A stochastic modeling methodology based on weighted Wiener chaos and Malliavin calculus

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In many stochastic partial differential equations (SPDEs) involving random coefficients, modeling the randomness by spatial white noise may lead to ill-posed problems. Here we consider an elliptic problem with spatial Gaussian coefficients and present a methodology that resolves this issue. It is based on stochastic convolution implemented via generalized Malliavin operators in conjunction with weighted Wiener spaces that ensure the ellipticity condition. We present theoretical and numerical results that demonstrate the fast convergence of the method in the proper norm. Our approach is general and can be extended to other SPDEs and other types of multiplicative noise.

Elliptic partial differential equations perturbed by spatial noise provide an important stochastic model in applications, such as diffusion through heterogeneous random media (1), stationary Schrodinger equation with a stochastic potential (2), etc. A typical model of interest is the following stochastic partial differential equation (SPDE):

\[ -\nabla \cdot (a(x) \nabla u(x)) = f(x), \quad x \in D \subset \mathbb{R}^d, \quad u_{|\partial D} = g(x), \]

where \( \omega \) indicates randomness. Recently, this model has been actively investigated in the context of uncertainty quantification (UQ) for mathematical and computational models, see, e.g., refs. 3 and 4 and the references therein. The diffusion coefficient \( a(x, \omega) \) is typically modeled by the Karhunen–Loève type expansion \( a(x, \omega) = \bar{a}(x) + \varepsilon(x) \). Here, \( \bar{a}(x) \) is the mean and \( \varepsilon(x) = \sum_{k \geq 1} \sigma_k(x)\xi_k \) is the noise term, where \( \sigma_k(x) \) are deterministic matrices and \( \xi := \{\xi_k\}_{k \geq 1} \) is a set of uncorrelated random variables with zero mean and unit variance. For the problem of Eq. 1 to be well posed, the ellipticity condition is required. However, if the noise \( \varepsilon(x) \) is Gaussian, (or any other type with infinite negative part), the standard ellipticity condition would not hold, no matter how small the variance of \( \varepsilon(x) \) is. Various modifications of the aforementioned setting have been used to mitigate this problem. For example, Gaussian models were replaced by distributions with finite range (e.g., uniform distribution). We note that from the statistical and, by implication, UQ perspectives, Gaussian perturbations are of paramount importance and should be modeled judiciously.

In this article, we propose a well-posed modification of the model of Eq. 1 with Gaussian noise \( \varepsilon = \varepsilon(x) \). More specifically, we replace Eq. 1 by the following SPDE:

\[ \nabla \cdot (a(x) \nabla u(x)) = \nabla \cdot (\delta_u(\varepsilon(x)) \nabla u(x)) = f(x), \quad x \in D, \quad u_{|\partial D} = g(x), \]

where \( \delta_u(\varepsilon(x)) \) denotes the Malliavin divergence operator (MDO) with respect to Gaussian noise \( \varepsilon(x) \) (see, e.g., refs. 5 and 6 and the next section). The MDO is a stochastic integral and specifically, in the present setting, a convolution of the integrand \( \varepsilon \) with the driving Gaussian noise \( \varepsilon \). Although replacing the ordinary product by stochastic convolution may be surprising at first, it should be appreciated that physical systems rarely produce an instant response to sharp inputs such as multiplicative forcing.

Therefore, in modeling systems subject to sharp perturbations, convolutions often become a model of choice. Historically, stochastic convolution-based models were used to bypass the exceeding singularity of models with (literally understood) multiplicative noise. In fact, the idea of reduction to MDO could be traced back to the pioneering work of K. Itô on stochastic calculus and stochastic differential equations. Specifically, in his seminal work (7), Itô has replaced the “product” model

\[ \dot{u}(t) = a(u(t)) + b(u(t)) \cdot \dot{W}(t) \]

by the “stochastic convolution” model

\[ u(t) = u_0 + \int_0^t a(u(s))ds + \int_0^t b(u(s))dW(s), \]

where \( \int_0^t b(u(s))dW(s) \) is the famous Itô’s stochastic integral. In this article, we extend Itô’s idea to the infinite dimensional stationary system in Eq. 1 and replace it with Eq. 2. In contrast to Itô’s SDEs, elliptic equations like 1 or 2 are not causal. By this reason, we replace Itô’s integral by the MDO \( \delta_x \). The latter, in this instance, is equivalent to anticipating (noncausal) Skorohod integral, (see ref. 6).

