SOLUTION OF HW1

- Problem 1.3. Determine which of the following assertions are true:

(1) $e^x - 1 = O(x^2)$ as $x \to 0$.

solution: Note

$$\lim_{x\to 0} \frac{e^x - 1}{x^2} = \lim_{x\to 0} \frac{e^x}{2x} \quad \text{d.n.e.}$$

By definition, it is false.

(2) $x = o(\ln|x|)$ as $x \to 0$.

solution: When $x \to 0$, $\ln|x| \to -\infty$, meaning that

$$\lim_{x\to 0} \frac{x}{\ln|x|} = 0.$$

By definition, it is true.

(3) $e^{-x^2} = o(x^{-3})$ as $x \to \infty$.

solution: Consider

$$\lim_{x\to\infty} \frac{e^{-x^2}}{x^{-3}} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x}{2e^{x^2}} = \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = 0.$$

By definition, it is true.

(4) $x^{-2} = O(\cot x)$ as $x \to 0$.

solution: We have

$$\lim_{x\to 0} \frac{x^{-2}}{\cot x} = \lim_{x\to 0} \frac{\sin x}{x^2 \cos x} = \lim_{x\to 0} \frac{\cos x}{2x\cos x - x^2 \sin x} \quad \text{d.n.e.}$$

By definition, it is false.

- Problem 1.4. **solution**: According to the mean value theorem, we have

$$f(b) - f(a) = f'(\xi)(b-a) = (b-a)(2\alpha\xi + \beta),$$

which results in

$$\xi = \frac{f(b) - f(a) - \beta(b-a)}{2\alpha(b-a)} = \frac{b+a}{2}.$$

- Problem 1.5. **solution**. Note that

$$f'(x) = \frac{1}{1+x},\ f''(x) = -\frac{1}{(1+x)^2},\ f'''(x) = \frac{2}{(1+x)^3}, \dots, f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

So we have

$$\begin{aligned} f(x) &= \sum_{i=1}^{n} \frac{(-1)^{n-1}(n-1)!}{n!} x^n + \frac{(-1)^n n!}{(n+1)!(1+\xi)^{n+1}} x^{n+1} \\ &= \sum_{i=1}^{n} \frac{(-1)^{n-1}}{n} x^n + \frac{(-1)^n}{(n+1)(1+\xi)^{n+1}} x^{n+1}, \end{aligned}$$

where ξ is between 0 and x.

- Problem 2.2. **solution**: First, the exact solution is

$$Y = \frac{1}{2}x^2.$$

According to the definition of Euler's method, we have

$$\begin{aligned} y_n = y_{n-1} + hx_{n-1} &= y_{n-2} + hx_{n-2} + hx_{n-1} \\ &= \dots \\ &= y_0 + h\sum_{i=0}^{n-1} x_i = h^2 \sum_{i=0}^{n-1} i \\ &= \frac{(n-1)nh^2}{2} = \frac{1}{2}(nh)^2 - \frac{1}{2}nh^2 = \frac{1}{2} - \frac{1}{2}h. \end{aligned}$$

So $y_n \to \frac{1}{2} = Y(1)$, as $h \to 0$.

Problem 2.3. **solution**: We have

$$y_i = y_{i-1} + h = y_{i-2} + 2h = \dots = y_0 + ih = x_i.$$

The exact soluton is recovered.

Problem 2.4. **solution**: Let us first estimate $|Y(x)|$ using theorem 2.10. Taking $\beta_* = \gamma_* = 1$, we have

$$|Y(x)| \le (e^x - 1).$$

We then have

$$Y''(x) = Y'(x) = Y + 1.$$

Then

$$|Y''(x)| \leq |Y(x)| + 1 \leq e^x - 1 + 1 \leq e^2, \quad \forall x \in [0, 2].$$

Using theorem 2.4, we have

$$|Y(x_i) - y_i| \leq (e^{x_i} - 1)\frac{e^2}{2}h,$$

where we let $|\frac{\partial f}{\partial y}| \leq L = 1$.