hat{a}_0 \nabla u_{n,1}^{\text{II}}) = \nabla \cdot (\hat{a}_1 \nabla u_{n-1,0}^{\text{II}}) + c_{n,1,n+1} \nabla \cdot (\hat{a}_1 \nabla u_{n+1,0}^{\text{II}}), \tag{3.28}$$

where all known entities were moved to the right-hand side, as the force term.

For $n = 0$, (3.28) becomes

$$-\nabla \cdot (\hat{a}_0 \nabla u_{0,1}^{\text{II}}) = c_{0,1,1} \nabla \cdot (\hat{a}_1 \nabla u_{1,0}^{\text{II}}).$$

Since $u_{1,0}^{\text{II}} = 0$, we have $u_{0,1}^{\text{II}} = 0$.

For $n = 1$, we have

$$-\nabla \cdot (\hat{a}_0 \nabla u_{1,1}^{\text{II}}) = \nabla \cdot (\hat{a}_1 \nabla u_{0,0}^{\text{II}}) + c_{1,1,2} \nabla \cdot (\hat{a}_1 \nabla u_{2,0}^{\text{II}}).$$

Since $u_{0,0}^{\text{II}} \neq 0$, we have $u_{1,1}^{\text{II}} \neq 0$.

For $n > 1$, we have $u_{n,1}^{\text{II}} = 0$, due to the fact that $u_{n-1,0}^{\text{II}} = u_{n+1,0}^{\text{II}} = 0$.

For σ^2, we have equations

$$\begin{aligned} -\nabla \cdot (\hat{a}_0 \nabla u_{n,2}^{\text{II}}) &= \nabla \cdot (\hat{a}_1 \nabla u_{n-1,1}^{\text{II}}) + \nabla \cdot (\hat{a}_2 \nabla u_{n-2,0}^{\text{II}}) \\ &\quad + c_{n,1,n+1} \nabla \cdot (\hat{a}_1 \nabla u_{n+1,1}^{\text{II}}) + c_{n,2,n} \nabla \cdot (\hat{a}_2 \nabla u_{n,0}^{\text{II}}). \end{aligned} \tag{3.29}$$

For $n = 0$, (3.29) takes the form

$$-\nabla \cdot (\hat{a}_0 \nabla u_{0,2}^{\text{II}}) = c_{0,1,1} \nabla \cdot (\hat{a}_1 \nabla u_{1,1}^{\text{II}}) + c_{0,2,0} \nabla \cdot (\hat{a}_2 \nabla u_{0,0}^{\text{II}}),$$

which yields that $u_{0,2}^{\text{II}} \neq 0$ since $u_{1,1}^{\text{II}} \neq 0$ and $u_{0,0}^{\text{II}} \neq 0$.

For $n = 1$, (3.29) becomes

$$-\nabla \cdot (\hat{a}_0 \nabla u_{1,2}^{\text{II}}) = \nabla \cdot (\hat{a}_1 \nabla u_{0,1}^{\text{II}}) + c_{1,1,2} \nabla \cdot (\hat{a}_1 \nabla u_{2,1}^{\text{II}}) + c_{1,2,1} \nabla \cdot (\hat{a}_2 \nabla u_{1,0}^{\text{II}}),$$

which yields that $u_{1,2}^{\text{II}} = 0$, since $u_{0,1}^{\text{II}} = u_{2,1}^{\text{II}} = u_{1,0}^{\text{II}} = 0$.

For $n = 2$, we have

$$-\nabla \cdot (\hat{a}_0 \nabla u^{\shortparallel}_{2,2}) = \nabla \cdot (\hat{a}_1 \nabla u^{\shortparallel}_{1,1}) + \nabla \cdot (\hat{a}_2 \nabla u^{\shortparallel}_{0,0}) + c_{2,1,3} \nabla \cdot (\hat{a}_1 \nabla u^{\shortparallel}_{3,1}) + c_{2,2,2} \nabla \cdot (\hat{a}_2 \nabla u^{\shortparallel}_{2,0}),$$

which implies that $u^{\shortparallel}_{2,2} \neq 0$ since the form term is not equal to zero.

For $n > 2$, we have $u^{\shortparallel}_{n,2} = 0$, due to the fact that $u^{\shortparallel}_{n-1,1} = u^{\shortparallel}_{n-2,0} = u^{\shortparallel}_{n+1,1} = u^{\shortparallel}_{n,0} = 0$.

Let us assume that (3.24) holds for $u^{\shortparallel}_{n,m}$, $m < \hat{m} \in \mathbb{N}$. We now examine the terms corresponding to $\sigma^{\hat{m}}$. The related equations are as follows:

$$-\nabla \cdot (\hat{a}_0 \nabla u^{\shortparallel}_{n,\hat{m}}) = \sum_{m=0}^{\hat{m}-1} \nabla \cdot (\hat{a}_{\hat{m}-m} \nabla u^{\shortparallel}_{n-\hat{m}+m,m}) + \sum_{m=0}^{\hat{m}-1} c_{n,\hat{m}-m,n+2-\hat{m}+m} \nabla \cdot (\hat{a}_{\hat{m}-m} \nabla u^{\shortparallel}_{n+2-\hat{m}+m,m}).$$

If $\hat{m} < n$, then $u^{\shortparallel}_{n-\hat{m}+m,m} = 0$, because $(n - \hat{m} + m) - m = n - \hat{m} > 0$, and $u^{\shortparallel,m}_{n+2-\hat{m}+m,m} = 0$ for the same reason. Then $u^{\shortparallel}_{n,\hat{m}} = 0$ for $\hat{m} < n$. If $n = \hat{m}$, $u^{\shortparallel}_{n-\hat{m}+m,m} = u^{\shortparallel}_{0,0}$ when $m = 0$. Thus $u^{\shortparallel}_{n,n} \neq 0$. If $\hat{m} > n$, we consider two cases: (1) $\hat{m} - n$ is odd, and (2) $\hat{m} - n$ is even. If $\hat{m} - n$ is odd, then $m - (n - \hat{m} + m) = \hat{m} - n$ and $m - (n + 2 - \hat{m} + m) = \hat{m} - n - 2$ are also odd. This implies that $u^{\shortparallel}_{n-\hat{m}+m,m} = u^{\shortparallel}_{n+2-\hat{m}+m,m} = 0$, since $m < \hat{m}$. Then $u^{\shortparallel}_{n,\hat{m}} = 0$, when $\hat{m} - n > 0$ is odd. Similarly, we have $u^{\shortparallel}_{n,\hat{m}} \neq 0$, when $\hat{m} - n > 0$ is even. This concludes the proof. ∎

