

SOLUTION OF HW8

- Problem 7.1. From example 4.4, we know that

$$y_{i+1} = \left(1 + z + \frac{z^2}{2}\right) y_i,$$

which is the same as the second order Taylor series method. The region of absolute stability is then defined as

$$R_a = \left\{z : \left|1 + z + \frac{1}{2}z^2\right| < 1\right\}.$$

- Problem 7.3. For an explicit multistep method, we have from equation (7.20)

$$z = \frac{\sum_{k=0}^{\ell} a_k r^k}{\sum_{k=0}^{\ell-1} b_k r^k},$$

since $b_\ell = 0$. Divided both the numerator and denominator by $r^{\ell-1}$, we have

$$z = \frac{a_\ell r + \sum_{k=0}^{\ell-1} a_k r^{k-\ell+1}}{\sum_{k=0}^{\ell-1} b_k r^{k-\ell+1}}.$$

When $|r| \gg 1$ is large, we have

$$z \approx \frac{a_\ell r + a_{\ell-1}}{b_{\ell-1}},$$

which implies that z can have a large negative real part. Thus no explicit multistep methods are A -stable.

- Problem 7.4. From exercise 6.23, we know that all multistep methods should satisfy

$$\sum_{k=0}^{\ell} a_k = 0$$

for consistency. This condition corresponds to the fact that equation (7.19) has a root $r = 1$ when $z = 0$. However, by definition, $z = 0$ belongs the region of absolute stability when all roots of (7.19) are strictly less than 1. Thus, $z = 0$ does not belong to the region of absolute stability of any multistep method.

- Problem 7.5. For A-M-I, the boundary of the region of absolute stability is given by

$$z = \frac{r-1}{r/2 + 1/2}.$$

Let $r = e^{i\theta}$. We have

$$\begin{aligned} z &= 2 \frac{r-1}{r+1} \\ &= 2 \frac{\cos \theta - 1 + i \sin \theta}{\cos \theta + 1 + i \sin \theta} \\ &= 2 \frac{-\sin^2 \frac{\theta}{2} + i \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + i \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\ &= 2 \tan \frac{\theta}{2} \left(\frac{-\sin \frac{\theta}{2} + i \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}} \right) \\ &= 2 \tan \frac{\theta}{2} \left(\frac{-\sin \frac{\theta}{2} \cos \frac{\theta}{2} + i \sin^2 \frac{\theta}{2} + i \cos^2 \frac{\theta}{2} + \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}} \right) \\ &= 2i \tan \frac{\theta}{2}, \end{aligned}$$

which is the imaginary axis for $0 \leq \theta \leq 2\pi$.

- Problem 7.6. For A-B-II, we have

$$\begin{aligned} a_2 &= 1, a_1 = -1, a_0 = 0, \\ b_2 &= 0, b_1 = \frac{3}{2}, b_0 = -\frac{1}{2}. \end{aligned}$$

If $z = -1$ belongs to the region of absolute stability, we need that the roots of

$$r^2 - r + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$$

satisfy $|r| < 1$. The equation has two roots $r = -1, \frac{1}{2}$, which does not satisfy the requirement.

- Problem 7.7. The region R_a satisfies

$$R_a = \{z : |1 + z + z^2/2| < 1\}.$$

For $z = x + iy$, the boundary of R_a satisfies

$$\left(1 + x + \frac{x^2 - y^2}{2}\right)^2 + (y + xy)^2 = 1.$$

We may let

$$1 + x + \frac{x^2 - y^2}{2} = \cos \theta,$$
$$y(1 + x) = \sin \theta.$$

From the second equation, we have $y = \frac{\sin \theta}{1+x}$, which yields

$$2(1 + x) + x^2 - \frac{\sin^2 \theta}{(1 + x)^2} = 2 \cos \theta.$$

Let $w = (1 + x)^2$. We have

$$1 + w - \frac{\sin^2 \theta}{w} = 2 \cos \theta,$$

i.e.,

$$w^2 + (1 - 2 \cos \theta)w - \sin^2 \theta = 0.$$

Since $w = (1 + x)^2 > 0$, we have

$$w = \frac{(2 \cos \theta - 1) + \sqrt{5 - 4 \cos \theta}}{2},$$

from which we obtain x and then y . Let θ go from 0 to 2π . We obtain the boundary of R_a .