SOLUTION OF HW3

- Problem 3.1. The exact solution of $Y(x)$ is

$$Y(x) = 2e^x - 1.$$

Compare the numerical solution with the exact one.

- Problem 3.2. The exact solution of $Y(x)$ is

$$Y(x) = \frac{2}{2+x^2}.$$

Compare the numerical solution with the exact one.

- Problem 3.4. The second order Taylor series method is

$$\frac{y_{i+1} - y_i}{h} = f(x_i, y_i) + \frac{h}{2}\left[f_x(x_i, y_i) + f_y(x_i, y_i) f(x_i, y_i)\right].$$

For this problem,

$$f_x(x, y) = 1$$
$$f_y(x, y) = 2y.$$

We then have

$$\frac{y_{i+1} - y_i}{h} = (x_i + y_i^2) + \frac{h}{2}(1 + 2y_i(x_i + y_i^2)).$$

- Problem 3.7. For the third-order Taylor series method, we consider

$$Y(x_{i+1}) = Y(x_i) + Y'(x_i)h_i + \frac{Y''(x_i)}{2}h_i^2 + \frac{Y'''(x_i)}{6}h_i^3 + O(h_i^4).$$

Note that

$$
\begin{aligned}
Y'(x_i) =& f(x_i, Y_i) \\
Y''(x_i) =& f_x(x_i, Y_i) + f_y(x_i, Y_i) f(x_i, Y_i) \\
Y'''(x_i) =& f_{xx}(x_i, Y_i) + f_{xy}(x_i, Y_i) f(x_i, Y_i) + (f_{yx}(x_i, Y_i) \\
& + f_{yy}(x_i, Y_i) f(x_i, Y_i)) f(x_i, Y_i) + (f_x(x_i, Y_i) + f_y(x_i, Y_i) f(x_i, Y_i)) f_y(x_i, Y_i) \\
=& f_{xx}(x_i, Y_i) + 2 f_{xy}(x_i, Y_i) f(x_i, Y_i) + f_{yy}(x_i, Y_i) f^2(x_i, Y_i) \\
& + f_x(x_i, Y_i) f_y(x_i, Y_i) + f_y^2(x_i, Y_i) f(x_i, Y_i).
\end{aligned}
$$

Replacing Y_i with y_i and neglecting the remainder term, we have

$$
\begin{aligned}
\frac{y_{i+1} - y_i}{h_i} =& f(x_i, y_i) + \frac{h_i}{2}(f_x(x_i, y_i) + f_y(x_i, y_i) f(x_i, y_i)) \\
& + \frac{h_i^2}{6}(f_{xx}(x_i, y_i) + 2 f_{xy}(x_i, y_i) f(x_i, y_i) + f_{yy}(x_i, y_i) f^2(x_i, y_i) \\
& + f_x(x_i, y_i) f_y(x_i, y_i) + f_y^2(x_i, y_i) f(x_i, y_i)).
\end{aligned}
$$

It is seen the remainder term $O(h_i^4)$ suggests that $|Y^{(4)}(x)|$ is bounded and the stability requires that $|\partial_y \Phi_f| \leq L$. We can assume that $f(x, y)$ is a C^3 function on a compact subset in $\mathbb{R}^2$.

Problem 3.10. The exact solution is

$$Y(x) = x - x^2$$

Compare the numerical solution with the exact one. The second-order Taylor series method can exactly recover the true solution. The numerical scheme is:

$$\frac{y_{i+1} - y_i}{h} = f(x_i, y_i) + \frac{h}{2}(f_x + f_y f) = (1 - 2x_i) - h.$$

or

$$y_{i+1} = y_i + (1 - 2x_i)h - h^2$$

It is easy to check:

$$y_1 = h - h^2,$$

which is the exact solution. We argue by induction. Assume that $y_i = x_i - x_i^2 = ih - i^2h^2$. We have

$$\begin{aligned} y_{i+1} &= ih - i^2h^2 + (1-2ih)h - h^2 \\ &= (i+1)h - (i^2 + 2i + 1)h^2 \\ &= (i+1)h - (i+1)^2h^2. \end{aligned}$$

Problem 3.11. Note that

$$\frac{dY}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Instead of considering numerical integration directly, we may solve the ODE numerically

$$\frac{dY}{dx} = f(x), \quad Y(a) = 0, \quad \text{on } [a, b].$$

The numerical solution of $Y(x)$ at $x = b$ yields the approximate integration since

$$Y(b) = \int_a^b f(x)dx.$$