Now we shall explain, in more detail, why the high-order terms $O(\sigma^{n+2k})$, $k > 1$, are necessary. Assume that (3.24) is satisfied for chaos expansion up to polynomial order p. Then, we consider the chaos expansion of order $p + 1$. We shall concentrate on the last two equations ($n = p, p + 1$) of the corresponding uncertainty propagator, which are related to $u^{\shortparallel}_{p+1}$:

$$-\nabla \cdot \left(a_0 \nabla u^{\shortparallel}_p\right) - \underline{c_{p,1,p+1} \nabla \cdot \left(a_1 \nabla u^{\shortparallel}_{p+1}\right)} - \sum_{i=1}^{p} \nabla \cdot \left(a_i \nabla u^{\shortparallel}_{p-i}\right) - \sum_{i=2}^{p+1} c_{p,i,p+2-i} \nabla \cdot \left(a_i \nabla u^{\shortparallel}_{p+2-i}\right) = 0, \tag{3.30}$$

$$-\nabla \cdot \left(a_0 \nabla u^{\shortparallel}_{p+1}\right) - c_{p+1,2,p+1} \nabla \cdot \left(a_2 \nabla u^{\shortparallel}_{p+1}\right) - \sum_{i=1}^{p+1} \nabla \cdot \left(a_i \nabla u^{\shortparallel}_{p+1-i}\right) - \sum_{i=3}^{p+2} c_{p+1,i,p+3-i} \nabla \cdot \left(a_i \nabla u^{\shortparallel}_{p+3-i}\right) = 0. \tag{3.31}$$

It is easy to see that if the underlined term in (3.30) is dropped, then (3.30) becomes the equation for $u^{\shortparallel}_p$ in the uncertainty propagator for chaos expansion of polynomial order p. Assume that

$$F(u^{\shortparallel}_0, \ldots, u^{\shortparallel}_p) = 0$$

is the uncertainty propagator for chaos expansion of polynomial order p. Then, the uncertainty propagator for chaos expansion of polynomial order $p+1$ can be written as

$$F(u^{\shortparallel}_0, \ldots, u^{\shortparallel}_p) + \epsilon(u^{\shortparallel}_{p+1}) = 0, \tag{3.32}$$

$$G(u^{\shortparallel}_{p+1}) = f(u^{\shortparallel}_0, \ldots, u^{\shortparallel}_p), \tag{3.33}$$

where $\epsilon(u^{\mathsf{II}}_{p+1})$ is the underlined term in (3.30) and serves as a correction term for $F(u^{\mathsf{II}}_0,\ldots,u^{\mathsf{II}}_p)$. Here (3.33) corresponds to (3.31), where all terms related to $u^{\mathsf{II}}_0,\ldots,u^{\mathsf{II}}_p$ are collected on the right-hand side of (3.33) as the force term.

Next, we shall consider an iterative procedure for solving (3.32) and (3.33). Starting from $u^{\mathsf{II},\hat{k}}_{p+1}=0$, we solve (3.32), where $\hat{k}$ indicates the iteration step. Then (3.24) holds for $u^{\mathsf{II},\hat{k}}_n$, $n=0,\ldots,p$. Next, to obtain an intermediate term $u^{\mathsf{II},\hat{k}}_{p+1}$, we solve (3.33). Let

$$u^{\mathsf{II},\hat{k}}_{p+1}=\sum_{m=0}^{\infty}u^{\mathsf{II},\hat{k}}_{p+1,m}\sigma^m.$$

Note that in (3.33), i.e., (3.31),

$$-\sum_{i=1}^{p+1}\nabla\cdot\left(a_i\nabla u^{\mathsf{II},\hat{k}}_{p+1-i}\right)-\sum_{i=3}^{p+2}c_{i,p+3-i,p+1}\nabla\cdot\left(a_i\nabla u^{\mathsf{II},\hat{k}}_{p+3-i}\right)\sim\sum_{k=0}^{\infty}O(\sigma^{p+1+2k}).$$

Comparing the coefficients related to the powers of σ, we see that the nonzero terms are

$$u^{\mathsf{II},\hat{k}}_{p+1,m}\neq 0,\quad m=p+1+2l,\, l\in\mathbb{N}_0.$$

This is consistent with (3.30), because the (underlined) correction term

$$c_{p,1,p+1}\nabla\cdot\left(a_1\nabla u^{\mathsf{II},\hat{k}}_{p+1}\right)\sim\sum_{k=0}^{\infty}O(\sigma^{p+2(k+1)})\sim O(\sigma^{p+2}).$$

Note that the leading order of (3.30) is $O(\sigma^p)$. Thus, in the iterative procedure for solving the uncertainty propagator for the chaos expansion of polynomial order $p+1$, (3.24) holds on each iteration step. Thus the high-order terms $O(\sigma^{n+2k})$, $k>1$, are required by the first-order Malliavin derivative. If we consider only the zeroth-order Malliavin derivative in the Wick-type model, then $u^{\mathsf{II}}_n\sim O(\sigma^n)$.

Using the properties of the tensor product and the M-R formula, one could generalize (3.24) to the multidimensional setting.

PROPOSITION 3.8. *For the Wiener chaos expansion of* $u^{\mathsf{II}}=\sum_{\boldsymbol{\alpha}\in\mathcal{J}_{N,p}}u^{\mathsf{II}}_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, *subject to* N*-dimensional random coefficient* $a_N(\boldsymbol{x},\boldsymbol{\xi})$, *we have that for Model* II,

$$u^{\mathsf{II}}_{\boldsymbol{\alpha}}\sim\sum_{k=0}^{\infty}O(\sigma^{|\boldsymbol{\alpha}|+2k})\sim O(\sigma^{|\boldsymbol{\alpha}|}),\tag{3.34}$$

where $p\geq 1$.

Proof. Let us consider the following uncertainty propagator:

$$-\sum_{\boldsymbol{\alpha}\leq\boldsymbol{\gamma}}\nabla\cdot\left(a_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\alpha}}\right)-\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}_{N,p}}c_{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}}\nabla\cdot\left(a_{\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\beta}}\right)=f(\boldsymbol{x})1_{\boldsymbol{\gamma}=0}$$

for $\boldsymbol{\gamma}\in\mathcal{J}_{N,p}$. Define

$$u^{\mathsf{II}}_{\boldsymbol{\alpha}}=\sum_{m=0}^{\infty}u^{\mathsf{II}}_{\boldsymbol{\alpha},m}\sigma^m,\quad a_{\boldsymbol{\alpha}}=\hat{a}_{\boldsymbol{\alpha}}\sigma^{|\boldsymbol{\alpha}|}.$$

Then

$$
\begin{aligned}
&-\sum_{m=0}^{\infty}\left(\sum_{\boldsymbol{\alpha}\le\boldsymbol{\gamma}}\nabla\cdot(\hat{a}_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\alpha},m})\sigma^{|\boldsymbol{\gamma}|-|\boldsymbol{\alpha}|+m}-\sum_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|\ge 1}c_{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}}\nabla\cdot(\hat{a}_{\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\beta},m})\sigma^{|\boldsymbol{\alpha}|+m}\right)\\
&\quad= f(\boldsymbol{x})1_{\boldsymbol{\gamma}=0}.
\end{aligned}
\tag{3.35}
$$

Comparing (3.35) and (3.26), we see that equations for σ^i have the same pattern as in the one-dimensional case. Recall that $|\boldsymbol{\gamma}|+2=|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|$ corresponds to $n+2=i+j$ in (3.26). For example, the equation for $\sigma^{\hat{m}}$ takes the form

$$
\begin{aligned}
-\nabla\cdot(\hat{a}_{(\mathbf{0})}\nabla u^{\mathsf{II}}_{\boldsymbol{\gamma},\hat{m}}) &= \sum_{m=0}^{\hat{m}-1}\sum_{|\boldsymbol{\gamma}-\boldsymbol{\alpha}|=\hat{m}-m}\nabla\cdot(\hat{a}_{\boldsymbol{\gamma}-\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\alpha},m})\\
&\quad+\sum_{m=0}^{\hat{m}-1}\sum_{|\boldsymbol{\alpha}|=\hat{m}-m,|\boldsymbol{\beta}|\ge 1}c_{\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}}\nabla\cdot(\hat{a}_{\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\beta},m}).
\end{aligned}
\tag{3.36}
$$

All the information related to the right-hand side is given by the coefficients of σ^i, $i<\hat{m}$. Then (3.34) can be verified by the same procedure and the statement will follow by induction. ∎

Using Proposition 3.8, we can estimate $(\mathcal{L}^{\mathsf{I}}-\mathcal{L}^{\mathsf{II}})u^{\mathsf{II}}$ in terms of σ.

