CATEGORIES OF MODULES OVER AN AFFINE KAC–MOODY ALGEBRA AND FINITENESS OF THE KAZHDAN–LUSZTIG TENSOR PRODUCT

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ABSTRACT. To each category $\mathcal{C}$ of modules of finite length over a complex simple Lie algebra $\mathfrak{g}$, closed under tensoring with finite dimensional modules, we associate and study a category $\mathcal{AFF}(\mathcal{C})_\kappa$ of smooth modules (in the sense of Kazhdan and Lusztig [13]) of finite length over the corresponding affine Kac–Moody algebra in the case of central charge less than the critical level. Equivalent characterizations of these categories are obtained in the spirit of the works of Kazhdan–Lusztig [13] and Lian–Zuckerman [18, 19]. In the main part of this paper we establish a finiteness result for the Kazhdan–Lusztig tensor product which can be considered as an affine version of a theorem of Kostant [17]. It contains as special cases the finiteness results of Kazhdan, Lusztig [13] and Finkelberg [7], and states that for any subalgebra $\mathfrak{f}$ of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$ the "affinization" of the category of finite length admissible $(\mathfrak{g}, \mathfrak{f})$ modules is stable under Kazhdan–Lusztig's tensoring with the "affinization" of the category of finite dimensional $\mathfrak{g}$ modules (which is $\mathcal{O}_\kappa$ in the notation of [13, 14, 15]).

1. INTRODUCTION

Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\tilde{\mathfrak{g}}$ be the corresponding untwisted affine Kac–Moody algebra, see [12]. It is the central extension of the loop algebra $\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ by

$$[xt^n, yt^m] = [x, y]t^{n+m} + n\delta_{n,-m}(x, y)K, \; x, y \in \mathfrak{g} \tag{1.1}$$

where $(.,.)$ denotes the invariant bilinear form on $\mathfrak{g}$ normalized by $(\alpha, \alpha) = 2$ for long roots α. The affine Kac–Moody algebra $\tilde{\mathfrak{g}}$ is a $\mathbb{Z}$ graded Lie algebra by

$$\deg xt^n = n, \; \deg K = 0. \tag{1.2}$$

Set for shortness $\tilde{\mathfrak{g}}_+ = t\mathfrak{g}[t] \hookrightarrow \tilde{\mathfrak{g}}$. Denote the graded components of $U(\tilde{\mathfrak{g}})$ and $U(\tilde{\mathfrak{g}}_+)$ of degree N by $U(\tilde{\mathfrak{g}})^N$ and $U(\tilde{\mathfrak{g}}_+)^N$, respectively.

Definition 1.1. (Kazhdan–Lusztig) For a $\tilde{\mathfrak{g}}$ module V define

$$V(N) = \mathrm{Ann}_{U(\tilde{\mathfrak{g}}_+)^N} V \subset V, \; N \in \mathbb{Z}_{>0}.$$

A $\tilde{\mathfrak{g}}$ module V is called *strictly smooth* if

$$\cup_N V(N) = V.$$

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Recall also that a $\tilde{\mathfrak{g}}$ module V is called *smooth* if each vector in V is annihilated by $t^N\mathfrak{g}[t]$ for all sufficiently large integers N.

Clearly

$$V(1) \subset V(2) \subset \dots$$

and each $V(N)$ is a $\mathfrak{g}$ module since $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ normalizes $U(\tilde{\mathfrak{g}}_+)^N$. Each strictly smooth $\tilde{\mathfrak{g}}$ module is a module for the topological algebra $\hat{\mathfrak{g}}$ which is the central extension of $\mathfrak{g}((t))$ by (1.1).

The main objects of our consideration are the following categories of $\tilde{\mathfrak{g}}$ modules.

Definition 1.2. Let $\mathcal{C}$ be a full subcategory of the category of $\mathfrak{g}$ modules and $\kappa \in \mathbb{C}$. Define $\mathcal{AFF}(\mathcal{C})_\kappa$ to be the full subcategory of the category $\tilde{\mathfrak{g}}$ modules of central charge $\kappa - h^\vee$ consisting of strictly smooth, finitely generated $\tilde{\mathfrak{g}}$ modules V such that

$$V(N) \in \mathcal{C}, \text{ for all } N = 1, 2, \dots$$

Here $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$.

We will restrict our attention only to categories $\mathcal{C}$ of finite length $\mathfrak{g}$ modules which are closed under tensoring with the adjoint representation and taking subquotients. The following are some important examples.

Example 1.3. (1) Let $\mathcal{C} = \mathcal{F}in_{\mathfrak{g}}$ be the category of finite dimensional $\mathfrak{g}$ modules. Then $\mathcal{AFF}(\mathcal{F}in_g)_\kappa$ is Kazhdan–Lusztig's category $\mathcal{O}_\kappa$ defined in [13].

(2) Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$, i.e. $\mathfrak{g}$ is completely reducible as an $\mathfrak{f}$ module under the adjoint representation, see [6, section 1.7] for details. Consider the category of finite length admissible $(\mathfrak{g}, \mathfrak{f})$ modules – $\mathfrak{g}$ modules which restricted to $\mathfrak{f}$ decompose to a direct sum of finite dimensional irreducible $\mathfrak{f}$ modules, each of which occurs with finite multiplicity. It will be denoted by $\mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$. Kostant's theorem states that $\mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ is closed under tensoring with finite dimensional $\mathfrak{g}$ modules, [17, Theorem 3.5]. Clearly $\mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ is also closed under taking subquotients. The example (1) is obtained from (2) when one specializes $\mathfrak{f} = \mathfrak{g}$.

(3) Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ and $\mathfrak{k}$ be the complexification of a maximal compact subalgebra of $\mathfrak{g}_0$. Then $\mathfrak{k}$ is reductive in $\mathfrak{g}$ and the category from part 2 specializes to the category of Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$ modules to be denoted by $\mathcal{HC}_{(\mathfrak{g},\mathfrak{k})}$.

Lian and Zuckerman [18, 19] studied the category of $\mathbb{Z}$-graded $\tilde{\mathfrak{g}}$ modules

$$V = \oplus_{n\in\mathbb{Z}} V_n \tag{1.3}$$

for the grading (1.2) of $\tilde{\mathfrak{g}}$ for which V_n is a Harish-Chandra $\mathfrak{g}$-module and $V_n = 0$ for $n \gg 0$. Later we will show that this is exactly the category $\mathcal{AFF}(\mathcal{HC}_{(\mathfrak{g},\mathfrak{k})})$ (see Proposition 3.8 below) and will obtain a characterization of all categories $\mathcal{AFF}(\mathcal{C})_\kappa$ in this spirit. At the same time the above $\mathbb{Z}$-grading is not canonical and is not preserved in general by $\tilde{\mathfrak{g}}$ homomorphisms. In the general case there is a canonical $\mathbb{C}$ grading with similar properties, obtained from the generalized eigenvalues of the Sugawara operator L_0, see Proposition 1.5. It is preserved by arbitrary $\tilde{\mathfrak{g}}$ homomorphisms.

I. Frenkel and Malikov [8] defined a category of affine Harish-Chandra bimodules from the point of view of the construction of Bernstein and S. Gelfand [4], treating case when $\mathfrak{g}_0$ is a complex simple Lie algebra considered as a real algebra. It is unclear how in this case [8] is related to [18] and the constructions of this paper.

(4) Finally as another specialization of (2) one can choose $\mathfrak{f}$ to be a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and obtain the category of weight modules for $\mathfrak{g}$. Its subcategory

$\mathcal{O}$ of Bernstein–Gelfand–Gelfand (for a fixed Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ of $\mathfrak{g}$) is also closed under tensoring with finite dimensional $\mathfrak{g}$ modules and taking subquotients. From the results in Section 3, in particular Theorem 3.5, it follows that the category $\mathcal{AFF}(\mathcal{O})_\kappa$ for $\kappa \notin \mathbb{R}_{\geq 0}$ is essentially the affine BGG category $\mathcal{O}$ of $\tilde{\mathfrak{g}}$ modules with central charge $\kappa - h^\vee$, see [12]. The generator d for the extended affine Kac–Moody algebra acts on $\tilde{\mathfrak{g}}$ modules from $\mathcal{AFF}(\mathcal{O})_\kappa$ by $const - L_0$ where L_0 is the 0th Sugawara operator, see [12] and Section 2 below.

The following Theorem summarizes some of the main properties of the categories $\mathcal{AFF}(\mathcal{C})_\kappa$.

Theorem 1.4. *Assume that the category $\mathcal{C}$ of finite length $\mathfrak{g}$ modules is closed under tensoring with the adjoint representation and taking subquotients, and that $\kappa \notin \mathbb{R}_{\geq 0}$. Then the following hold:*

(1) For any $M \in \mathcal{C}$ the induced module

$$\operatorname{Ind}(M)_\kappa = U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} M$$

belongs to $\mathcal{AFF}(\mathcal{C})_\kappa$. (In the definition of the tensor product the central element K acts on M by $(\kappa - h^\vee)$ and $\tilde{\mathfrak{g}}_+ = t\mathfrak{g}[t]$ annihilates M.)