A general theory of bilinear elliptic SPDEs that includes, in particular, Eq. 2 was developed recently in ref. 8. In particular, it was shown that to ensure well-posedness of Eq. 2 it suffices to assume positivity of \( a(x) \) rather than of \( a(x, \omega) = \bar{a}(x) + \varepsilon(x) \). This approach is based on Malliavin calculus and Wiener chaos expansion (WCE) with respect to Cameron–Martin basis (see ref. 9 and next section).

The Cameron–Martin basis consists of random variables \( \xi_k \), where \( \alpha = (\alpha_1, \alpha_2, \ldots) \) is a multiindex with nonnegative integer entries (see the next section for more detail). The WCE solution of Eq. 2 is given by the series \( u(x) = \sum_{k \in \mathbb{N}} u_k(x)\xi_k \), where \( u_k = \mathbb{E}[u|\xi_k] \). One can view the Cameron–Martin expansion as a Fourier expansion that separates randomness from the deterministic “backbone” of the equation. It was shown in ref. 8 that the WCE coefficients \( u_k(x) \) for Eq. 2 verify a lower triangular system of linear deterministic elliptic equations (see Eq. 19 below): We refer to this system as the (uncertainty) propagator. Under very general assumptions, the propagator is equivalent to SPDE 2 in the same way as the set of coefficients of a Fourier expansion is equivalent to the underlying function.

In this article, we develop a numerical method for solving Eq. 2 with high-order discretization in the physical space and weighted WCE in the probability space. We conduct a theoretical and numerical investigation of the propagation of the truncation errors and illustrate the results by numerical simulations. In particular, an a priori error estimate is presented to demonstrate the

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convergence of the developed numerical algorithm. We also carry out numerical and theoretical comparisons of the model of Eq. 2 with direct multiplication model of Eq. 1 for positive (lognormal) coefficient $a(t)$. To facilitate the theoretical comparison of these models, we have developed a generalization of the MDO $\delta_k$ based on Gaussian noise $\epsilon$ to a divergence operator with respect to a class of noises including nonlinear transformations of Gaussian noise (e.g., lognormal).

In addition to its physical meaning, there are other mathematical and computational reasons for employing the “convolution” model of Eq. 2 instead of the “multiplication” model of Eq. 1, specifically: (i) The convolution model is computationally efficient due to the lower triangular (in fact, bidiagonal) structure of the propagator; its computational complexity is linear. One could also formally define an approximate WCE solution for the multiplication model of Eq. 1; however, its propagator would be a full system with quadratic computational complexity. (ii) The variance of WCE solutions for both models is infinite. However, could also formally define an approximate WCE solution for the adequate due to the lower triangular (in fact, bidiagonal) structure specifically: $(i)$

\[ D_{\delta} (\xi, \eta) := \sqrt{\alpha} \xi_{\alpha} \eta_{\alpha}, \quad \delta_{\delta} (\xi, \eta) := \sqrt{\alpha} + 1 \xi_{\alpha} \eta_{\alpha}. \]  

[5]

In the literature on quantum physics (see, e.g., ref. 18) the operators $\delta_k$ and $D_{\delta}$ are often called creation and annihilation operators, respectively. As intuition suggests, the MDO of $D_{\delta} (\xi, \eta)$ recovers the integrand $\xi_\alpha$. Indeed, for $\alpha > 0$, $D_{\delta} (\xi, \eta) = \xi_\alpha$. Also, MD and MDO can be extended to $L_2 (F, X)$ (see, e.g., ref. 17).

Before proceeding with the SPDE 2, let us consider a simple example to shed some light on the structure of the spaces within which we could expect existence of solutions of this equation.