PROPOSITION 3.9. *In Model* II*,*

$$(\mathcal{L}^{\mathsf{I}}-\mathcal{L}^{\mathsf{II}})u^{\mathsf{II}}\sim O(\sigma^4).$$

Proof. It follows from Lemma 3.2 that

$$(\mathcal{L}^{\mathsf{I}}-\mathcal{L}^{\mathsf{II}})u^{\mathsf{II}}=\sum_{|\boldsymbol{\alpha}|>1,|\boldsymbol{\beta}|>1}a_{\boldsymbol{\alpha}}\nabla u^{\mathsf{II}}_{\boldsymbol{\beta}}\left(\mathrm{H}_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\beta}}-\sum_{n=0}^{1}\mathcal{D}^n\mathrm{H}_{\boldsymbol{\alpha}}\diamond\mathcal{D}^n\mathrm{H}_{\boldsymbol{\beta}}\right).$$

Therefore, the leading term on the right-hand side is $O(\sigma^4)$. Thus, the conclusion follows from Proposition 3.8. ∎

Using (3.22) and Proposition 3.9, we obtain the following result.

LEMMA 3.10. *Assume that the coefficient $a_N(\boldsymbol{x},\boldsymbol{\xi})$ remains the same for u^{I} and u^{II} as in the discussion above. Then,*

$$\|u^{\mathsf{I}}-u^{\mathsf{II}}\|\sim O(\sigma^4),\tag{3.37}$$

where the norm in (3.37) *is determined by the norm of u^{II}.*

Thus, Model II is a fourth-order approximation of Model I in terms of σ.

Remark 3.11. If one considers the approximation of $a(\boldsymbol{x},\omega)$

$$e^{G_N(\boldsymbol{x},\omega)-\frac{1}{2}\sigma^2}=e^{-\frac{1}{2}\sigma^2\sum_{k=N+1}^{\infty}\lambda_i\phi_i^2(\boldsymbol{x})}a_N(\boldsymbol{x},\omega),$$

then the chaos coefficients will be determined by

$$e^{-\frac{1}{2}\sigma^2\sum_{k=N+1}^{\infty}\lambda_i\phi_i^2(\boldsymbol{x})}\frac{\sigma^{|\boldsymbol{\alpha}|}\Phi^{\boldsymbol{\alpha}}(\boldsymbol{x})}{\boldsymbol{\alpha}!}=\frac{\sigma^{|\boldsymbol{\alpha}|}\Phi^{\boldsymbol{\alpha}}(\boldsymbol{x})}{\boldsymbol{\alpha}!}(1+O(\sigma^2)),\quad \boldsymbol{\alpha}\in\mathcal{J}_N,$$

where higher-order terms, with respect to σ, are involved. Although the analysis will become more tedious, Proposition 3.8 and Lemma 3.10 will still hold.

4. High-order Wick–Malliavin approximation of Model I. Next, we will generalize the setting of section 3.4 by considering the following high-order approximation of Model I:

$$\text{(4.1)} \qquad \text{Model G:} \quad \begin{cases} -\nabla \cdot \left(\sum_{q=0}^{Q} \frac{\mathcal{D}^q a(\boldsymbol{x},\omega) \diamond \nabla \mathcal{D}^q u}{q!} \right) = f(\boldsymbol{x}), & \boldsymbol{x} \in D, \\ u(\boldsymbol{x},\omega) = 0, & \boldsymbol{x} \in \partial D, \end{cases}$$

where $Q \in \mathbb{N}$. Let u^G be the solution of Model G. Now, we shall discuss the chaos coefficients of u^G. For simplicity, we only consider the one-dimensional case, i.e., the random coefficient takes the form given in (3.23), and provide numerical evidence for high-dimensional cases.

For $i \in \mathbb{N}$, the qth-order Malliavin derivative verifies the following relations:

$$\mathcal{D}^q \mathrm{H}_i = \frac{i!}{(i-q)!} \mathrm{H}_{i-q} \mathfrak{u}^{(q)}, \quad i \geq q,$$

and

$$\mathcal{D}^q \mathrm{H}_i \diamond \mathcal{D}^q \mathrm{H}_j = \frac{i!j!}{(i-q)!(j-q)!} \mathrm{H}_{i+j-2q} = \frac{i!j!1_{i+j-2q=k}}{(i-q)!(j-q)!} \mathrm{H}_k = c_{q,k,i,j} \mathrm{H}_k,$$

where $i, j \geq q$, and $i+j-2q = k \geq 0$, and $\mathfrak{u}^{(q)} = \mathfrak{u}_1^{\otimes q}$. Therefore,

$$\begin{aligned} \mathcal{D}^q a_{N=1} \diamond \nabla \mathcal{D}^q u &= \sum_{i \geq q, j \geq q} a_i \nabla u_j \mathcal{D}^q \mathrm{H}_i \diamond \mathcal{D}^q \mathrm{H}_j \\ &= \sum_k \sum_{i \geq q, j \geq q} c_{q,k,i,j} a_i \nabla u_j \mathrm{H}_k. \end{aligned}$$

Proposition 4.1. *Given the Wiener chaos expansion* $u^G = \sum_{n=0}^{p} u_n^G \mathrm{H}_n(\xi_1)$ *with one-dimensional random coefficient given by* (3.23) *and* $p \geq Q$, *the following asymptotics hold:*

$$\text{(4.2)} \qquad u_n^G \sim \sum_{k=0}^{\infty} O(\sigma^{n+2k}) \sim O(\sigma^n).$$

Proof. Since the Qth-order Malliavin derivative is involved in Model G, one shall consider the pth-order chaos expansion $u^G = \sum_{i=0}^{p} u_i^G \mathrm{H}_i$ with $p \geq Q$. The corresponding uncertainty propagator is

$$\text{(4.3)} \qquad -\sum_{q=0}^{Q} \sum_{i,j \geq q} c_{q,n,i,j} \nabla \cdot (a_i \nabla u_j^G) = f(\boldsymbol{x}) 1_{n=0}, \quad n = 0, 1, \ldots, p.$$

Assume that

$$\text{(4.4)} \qquad u_i^G = \sum_{m=0}^{\infty} u_{i,m}^G \sigma^m.$$

Substituting (4.4) into uncertainty propagator (4.3), we obtain

$$\text{(4.5)} \qquad -\sum_{m=0}^{\infty} \sum_{q=0}^{Q} \sum_{i,j \geq q} c_{q,n,i,j} \nabla \cdot (\hat{a}_i \nabla u_{j,m}) \, \sigma^{m+i} = f(\boldsymbol{x}) 1_{n=0}, \quad n = 0, \ldots, p,$$

where we let $a_i = \hat{a}_i \sigma^i$ since $a_i = O(\sigma^i)$.

