SOLUTION OF HW5

- Problem 5.1. The second-order Taylor series method is:

$$\frac{y_{i+1}-y_i}{h} - f(x_i, y_i) - \frac{h}{2}\left[f_x(x_i, y_i) + f_y(x_i, y_i) f(x_i, y_i)\right] = 0.$$

The truncation error is:

$$\begin{aligned}
\tau_{i+1} &= \frac{Y_{i+1}-Y_i}{h} - f(x_i, Y_i) - \frac{h}{2}\left[f_x(x_i, Y_i) + f_y(x_i, Y_i) f(x_i, Y_i)\right] \\
&= \frac{Y_{i+1}-Y_i}{h} - f(x_i, Y_i) - \frac{h}{2} Y_i'' \\
&= \frac{Y_i + Y_i' h + \frac{h^2}{2} Y_i'' + \frac{h^3}{6} Y_i''' + O(h^4) - Y_i}{h} - f(x_i, Y_i) - \frac{h}{2} Y_i'' \\
&= \frac{h^2}{6} Y_i''' + O(h^3) \\
&= \frac{h^2}{6}(f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_x f_y + f_y^2 f)|_{x_i, Y_i} + O(h^3).
\end{aligned}$$

So

$$T(x, y) = \frac{1}{6}(f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_x f_y + f_y^2 f).$$

- Problem 5.2. The RK-2 is:

$$\begin{aligned}
y_{i+1} &= y_i + h_i\left(\frac{k_1 + k_2}{2}\right), \\
k_1 &= f(x_i, y_i), \\
k_2 &= f(x_i + h, y_i + h k_1).
\end{aligned}$$

The truncation error is:

$$\tau_{i+1} = \frac{Y_{i+1} - Y_i}{h} - \frac{1}{2}(f(x_i, Y_i) + f(x_i + h, Y_i + h f(x_i, Y_i)))$$

We only need to look at the third-order in the Taylor expansion of Y_{i+1} and the second-order term in the Taylor expansion of $f(x_i + h, Y_i + h f(x_i, Y_i))$ since the low-order terms will be cancelled out by definition. We then have

$$\begin{aligned}\tau_{i+1} &= \frac{h^2}{6}Y_i''' - \frac{h^2}{4}(f_{xx} + 2f_{xy}f + f_{yy}f^2)|_{x_i,Y_i} + O(h^3)\\ &= \frac{h^2}{12}(2f_xf_y + 2f_y^2f - f_{xx} - 2f_{xy}f - f_{yy}f^2)|_{x_i,Y_i} + O(h^3).\end{aligned}$$

So

$$T(x,y) = \frac{1}{12}(2f_xf_y + 2f_y^2f - f_{xx} - 2f_{xy}f - f_{yy}f^2).$$

- Problem 5.3. From Example 5.2, we have, for Euler's method,

$$T(x,y) = \frac{1}{2}(f_x + f_yf).$$

Let $\tilde{\Phi}_f$ and Φ_f correspond to RK-2 and Euler's methods respectively. We have from equation (5.14) that

$$\begin{aligned}\tilde{\Phi}_f - \Phi_f &= \frac{1}{2}(f(x,y) + f(x+h, y+hf)) - f(x,y)\\ &= \frac{1}{2}(f(x,y) + f(x,y) + f_xh + f_yfh + O(h^2)) - f(x,y)\\ &= \frac{h}{2}(f_x + f_yf) + O(h^2)\\ &= T(x,y)h + O(h^2).\end{aligned}$$

- Problem 5.4. For the Euler's method, we have

$$\tau_{i+1} = T(x_i, Y_i)h + O(h^2) = \frac{h}{2}(f_x(x_i,Y_i) + f_y(x_i,Y_i)f(x_i,Y_i)) + O(h^2).$$

If $f_x(a,\alpha)$ and $f_y(a,\alpha)$ is known, we have

$$|\tau_1| \approx \left|\frac{h_0}{2}(f_x(a,\alpha)) + f_y(a,\alpha)f(a,\alpha)\right| \leq \text{Tol},$$

which yields an estimate of h_0.

- Problem 5.5. The truncation error of Euler's method is

$$\begin{aligned}\tau_{i+1} &= \frac{h}{2}(f_x + f_yf)|_{x_i,Y_i} + O(h^2)\\ &= \frac{h}{2}f'(x_i) + O(h^2).\end{aligned}$$

If the conclusion holds, we should have

$$O(h^2) = \frac{f(x_i+h) - f(x_i)}{2} - \frac{h}{2}f'(x_i) = \frac{1}{2}(f(x_i) + f'(x_i)h + O(h^2) - f(x_i)) - \frac{h}{2}f'(x_i) = O(h^2).$$