SOLUTION OF HW7

- Problem 6.23

The general multistep method is:

$$\frac{1}{h}\sum_{k=0}^{\ell} a_{\ell-k}y_{i-k} - \sum_{k=0}^{\ell} b_{\ell-k}f(x_{i-k}, y_{i-k}) = 0.$$

The truncation error is

$$\tau = \frac{1}{h}\sum_{k=0}^{\ell} a_{\ell-k}Y_{i-k} - \sum_{k=0}^{\ell} b_{\ell-k}f(x_{i-k}, Y_{i-k}).$$

To make the scheme consistent, we need that τ is at least $O(h)$ such that $\tau \to 0$ as $h \to 0$. We then have

$$\begin{aligned}
\tau =& \frac{1}{h}\sum_{k=0}^{\ell} a_{\ell-k}Y_{i-k} - \sum_{k=0}^{\ell} b_{\ell-k}Y'_{i-k} \\
=& \frac{1}{h}a_\ell Y_i + \frac{1}{h}\sum_{k=1}^{\ell} a_{\ell-k}(Y_i + Y_i'(-kh) + O(h^2)) \\
& - b_\ell Y_i' - \sum_{k=1}^{\ell} b_{\ell-k}(Y_i' + O(h)) \\
=& \frac{1}{h}Y_i\sum_{k=0}^{\ell} a_{\ell-k} - Y_i'\left(\sum_{k=1}^{\ell} k a_{\ell-k} + \sum_{k=0}^{\ell} b_{\ell-k}\right) + O(h)
\end{aligned}$$

The first two terms must be equal to zero, which implies that

$$\sum_{k=0}^{\ell} a_{\ell-k} = \sum_{k=0}^{\ell} a_k = 0,$$

and

$$\begin{aligned}\sum_{k=0}^{\ell} b_{\ell-k} &= \sum_{k=0}^{\ell} b_k \\ &= -\sum_{k=1}^{\ell} k a_{\ell-k} = -\sum_{k=0}^{\ell-1}(\ell-k)a_k \\ &= \sum_{k=1}^{\ell-1} k a_k - \ell\sum_{k=0}^{\ell-1} a_k \\ &= \sum_{k=1}^{\ell-1} k a_k + \ell a_\ell = \sum_{k=1}^{\ell} k a_k,\end{aligned}$$

where in the second last step we applied the fact that

$$\sum_{k=0}^{\ell} a_k = 0 \Rightarrow \sum_{k=0}^{\ell-1} a_k = -a_\ell.$$

By the definition, 1 is a root of the characteristic equation given by (6.55).

- Problem 6.25

We approximate $Y'(x)$ using polynomial interpolation on $(x_{i-1}, Y'(x_{i-1}))$, $(x_{i-2}, Y'(x_{i-2}))$, and $(x_{i-3}, Y'(x_{i-3}))$. We have

$$\begin{aligned}P_2(x) =& Y'_{i-1}\frac{(x-x_{i-2})(x-x_{i-3})}{(x_{i-1}-x_{i-2})(x_{i-1}-x_{i-3})} \\ &+ Y'_{i-2}\frac{(x-x_{i-1})(x-x_{i-3})}{(x_{i-2}-x_{i-1})(x_{i-2}-x_{i-3})} \\ &+ Y'_{i-3}\frac{(x-x_{i-1})(x-x_{i-2})}{(x_{i-3}-x_{i-1})(x_{i-3}-x_{i-2})} \\ =& \frac{1}{2h^2}Y'_{i-1}(x-x_{i-2})(x-x_{i-3}) - \frac{1}{h^2}Y'_{i-2}(x-x_{i-1})(x-x_{i-3}) \\ &+ \frac{1}{2h^2}Y'_{i-3}(x-x_{i-1})(x-x_{i-2}).\end{aligned}$$

We then have

$$\begin{aligned} Y_i - Y_{i-1} &= \int_{x_{i-1}}^{x_i} Y'(x)dx \\ &\approx \int_{x_{i-1}}^{x_i} P_2(x)dx \\ &= \frac{Y'_{i-1}}{2h^2}\int_0^h (x+h)(x+2h)dx - \frac{Y'_{i-2}}{h^2}\int_0^h x(x+2h)dx + \frac{Y'_{i-3}}{2h^2}\int_0^h x(x+h)dx \\ &= \frac{23}{12}hY'_{i-1} - \frac{4}{3}hY'_{i-2} + \frac{5}{12}hY'_{i-3} \\ &= \frac{23}{12}hf(x_{i-1},Y_{i-1}) - \frac{4}{3}hf(x_{i-2},Y_{i-2}) + \frac{5}{12}hf(x_{i-3},Y_{i-3}), \end{aligned}$$

which suggests the numerical scheme:

$$\frac{y_i - y_{i-1}}{h} = \frac{23}{12}f(x_{i-1},y_{i-1}) - \frac{4}{3}f(x_{i-2},y_{i-2}) + \frac{5}{12}f(x_{i-3},y_{i-3}),$$

i.e., the A-B-III.

- Problem 6.28

We approximate $Y'(x)$ using polynomial interpolation on $(x_i, Y'(x_i))$, $(x_{i-1}, Y'(x_{i-1}))$, and $(x_{i-2}, Y'(x_{i-2}))$. We have

$$\begin{aligned} P_2(x) =& Y'_i \frac{(x-x_{i-1})(x-x_{i-2})}{(x_i-x_{i-1})(x_i-x_{i-2})} \\ &+ Y'_{i-1}\frac{(x-x_i)(x-x_{i-2})}{(x_{i-1}-x_i)(x_{i-1}-x_{i-2})} \\ &+ Y'_{i-2}\frac{(x-x_i)(x-x_{i-1})}{(x_{i-2}-x_i)(x_{i-2}-x_{i-1})} \\ =& \frac{1}{2h^2}Y'_i(x-x_{i-1})(x-x_{i-2}) - \frac{1}{h^2}Y'_{i-1}(x-x_i)(x-x_{i-2}) \\ &+ \frac{1}{2h^2}Y'_{i-2}(x-x_i)(x-x_{i-1}). \end{aligned}$$

We then have

$$\begin{aligned}
Y_i - Y_{i-1} &= \int_{x_{i-1}}^{x_i} Y'(x)dx \\
&\approx \int_{x_{i-1}}^{x_i} P_2(x)dx \\
&= \frac{Y_i'}{2h^2}\int_0^h x(x+h)dx - \frac{Y_{i-1}'}{h^2}\int_0^h (x-h)(x+h)dx + \frac{Y_{i-2}'}{2h^2}\int_0^h (x-h)xdx \\
&= \frac{5}{12}hY_i' + \frac{2}{3}hY_{i-1}' - \frac{1}{12}hY_{i-2}' \\
&= \frac{5}{12}hf(x_i, Y_i) + \frac{2}{3}hf(x_{i-1}, Y_{i-1}) - \frac{1}{12}hf(x_{i-2}, Y_{i-2}),
\end{aligned}$$

which suggests the numerical scheme:

$$\frac{y_i - y_{i-1}}{h} = \frac{5}{12}f(x_i, y_i) + \frac{2}{3}f(x_{i-1}, y_{i-1}) - \frac{1}{12}f(x_{i-2}, y_{i-2}),$$

i.e., A-M-II.