We now look at the coefficient of the zeroth-order term σ^0, i.e., $m+i=0$. Then $m=i=0$. Since we require that $i \geq q$, then $q=0$. Using the fact that $i+j-2q=n$, we have $j=n$. Then the coefficients $u^G_{n,0}$ will be given by the following system:

$$-c_{0,n,0,n}\nabla\cdot\left(\hat{a}_0\nabla u^G_{n,0}\right)=f(\boldsymbol{x})1_{n=0}. \tag{4.6}$$

Since $\hat{a}_0=a_0=\mathbb{E}[a]>0$, the above system is well-posed and

$$u^G_{0,0}\neq 0,\quad u^G_{n,0}=0\quad \text{if } n\geq 1.$$

For σ^1, we have $m+i=1$, which implies that $m=1, i=0$, or $m=0, i=1$. If $i=0$, we have $q=0$ and $j=n$. If $i=1$, we have $j=n+2q-1$, where $q=0,1$. In other words, only the zeroth and first-order Malliavin derivatives contribute to the coefficient of σ^1. Then the corresponding equations are

$$-c_{0,n,0,n}\nabla\cdot\left(\hat{a}_0\nabla u^G_{n,1}\right)=c_{0,n,1,n-1}\nabla\cdot\left(\hat{a}_1\nabla u^G_{n-1,0}\right)+c_{1,n,1,n+1}\nabla\cdot\left(\hat{a}_1\nabla u^G_{n+1,0}\right). \tag{4.7}$$

Note that $u^G_{j,m}=0$ when $j<0$. For convenience, we keep them on the right-hand side of the equation. Since $u^G_{n,0}=0$ when $n\geq 1$, we have

$$u^G_{n,1}=\begin{cases}\neq 0, & n=1,\\ =0, & n\neq 1.\end{cases}$$

For σ^2, we need $m+i=2$, where we have the following three possible cases:

$$i=0, m=2;\ i=1, m=1;\ i=2, m=0.$$

The corresponding equations are

$$\begin{aligned}-c_{0,n,0,n}\nabla\cdot\left(\hat{a}_0\nabla u^G_{n,2}\right)&=c_{0,n,1,n-1}\nabla\cdot\left(\hat{a}_1\nabla u^G_{n-1,1}\right)+c_{1,n,1,n+1}\nabla\cdot\left(\hat{a}_1\nabla u^G_{n+1,1}\right)\\&\quad+c_{0,n,2,n-2}\nabla\cdot\left(\hat{a}_2\nabla u^G_{n-2,0}\right)+c_{1,n,2,n}\nabla\cdot\left(\hat{a}_2\nabla u^G_{n,0}\right)\\&\quad+c_{2,n,2,n+2}\nabla\cdot\left(\hat{a}_2\nabla u^G_{n+2,0}\right).\end{aligned} \tag{4.8}$$

Using (4.8) and properties of $u^G_{n,0}$ and $u^G_{n,1}$, we have

$$u^G_{n,2}=\begin{cases}\neq 0, & n\in\{0,2\},\\ =0, & n\in\mathbb{N}\backslash\{0,2\}.\end{cases}$$

Assuming that the coefficients $u^G_{n,m}$ with $m<\hat{m}\in\mathbb{N}$ satisfy (4.2), we consider a general case $m+i=\hat{m}$, corresponding to equations

$$-c_{0,n,0,n}\nabla\cdot\left(\hat{a}_0\nabla u^G_{n,\hat{m}}\right)=\sum_{i=1}^{\hat{m}}\sum_{q=0}^{i}c_{q,n,i,n+2q-i}\nabla\cdot\left(\hat{a}_i\nabla u^G_{n+2q-i,\hat{m}-i}\right).$$

We first check the case $\hat{m}>n$, which includes two cases: (1) $\hat{m}-n$ is odd, and (2) $\hat{m}-n$ is even. Let $\hat{m}-n=2l+1$. Then $\hat{m}-i=n+2l-i+1$. We have $u^G_{n+2q-i,\hat{m}-i}=0$ since $\hat{m}-i<\hat{m}$ and the difference between $n+2q-i$ and $\hat{m}-i$ is odd, which implies that $u^G_{n,\hat{m}}=0$ for case (1). For case (2), let $\hat{m}-n=2l$. Then for

$\hat{m} - i < \hat{m}$, $u_{n+2q-i,\hat{m}-i} \neq 0$ because the difference between $n + 2q - i$ and $\hat{m} - i$ is even, which implies that $u^G_{n,\hat{m}} \neq 0$ for case (2).

We subsequently check the case $\hat{m} = n$. When $\hat{m} = n$, we note that the difference between $n + 2q - i$ and $\hat{m} - i$ is always even, which implies that $u^G_{n,\hat{m}=n} \neq 0$.

Finally we examine the case $\hat{m} < n$. Noting the fact that $n+2q-i > \hat{m}+2q-i > \hat{m} - i$, we have $u^G_{n+2q-i,\hat{m}-i} = 0$, which implies that $u^G_{n,\hat{m}} = 0$ if $n > \hat{m}$.

Since all these observations are consistent with (4.2), we can conclude by induction. ☐

Similarly to Model II, Proposition 4.1 can be generalized to N-dimensional cases quite straightforwardly. We then have the next lemma.

LEMMA 4.2. *In Model G, for the Wiener chaos expansion* $u^G = \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{N,p}} u^G_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, *subject to* N*-dimensional random coefficient* $a_N(\boldsymbol{x}, \boldsymbol{\xi})$*, we have*

$$u^G_{\boldsymbol{\alpha}} \sim \sum_{k=0}^{\infty} O(\sigma^{|\boldsymbol{\alpha}|+2k}) \sim O(\sigma^{|\boldsymbol{\alpha}|}), \tag{4.9}$$

where $p \geq Q$. *For the same random coefficient* $a_N(\boldsymbol{x}, \boldsymbol{\xi})$*, the difference between* u^{II} *and* u^G *is as follows:*

$$\|u^{\mathsf{I}} - u^G\| \sim O(\sigma^{2(Q+1)}), \tag{4.10}$$

where the norm should be determined by the norm of u^G.

Proof. Here we will only check the difference between $\mathcal{L}^{\mathsf{II}}$ and $\mathcal{L}^G$, where

$$\mathcal{L}^G u^G = \sum_{q=0}^{Q} \frac{\mathcal{D}^q a \diamond \mathcal{D}^q u^G}{q!}.$$

Using Lemma 3.2, one gets the following asymptotics:

$$\begin{aligned}(\mathcal{L}^{\mathsf{I}} - \mathcal{L}^G) u^G &= \sum_{|\boldsymbol{\alpha}|>Q, |\boldsymbol{\beta}|>Q} a_{\boldsymbol{\alpha}} \nabla u^G_{\boldsymbol{\beta}} \left(\mathrm{H}_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\beta}} - \sum_{q=0}^{Q} \frac{\mathcal{D}^q \mathrm{H}_{\boldsymbol{\beta}} \diamond \mathcal{D}^q \mathrm{H}_{\boldsymbol{\beta}}}{q!} \right) \\ &\sim \sum_{|\boldsymbol{\alpha}|>Q, |\boldsymbol{\beta}|>Q} O(\sigma^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}) \sim O(\sigma^{2(Q+1)}). \qquad \square\end{aligned}$$

5. Numerical comparison of different models.

To illustrate our theoretical results, one-dimensional elliptic problems in physical space will be considered.