**Example 1.** Let $\eta$ be a solution of equation

\[ u = 1 + \delta_{\eta} (u), \]  

where $\xi \sim N(0, 1)$. Simple calculations show that $\| u \|^2_{L_2 (F, X)} = \sum_{k=1}^{\infty} \| \eta_k \|^2_{L_2 (F, X)} = \| \eta \|^2_{L_2 (F, X)}$. On the other hand, taking a weighted norm with weights $r_n = (n!)^{-1/2} \cdot \nu^{1/2}$, we get

\[ \| u \|^2_{R L_2 (F, X)} := \sum_{n=0}^{\infty} r_n^2 \| \eta_k \|^2_{L_2 (F, X)} = (1 - 2^{-q})^{-1}. \]  

[7]

The above example demonstrates that even simple stationary equations do not have solutions with finite second moments. To overcome this obstacle one should introduce an appropriate weighted version of the solution space. Clearly, introduction of weights amounts to rescaling of the stochastic Fourier (Wiener chaos) representation of the solution. Below we discuss briefly the construction of weighted spaces.

Let $R$ be a bounded linear operator on $L_2 (F, X)$ defined by $R \xi = r_\alpha \xi_\alpha$ for every $\alpha \in J$, where the weights $r_\alpha$ are positive numbers. In what follows, we will identify the operator $R$ with the corresponding collection $\{ r_\alpha, \alpha \in J \}$. The inverse operator $R^{-1}$ is defined as $R^{-1} \xi = r_{-\alpha} \xi_{-\alpha}$. The elements of $R L_2 (F, X)$ can be identified with a formal series $\sum_{\alpha \in J} r_\alpha \xi_\alpha$, where $\sum_{\alpha \in J} \| \xi \|^2_{L_2 (F, X)} < \infty$. Clearly, $R L_2 (F, X)$ is a Hilbert space with respect to the norm $\| f \|^2_{R L_2 (F, X)} := \| f \|^2_{L_2 (F, X)}$. We define the space $R^{-1} L_2 (F, X)$ as the dual of $R L_2 (F, X)$ relative to the inner product in the space $L_2 (F, X)$, where $\langle f, g \rangle := \sum_{\alpha \in J} f_\alpha g_\alpha$, with $g = \sum_{\alpha \in J} g_\alpha \xi_\alpha$.

**Remark 1.** One could readily check that $u = 1 + \xi \cdot u$, i.e., the multiplication version of Eq. 6, is much more complicated and blows up much faster than Eq. 6.

To address SPDEs driven by lognormal and other important types of random perturbations, we need to develop a bilinear symmetric version of MDO with white noise $W$ replaced by an arbitrary nonlinear transformation of Gaussian random variables. To begin with, assume that vector $\xi$ consists of a single Gaussian random variable $\xi \sim N(0, 1)$. Let $\xi_{\alpha} = H_{\alpha} (\xi) / \sqrt{\alpha}$. Define a n-tuple MDO by induction: $\xi_{\alpha}^{(n)} (\xi_{\alpha}) := \delta_{\alpha} (\xi_{\alpha})$ and $\xi_{\alpha}^{(n)} (\xi_{\alpha}) := $
\[\delta_t \delta^{(n-1)}(\xi_m) \] for \( n > 1 \). Let 
\[\delta_{0n}(\xi_m) := \delta^{(n)}(\xi_m)/\sqrt{n!}.\] It is readily checked that
\[\delta_{0n}(\xi_m) = \sqrt{(m+n)!}\xi_m = \frac{1}{\sqrt{m!}}H_{m+n}(\xi).\] [9]

Now, for \( \xi := (\xi_1, \xi_2, \ldots) \) and multiindices \( \alpha \) and \( \beta \), define
\[\delta_{\alpha\beta}(\xi) := \prod_{k \in \mathbb{N}} \delta_{\alpha_k}(\xi_k).\] The formula of Eq. 9 translates into the multidimensional case as follows:
\[\delta_{\alpha\beta}(\xi) = \sqrt{(\beta + \alpha)!/\alpha!\beta!}\delta_{\alpha\beta}.\] [10]

**Remark 2. (Wick product)** The Wick product \((\diamond)\), can be defined as follows: for \( \alpha, \beta \in J, H_\alpha(\xi) \diamond H_\beta(\xi) := H_{\alpha+\beta}(\xi) \), where \( H_\alpha(\xi) := \prod_{k \in \mathbb{N}} \xi_k^{\alpha_k} \). It follows from Eq. 10 that \( H_\alpha(\xi) \diamond H_\beta(\xi) = \delta_{\alpha\beta}(\xi) \).