(2) The $\tilde{\mathfrak{g}}$ modules in $\mathcal{AFF}(\mathcal{C})_\kappa$ have finite length and are exactly the quotients of the induced modules from $U(\mathfrak{g}[t])$ modules $\mathcal{N}$ annihilated by the degree n component $U(\tilde{\mathfrak{g}}_+)^n$ of $U(\tilde{\mathfrak{g}}_+)$ for some $n \gg 0$, recall (1.2).

(3) $\mathcal{AFF}(\mathcal{C})_\kappa$ is closed under taking subquotients.

(4) If the original category of $\mathfrak{g}$ modules $\mathcal{C}$ is closed under extension then the category $\mathcal{AFF}(\mathcal{C})_\kappa$ is closed under extension inside the category of $\tilde{\mathfrak{g}}$ modules of central charge $\kappa - h^\vee$.

Let us also note that every irreducible module in $\mathcal{AFF}(\mathcal{C})_\kappa$ is (the) unique irreducible quotient of $\operatorname{Ind}(M)_\kappa$ for some irreducible $\mathfrak{g}$ module M. In addition for two nonisomorphic irreducible $\mathfrak{g}$ modules the related irreducible $\tilde{\mathfrak{g}}$ modules are nonisomorphic.

Each smooth $\tilde{\mathfrak{g}}$ module of fixed central charge, different from the critical level $-h^\vee$, canonically gives rise to a representation of the Virasoro algebra by the Sugawara operators L_k, see [12] or the review in Section 2. The generalized eigenspaces V^ξ of the operator L_0 ($\xi \in \mathbb{C}$) are naturally $\mathfrak{g}$ modules. One has the following characterization of $\mathcal{AFF}(\mathcal{C})_\kappa$.

Proposition 1.5. *In the setting of Theorem 1.4 the category $\mathcal{AFF}(\mathcal{C})_\kappa$ consists exactly of those finitely generated smooth $\tilde{\mathfrak{g}}$ modules of central charge $\kappa - h^\vee$ for which*

$$V = \bigoplus_{\xi:\ \xi - \xi_i \in \mathbb{Z}_{\geq 0}} V^\xi \quad \text{for some} \quad \xi_1, \ \dots, \ \xi_n \in \mathbb{C} \tag{1.4}$$

and $V^\xi \in \mathcal{C}$.

Kazhdan and Lusztig [13, 14, 15] defined a fusion tensor product $V_1 \dot{\otimes} V_2$ of any two strictly smooth $\tilde{\mathfrak{g}}$ modules V_1 and V_2, motivated by developments in conformal field theory [3, 16, 20, 21, 1]. In a related series of works Huang and Lepowsky developed a theory of tensor products for modules over vertex operator algebras. The modules $V_1 \dot{\otimes} V_2$ obtained from the Kazhdan–Lusztig tensor product are strictly smooth but in general it is hard to check under what conditions they have finite length. Kazhdan and Lusztig proved in [13] that the category $\mathcal{O}_\kappa$ is closed under the fusion tensor product and further used it to construct functors to representations of quantized universal enveloping algebras. In the case of positive integral central charge Cherednik defined in [5] a version of the fusion product for a category of integrable modules and showed that the latter is invariant under that tensor product.

For the affine algebra $\tilde{\mathfrak{g}}$ and the fusion tensor product the category $\mathcal{O}_\kappa$ plays the role of the category of finite dimensional modules for the algebra $\mathfrak{g}$. *We prove the following finiteness property of the Kazhdan–Lusztig tensor product. It is an affine version of Kostant's theorem* [13] *that tensoring with finite dimensional* $\mathfrak{g}$ *modules preserves the category of* $\mathfrak{g}$ *modules* $\mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$.

Theorem 1.6. *Let* $\kappa \notin \mathbb{R}_{\geq 0}$ *and* $\mathfrak{f}$ *be a subalgebra of the complex simple Lie algebra* $\mathfrak{g}$ *which is reductive in* $\mathfrak{g}$. *Then the Kazhdan–Lusztig fusion tensor product of a* $\tilde{\mathfrak{g}}$ *module in* $\mathcal{O}_\kappa$ *and a* $\tilde{\mathfrak{g}}$ *module in* $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ *belongs to* $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$:

$$\dot{\otimes}\colon \mathcal{O}_\kappa \times \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa \to \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa, \quad \dot{\otimes}\colon \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa \times \mathcal{O}_\kappa \to \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa.$$

Thus $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ become (bi)module categories for the ring category $\mathcal{O}_\kappa$ using the left and right Kazhdan–Lusztig tensoring with objects from $\mathcal{O}_\kappa$. The associativity is defined by a straightforward generalization of [14]. There are also braiding isomorphisms, intertwining the two tensor products, again defined as in [14].

In the special case $\mathfrak{f} = \mathfrak{g}$ one has $\mathcal{O}_\kappa = \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{g})})_\kappa$ and we get just another proof of one of the main results of Kazhdan and Lusztig in [13] that

$$\dot{\otimes}\colon \mathcal{O}_\kappa \times \mathcal{O}_\kappa \to \mathcal{O}_\kappa.$$

The novelty in this paper is a direct proof of Theorem 1.6 which in the main part is independent of the one of Kazhdan and Lusztig [13, Section 3] who use Soergel's generalized Bernstein–Gelfand–Gelfand (Brauer) reciprocity [2, Section 3.2]. They show that if V is a strictly smooth $\tilde{\mathfrak{g}}$ module of central charge less than the critical level and $V(1)$ is finite dimensional then $V \in \mathcal{O}_\kappa$. Unfortunately BGG reciprocity does not to hold in the categories of $\tilde{\mathfrak{g}}$ modules $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ and even in the standard category of Harish-Chandra $\mathfrak{g}$ modules. (This was communicated to us by G. Zuckerman.) Moreover it seems that in general strict smoothness of a $\tilde{\mathfrak{g}}$ module V and $V(1) \in \mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ do not imply that $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ (the problem being that V might have infinite length) but we do not know a counterexample at this time.

As another consequence of the special case when $\mathfrak{f}$ is a Cartan subalgebra of $\mathfrak{g}$ (see part 4 of Example 1.3) one easily obtains that the fusion tensor product of a module in $\mathcal{O}_\kappa$ and a module in the affine category $\mathcal{O}$ with central charge $\kappa - h^\vee$ is again a module in the affine category $\mathcal{O}$. This was previously proved by Finkelberg [7].

Theorem 1.6 opens up the possibility for defining translation functors [22, 11, 10] in the categories $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ using the Kazhdan–Lusztig fusion tensor product. It is also interesting to investigate if tensoring with $\mathcal{O}_\kappa$ can be used to define functors from the categories $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ to some categories of representations of the corresponding quantum group $U_q(\mathfrak{g})$ in the spirit of Kazhdan and Lusztig [15]. This can be viewed as a procedure of "quantizing categories of modules over a complex simple Lie algebra" by considering first categories of modules for the related affine Kac–Moody algebra.

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2. Properties of induced modules

Recall that on any smooth $\tilde{\mathfrak{g}}$ module V of central charge $\kappa - h^\vee$ ($\kappa \neq 0$) there is a well defined action of the Sugawara operators (see [12] for details)

$$L_k = \frac{1}{2\kappa}\sum_p \sum_{j\in\mathbb{Z}} :(x_p t^{-j})(x_p t^{j+k}): \tag{2.1}$$

where the first sum is over an orthonormal basis $\{x_p\}$ of $\mathfrak{g}$ with respect to the bilinear form $(.,.)$. In (2.1) the standard normal ordering is used, prescribing pulling to the right the term xt^n with larger n.

The Sugawara operators define a representation of the Virasoro algebra on V [12] for which

$$[L_k, xt^n] = -n(xt^{n+k}). \tag{2.2}$$

For a $\tilde{\mathfrak{g}}$ module V consider the generalized eigenspaces of the operator L_0

$$V^\xi = \{v \in V \mid (L_0 - \xi)^n v = 0 \text{ for some integer } n\},\ \xi \in \mathbb{C}.$$

Since L_0 commutes with $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ (see (2.2)) each V^ξ is a $\mathfrak{g}$ module.

Definition 2.1. For a $\mathfrak{g}$ module M and $\kappa \in \mathbb{C}$ define the Weyl module

$$\operatorname{Ind}(M)_\kappa = U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} M$$

of $\tilde{\mathfrak{g}}$ where $\mathfrak{g}[t]$ acts through the quotient map $\mathfrak{g}[t] \to \mathfrak{g}[t]/t\mathfrak{g}[t] \cong \mathfrak{g}$ and K acts by $(\kappa - h^\vee)\mathrm{id}$.