5.1. Numerical solution of uncertainty propagator of Model II.

We present a numerical procedure for Model G. Consider the chaos expansion $u^G = \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{N,p}} u^G_{\boldsymbol{\alpha}} \mathrm{H}_{\boldsymbol{\alpha}}$. The corresponding uncertainty propagator takes the form

$$-\sum_{q=0}^{Q} \sum_{|\boldsymbol{\alpha}|, |\boldsymbol{\beta}| \geq q} c_{q,\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}} \nabla \cdot (a_{\boldsymbol{\beta}} \nabla u^G_{\boldsymbol{\alpha}}) = f(\boldsymbol{x}) 1_{\boldsymbol{\gamma}=0} \quad \forall \boldsymbol{\gamma} \in \mathcal{J}_{N,p}, \tag{5.1}$$

where $|\boldsymbol{\gamma}| + 2q = |\boldsymbol{\alpha}| + |\boldsymbol{\beta}|$ and

$$c_{q,\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\boldsymbol{\theta}\leq\boldsymbol{\gamma}} \sum_{|\boldsymbol{\kappa}|=q} 1_{\boldsymbol{\kappa}+\boldsymbol{\gamma}-\boldsymbol{\theta}=\boldsymbol{\alpha}} 1_{\boldsymbol{\kappa}+\boldsymbol{\theta}=\boldsymbol{\beta}} \frac{\boldsymbol{\alpha}!\boldsymbol{\beta}!}{\boldsymbol{\kappa}!\boldsymbol{\theta}!(\boldsymbol{\gamma}-\boldsymbol{\theta})!}. \tag{5.2}$$

For any given $\boldsymbol{\gamma}, \boldsymbol{\alpha} \in \mathcal{J}_{N,p}$, $c_{q,\boldsymbol{\gamma},\boldsymbol{\alpha},\boldsymbol{\beta}}$ can be determined in a preprocessing stage, where for any $|\boldsymbol{\kappa}| = q$, the index $\boldsymbol{\beta}$ is either not involved or defined by $\boldsymbol{\beta} = 2\boldsymbol{\kappa} + \boldsymbol{\gamma} - \boldsymbol{\alpha}$. Consider the equation for $u^G_{\boldsymbol{\alpha}=\boldsymbol{\gamma}}$. We have

$$-\nabla \cdot \left(\left(a_0 + \sum_{q=1}^{Q} \sum_{|\boldsymbol{\beta}|=2q} 1_{|\boldsymbol{\gamma}|\geq q} c_{q,\boldsymbol{\gamma},\boldsymbol{\gamma},\boldsymbol{\beta}} a_{\boldsymbol{\beta}} \right) \nabla u^G_{\boldsymbol{\gamma}} \right) = F, \tag{5.3}$$

where F indicates the contribution from the force term $f(\boldsymbol{x})$ and other chaos coefficients $u^G_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \neq \boldsymbol{\gamma}$. It is quite clear that due to the action of the Malliavin derivative, the coefficient varies in terms of $\boldsymbol{\gamma}$. In this paper, we consider the following Gauss–Seidel iteration:

$$-\nabla \cdot \left(\left(a_0 + \sum_{q=1}^{Q} \sum_{|\boldsymbol{\beta}|=2q} 1_{|\boldsymbol{\gamma}|\geq q} c_{q,\boldsymbol{\gamma},\boldsymbol{\gamma},\boldsymbol{\beta}} a_{\boldsymbol{\beta}} \right) \nabla u^{G,k+1}_{\boldsymbol{\gamma}} \right) = F_1^{k+1} + F_2^k, \tag{5.4}$$

where the superscript k indicates the iteration step, F_1^{k+1} the contribution from coefficients $u^{G,k+1}_{\boldsymbol{\alpha}}$ obtained before $u^{G,k+1}_{\boldsymbol{\gamma}}$, and F_2^k the contribution from other chaos coefficients $u^{G,k}_{\boldsymbol{\alpha}}$ obtained from the previous iteration step. Note that for each $u^G_{\boldsymbol{\gamma}}$, a corresponding stiffness matrix needs to be constructed when the finite element method is employed.

When σ is relatively small, the following Gauss–Seidel-like procedure can also be considered:

$$-\nabla \cdot (a_0 \nabla u^{G,k+1}_{\boldsymbol{\gamma}}) = \nabla \cdot \left(\left(\sum_{q=1}^{Q} \sum_{|\boldsymbol{\beta}|=2q} 1_{|\boldsymbol{\gamma}|\geq q} c_{q,\boldsymbol{\gamma},\boldsymbol{\gamma},\boldsymbol{\beta}} a_{\boldsymbol{\beta}} \right) \nabla u^{G,k}_{\boldsymbol{\gamma}} \right) + F_1^{k+1} + F_2^k, \tag{5.5}$$

where one takes out the contribution of the Malliavin derivative to the diagonal terms such that only one stiffness matrix with respect to a_0 is needed.

Remark 5.1. The Gauss–Seidel iteration works well when σ is relatively small. As σ becomes large, the Gauss–Seidel iteration will become less efficient. A more effective preconditioner will be required, especially for high-dimensional problems in the physical space. One possible strategy is to use the Wick-type models (3.9) or (3.13) as a preconditioner, which can be inverted efficiently due to the lower-triangular structure of their uncertainty propagators. In particular, the solution of Wick-type model (3.13) converges to u^{I} as the correlation length of $G(\boldsymbol{x}, \omega)$ goes to infinity, which may alleviate the slow convergence of Gauss–Seidel iteration for a relatively large σ. Furthermore, we would also like to take advantage of the sparseness in the upper-triangular part of the uncertainty propagator. Such issues are beyond the scope of this paper and left for further study.

5.2. Spatially independent noise. We consider the following simplest cases:

$$\text{Model I:} \quad -\frac{d}{dx}\left(e^{\sigma\xi - \frac{1}{2}\sigma^2} \frac{du^{\mathsf{I}}}{dx} \right) = f(x), \tag{5.6}$$

$$\text{Model G:} \quad -\frac{d}{dx}\left(\sum_{q=0}^{Q} \frac{1}{q!} \mathcal{D}^q e^{\sigma\xi - \frac{1}{2}\sigma^2} \diamond \mathcal{D}^q \left(\frac{du^G}{dx} \right) \right) = f(x) \tag{5.7}$$

with homogeneous boundary conditions on the domain $x \in [0, 1]$, where $\xi \sim \mathcal{N}(0, 1)$ is a normal Gaussian random variable. We choose $f(x) = 1$.

Since the random coefficient is spatially independent, the uncertainty propagators for $u^G(x) = \sum_{i=0}^{p} u_i^G(x)\mathrm{h}_i(\xi)$ have the following forms:

$$-\sum_{i=0}^{p} \frac{d^2 u_i^G}{dx^2} \sum_{q=0}^{Q} \frac{1}{q!}\mathbb{E}\left[\mathcal{D}^q e^{\sigma\xi-\frac{1}{2}\sigma^2} \diamond \mathcal{D}^q \mathrm{h}_i)\mathrm{h}_j\right] = 1_{j=0},$$

where $j = 0, 1, \ldots, p$.

