SOLUTION OF HW2

- Problem 2.7. This problem takes the form of equation (2.25) subject to

$$\beta(x) = x, \quad \gamma(x) = \frac{1}{1+x^2}.$$

For $x \in [0,1]$, we have

$$\beta(x) \le \beta_* = 1, \quad |\gamma(x)| \le \gamma_* = 1.$$

From Theorem 2.10, we have

$$|Y(x)| \le e^x - 1 < e - 1.$$

Note

$$\begin{aligned} Y''(x) &= Y + x\left(xY + \frac{1}{1+x^2}\right) - \frac{2x}{(1+x^2)^2} \\ &= Y + x^2 Y + \frac{x^3 - x}{(1+x^2)^2} \end{aligned}$$

We have

$$\begin{aligned} |Y''(x)| &\le (1+x^2)|Y| + \left|\frac{x^3-x}{(1+x^2)^2}\right| \\ &\le 2|Y| + |x^3 - x| \\ &\le 2|Y| + \max_{x\in[0,1]} |x^3 - x| \\ &\le 2(e-1) + \frac{\sqrt{2}}{4} = N \end{aligned}$$

Also

$$|f_y(x,y)| = |x| \le 1 = L$$

The error bound can then be obtained from Theorem 2.4.

- Problem 2.8. Note that

$$\begin{aligned}\frac{d}{dt}(e^{5x}Y) &= 5e^{5x}Y + e^{5x}Y' \\ &= 5e^{5x}Y + e^{5x}(-5Y + xe^{-5x}) \\ &= x,\end{aligned}$$

which yield that

$$e^{5x}Y = \frac{1}{2}x^2 + C.$$

Since $Y(0) = 0$, we have $C = 0$. Thus,

$$Y = \frac{1}{2}x^2e^{-5x}.$$

Problem 2.10.

(a) Note that $f(x, y) = -xY^2 \leq 0$ on $[0, 1]$, which means that Y decreases from 1 as x increases from 0. However, Y can at most reach zero, since $Y'(x) = 0$ when $Y = 0$. Actually, $Y(x) = \frac{2}{x^2+2}$.

(b) We have

$$y_{i+1} = y_i + h(-x_iy_i^2) = y_i - \frac{i}{n^2}y_i^2 < y_i < y_0 = 1, \quad \forall i = 1, \ldots, n.$$

and

$$\begin{aligned}y_n &= y_{n-1} - \frac{n-1}{n^2}y_{n-1}^2 \\ &> y_{n-1} - \frac{n-1}{n^2}y_0^2 = y_{n-1} - \frac{n-1}{n^2} \\ &> y_{n-2} - \frac{n-2}{n^2} - \frac{n-1}{n^2} \\ &\ldots \\ &> y_1 - \frac{1}{n^2} - \frac{2}{n^2} - \ldots \frac{n-1}{n^2} \\ &= 1 - \frac{1}{n^2}(1 + \ldots + (n-1)) \\ &= 1 - \frac{1}{n^2}\frac{n}{2}(n-1) \\ &= 1 - \frac{1}{2}\frac{n-1}{n} > 1/2 > 0.\end{aligned}$$

(c) Note that

$$|f_y(x,y)| = |2xy| \leq 2 = L,$$

since $x \in [0,1]$ and $y \in (0,1]$, and

$$\begin{aligned} |Y''(x)| &= |Y^2 + 2xY| \\ &\leq Y^2 + 2|xY| \leq 3 = N. \end{aligned}$$

The error estimate can then be obtained from Theorem 2.4.

Problem 2.13. When $\beta_* = 0$, we have

$$\zeta'(x) \leq \gamma_*,$$

which yields that

$$\zeta(x) - \zeta(a) = \int_a^x \zeta'(x)dx \leq \int_a^x \gamma_* dx = \gamma_*(x-a).$$

This concludes the proof.

Problem 2.14. Using the definition

$$\zeta(x) = (Y^2(x) + \epsilon)^{1/2}.$$

We have from the proof of Theorem 2.10,

$$\zeta'(x) \leq \gamma_*,$$

for $\beta_* = 0$. Thus

$$\zeta(x) \leq \zeta(a) + \gamma_*(x-a)$$

or

$$\sqrt{Y^2(x) + \epsilon} \leq \sqrt{Y^2(a) + \epsilon} + \gamma_*(x-a).$$

Letting $\epsilon \to 0$, we have

$$|Y(x)| \leq |\alpha| + \gamma_*(x-a).$$