**Theorem 2.** If \( \theta, \eta \in \mathcal{R}L_2(\mathbb{R}; X) \), set \( \delta_{\alpha\beta}(\theta) := \sum_{\beta' \leq \beta} \theta_{\alpha\beta'} \) and \( \delta_{\alpha\beta}(\eta) := \sum_{\alpha' \leq \alpha} \eta_{\alpha'\beta} \). Then \( \delta_{\alpha\beta} \) is a bounded linear operator on \( \mathcal{R}L_2(\mathbb{R}; X) \) and
\[\delta_{\alpha\beta}(\theta) = \delta_{\alpha\beta}(\eta).\] [11]

**Numerical Method.** Let us rewrite Eq. 2 as an SPDE in the sense of Malliavin calculus:
\[\{Au(x) + \delta_{\beta\alpha}(Mu(x)) = f(x) \text{ on } D, \]
\[u(x) = g(x) \text{ on } \partial D,\] [12]

where \( D \) denotes the physical domain, \( \mathcal{W} \) is white noise on \( \mathcal{U} = L_2(D) \), the Hilbert space of square summable sequences of real numbers,
\[A u(x) := -\sum_{ij} D_i(a_{ij}(x)D_ju(x)),\] [13a]
\[M u(x) := \sum_{ij} D_i(\sigma_{ij}(x))D_ju(x),\] [13b]

and \( M u = \sum k \mathcal{M}_k \otimes u_k \). We note that \( u_k \) in \( L_2(D) \) can be identified with \( \mathcal{E}_k \), see Eq. 5. For simplicity, we assume that \( g = 0 \) and that \( f \) is deterministic. We can now rewrite the term \( \delta_{\beta\alpha} \) as
\[\delta_{\beta\alpha}(Mu) = \sum_{k \geq 1} \delta_{\beta\alpha}(M_k u).\] [14]

**Definition 1.** A solution to Eq. 12 is a random element \( u \in H_0^1(D) \) such that the equality
\[\{Au(v) + \delta_{\beta\alpha}(Mu,v) = \langle f, v \rangle\] [15]

where \( f \in H^{-1}(D) \) and
\[v \in \mathcal{R}^{-1}L_2(\mathbb{R}) \text{ and } D_{\beta\alpha}v \in \mathcal{R}^{-1}L_2(\mathbb{R}; \mathbb{C});\] [16]

Analytical issues related to Eq. 15 have been investigated in ref. 8. In particular, the following result holds:

**Theorem 3.** There exists an operator \( R \) and a unique solution \( u \in \mathcal{R}L_2(\mathbb{R}; H_0^1(D)) \) of Eq. 12 such that:
(i) The operator \( R \) is defined by the weights \( r_{\alpha\beta} \) given by
\[r_{\alpha\beta} = \frac{q_{\alpha\beta}}{\sqrt{\alpha!\beta!}}, \] [17]

where the numbers \( q_{\alpha\beta} \) are chosen so that the “renormalization condition”
\[\sum_{k \geq 1} C_k q_{k \alpha\beta} < 1\] [18]

holds, and \( C_k \) are defined by
\[\|A^{-1}M_k \|_2^2 \leq C_k \|\eta\|_2^2 \forall \eta \in H_0^1(D).\] [19]

Because the expectation of the MDO is zero, we have
\[E[Au + \delta_{\beta\alpha}(Mu)] = A\mathbb{E}[u] = f(x),\]

which is the unperturbed (deterministic) version of elliptic Eq. 1. By substituting the WCE \( u = \sum_{k \geq 1} u_k \mathcal{E}_k \) into Eq. 15 and performing a Galerkin projection in the probability space, we can establish the equivalence between Eqs. 2 and 15 and derive the following uncertainty propagator:
\[Au + \sum_{k \geq 1} \sqrt{\alpha_k}M_k u_{\alpha\beta} - v_k = f_u,\] [20]

which is a system of deterministic partial differential equations (PDEs). In the next section, we will develop an efficient numerical algorithm to solve Eq. 19.
on piecewise polynomials. There exist many choices of basis functions on the reference elements, such as $h$-type finite elements (19), spectral/ hp elements (20, 21), etc.