Noting that

$$e^{\sigma\xi-\frac{1}{2}\sigma^2} = \sum_{n=0}^{\infty} \frac{\sigma^n}{\sqrt{n!}}\mathrm{h}_n(\xi),$$

we obtain that

$$\begin{aligned}\mathcal{D}e^{\sigma\xi-\frac{1}{2}\sigma^2} &= \sum_{i=0}^{\infty} \frac{\sigma^i}{\sqrt{i!}}\mathcal{D}\mathrm{h}_i(\xi) = \sum_{i=1}^{\infty} \frac{\sigma^i}{\sqrt{i!}}\sqrt{i}\mathrm{h}_{i-1}(\xi)\\ &= \sum_{i=0}^{\infty} \frac{\sigma^{i+1}}{\sqrt{i!}}\mathrm{h}_i(\xi) = \sigma e^{\sigma\xi-\frac{1}{2}\sigma^2}.\end{aligned}$$

Therefore,

$$\mathcal{D}^q e^{\sigma\xi-\frac{1}{2}\sigma^2} = \sigma^q e^{\sigma\xi-\frac{1}{2}\sigma^2}. \tag{5.8}$$

Thus,

$$\mathbb{E}\left[\left(\mathcal{D}^q e^{\sigma\xi-\frac{1}{2}\sigma^2} \diamond \mathcal{D}^q \mathrm{h}_i\right)\mathrm{h}_j\right] = \frac{\sigma^{j-i+2q}\sqrt{i!j!}}{(j-i+q)!(i-q)!} 1_{i\geq q}1_{j-i+q\geq 0}.$$

Then, we have

$$\begin{aligned}&\sum_{q=0}^{Q} \frac{1}{q!}\mathbb{E}\left[\left(\mathcal{D}^q e^{\sigma\xi-\frac{1}{2}\sigma^2} \diamond \mathcal{D}^q \mathrm{h}_i\right)\mathrm{h}_j\right]\\ &\quad= \sum_{q=0}^{Q} \frac{\sigma^{j-i+2q}\sqrt{i!j!}}{q!(j-i+q)!(i-q)!} 1_{i\geq q}1_{j-i+q\geq 0}.\\ &\quad= \sum_{q=0}^{Q} \frac{\sigma^{j-i+2q}}{\sqrt{q!(j-i+q)!}} \binom{i}{q}^{1/2} \binom{j}{i-q}^{1/2} 1_{i\geq q}1_{j-i+q\geq 0}.\end{aligned}$$

Wiener chaos expansions of polynomial order 10 are considered for u^{I} and u^G. In Figure 5.1, we plot the relative difference between u^{I} and u^G with respect to the $L_2(\mathbb{F}; H_0^1(D))$ norm defined as

$$\epsilon(Q) = \frac{\|u^{\mathsf{I}} - u^G\|_{L_2(\mathbb{F};H_0^1(D))}}{\|u^{\mathsf{I}}\|_{L_2(\mathbb{F};H_0^1(D))}},$$

where the straight solid and dashed lines have slopes 4 and 6, respectively. It is readily seen that the numerical results are consistent with our theoretical prediction $\epsilon(Q) \sim O(\sigma^{2(Q+1)})$, when σ is small enough.

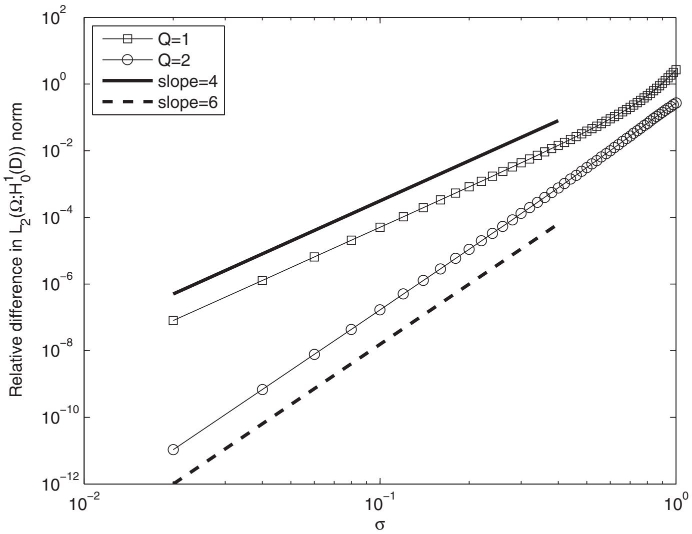

FIG. 5.1. *Relative difference $\epsilon(Q)$ between u^{I} and u^{G} with respect to σ for a spatially independent one-dimensional random coefficient.*

5.3. Spatially dependent noise. Let us examine now the setting where the random coefficient is a log-normal random process. For Model I, the numerical approximation is obtained from the uncertainty propagator corresponding to the chaos expansion $u^{\mathsf{I}} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}_{N,p}} u^{\mathsf{I}}_{\boldsymbol{\alpha}}\mathrm{h}_{\boldsymbol{\alpha}}$. Since

$$\mathrm{h}_{\boldsymbol{\gamma}}\mathrm{h}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\kappa}\le\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}} \frac{\sqrt{\boldsymbol{\gamma}!\boldsymbol{\alpha}!(\boldsymbol{\gamma}+\boldsymbol{\alpha}-2\boldsymbol{\kappa})!}}{\boldsymbol{\kappa}!(\boldsymbol{\gamma}-\boldsymbol{\kappa})!(\boldsymbol{\alpha}-\boldsymbol{\kappa})!}\mathrm{h}_{\boldsymbol{\gamma}+\boldsymbol{\alpha}-2\boldsymbol{\kappa}},$$

where $\kappa_i = \gamma_i \wedge \alpha_i = \min(\gamma_i, \alpha_i)$, it follows that

$$\mathbb{E}[a(x,\omega)\mathrm{h}_{\boldsymbol{\alpha}}\mathrm{h}_{\boldsymbol{\gamma}}] = \sum_{\boldsymbol{\kappa}\le\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}} \frac{\sigma^{|\boldsymbol{\gamma}+\boldsymbol{\alpha}-2\boldsymbol{\kappa}|}\sqrt{\boldsymbol{\gamma}!\boldsymbol{\alpha}!}}{\boldsymbol{\kappa}!(\boldsymbol{\gamma}-\boldsymbol{\kappa})!(\boldsymbol{\alpha}-\boldsymbol{\kappa})!}\Phi^{\boldsymbol{\gamma}+\boldsymbol{\alpha}-2\boldsymbol{\kappa}}(x).$$

The uncertainty propagator of Model I will be solved using the Gauss–Seidel iteration and the spectral element method.

We still employ the physical domain $x \in [0,1]$, homogeneous boundary conditions, and $f(x) = 1$. We consider the exponential covariance kernel $K(|x_1 - x_2|) = e^{-\frac{|x_1-x_2|}{l_c}}$ with l_c being the correlation length, for which the eigen-pairs $(\lambda_i, \phi_i(x))$ are given,

$$\phi_i(x) = \frac{\hat{\lambda}_i l_c \cos(\hat{\lambda}_i x) + \sin(\hat{\lambda}_i x)}{\sqrt{\frac{1}{2}(1+\hat{\lambda}_i^2 l_c^2) + (\hat{\lambda}_i^2 l_c^2 - 1)\frac{\sin(2\hat{\lambda}_i)}{4\hat{\lambda}_i} + \frac{1}{2}l_c(1-\cos(2\hat{\lambda}_i))}}, \tag{5.9}$$

where

$$\hat{\lambda}_i^2 = \frac{2/l_c - \lambda_i/l_c^2}{\lambda_i}, \quad (\hat{\lambda}_i^2 - 1/l_c^2)\tan(\hat{\lambda}_i) - 2\hat{\lambda}_i/l_c = 0 \tag{5.10}$$

(see [12]).