Proposition 2.2. *Assume that M is a $\mathfrak{g}$ module on which the Casimir Ω of $\mathfrak{g}$ acts by $a.\mathrm{id}$ for some $a \in \mathbb{C}$ and that κ is a nonzero complex number. Then:*

(1) $\operatorname{Ind}(M)_\kappa = \bigoplus_{\xi\in\mathbb{C}} \operatorname{Ind}(M)^\xi_\kappa$ and $\operatorname{Ind}(M)^\xi_\kappa$ are actual (not generalized) eigenspaces of L_0.

(2) $\operatorname{Ind}(M)^\xi_\kappa = 0$ unless $\xi \in a/2\kappa + \mathbb{Z}_{\geq 0}$ and as $\mathfrak{g}$ modules

$$\operatorname{Ind}(M)^{(a/2\kappa+n)}_\kappa \cong M \otimes S(\mathrm{ad})^n \tag{2.3}$$

where $S(\mathrm{ad})^n$ denotes the degree n component of the symmetric algebra of the graded vector space $\mathfrak{g}\oplus\mathfrak{g}\oplus\ldots$ with k-th term sitting in degree k ($k = 1, 2, \ldots$) considered as a $\mathfrak{g}$ module under the adjoint action.

(3) If M is an irreducible $\mathfrak{g}$ module and V is a nontrivial $\tilde{\mathfrak{g}}$ submodule of $\operatorname{Ind}(M)_\kappa$ then

$$V \cap \operatorname{Ind}(M)^{a/2\kappa}_\kappa = V^{a/2\kappa} = 0.$$

Part 2 follows from (2.2) and the Poincare–Birkhoff–Witt lemma. Parts 1 and 3 are straightforward.

Corollary 2.3. *If M is an irreducible $\tilde{\mathfrak{g}}$ module then $\operatorname{Ind}(M)_\kappa$ has a unique maximal $\tilde{\mathfrak{g}}$ submodule $M_{\max}$. It satisfies $M_{\max} \cap \operatorname{Ind}(M)^{a/2\kappa}_\kappa = 0$. The corresponding irreducible quotient will be denoted by*

$$\operatorname{Irr}(M)_\kappa = \operatorname{Ind}(M)_\kappa / M_{\max}.$$

We now use a theorem of Kostant [17]. Recall that through the Harish-Chandra isomorphism the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is identified with $S(\mathfrak{h})^W$ for a given Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, see e.g. [6, Chapter 7.4]. Thus the characters of $Z(\mathfrak{g})$ are parametrized by $\mathfrak{h}^*/W$. The character corresponding to the W orbit of $\lambda \in \mathfrak{h}^*$ will be denoted by $\chi_\lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$. Recall also that for the Casimir element $\Omega \in Z(\mathfrak{g})$

$$\chi_\lambda(\Omega) = |\lambda|^2 - |\rho|^2$$

where $\rho \in \mathfrak{h}^*$ is the half-sum of the positive roots of $\mathfrak{g}$ for the Borel subalgebra used to define the Harish-Chandra isomorphism.

Theorem 2.4. *(Kostant) Let M be a $\mathfrak{g}$-module with infinitesimal character χ_λ, $\lambda \in \mathfrak{h}^*$ and U be a finite dimensional $\mathfrak{g}$ module with weights $\mu_1, \cdots, \mu_n$, counted with their multiplicities. Then*

$$\prod_{i=1}^{n} (z - \chi_{\lambda+\mu_i}(z).\mathrm{id})$$

annihilates $U \otimes M$ for all $z \in Z(\mathfrak{g})$, in particular

$$\prod_{i=1}^{n} \big(\Omega - (|\lambda + \mu_i| - |\rho|^2).\mathrm{id}\big)$$

annihilates $U \otimes M$.

Lemma 2.5. *Let M be an irreducible $\mathfrak{g}$ module with infinitesimal character χ_λ, $\lambda \in \mathfrak{h}^*$. Then for any two $\tilde{\mathfrak{g}}$ submodules V and V' of* $\mathrm{Ind}(M)_\kappa$ *such that*

$$V \subset V', \quad V \neq V'$$

there exists an element μ of the root lattice Q of $\mathfrak{g}$ such that

(a) the operator $L_0 : V'/V \to V'/V$ has the eigenvalue

$$\frac{1}{2\kappa}\big(|\lambda + \mu|^2 - |\rho|^2\big)$$

and

(b)

$$\frac{1}{2\kappa}\big(|\lambda + \mu|^2 - |\lambda|^2\big) \in \mathbb{Z}_{\geq 0}.$$

Proof. The subspaces V and V' of $\mathrm{Ind}(M)_\kappa$ are invariant under L_0 and therefore L_0 induces a well defined endomorphism of V/V'. Choose the eigenvalue ξ_0 of L_0 on V/V' with minimal real part. (It exists due to part 2 of Proposition 2.2.) Then $\tilde{\mathfrak{g}}_+$ annihilates $(V/V')^{\xi_0}$ because of (2.2). This implies that the Casimir of $\mathfrak{g}$ acts on $(V/V')^{\xi_0}$ by $2\kappa\xi_0$.id. On the other hand $(V/V')^{\xi_0}$ considered as a $\mathfrak{g}$ module is a subquotient of the module $\mathrm{Ind}(M)_\kappa^{(|\lambda|^2-|\rho|^2)/2\kappa+n}$ for some $n \in \mathbb{Z}_{\geq 0}$, see (2.3) in Proposition 2.2.

According to Kostant's theorem

$$\xi_0 = \frac{1}{2\kappa}\big(|\lambda + \mu|^2 - |\rho|^2\big)$$

for some μ in the root lattice Q of $\mathfrak{g}$. At the same time

$$\xi_0 = \frac{1}{2\kappa}\big(|\lambda|^2 - |\rho|^2\big) + n$$

for the nonnegative integer n above. This weight μ satisfies properties (a) and (b) above. ∎

Lemma 2.5 motivates the following definition. Assume that M is a $\mathfrak{g}$ module of finite length and that all tensor products of M with powers of the adjoint representation of $\mathfrak{g}$ have finite length as well. Then for every $\tilde{\mathfrak{g}}$ submodule V of $\mathrm{Ind}(M)_\kappa$ set

$$\delta(V) = \sum_{\substack{\xi\colon \xi \in (|\lambda|^2 - |\rho|^2)/2\kappa + \mathbb{Z}_{\geq 0} \text{ and} \\ \exists \mu \in Q, \xi = (|\lambda+\mu|^2 - |\rho|^2)/2\kappa}} l(V \cap \mathrm{Ind}(M)^\xi_\kappa) \tag{2.4}$$

where $l(.)$ denotes the length of a $\mathfrak{g}$ module. The $\mathfrak{g}$ modules $\mathrm{Ind}(M)^\xi_\kappa$ have finite lengths because of part 2 of Proposition 2.2. Clearly $\delta(V) \in \mathbb{Z}_{>0} \cup \{\infty\}$.

The following lemma contains the major property of the function $\delta(.)$.

Lemma 2.6. *Let M be a $\mathfrak{g}$ module for which the tensor products*

$$M \otimes (\mathrm{ad})^{\otimes n}$$

have finite length for $n \in \mathbb{Z}_{\geq 0}$. If $V \subset V'$ are two $\tilde{\mathfrak{g}}$ submodules of $\mathrm{Ind}(M)_\kappa$ *such that $V \neq V'$ then either $\delta(V) = \delta(V') = \infty$ or $\delta(V) < \delta(V')$.*

In the case of an irreducible $\mathfrak{g}$ module M, Lemma 2.6 is a direct consequence of Lemma 2.5, analogously to the proof of [13, Proposition 2.14]. The general case follows from the exactness of the functor $M \mapsto \mathrm{Ind}(M)_\kappa$.

Next, for some $\mathfrak{g}$ modules M, we establish bonds on $\delta(V)$ for all $\tilde{\mathfrak{g}}$ submodules V of $\mathrm{Ind}(M)_\kappa$. First note that

$$C = \min_{\mu \in Q} \mathrm{Re}\left(|\mu|^2 + 2(\lambda,\mu)\right) \tag{2.5}$$

exists and is finite because for a fixed $\lambda \in \mathfrak{h}^*$, $\mathrm{Re}\,(|\mu|^2 + 2(\lambda,\mu)) = |\mu|^2 + 2\mathrm{Re}\,(\lambda,\mu)$ is a positive definite quadratic function on the root lattice Q of $\mathfrak{g}$. Moreover $C \leq 0$ because the above function of μ vanishes at $\mu = 0$.