Let $\mathcal{J}_{M,p}$ be a finite dimensional subset of $\mathcal{J}$ given by $\mathcal{J}_{M,p} := \{ \alpha | \alpha \in \mathbb{N}_{0}^{d}, |\alpha| \leq p, p, p \in \mathbb{N} \}$ and $R_{M,p}$ be a given set of weights $r_{\alpha}, \alpha \in \mathcal{J}_{M,p}$. We define

$$V_{c} := \left\{ f = \sum_{\alpha \in \mathcal{J}_{M,p}} f_{\alpha} \xi_{\alpha} : \| f \|_{R_{M,p}^{2}(\mathcal{F})} < \infty \right\},$$

$$V_{c}^{-1} := \left\{ f = \sum_{\alpha \in \mathcal{J}_{M,p}} f_{\alpha} \xi_{\alpha} : \| f \|_{R_{M,p}^{-1}(\mathcal{F})} < \infty \right\}.$$

Then, the stochastic finite element method (sFEM) can be defined as: Find $u_{h}^{M,p}, v \in V_{h} \otimes V_{c}$ such that

$$\left\langle \left( A_{h}^{M,p} + \sum_{k=1}^{M} \delta_{h_k} (M_{k}u_{h}^{M,p}), v \right) \right\rangle_{H_{h}^{1}(\mathcal{D})} = \langle f, v \rangle_{H_{h}^{1}(\mathcal{D})}, \quad (20)$$

for any $v \in V_{h} \otimes V_{c}^{-1}$ where

$$\langle g_{1}, g_{2} \rangle_{H_{h}^{1}(\mathcal{D})} = \mathbb{E}\{ (R_{q_{1}}, R_{n}^{-1}q_{2})_{H_{h}^{1}(\mathcal{D})} \}. $$

Note that we truncate the expansion of the white noise up to $M$ terms and the WCE up to polynomial order $p$. Let $H = H_{h}^{1}(\mathcal{D})$, and $r_{\alpha} = q^{p}/\sqrt{|\alpha|!}$ as in Theorem 3. The main result regarding convergence of the stochastic finite element method (sFEM) is given by the following theorem:

**Theorem 4.** For $u \in RL_{2}(\mathbb{P}; F) \cap RL_{2}(\mathbb{P}; H^{m+1}(\mathcal{D}))$, the error of approximation of the sFEM is given by

$$\| u - u_{h}^{M,p} \|_{RL_{2}(\mathbb{P}; H)} \leq C \left( h^{m} \| u \|_{RL_{2}(\mathbb{P}; H^{m+1})} + \sqrt{\hat{q}_{W}} \left( 1 - \hat{q}^{-1} \right)^{1/2} + \hat{q}_{\alpha}^{p+1} \right), \quad (21)$$

where the constant $C$ is independent of $h$, $\hat{q} = \sum_{k=1}^{M} C_{k}^{2}q_{k}^{2} < 1$, $\hat{q}_{W} = \sum_{k=1}^{M} M_{k}^{2}C_{k}^{2}q_{k}^{2}$, and $C_{k}$ are constants defined in Theorem 3.

We remark that the 3 main components of the error $O(h^{m})$, $O(\hat{q}_{W})$, and $O(\hat{q}_{\alpha}^{p+1})$ are due to the finite element discretization, the approximation of white noise, and truncation of the WCE, respectively. We also note that spectral convergence is obtained in the weighted norm $\| \cdot \|_{RL_{2}(\mathbb{P}; H)}$. Next, we present a sketch of the proof whereas all technical details can be found in supporting information (SI) Appendix.

The approximation error can be decomposed as

$$u - u_{h}^{M,p} = \sum_{\alpha \in \mathcal{J}_{M,p}} (u_{\alpha} - \hat{u}_{\alpha}) \xi_{\alpha} + \sum_{\alpha \in \mathcal{J}_{M,p}} u_{\alpha} \xi_{\alpha},$$

where $u_{\alpha}$ and $\hat{u}_{\alpha}$ are the coefficients of chaos expansions of $u$ and $u_{h}^{M,p}$, respectively. Correspondingly, we obtain

$$\| u - u_{h}^{M,p} \|_{RL_{2}(\mathbb{P}; H)}^{2} \leq \sum_{\alpha \in \mathcal{J}_{M,p}} \| u_{\alpha} - \hat{u}_{\alpha} \|_{RL_{2}(\mathbb{P})}^{2} + \sum_{\alpha \in \mathcal{J}_{M,p}} \| u_{\alpha} \|_{H_{h}^{1}(\mathcal{D})} \| \xi_{\alpha} \|_{RL_{2}(\mathbb{P})}^{2} = I_{1} + I_{2}.$$