Let us take $N = 3$ and $p = 8$. The correlation length is set to be 5 such that $\sum_{k=4}^{\infty} \lambda_i / \sum_{k=1}^{\infty} \lambda_i \approx 1.6\%$. The relative difference between u^{I} and u^{G} is plotted in Figure 5.2, where in the left plot the random coefficient is approximated by $a_N(\boldsymbol{x}, \boldsymbol{\xi})$ and in the right plot the random coefficient is approximated as $e^{G_N(\boldsymbol{x},\boldsymbol{\xi})-\frac{1}{2}\sigma^2}$. It is seen that the convergence behaviors for both types of approximation of $a(\boldsymbol{x}, \omega)$ are almost the same and the convergence rates with respect to σ are consistent with the theoretical prediction for both cases. Furthermore, for $Q = 1$, $\epsilon(Q) \approx 1\%$ when $\sigma = 0.40$; for $Q = 2$, $\epsilon(Q) \approx 2\%$ when $\sigma = 0.69$. The corresponding mean and standard deviation of u^{I} and u^{G} are plotted in Figures 5.3 and 5.4. In other words, Model G with a relatively small Q can provide accurate statistics for a relatively large degree of perturbation in the random coefficient. In Figure 5.5, we demonstrate the coupling between chaos coefficients $u^{G}_{\boldsymbol{\alpha}}$ in the uncertainty propagator of Model G. The nonzero entry (i, j) of the matrix indicates the jth chaos coefficient contributes in the ith equation. It is seen that the coupling is relatively sparse in contrast to the full coupling for Model I. When $N = 3$, $p = 8$, there are 165 chaos coefficients. For $Q = 1$, only 25% of the entries are nonzero; for $Q = 2$, the coupling is stronger and around 40% of the entries are nonzero.

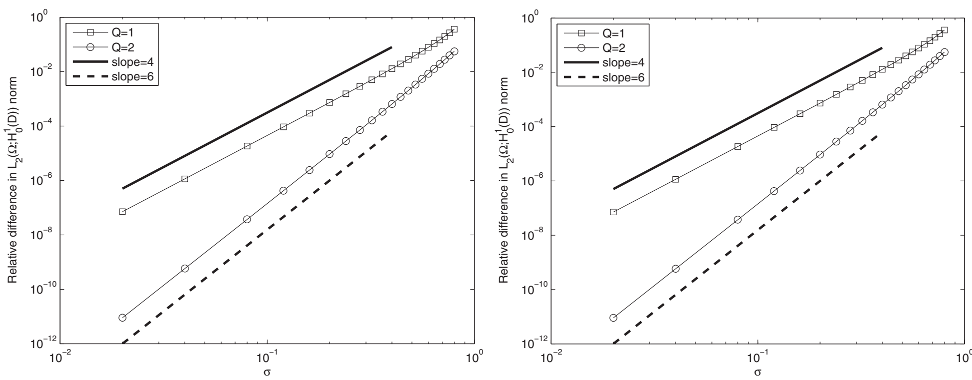

FIG. 5.2. *Relative difference $\epsilon(Q)$ between u^{I} and u^{G} with respect to σ for a spatially dependent three-dimensional random coefficient. Left: the random coefficient is approximated as $a_N(x, \boldsymbol{\xi})$. Right: the random coefficient is approximated as $e^{G_N(x,\boldsymbol{\xi})-\frac{1}{2}\sigma^2}$.*

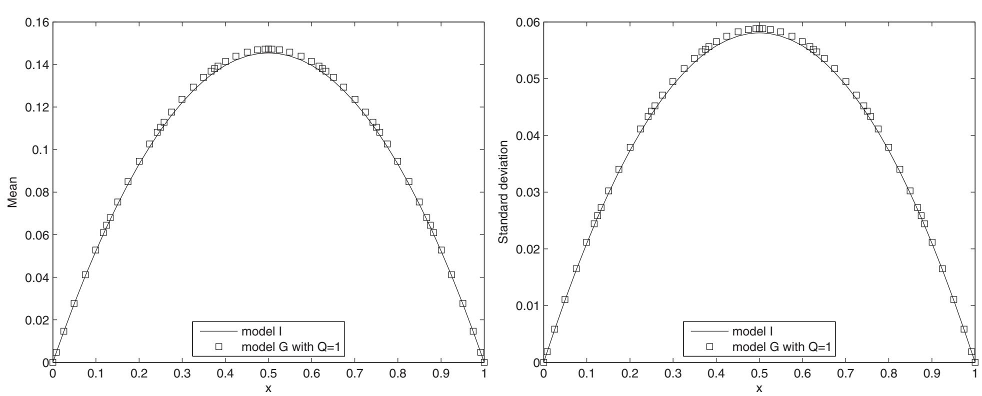

FIG. 5.3. *Mean and standard deviation of u^{I} and u^{G} when $Q = 1$ and $\sigma = 0.40$.*

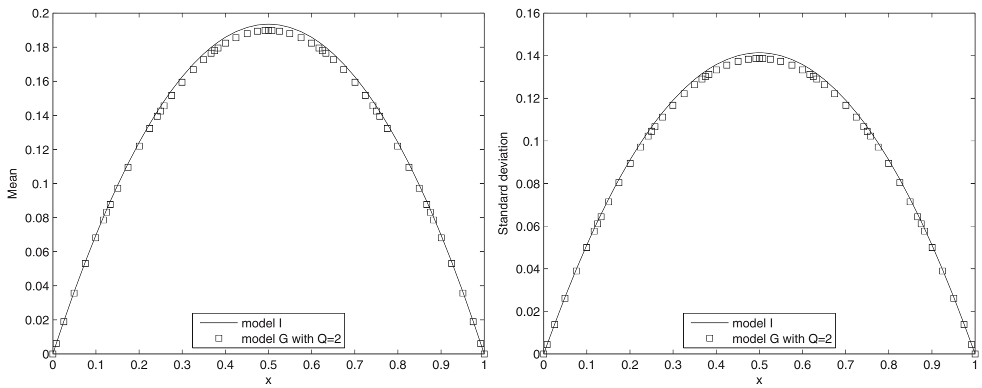

FIG. 5.4. *Mean and standard deviation of u^I and u^G when $Q = 2$ and $\sigma = 0.69$.*