Lemma 2.7. *(1) If $\kappa \notin \mathbb{R}_{\geq 0}$ then*

$$\delta(\mathrm{Ind}(M)_\kappa) < \infty. \tag{2.6}$$

(2) Define the set

$$X_\lambda = \left\{ \frac{|\mu|^2 + 2(\lambda,\mu)}{2n} \mid \mu \in Q, n \in \mathbb{Z}_{>0} \right\} \subset \mathbb{C}. \tag{2.7}$$

If $\kappa \notin X_\lambda$ and in particular if

$$\kappa \notin Y_\lambda = \mathbb{Q} + \mathbb{Q}(\lambda,\alpha_1) + \dots \mathbb{Q}(\lambda,\alpha_r) \supset X_\lambda \tag{2.8}$$

then

$$\delta(\mathrm{Ind}(M)_\kappa) = l(M).$$

(3) If $\mathrm{Re}\,\kappa < \frac{C}{2}$ then

$$\delta(\mathrm{Ind}(M)_\kappa) = l(M).$$

Proof. Since the sum (2.4) is over those $\xi \in \mathbb{C}$ for which

$$\xi = (|\lambda|^2 - |\rho|^2)/2\kappa + n = (|\lambda+\mu|^2 - |\rho|^2)/2\kappa$$

for some $\mu \in Q$, $n \in \mathbb{Z}_{\geq 0}$, for each ξ in (2.4) there exists a pair $(\mu, n) \in Q \times \mathbb{Z}_{\geq 0}$ such that

$$|\mu|^2 + 2(\lambda,\mu) = 2\kappa n. \tag{2.9}$$

(1) We claim that if $\kappa \notin \mathbb{R}_{\geq 0}$ then the sum in (2.4) is finite for any $\tilde{\mathfrak{g}}$ submodule of $\mathrm{Ind}(M)_\kappa$. This follows from the fact that if $\kappa \notin \mathbb{R}_{\geq 0}$, then the set of pairs (2.9) is finite because for each $\epsilon > 0$ there exists $R > 0$ such that

$$\frac{\mathrm{Im}\,(|\mu|^2 + 2(\lambda,\mu))}{\mathrm{Re}\,(|\mu|^2 + 2(\lambda,\mu))} < \epsilon \text{ for } |\mu| > R.$$

The statement now follows from the fact that for each of those finitely many ξ's the $\mathfrak{g}$ module $\mathrm{Ind}(M)_\kappa^\xi$ has finite length.

(2) In this case the set of pairs (2.9) consists only of the pair $(\mu, n) = (0,0)$, i.e.

$$\delta(\mathrm{Ind}(M)_\kappa) = l\big(\mathrm{Ind}(M)_\kappa^{|\lambda|^2/2\kappa}\big) = l(M).$$

(3) If $\mathrm{Re}\,\kappa < \frac{C}{2}$ then for every $\mu \in Q$ and $n \in \mathbb{Z}_{>0}$

$$\mathrm{Re}\,\kappa < \frac{C}{2} \leq \frac{C}{2n} \leq \mathrm{Re}\,\frac{|\mu|^2 + 2(\lambda,\mu)}{2n}$$

because $C \leq 0$, as noted before the statement of Lemma 2.7. Therefore $\mathrm{Re}\,\kappa < \frac{C}{2}$ implies that $\kappa \notin X_\lambda$ and part (3) follows from part (2). $\square$

Theorem 2.8. *Let M be a $\mathfrak{g}$ module as in Lemma 2.6.*

(1) If $\kappa \notin \mathbb{R}_{\geq 0}$ then $\mathrm{Ind}(M)_\kappa$ has finite composition series with quotients of the type $\mathrm{Irr}(M')_\kappa$ for some irreducible subquotients M' of $M \otimes S(\mathrm{ad})^n$, see (2.3).

(2) If $\mathrm{Re}\,\kappa < C/2$ and M is irreducible then $\mathrm{Ind}(M)_\kappa$ is an irreducible $\tilde{\mathfrak{g}}$ module.

(3) If $\kappa \notin Y_\lambda$ or more generally $\kappa \notin X_\lambda$ and M is irreducible then $\mathrm{Ind}(M)_\kappa$ is again an irreducible $\tilde{\mathfrak{g}}$ module (see (2.7), (2.8) for the definitions of the sets $X_\lambda \subset Y_\lambda \subset \mathbb{C}$).

Proof. The first statement in part 1 and parts 2-3 follow from Lemma 2.6 and Lemma 2.7.

To prove the second statement in part 1, assume that $V \subset V' \subset \mathrm{Ind}(M)_\kappa$ are two submodules such that V'/V is a nontrivial irreducible $\tilde{\mathfrak{g}}$ module. Choose the eigenvalue ξ_0 of L_0 acting on V'/V with minimal real part. Then $(V'/V)^{\xi_0}$ is annihilated by $\tilde{\mathfrak{g}}_+$ and is an irreducible $\mathfrak{g}$ module, otherwise if M_0 is a submodule of $(V'/V)^{\xi_0}$ we obtain a morphism $\mathrm{Ind}(M_0)_\kappa \to V'/V$ whose image is a nontrivial $\tilde{\mathfrak{g}}$ submodule of V'/V because of Proposition 2.2.

Next we obtain a homomorphism $\mathrm{Ind}\,\big((V'/V)^{\xi_0}\big)_\kappa \to V'/V$ which needs to be surjective and consequently we obtain that

$$V'/V \cong \mathrm{Irr}\,\big((V'/V)^{\xi_0}\big)_\kappa.$$

$\square$

Remark 2.9. Part 2 of Theorem 2.8 generalizes a result of Lian and Zuckerman [19, Proposition 2.2] in the case when M is a Harish-Chandra module which they obtained, using the Jacquet functor.

3. The categories $\mathcal{AFF}(\mathcal{C})_\kappa$

Throughout this section we will assume that $\mathcal{C}$ is a full subcategory of the category of $\mathfrak{g}$ modules of finite length which is closed under tensoring with the adjoint representation of $\mathfrak{g}$ and taking subquotients, see Example 1.3. We will also assume that

$$\kappa \notin \mathbb{R}_{\geq 0}.$$

Proposition 3.1. *Under the above assumptions for any $M \in \mathcal{C}$*

(1) $\mathrm{Ind}(M)_\kappa$ *has finite composition series with quotients of the type* $\mathrm{Irr}(M')_\kappa$ *for some subquotients M' of $M \otimes S(\mathrm{ad})^n$, see* (2.3).

(2) $\mathrm{Ind}(M)_\kappa \in \mathcal{AFF}(\mathcal{C})_\kappa$.

Proof. Part 1 follows from the exactness of the functor $\mathrm{Ind}(.)_\kappa$ and part 1 of Theorem 2.8.

Since the functor $V \mapsto V(N)$ (from the category of $\tilde{\mathfrak{g}}$ modules to the category $\mathfrak{g}$ modules) is left exact to prove part 2 it is sufficient to prove that

$$\mathrm{Irr}(M)_\kappa \in \mathcal{AFF}(\mathcal{C})_\kappa \tag{3.1}$$

whenever $M \in \mathcal{C}$ is an irreducible $\mathfrak{g}$ module. Indeed the left exactness shows that $\mathrm{Ind}(M)_\kappa(N)$ is finitely generated by induction on the length of $\mathrm{Ind}(M)_\kappa$. But $\mathrm{Ind}(M)_\kappa(N)$ is also a submodule of $M \otimes \left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} S(\mathrm{ad})^n\right)$ and thus of the truncated tensor product for $n \leq k$ for some integer k. This implies that $\mathrm{Ind}(W)_\kappa \in \mathcal{C}$.

To show (3.1) we note that $\mathrm{Irr}(M)_\kappa(1) \cong M$ for an irreducible $\mathfrak{g}$ module M. (Indeed $\mathrm{Irr}(M)_\kappa(1)$ is an irreducible $\mathfrak{g}$ module because if M' is a submodule of it then there would exist a homomorphism of $\tilde{\mathfrak{g}}$ modules $\mathrm{Ind}(M')_\kappa \to \mathrm{Irr}(M)_\kappa$ whose image would be a nontrivial submodule of $\mathrm{Irr}(M)_\kappa$. Now Corollary 2.3 gives $\mathrm{Irr}(M)_\kappa(1) \cong M$.) Thus $\mathrm{Irr}(M)_\kappa(1) \in \mathcal{C}$. Finally we use the following result of Kazhdan and Lusztig [13, Lemma 1.10(d)]:

For any $\tilde{\mathfrak{g}}$ module V there is an exact sequence of $\mathfrak{g}$ modules

$$0 \to V(1) \to V(N) \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, V(N-1)) \cong \mathrm{ad} \otimes V(N-1), \text{ for } N \geq 2$$

where the map i is given by

$$i(v)(x) = (tx).v \in V(N-1),\ v \in V(N),\ x \in \mathfrak{g},$$

By induction on N one shows that $\mathrm{Irr}(M)_\kappa(N)$ is a finitely generated $\mathfrak{g}$ module. Hence $\mathrm{Irr}(M)_\kappa(N)$ is a subquotient of $M \otimes \left(\bigoplus_{n=0}^{k} S(\mathrm{ad})^n\right)$ for some sufficiently large integer k and thus belongs to $\mathcal{C}$. □

Definition 3.2. A $\mathfrak{g}[t]$ module $\mathcal{N}$ is called a nil-$\mathcal{C}$-type module if $U(\tilde{\mathfrak{g}}_+)^n$ annihilates it for a sufficiently large integer n and considered as a $\mathfrak{g}$ module $\mathcal{N} \in \mathcal{C}$.