To obtain the error contribution $I_{1}$, we need to estimate the finite element approximation error $\| u_{\alpha} - \hat{u}_{\alpha} \|$. Since $u_{\alpha}$ depends on $u_{\beta}$ with $|\alpha - \beta| = 1$, we should consider the error propagation in the uncertainty propagator. The key observation is that for $|\alpha| > 0$, the following equations are satisfied in the weak sense

$$A\hat{u}_{a} + \sum_{k=1}^{M} \sqrt{\alpha_{k}} M_{k} \hat{u}_{a-k} = 0, \quad \forall \hat{v} \in V_{h},$$

$$A_{h}u_{a} + \sum_{k=1}^{M} \sqrt{\alpha_{k}} M_{k} u_{a-k} = 0, \quad \forall v \in H,$$

from which we can obtain a recursive inequality for the error propagation as

$$\| u_{a} - \hat{u}_{a} \|_{H} \leq C \inf_{v_{h} \in V_{h}} \| u_{a} - v_{h} \|_{H} + \sum_{k=1}^{M} c_{k} \| u_{a-k} - \hat{u}_{a-k} \|_{H}.$$

We then use results from the approximation theory to bound the finite element approximation error. The error contribution from $I_{2}$ is from the truncation of WCE and the approximation of white noise, which can be estimated using Theorem 3. More details are given in SI Appendix.

**Numerical Results**

We consider the 2-dimensional stochastic elliptic problem

$$-\nabla \cdot (\mathbb{E}[a](x) + \delta_{h}(\nabla u(x))) = f(x), \quad x \in D,$$

$$u(x) = 0, \quad x \in \partial D,$$

where $\mathbb{E}[a](x)$ denotes the mean field of coefficient and $\bar{W}(x)$ the random perturbation. For simplicity, we choose the physical domain $D = (0, 1)^{2}$, $\mathbb{E}[a](x) = 1$, and $f(x) = 1$.

To represent the white noise on $L_{2}(D)$, we select the following orthonormal basis

$$\bar{W}(x) = \sum_{m=1}^{M} w_{m,n}(x) \xi_{m,n}.$$ 

For convenience, we use an 1-dimensional index and rewrite the above equation as

$$\bar{W}(x) = \sum_{k=1}^{M} w_{k}(x) \xi_{k}.$$ 

Let $u(x) = \sum_{k=0}^{M} u_{k} \xi_{k}$ be the truncated WCE of the solution up to polynomial order $p$. The uncertainty propagator takes the form

$$-\nabla \cdot (\mathbb{E}[a](x) \nabla u_{a}) - \sum_{k=1}^{M} \sqrt{\alpha_{k}} \nabla \cdot (w_{k}(x) \nabla u_{a-k}) = f_{a},$$

where $f_{a} = 0$ if $|\alpha| > 0$, because $f(x)$ is assumed to be deterministic, and the operator $M_{k}$ is defined as $M_{k} u_{a} = -\nabla \cdot (w_{k}(x) \nabla u_{a})$. To choose proper $r_{\alpha}$ for the weighted Wiener chaos space, we need to specify the constant $C_{k}$ defined in Theorem 3. For our case, we have

$$C_{k} = \| w_{k}(x) \|_{L_{2}(D)}/A_{1}.$$ 

Here, $\| w_{k}(x) \|_{L_{2}(0)}$ is uniformly bounded and $A_{1} = 1$ because $\mathbb{E}[a](x) = 1$. Hence, the weights can be defined as $r_{\alpha} = q^{p}/\sqrt{|\alpha|!}$ with $q^{p} = \prod_{k=1}^{M} q_{k}^{k}$, if $q_{k}$ is chosen as $q_{k} = 1/(k + 1)C_{k}$.

