SOLUTION OF HW4

- Problem 4.1. For $f(x,y) = \cos(xy)$, we have

$$\begin{aligned} f_x &= -y\sin(xy) \\ f_y &= -x\sin(xy) \\ f_{xx} &= -y^2\cos(xy) \\ f_{xy} &= -\sin(xy) - xy\cos(xy) \\ f_{yy} &= -x^2\cos(xy) \\ f_{xxx} &= y^3\sin(xy) \\ f_{xxy} &= -2y\cos(xy) + xy^2\sin(xy) \\ f_{xyy} &= -2x\cos(xy) + x^2y\sin(xy) \\ f_{yyy} &= x^3\sin(xy). \end{aligned}$$

We then have the Taylor series of $f(x,y)$ as

$$\begin{aligned} \cos(xy) = 1 + \frac{1}{6}[&\theta^3y^3\sin(\theta^2xy)x^3 + (\theta^3xy^2\sin(\theta^2xy) - 2\theta y\cos(\theta^2xy))x^2y \\ &+ (\theta^3xy^2\sin(\theta^2xy) - 2\theta x\cos(\theta^2xy))xy^2 + \theta^3x^3\sin(\theta^2xy)y^3], \end{aligned}$$

with $\theta \in (0,1)$, where the first- and second-order terms are zero at $(0,0)$.

- Problem 4.2. For (4.9), the requirement for the 2nd-order accuracy is given by (4.19):

$$\begin{aligned} \eta_i + \beta_i &= 1, \\ 2\beta_i\gamma_i &= 1, \\ 2\beta_i\delta_i &= f(x_i, Y_i). \end{aligned}$$

Let $\beta_i = \beta$, and $\eta_i = 1-\beta$, $\gamma_i = \frac{1}{2\beta}$, and $\delta_i = \frac{f(x_i,Y_i)}{2\beta}$. We have RK2 as

$$y_{i+1} = y_i + h[(1-\beta)k_1 + \beta k_2],$$

where $k_1 = f(x_i, y_i)$ and $k_2 = f(x_i + \frac{h_i}{2\beta}, y_i + \frac{h_i}{2\beta}k_1)$. When $\beta = 0$, we have the Euler's method:

$$y_{i+1} = y_i + h_i f(x_i, y_i).$$

When $\beta = 1$, we have the modified Euler's method

$$y_{i+1} = y_i + h_i f(x_i + h_i/2, y_i + (h_i/2)k_1).$$

When $\beta = 1/2$, we obtain RK2 (4.21).

- Problem 4.3.The 2nd-order Taylor series method takes the form:

$$\frac{y_{i+1} - y_i}{h_i} = f(x_i, y_i) + \frac{h_i}{2}[f_x + f_y f].$$

We have $f(x, y) = x + y^2$ with

$$f_x = 1, \quad f_y = 2y.$$

So the 2nd-order Taylor series method for problem studied is

$$\frac{y_{i+1} - y_i}{h_i} = x_i + y_i^2 + \frac{h_i}{2}(1 + 2x_i y_i + 2y_i^3).$$

The RK2 takes the form:

$$\begin{aligned}
\frac{y_{i+1} - y_i}{h_i} &= \frac{1}{2}(k_1 + k_2) \\
k_1 &= f(x_i, y_i) = x_i + y_i^2 \\
k_2 &= f(x_i + h_i, y_i + h_i k_1) = (x_i + h_i) + (y_i + h_i(x_i + y_i^2))^2 \\
&= x_i + y_i^2 + h_i(1 + 2y_i(x_i + y_i^2)) + h_i^2(x_i + y_i^2)^2.
\end{aligned}$$

More specifically,

$$\frac{y_{i+1} - y_i}{h_i} = x_i + y_i^2 + \frac{h_i}{2}(1 + 2x_i y_i + 2y_i^3) + h_i^2(x_i + y_i^2)^2.$$

So the difference between RK2 and 2nd-order Taylor series method is the second-order term

$$h_i^2(x_i + y_i^2),$$

which does not affect the order of the truncation error.

- Problem 4.6. Compare equation (4.26), (4.27) and the scheme in problem 4.6. We have

$$\alpha_1 = 1/4,\ \alpha_2 = 3/8,\ \alpha_3 = 3/8,\ \beta_2 = 2/3,\ \beta_3 = 2/3.$$

It is easy to verify that these coefficients satisfy (4.40), which is required by RK3.