Lemma 3.3. *A module $\mathcal{N}$ over $\mathfrak{g}[t]$ is a nil-$\mathcal{C}$-type module if and only if it admits a filtration by $\mathfrak{g}[t]$ submodules*

$$\mathcal{N} = \mathcal{N}_m \supset N_{m-1} \supset \ldots \mathcal{N}_1 \supset \mathcal{N}_0 = 0$$

such that $\tilde{\mathfrak{g}}_+ \mathcal{N}_i \subset \mathcal{N}_{i-1}$ and $\mathcal{N}_i/\mathcal{N}_{i-1}$ are irreducible $\mathfrak{g}$ modules which belong to $\mathcal{C}$.

For a $\mathfrak{g}[t]$ module M we define the induced $\tilde{\mathfrak{g}}$ module

$$\mathrm{I}(M)_\kappa = U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} M \tag{3.2}$$

where M is extended to a $\mathfrak{g}[t] \oplus \mathbb{C}K$ module by letting K act by $\kappa - h^\vee$.

Definition 3.4. A generalized Weyl module over $\tilde{\mathfrak{g}}$ of type $\mathcal{C}$ and central charge $\kappa - h^\vee$ is an induced module

$$\mathrm{I}(M)_\kappa$$

for some nil-$\mathcal{C}$-type module $\mathcal{N}$ over $\mathfrak{g}[t]$.

Theorem 3.5. *Let V be a $\tilde{\mathfrak{g}}$ module of central charge $\kappa - h^\vee$, $\kappa \notin \mathbb{R}_{\geq 0}$.*

(1)The following (a)-(c) are equivalent

(a) $V \in \mathcal{AFF}(\mathcal{C})_\kappa$,

(b) There exists a positive integer N such that $V(N) \in \mathcal{C}$ and $V(N)$ generates V as a $\tilde{\mathfrak{g}}$ module,

(c) V is a quotient of a generalized Weyl module of type $\mathcal{C}$.

(2) The irreducible objects in $\mathcal{AFF}(\mathcal{C})_\kappa$ are the modules $\operatorname{Irr}(M)_\kappa$ for irreducible $\mathfrak{g}$ modules M. In addition for two nonisomorphic irreducible $\mathfrak{g}$ modules M and M' the $\tilde{\mathfrak{g}}$ modules $\operatorname{Irr}(M)_\kappa$ and $\operatorname{Irr}(M')_\kappa$ are not isomorphic.

(3) The category $\mathcal{AFF}(\mathcal{C})_\kappa$ is closed under taking subquotients. Any module in $\mathcal{AFF}(\mathcal{C})_\kappa$ has finite length and thus has a filtration with quotients of the type $\operatorname{Irr}(M)_\kappa$ for some irreducible $\mathfrak{g}$ modules M.

Proof. Part 1: Obviously (a) implies (b).

Condition (b) implies (c) because assuming (b), $V(N)$ is a naturally a nil-$\mathcal{C}$-type module over $\mathfrak{g}[t]$ and thus V is a quotient of the corresponding generalized Weyl module.

Condition (c) implies (a) as follows. Because of Lemma 3.3 and the exactness of the induction functor any generalized Weyl module for $\tilde{\mathfrak{g}}$ has a filtration with quotients of the type $\operatorname{Ind}(M)$ for some irreducible $\mathfrak{g}$ modules $M \in \mathcal{C}$. Now Proposition 3.1 implies that it also has a filtration with quotients of the type $\operatorname{Irr}(M)_\kappa$ (again for some irreducible $\mathfrak{g}$ modules M). Thus any quotient V of a generalized Weyl module has a filtration of the same type. The left exactness of the functor $V \to V(N)$ implies by induction that $V(N)$ are finitely generated $\mathfrak{g}$ modules. Using (2.3) as in the proof of Proposition 3.1 we see that $V(N) \in \mathcal{C}$.

Part 2: If V is an irreducible $\tilde{\mathfrak{g}}$ module which belongs to $\mathcal{AFF}(\mathcal{C})_\kappa$ then $V(1)$ should be an irreducible $\mathfrak{g}$ module. Otherwise, since it has finite length, it would contain an irreducible $\mathfrak{g}$ module M and one would obtain a homomorphism $\operatorname{Ind}(M)_\kappa \to V$ which should factor through an isomorphism $\operatorname{Irr}(M)_\kappa \cong V$. But this is a contradiction since $\operatorname{Irr}(M)_\kappa(1) \cong M$, see the proof of Proposition 3.1.

If $\operatorname{Ind}(M)_\kappa$ and $\operatorname{Irr}(M')_\kappa$ are isomorphic $\tilde{\mathfrak{g}}$ modules for two irreducible $\mathfrak{g}$ modules M and M' then $\operatorname{Ind}(M)_\kappa(1) \cong M$ and $\operatorname{Ind}(M')_\kappa(1) \cong M'$ are isomorphic $\mathfrak{g}$ modules and thus $M \cong M'$.

Part 3 follows from the characterization (c) of $\mathcal{AFF}(\mathcal{C})_\kappa$ by generalized Weyl modules. It can be easily proved directly. □

As a consequence of part 1, condition (b) of Theorem 3.5 one obtains:

Corollary 3.6. *Any module $V \in \mathcal{AFF}(\mathcal{C})_\kappa$ is finitely generated over $U(\mathfrak{g}[t^{-1}])$.*

In the case when the category $\mathcal{C}$ of $\mathfrak{g}$ modules is closed under extension we get that the category of $\tilde{\mathfrak{g}}$ modules is closed under extensions too. This is the case for the categories $\mathcal{F}in_{\mathfrak{g}}$ and more generally $\mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ in Example 1.3 when $\mathfrak{f}$ is a semisimple Lie algebra.

Theorem 3.7. *Assuming that the category $\mathcal{C}$ is closed under extensions and $\kappa \notin \mathbb{R}_{\geq 0}$ the following hold:*

(1) The category of $\tilde{\mathfrak{g}}$ modules $\mathcal{AFF}(\mathcal{C})_\kappa$ is closed under extension inside the category of $\tilde{\mathfrak{g}}$ modules of central charge $\kappa - h^\vee$ and

(2) A $\tilde{\mathfrak{g}}$ module of central charge $\kappa - h^\vee$ belongs to $\mathcal{AFF}(\mathcal{C})_\kappa$ if and only if it has a finite composition series with quotients of the type $\operatorname{Irr}(M)_\kappa$ for some irreducible $\mathfrak{g}$ modules $M \in \mathcal{C}$.

The proof of Theorem 3.7 mimics the one of Theorem 3.5.

Finally we return to Proposition 1.5

Proof of Proposition 1.5. If $V \in \mathcal{AFF}(\mathcal{C})_\kappa$ then it is a quotient of a generalized Weyl module, (1.4) and $V^\xi \in \mathcal{C}$ holds because of Proposition 2.2.

In the other direction – assume that V is a finitely generated $\tilde{\mathfrak{g}}$ module for which (1.4) holds and $V^\xi \in \mathcal{C}$. Then V is generated as a $U(\tilde{\mathfrak{g}})$ module by $v_j \in V^{\zeta_j}$ for some $\zeta_j \in \mathbb{C}$, $j = 1, \dots, k$. Thus it is generated by

$$\bigoplus_{\xi:\, \xi=\zeta_j-n,\, j=1,\dots,k,\, n\in\mathbb{Z}_{\geq 0}} V^\xi. \tag{3.3}$$

(Note that the above sum is finite because of (1.4)). Eq. (3.3) defines a $\mathfrak{g}[t]$ submodule of V because of (2.2) and as a $\mathfrak{g} \hookrightarrow \mathfrak{g}[t]$ module it belongs to $\mathcal{C}$ since the sum in (3.3) is finite. Therefore V is a quotient of the corresponding generalized Weyl module of type $\mathcal{C}$ and belongs to $\mathcal{AFF}(\mathcal{C})_\kappa$. □

There exists also a characterization of the categories $\mathcal{AFF}(\mathcal{C})_\kappa$ in the spirit of Lian and Zuckerman [18, 19]:

Proposition 3.8. *The category $\mathcal{AFF}(\mathcal{C})_\kappa$ consists exactly of those $\tilde{\mathfrak{g}}$ modules V of central charge $\kappa - h^\vee$ which are $\mathbb{Z}$ graded*

$$V = \bigoplus_{n\in\mathbb{Z}} V_n \tag{3.4}$$

with respect to the grading (1.2) *and*

$$V_n \in \mathcal{C}, \quad V_n = 0 \textit{ for } n \ll 0.$$

Sketch of the proof of Proposition 3.8. Let $V \in \mathcal{AFF}(\mathcal{C})_\kappa$. Then (1.4) holds for some $\xi_1, \dots, \xi_n \in \mathbb{C}$ and we can assume that $\xi_i - \xi_j \notin \mathbb{Z}$. To get the grading (3.4) we can set e.g.

$$V_m := V_{\xi_1+m} \oplus \dots \oplus V_{\xi_n+m}.$$

The opposite statement is proved similarly to Proposition 1.5. □

4. Duality in the categories $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$

Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$ and $\kappa \notin \mathbb{R}_{\geq 0}$. In this section, completely analogously to [13], we define a natural duality in the categories $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$. We will only state the results.