We now examine the convergence of the sFEM. We employ the spectral/hp element method to solve the uncertainty...
that spectral convergence is obtained, which is consistent with the
the dominant error in the error estimate of Eq. 5.1. In Fig. 1, we plot the convergence
approximation of white noise and increase the polynomial order of WCE,
negligible. If we fix the number of random variables in the approx-
lateral elements, where 12th-order piecewise polynomials are used
truncation error of the WCE. In Fig. 1, we plot the convergence
where “*” indicates stochastic convolution or ordinary multiplication.
we choose \( f(x) = 1 \), and we consider first the simple model
that given by the convolution model. For
lognormal noise \( e^W \). In addition, we consider a second model \( K_H \)
with space-dependent lognormal noise of the type
\[
K_H(x, \xi) = \exp \left[ c \left( \xi_1 + \sqrt{2} \cos(\pi \xi) \xi_2 + \sqrt{2} \cos(2\pi \xi) \xi_3 \right) \right] \\
- \frac{1}{2} c^2 \left( 1 + 2 \cos^2(\pi \xi) + 2 \cos^2(2\pi \xi) \right),
\]
where \( \xi_i, i = 1, 2, 3 \), are normal random variables. In other
words, we take the first 3 modes of white noise \( W(x) = \xi_1 + \sum_{i=2}^{\infty} \sqrt{2} \cos((i - 1)\pi x) \xi_i \), on \( L_2(0,1) \). It is easy to show that
\( \mathbb{E}[K_H(x, \xi)] = 1 \), and that \( \text{Var}(K_H(x, \xi)) = \exp(c^2 + 2c^2 \cos^2(\pi x) + 2c^2 \cos^2(2\pi x)) - 1 \). In Fig. 2, we plot the variances of solutions
concerning Example 2. Specifically, we consider the simple problem
perturbation can be explained by referring to the aforemen-
tioned Example 2. Specifically, we consider the simple problem
error of the WCE. In Fig. 1, we plot the convergence
behavior of the approximate white noise in the weighted norm
\( \mathcal{R}L_2(\mathbb{F},L_2(D)) \) with respect to \( M \). We also plot the errors of the
approximate solution in the weighted norm \( \mathcal{R}L_2(\mathbb{F},H_0^1(D)) \) with
respect to the polynomial order of WCE. For the numerical sim-
ulation, we choose \( M = 21 \), corresponding to \( m + n \leq 5 \). The
errors are approximated as
\[
\|u_h^{M,n+1} - u_{\text{approx}}\|_{\mathcal{R}L_2(\mathbb{F},H_0^1(D))}.
\]
We see that spectral convergence is obtained, which is consistent with the error estimate of Eq. 21.

**Model Comparison.** Next, we present a comparison of the con-
volution and multiplication models by considering the following
1-dimensional problem
\[
\begin{aligned}
-\frac{d}{dx} \left( K(x) \frac{d}{dx} u(x) \right) &= f(x), \\
\quad u(0) = 0, \quad u(1) = 0,
\end{aligned}
\]  
where \( \theta \) is small, the
variances of solutions corresponding to 2 different models obtained from solving both
the convolution and the multiplication cases; results for both \( K_l \)
and \( K_H \) are shown. We observe that when the noise is small, the
variances given by the 2 models are about the same; however,
when the noise is large a large difference exists, with the variance
given by the multiplication model increasing much faster than
that given by the convolution model. For \( K_l \), when its standard
deviation increases from 0.1003 to 22.7379, the solution variance
increases from \( O(10^{-5}) \) to \( O(10^6) \) for the multiplication model,
and from \( O(10^{-5}) \) to \( O(10^5) \) for the convolution model—a 5-
difference in magnitude! For \( K_H \), the variance achieves larger
values as we include more terms in the expansion for the white
noise. More results and analysis for both models are included in
SI Appendix.