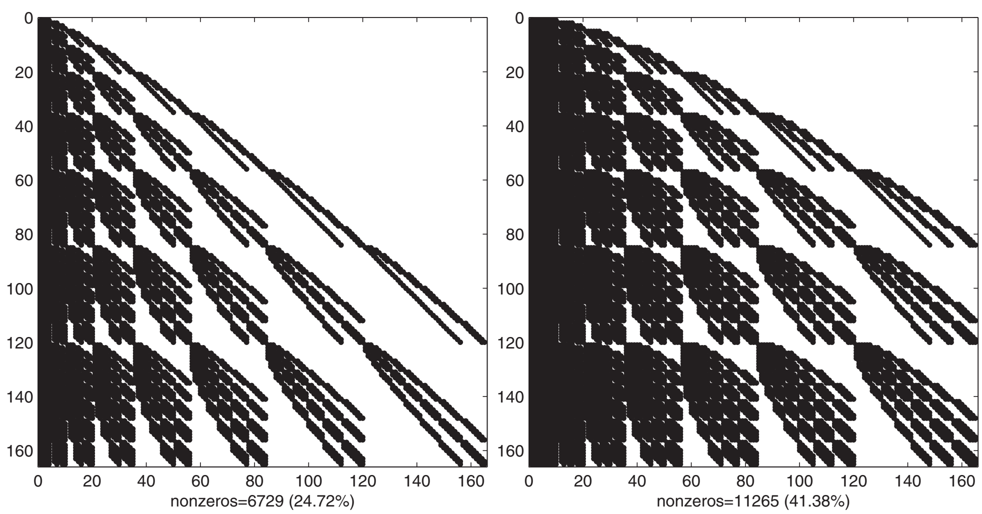

FIG. 5.5. *Coupling of chaos coefficients in the uncertainty propagator of Model G. $N = 3$, $p = 8$. Left: $Q = 1$. Right: $Q = 2$.*

6. Summary and discussions. In this work, we have generalized the Wick–type model (3.9) using the M-R formula, which theoretically converges to the classical Model I providing enough smoothness in the probability space. We focused on the difference between the solutions of Model I and G. When the standard deviation of the underlying Gaussian random process is small enough, we have proved that the error of truncation is $O(\sigma^{2(Q+1)})$, where Q is the level of truncation in the M-R formula (1.2) and σ characterizes the standard deviation of noise in the random coefficient $a(\boldsymbol{x}, \omega)$. We have demonstrated that the numerical results support the theoretical prediction. We also observed that the high-order terms in the M-R formula will introduce extra coupling in the upper-triangular part of the uncertainty propagator. The strength of the coupling increases as the truncation order Q becomes larger. However, the numerical experiments show that Model G with a relatively small Q can provide accurate statistics for a relatively large degree of perturbation in the random coefficient. When Q is relatively small, the upper-triangular part of the uncertainty propagator is sparse, which, of course, makes numerical approximation more efficient.

We have theoretically demonstrated also that the convergence rate with respect to σ is independent of the physical dimension. Therefore, it was reasonable, for our purposes, to limit the numerical experiments to one-dimensional problems in physical space. In view of our previous results [26, 27], we expect that Model G of low-order Q will remain effective for reasonably high-dimensional problems in physical space, where numerical efficiency for the uncertainty propagator becomes more important. The related study will be reported elsewhere.

REFERENCES

[1] I. Babuška, R. Tempone, and G. Zouraris, *Galerkin finite element approximations of stochastic elliptic differential equations*, SIAM J. Numer. Anal., 42 (2004), pp. 800–825.

[2] R. Cameron and W. Martin, *The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals*, Ann. Math., 48 (1947), p. 385.

[3] J. Charrier, *Strong and weak error estimates for elliptic partial differential equations with random coefficients*, SIAM J. Numer. Anal., 50 (2012), pp. 216–246.

[4] J. Charrier and A. Debussche, *Weak truncation error estimates for elliptic PDEs with lognormal coefficients*, Stoch. PDEs Anal. Comput., 1 (2013), pp. 63–93.

[5] X. Fernique, *Regularité des trajectoires des functions aleatoires Gaussiennes, École d'Eté de Probabilitées de Saint-Flour* IV-1794, P. Hennequin, ed., Lecture Notes in Math. 480, Springer-Verlag, New York, 1975, pp. 2–96.

[6] R. Ghanem and P. Spanos, *Stochastic Finite Element: A Spectral Approach*, Springer-Verlag, New York, 1991.

[7] C. Gittelson, *Stochastic Galerkin discretization of the log-normal isotropic diffusion problems*, Math. Models Methods Appl. Sci., 20 (2010), pp. 237–263.

[8] P. Frauenfelder, C. Schwab, and R. Todor, *Finite elements for elliptic problems with stochastic coefficients*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp 205–228.

[9] R. Todor and C. Schwab, *Convergence rates for sparse chaos approximations of elliptic problems with stochastic coefficients*, IMA J. Numer. Anal., 27 (2007), pp. 232–261.

[10] H. Holden, B. Oksendal, and T. Zhang, *Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach*, Birkhauser, Boston, 1996.

[11] H.-H. Kuo, *White Noise Distribution Theory*, CRC Press, Boca Raton, FL, 1996.

[12] M. Jardak, C.-H. Su, and G. Karniadakis, *Spectral polynomial chaos solutions of the stochastic advection equation*, J. Sci. Comput., 17 (2002), pp. 319–338.

[13] S. Lototsky and B. Rozovskii, *Stochastic differential equations driven by purely spatial noise*, SIAM J Math. Anal., 41 (2009), pp. 1295–1322.

[14] S. Lototsky, B. Rozovskii, and X. Wan, *Elliptic equations of higher stochastic order*, ESAIM Math. Model. Numer. Anal., 5 (2010), pp. 1135–1153.

[15] S. Lototsky, B. Rozovskii, and D. Seleši, *On generalized Malliavin calculus*, Stochastic Anal. Appl., 122 (2012), pp. 808–843.

[16] R. Mikulevicius and B. L. Rozovskii, *On unbiased stochastic Navier-Stokes equations*, Probab. Theory Related Fields, 154 (2012), pp. 787–834.

[17] D. Nualart, *Malliavin Calculus and Related Topics*, 2nd ed., Springer-Verlag, New York, 2006.

[18] J. Galvis and M. Sarkis, *Approximating infinity-dimensional stochastic Darcy's equations without uniform ellipticity*, SIAM J. Numer. Anal., 47 (2009), pp. 3624–3651.

[19] G. Papanicolaou, *Diffusion in random media*, in Surveys in Applied Mathematics, J. B. Keller, D. McLaughlin, and G. Papanicolaou, eds., Plenum Press, New York, 1995, pp. 205–255.

[20] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia Math. Appl. 44, Cambridge University Press, Cambridge, UK, 1992.

[21] A. Mugler and H.-J. Starkloff, *On elliptic partial differential equations with random coefficients*, Stud. Univ. Babeş-Bolyai Math., 56 (2011), pp. 473–487.

[22] T. Theting, *Solving Wick-stochastic boundary value problems using a finite element method*, Stoch. Stoch. Rep., 70 (2000), pp. 241–270.

[23] G. Våge, *Variational methods for PDEs applied to stochastic partial differential equations*, Math. Scand., 82 (1998), pp. 113–137.

[24] D. Venturi, X. Wan, R. Mikulevicius, B. Rozovskii, and G. Karniadakis, *Wick-Malliavin approximation to nonlinear stochastic PDEs: Analysis and simulations*, Proc. Roy. Soc. A, 469 (2013), 20130001.

[25] X. Wan, B. Rozovskii, and G. Karniadakis, *A stochastic modeling methodology based on weighted Wiener chaos and Malliavin calculus*, Proc. Natl. Acad. Sci. USA, 106 (2009), pp. 14189–14194.

[26] X. Wan, *A note on stochastic elliptic models*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 2987–2995.

[27] X. Wan, *A discussion on two stochastic modeling strategies for elliptic problems*, Comm. Comput. Phys., 11 (2012), pp. 775–796.