For any $\mathfrak{f}$ module M we define

$$M^d := (M^*)^{\mathfrak{f}-fin} \tag{4.1}$$

where $(.)^*$ stays for the full dual and $(.)^{\mathfrak{f}-fin}$ denotes the $U(\mathfrak{f})$-finite part, i.e. the set all η such that $\dim U(\mathfrak{f})\eta < \infty$.

It is well known that

$$M \mapsto M^d$$

is an involutive antiequivalence of $\mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$.

Recall [13] that $\tilde{\mathfrak{g}}$ has the following involutive automorphism

$$(xt^k)^\sharp = x(-t)^{-k}, \; k \in \mathbb{Z}; \quad (K)^\sharp = -K. \tag{4.2}$$

For a $\tilde{\mathfrak{g}}$ module V by $V^\sharp$ we will denote the twisting of V by this automorphism.

Recall also [13] that for any $\tilde{\mathfrak{g}}$ module V the strictly smooth part

$$V(\infty) = \cup_{N \in \mathbb{Z}_{>0}} V(N)$$

of V is a $\tilde{\mathfrak{g}}$ submodule.

For any $\tilde{\mathfrak{g}}$ module V define

$$D(V) := (V^d)^\sharp(\infty) = \cup_{N \geq 1} (V^d)^\sharp(N). \tag{4.3}$$

Here the restricted dual V^d is defined with respect to the action $\mathfrak{f}$ on V coming from the embedding $\mathfrak{f} \hookrightarrow \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$. It is clear that $D(V)$ is a strictly smooth $\tilde{\mathfrak{g}}$ module. If V has central charge $\kappa - h^\vee$ then $D(V)$ has the same central charge.

It is easy to see that if $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ then the generalized eigenspaces $(V^d)^\xi$ of the Sugawara operator L_0 for the $\tilde{\mathfrak{g}}$ module V^d are given by

$$(V^d)^\xi = \{\eta \in V^d \mid \eta(V^\zeta) = 0 \text{ for } \zeta \neq \xi\}. \tag{4.4}$$

Proposition 4.1. *(1) Fix $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ with decomposition* (1.4) *for some $\xi_1, \ldots, \xi_n \in \mathbb{C}$. Then as a subspace of V^d the dual module $D(V)$ is*

$$D(V) = \bigoplus_{\xi :\, \xi - \xi_1 \in \mathbb{Z}_{\geq 0}, \ldots, \xi - \xi_n \in \mathbb{Z}_{\geq 0}} (V^d)^\xi.$$

(2) The contravariant functor D is an involutive antiequivalence of the category $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$.

(3) The functor D transforms simple objects $\mathrm{Irr}(M)_\kappa \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ by

$$D(\mathrm{Irr}(M)_\kappa) \cong \mathrm{Irr}(M^d)_\kappa.$$

Parts 1 and 3 are proved analogously to Section 2.23 and Proposition 2.24 in [13]. Similarly to [13, Proposition 2.25] one shows that the functor D is exact. This implies that for any $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$, $D(V)$ has finite length and thus belongs to $\mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$, e.g. because of Proposition 1.5. Now part 2 of Proposition 4.1 is straightforward.

5. Finiteness properties of the Kazhdan–Lusztig tensor product

In this section we prove Theorem 1.6.

First we recall the definition of the Kazhdan–Lusztig fusion tensor product [13]. Consider the Riemann sphere $\mathbb{CP}^1$ with three fixed distinct points p_i, $i = 0, 1, 2$ on it. Choose local coordinates (charts) at each of them, i.e. isomorphisms $\gamma_i : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\gamma_i(p_i) = 0$ where the second copy of $\mathbb{CP}^1$ is equipped with a fixed coordinate function t vanishing at 0.

Set $R = \mathbb{C}[\mathbb{CP}^1 \backslash \{p_0, p_1, p_2\}]$ and denote by Γ the central extension of the Lie algebra $\mathfrak{g} \otimes R$ by

$$[f_1 x_1, f_2 x_2] := f_1 f_2 [x_1, x_2] + \mathrm{Res}_{p_0}(f_2 df_1)(x_1, x_2) K, \tag{5.1}$$

for $f_i \in R$ and $x_i \in \mathfrak{g}$. Here $(.,.)$ denotes the invariant bilinear form on $\mathfrak{g}$, fixed in Section 1. There is a canonical homomorphism

$$\Gamma \to \widehat{\mathfrak{g} \oplus \mathfrak{g}}, \quad xf \mapsto (x\mathrm{Ex}(\gamma_1^*)^{-1}(f), x\mathrm{Ex}(\gamma_2^*)^{-1}(f)),\ K \mapsto -K \tag{5.2}$$

where $\mathrm{Ex}(.)$ denotes the power series expansion of a rational function on $\mathbb{CP}^1$ at 0 in terms of the coordinate function t.

Define

$$G_N = \mathrm{span}\{(f_1 x_1) \ldots (f_N x_N) \mid f_i \text{ vanish at } p_0,\ x_i \in \mathfrak{g}\} \subset U(\Gamma).$$

Fix two smooth $\tilde{\mathfrak{g}}$ modules V_1 and V_2 of central charge $\kappa - h^\vee$. Equip $W = V_1 \otimes_{\mathbb{C}} V_2$ with a structure of Γ module with central charge $-\kappa + h^\vee$ using the homomorphism (5.2). Clearly

$$W \supset G_1 W \supset G_2 W \supset \ldots$$

and one can consider the projective limit of vector spaces

$$\widehat{W} = \varprojlim W/G_N W. \tag{5.3}$$

Define an action of $\tilde{\mathfrak{g}}$ on $\widehat{W}$ as follows. Fix $m \in \mathbb{Z}$ and for each $n \in \mathbb{Z}_{>0}$ choose $g_{n,m} \in R$ such that

$$(\gamma_0^*)^{-1}(g_{n,m}) - t^m \text{ vanishes of order at least } n \text{ at } 0.$$

Set $\overline{m} = \max\{-m, 0\}$ and

$$xt^m.(w_1, w_2, \ldots) = ((xg_{1,m})w_{\overline{m}+1}, (xg_{2,m})w_{\overline{m}+1}, \ldots) \tag{5.4}$$

for any sequence $(w_1, w_2, \ldots)$ in W representing an element of the projective limit (5.3) i.e. $w_N \in W$ and $w_{N+1} - w_N \in G_N W$. In [13] it is shown that this defines on $\widehat{W}$ a structure $\tilde{\mathfrak{g}}$ module of central charge $-\kappa + h^\vee$, independent of the choice of $g_{n,m} \in \Gamma$. Finally the Kazhdan–Lusztig tensor product [13] of V_1 and V_2 is defined by

$$V_1 \dot{\otimes} V_2 := (\widehat{W})^\sharp(\infty).$$

We show finiteness properties of a dual construction of the fusion tensor product. Let f_0 be a rational function on $\mathbb{CP}^1$ (unique up to a multiplication by a nonzero complex number) having only one (simple) zero at p_0 and only one (simple) pole at p_1. For instance when $\gamma_0(p_1)$ is finite $f_0(t) = a\gamma_0^*(t)/(\gamma_0^*(t) - \gamma_0(p_1))$, $a \neq 0$. Set

$$X_N = \operatorname{span}\{(f_0 x_1) \ldots (f_0 x_N) \mid x_i \in \mathfrak{g}\} \subset U(\Gamma). \tag{5.5}$$

Clearly $X_N \subset G_N$.

Kazhdan and Lusztig proved the following Lemma.

Lemma 5.1. [13, Proposition 7.4] *Assume that V_i are two strictly smooth $\tilde{\mathfrak{g}}$ modules of central charge $\kappa - h^\vee$, generated by $V_i(N_i)$, respectively. Then*

$$V_1 \otimes V_2 = \sum_{k=0}^{N-1} X_k(V_1(N_1) \otimes V_2(N_2)) + G_N(V_1 \otimes V_2)$$

for all $N \in \mathbb{Z}_{>0}$.

For a given category of $\mathfrak{g}$ modules $\mathcal{C}$ denote by $\overline{\mathcal{AFF}}(\mathcal{C})_\kappa$ the category of strictly smooth $\tilde{\mathfrak{g}}$ modules V such that

$$V(N) \in \mathcal{C}.$$

It differs from the category $\mathcal{AFF}(\mathcal{C})_\kappa$ in that we drop the condition for finite length.

Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$. Consider a module $U \in \mathcal{O}_\kappa$ and a module $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$. Using the homomorphism (5.2) $W = U \otimes_{\mathbb{C}} V$ becomes a Γ module of central charge $-\kappa + h^\vee$. Note that the restricted dual $(U \otimes V)^d$ (recall (4.1)) is naturally a Γ submodule of the full dual to $U \otimes V$, both of central charge $\kappa - h^\vee$. (The restricted dual is taken with respect to the embedding $\mathfrak{f} \hookrightarrow \mathfrak{g} \hookrightarrow \Gamma$ using constant functions on $\mathbb{CP}^1$.) Following [13] define the following Γ submodule of $(U \otimes V)^d$

$$T'(U, V) := \cup_{N \geq 1} T(U, V)\{N\} \tag{5.6}$$

where

$$T(U,V)\{N\} := \mathrm{Ann}_{G_N}(U \otimes V)^d.$$

Eq. (5.6) indeed defines a Γ submodule of $(U \otimes V)^d$ since for any $y \in \Gamma$ and any integer N there exists an integer i such that $G_{N+i}y \in \Gamma G_N$. In other words $T'(U,V)\{N\}$ is defined as

$$T'(U,V)\{N\} = \{\eta \in (U \otimes V)^* \mid \eta(G_N W) = 0,\ \dim U(\mathfrak{f})\eta < \infty\}.$$

Similarly to [13, Section 6.3] $T'(U,V)$ has a canonical action of $\tilde{\mathfrak{g}}$ ("the copy attached to p_0") defined as follows. Let $\eta \in T(U,V)\{N\}$. Fix $\omega \in \mathbb{C}[t,t^{-1}]$, $x \in \mathfrak{g}$ and choose $f \in R$ such that $f - \gamma_0^*(\omega)$ has a zero of order at least N at p_0. Then

$$(\omega x)\eta := (fx)\eta \tag{5.7}$$

correctly defines a structure of smooth $\tilde{\mathfrak{g}}$ module on $T'(U,V)$.

Lemma 5.2. *(1) For any two modules $U \in \mathcal{O}_\kappa$ and $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$, $T'(U,V)$ is a strictly smooth $\tilde{\mathfrak{g}}$ module of central charge $\kappa - h^\vee$ and*

$$T'(U,V)(N) = T'(U,V)\{N\} \in \mathcal{C}_{(\mathfrak{g},\mathfrak{f})} \tag{5.8}$$

considered as a $\mathfrak{g}$ module. Thus

$$T' \colon \mathcal{O}_\kappa \times \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa \to \overline{\mathcal{AFF}}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa, \quad U \times V \mapsto T'(U,V)$$

is a contravariant bifunctor.

(2) The bifunctor T' is right exact in each argument.

The proof of (5.8) is straightforward as [13, Lemma 6.5] for the case $\mathfrak{f} = \mathfrak{g}$ ($\mathcal{C}_{(\mathfrak{g},\mathfrak{f})} = \mathcal{O}_\kappa$). The part that $T'(U,V) \in \mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ as a $\mathfrak{g}$ module follows from Lemma 5.1 and the Kostant theorem [17] that $\mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ is closed under tensoring with finite dimensional $\mathfrak{g}$ modules (and taking subquotients).

Part 2, as in [15, Proof of Proposition 28.1], follows from the left exactness of the functors $M \mapsto Ann_A(M)$ and $M \mapsto M^{A-fin}$ for a given algebra A on the category of all A modules. (Here M^{A-fin} denotes the A finite part of M, i.e. the space of $m \in M$ such that $\dim A.m < \infty$, cf. section 4.)

Theorem 1.6 would be derived from the following Proposition.

Theorem 5.3. *For any modules $U \in \mathcal{O}_\kappa$ and $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ the $\tilde{\mathfrak{g}}$-module $T'(U,V)$ has finite length, i.e.*

$$T'(U,V) \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa.$$

Due to the right exactness of the bifunctor T' it is sufficient to prove the following Lemma. *This is the main part of the proof of Theorem 1.6.*

Lemma 5.4. *For any irreducible $\mathfrak{g}$ modules $U_0 \in \mathcal{F}in_{\mathfrak{g}}$ and $V_0 \in \mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ the $\tilde{\mathfrak{g}}$ module $T'(\mathrm{Irr}(U_0)_\kappa, \mathrm{Irr}(V_0)_\kappa)$ has finite length.*

To prove Lemma 5.4 we can further restrict ourselves to charts $\gamma_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ around p_0 such that

$$\gamma_0(p_1) = \infty. \tag{5.9}$$

This follows from the simple Lemma:

Lemma 5.5. *Assume that the $\tilde{\mathfrak{g}}$ module $T'(U,V)$ associated to one chart $\gamma_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ around p_0 has finite length. Then the respective module associated to any other chart $\alpha_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ around p_0 has finite length too.*

Proof. Notice that G_N does not depend on the choice of chart γ_0 or α_0 around p_0 and consequently the Γ module $T'(U,V)$ does not depend on such a choice either. Denote the actions of $\tilde{\mathfrak{g}}$ on the space $T'(U,V)$ associated to the charts γ_0 and α_0 by μ and ν, respectively. We claim that

$$\mu(U(\tilde{\mathfrak{g}}))\eta = \nu(U(\tilde{\mathfrak{g}}))\eta \quad \text{for any} \quad \eta \in T'(U,V). \tag{5.10}$$

Since $\alpha_0(\gamma_0)^{-1}$ is an automorphism of $\mathbb{CP}^1$ that preserves the origin

$$(\gamma_0^*)^{-1}\alpha_0^*(t) = \frac{at}{bt+d}$$

for some complex numbers $a \neq 0$, b, and $d \neq 0$. Let $\eta \in T'(U,V)\{N\}$. Then for each $n \in \mathbb{Z}$, $n \leq N$ there exist complex numbers $b_n, \ldots, b_N$ such that

$$\alpha_0^*(t^n) - \sum_{k=n}^{N} b_k \gamma_0^*(t^k) \,\text{vanishes of order at least}\, N \,\text{at}\, p_0.$$

This implies that

$$\nu(xt^n)\eta = \sum_{k=n}^{N} b_k \mu(xt^k) \quad \text{for any} \quad x \in \mathfrak{g}$$

and by induction (5.10).

Assuming that the $\tilde{\mathfrak{g}}$ module $(T'(U,V),\mu)$ has finite length we get that $(T'(U,V),\nu)$ is finitely generated $U(\tilde{\mathfrak{g}})$ module which belongs to $\overline{\mathcal{AFF}}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ and thus it also has finite length due to part 3 of Theorem 3.5. $\square$

In the rest of this section we prove Lemma 5.4 for a chart γ_0 around p_0 with the property (5.9). Let us fix such a chart. Then

$$\gamma_0^*(t^k) \in R = \mathbb{C}[\mathbb{CP}^1 \backslash \{p_0, p_1, p_2\}], \quad \text{for all} \quad k \in \mathbb{Z}.$$

As a consequence of this there exists an embedding

$$\tilde{\mathfrak{g}} \hookrightarrow \Gamma, \quad xt^k \mapsto x\gamma_0^*(t^k),\ K \to K. \tag{5.11}$$

In addition in (5.5) the function f_0 can be taken simply as $\gamma_0^*(t)$.

Fix two modules $U_0 \in \mathcal{F}in_{\mathfrak{g}}$ and $V_0 \in \mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$ and denote by

$$W = \mathrm{Irr}(U_0)_\kappa \otimes \mathrm{Irr}(V_0)_\kappa$$

the related Γ module. Using the homomorphism (5.11) it becomes a $\tilde{\mathfrak{g}}$ module of central charge $-\kappa + h^\vee$. We will denote by $W^\sharp$ the twisting of this $\tilde{\mathfrak{g}}$ module by the automorphism $(.)^\sharp$ of $\tilde{\mathfrak{g}}$ (see (4.2)). Note that $W^\sharp$ has central charge $\kappa - h^\vee$.

We claim that $\mathfrak{g}[t] \hookrightarrow \tilde{\mathfrak{g}}$ preserves

$$W_0^\sharp := \mathrm{Irr}(U_0)_\kappa(0) \otimes \mathrm{Irr}(V_0)_\kappa(0) \subset W^\sharp.$$

This follows from the facts that $xt^n \in \tilde{\mathfrak{g}}$ acts on $W_0^\sharp$ by

$$x\mathrm{Ex}(\gamma_1^*)^{-1}\gamma_0^*(-t)^{-n} \otimes \mathrm{id} + \mathrm{id} \otimes x\mathrm{Ex}(\gamma_1^*)^{-1}\gamma_0^*(-t)^{-n}$$

(see (5.2)), that $\gamma_0^*(t^{-n})$ are regular functions on $\mathbb{CP}^1 \backslash \{p_0\}$ for $n \in \mathbb{Z}_{\geq 0}$, and that $\mathfrak{g}[t] \hookrightarrow \tilde{\mathfrak{g}}$ preserves $\mathrm{Irr}(U_0)_\kappa(0)$ and $\mathrm{Irr}(V_0)_\kappa(0)$.