This exponential increase of the variance with respect to
perturbation can be explained by referring to the aforementioned Example 2. Specifically, we consider the simple problem
\( (d_n(u_t(x, \xi)))_h = f(x), x \in (x_0, x_1), u(0) = u(1) = 0 \) with log-
normal but space-independent coefficient \( a(\omega) = \exp(\xi - \frac{\omega^2}{2}) \);
the corresponding solution is \( u_t(x) = \theta \delta^{-1} f(x) \) with \( \delta \) being
the Dirichlet Laplacian on \( (x_0, x_1) \) and \( d_n(\theta) = 1 \). Example 2
shows that \( \theta \) is \exp(\xi - \frac{\omega^2}{2}) \). On the other hand, the solution
to the corresponding multiplication model \( (a \cdot v_t(x, \xi))_h = f(x) \) is \( v_t(x, \xi) = a^{-1} \delta^{-1} f(x) \). Hence, the rapid increase in the
variance observed in the computations is related to the ratio

![Fig. 1. Convergence of the proposed stochastic finite element method for the model problem. (A) Weighted \( L_2 \) norm of approximate white noise. (B) Spectral convergence of the stochastic finite element method with respect to the weighted norm. \( M = 21 \).](image)

![Fig. 2. Variance versus perturbation at \( x = 0.5 \). For model II, it is very expensive to compute the solution beyond \( c = 1 \) for the multiplication model.](image)
these models, we have developed a generalization of the MDO mal) coefficient. We are not aware of any systematic theoretical or numerical efforts to investigate such SPDEs. It appears though that the general methodology developed in this article could be extended to stochastic elliptic differential equations. We introduced the concept of “stochastic convolution model” and employed Malliavin calculus to implement it. The general methodology developed in this article also enjoys this important property of linear operators.

Summary and Discussion

In this article, we have developed an efficient numerical method for solving second-order elliptic SPDEs with Gaussian coefficients. We introduced the concept of “stochastic convolution model” and employed Malliavin calculus to implement it. The stochastic solution was represented by a weighted WCE in the appropriate norm. It was shown that the coefficients of the WCE can be obtained by solving a lower triangular system of deterministic elliptic PDEs, i.e., the uncertainty propagator. We derived a priori error estimates to measure the rate of convergence with respect to appropriately weighted $L_2$ norm. We have also carried out numerical and theoretical comparisons of the model of Eq. 2 with the direct multiplication model of Eq. 1 for positive (lognormal) coefficient $a_0(x)$. To facilitate the theoretical comparison of these models, we have developed a generalization of the MDO $\delta_0$ (based on Gaussian noise $\xi$) to a divergence operator with respect to a class of noises including nonlinear transformations of Gaussian noise (e.g., lognormal).

It might be instructive to reexamine the differences and similarities between the convolution model $Au(x) = \delta_0 (Mu(x))$ and its multiplication counterpart $Au(x) = Mu(x) \cdot W(x)$. It is well understood that the multiplication models are exceedingly singular. We are not aware of any systematic theoretical or numerical efforts to investigate such SPDEs. It appears though that the general methodology developed in this article could be extended to address elliptic equations of this type as well. However, in the multiplication setting, the propagator is not lower triangular, and the solution spaces are expected to be much larger than in the setting of this article. In contrast to multiplication models the convolution models are much more manageable analytically and numerically and have solid physical meaning. Both models become much easier if one replaces the white noise forcing by lognormal (which is a positive noise). We have shown that in the lognormal setting the SPDE has reasonable solutions for both models. Still, as the variance of the noise grows, the variance of the solution of the multiplication equation scales up exponentially as compared with the convolution model.

In spite of the aforementioned differences, the 2 models are closely related. In fact, the convolution models could be viewed as the highest stochastic order approximations to multiplication models. Indeed, by Remark 2, $\delta_0 (\mp (H_\beta (\xi))) = H_{\alpha+\beta} (\xi)$ while $H_\alpha (\xi) \cdot H_\beta (\xi) = H_{\alpha+\beta} (\xi) + R_{\alpha, \beta}$, where $R_{\alpha, \beta}$ is a linear combination of Hermite polynomials of orders less than $\alpha + \beta$.

Finally, we note that the mean in the multiplication model for the problems considered here deviate greatly from the corresponding deterministic solution (see SI Appendix). Clearly, linear systems with additive noise are unbiased perturbations of their deterministic counterparts with the statistical average of a solution to randomized system coinciding with the solution of the original unperturbed system. The convolution bilinear equations considered in this article also enjoy this important property of linear systems. However, we stress that this property does not hold and should not be expected from SPDEs with nonlinear operator $A$ or multiplication in the stochastic terms.

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