Consider the canonical induced homomorphism of $\tilde{\mathfrak{g}}$ modules

$$\rho\colon\ \mathrm{I}(W_0^\sharp)_\kappa \to W^\sharp$$

(Recall the definition (3.2) of an induced $\tilde{\mathfrak{g}}$ module $\mathrm{I}(.)_\kappa$.) Dually we obtain a homomorphism of $\tilde{\mathfrak{g}}$ modules

$$\rho^{d\sharp} : W^d \cong ((W^\sharp)^d)^\sharp \to ((\mathrm{I}(W_0^\sharp)_\kappa)^d)^\sharp, \tag{5.12}$$

see (4.1).

Notice that in the definition (5.7) of the action of $\tilde{\mathfrak{g}}$ on $T'(U,V)$ the function f can be simply taken as $\gamma_0^*(\omega)$ because all functions $\gamma_0^*(t^k)$, $k \in \mathbb{Z}$ are regular outside $\{p_0, p_1\}$. This means that the structure of $\tilde{\mathfrak{g}}$ module on the space $T'(\mathrm{Ind}(U_0)_\kappa, \mathrm{Ind}(V_0)_\kappa)$ is simply the one induced from the Γ action by the homomorphism (5.11). Thus $T'(\mathrm{Ind}(U_0)_\kappa, \mathrm{Ind}(V_0)_\kappa)$ is naturally a $\tilde{\mathfrak{g}}$ submodule of W^d.

Lemma 5.6. *The homomorphism $\rho^{d\sharp}$ (5.12) restricts to an inclusion*

$$\rho^{d\sharp} \colon T'(\mathrm{Ind}(U_0)_\kappa, \mathrm{Ind}(V_0)_\kappa) \hookrightarrow D(\mathrm{I}(W_0^\sharp)_\kappa).$$

(Recall that $D(\mathrm{I}(W_0^\sharp)_\kappa)$ is the smooth part of $((\mathrm{I}(W_0^\sharp)_\kappa)^d)^\sharp$.)

Proof. Since $T'(\mathrm{Ind}(U_0)_\kappa, \mathrm{Ind}(V_0)_\kappa)\{N\} \subset W^d(N)$

$$\rho^{d\sharp}(T'(\mathrm{Ind}(U_0)_\kappa, \mathrm{Ind}(V_0)_\kappa)\{N\}) \subset ((\mathrm{I}(W_0^\sharp)_\kappa)^d)^\sharp(N)$$

and

$$\rho^{d\sharp}(T'(\mathrm{Ind}(U_0)_\kappa, \mathrm{Ind}(V_0)_\kappa)) \subset ((\mathrm{I}(W_0^\sharp)_\kappa)^d)^\sharp(\infty) = D(\mathrm{I}(W_0^\sharp)_\kappa).$$

To show that this restricted $\rho^{d\sharp}$ is an inclusion assume that $\eta \in T'(\mathrm{Ind}(U_0)_\kappa, \mathrm{Ind}(V_0)_\kappa)\{N\}$ is such that $\rho^{d\sharp}(\eta) = 0$. Then

$$\begin{aligned}(-1)^N \eta(\pi_1(x_1\gamma_0^*(t)) \ldots \pi_1(x_n\gamma_0^*(t))w_0) &= \eta(\pi_2(x_1 t^{-1}) \ldots \pi_2(x_n t^{-1})w_0) \\ &= \rho^{d\sharp}(\eta)(\pi_3(x_1 t^{-1}) \ldots \pi_3(x_n t^{-1})w_0) = 0\end{aligned}$$

for all $x_i \in \mathfrak{g}$, $w_0 \in W_0$, $n \in \mathbb{Z}_{\geq 0}$. Here π_1 denotes the actions of Γ on W, and π_2, π_3 denote the actions of $\tilde{\mathfrak{g}}$ on $W^\sharp$, $I(W_0^\sharp)_\kappa$, respectively. This means that

$$\eta|_{X_n W_0} = 0, \; n \geq 0 \quad \text{and} \quad \eta|_{G_N W} = 0.$$

Because of Lemma 5.1 $\eta = 0$. $\square$

Recall the canonical isomorphisms

$$\mathrm{Irr}(U_0)_\kappa(0) \cong U_0, \; \mathrm{Irr}(V_0)_\kappa(0) \cong V_0.$$

Kostant's theorem [17, Theorem 3.5] implies that $U_0 \otimes V_0$ has finite length as a $\mathfrak{g}$ module, and thus $W_0^\sharp = \mathrm{Irr}(U_0)_\kappa(0) \otimes \mathrm{Irr}(V_0)_\kappa(0)$ is a finite length $\mathfrak{g}$ module.

Now Lemma 5.4 follows from the following fact.

Proposition 5.7. *Let M be a $\mathfrak{g}[t]$ module which is of finite length over $\mathfrak{g} \hookrightarrow \mathfrak{g}[t]$. Then the $\tilde{\mathfrak{g}}$ module $D(\mathrm{I}(M)_\kappa)$ has finite length for $\kappa \notin \mathbb{R}_{\geq 0}$.*

Proof. Denote by π the action of $\mathfrak{g}[t]$ on M. Define a new action $\overline{\pi}$ of $\mathfrak{g}[t]$ on the same space by

$$\overline{\pi}(xt^n) = \delta_{n,0}\pi(x).$$

(This defines a representation of $\mathfrak{g}[t]$ since $t\mathfrak{g}[t]$ is an ideal of $\mathfrak{g}[t]$.) This representation will be denoted by $\overline{M}$. The underlining vector spaces of M and $\overline{M}$ will be always identified.

Consider the two $\tilde{\mathfrak{g}}$ modules $\mathrm{I}(M)_\kappa$ and $\mathrm{I}(\overline{M})_\kappa$ and identify their underlining spaces with

$$\mathcal{M} = U(t^{-1}\mathfrak{g}[t^{-1}]) \otimes_{\mathbb{C}} M.$$

They are isomorphic as $\mathfrak{g}[t^{-1}]$ modules and are naturally graded as $\mathfrak{g}[t^{-1}]$ modules with respect to the grading (1.2) by

$$\deg u \otimes m = -k \quad \text{for} \quad u \in U(t^{-1}\mathfrak{g}[t^{-1}])^{-k}, m \in M.$$

By $(.)^k$ we denote the k-th graded component of a graded vector space (algebra).

Then

$$U(\mathfrak{g}[t])^{-k}\,\mathrm{I}(M)_\kappa = U(\mathfrak{g}[t])^{-k}\,\mathrm{I}(\overline{M})_\kappa = \sum_{j=k}^{\infty} \mathcal{M}^{-j}.$$

Denote

$$(\mathcal{M}^d)^{-k} = \{\eta \in \mathcal{M}^d \mid \eta(\mathcal{M}^{-j}) = 0 \text{ for } j \neq k\}.$$

In the definition of the restricted dual above, recall (4.1), we use the $\mathfrak{f}$ module structure on $\mathcal{M}$ coming from the identification of $\mathcal{M}$ with the isomorphic $\mathfrak{g}$ modules $\mathrm{I}(M)_\kappa$ and $\mathrm{I}(M)_\kappa$. In other words as an $\mathfrak{f} \hookrightarrow \mathfrak{g}$ module $\mathcal{M}$ is the tensor product of $U(t^{-1}\mathfrak{g}[t^{-1}])$ (under the adjoint action) and M (equipped with either the action π or $\overline{\pi}$ which coincide when restricted to $\mathfrak{g}$).

As subspaces of $\mathcal{M}^d$

$$D(I(M)_\kappa)(N) = D(I(\overline{M})_\kappa)(N) = \sum_{j=0}^{N-1} (\mathcal{M}^d)^{-j}. \tag{5.13}$$

This implies that the representation spaces of $D(I(M)_\kappa)$ and $D(I(\overline{M})_\kappa)$ can be identified with

$$\bigoplus_{j=0}^{\infty} (\mathcal{M}^d)^{-j}.$$

The actions of $\tilde{\mathfrak{g}}$ on this vector space related to $D(I(M)_\kappa)$ and $D(I(\overline{M})_\kappa)$ will be denoted by σ^* and $\overline{\sigma}^*$.

We claim that:

(5.14) If $\eta \in (\mathcal{M}^d)^{-j}$ and $g \in U(\mathfrak{g}[t^{-1}])^{-k}$ then $\sigma^*(g)\eta - \overline{\sigma}^*(g)\eta \in \bigoplus_{i=0}^{j+k-1} (\mathcal{M}^d)^{-i}$.

It suffices to check (5.14) for $g = xt^{-k}$. For this we need to show that if $u \in U(t^{-1}\mathfrak{g}[t^{-1}])^{-p}$ and $p \geq k+j$ then

$$\big(\sigma^*(xt^{-k})\eta - \overline{\sigma}^*(xt^{-k})\eta\big)(u \otimes m) = 0. \tag{5.15}$$

Let

$$(-1)^k (xt^k) u = \sum_i a_i b_i c_i \tag{5.16}$$

for some $a_i \in U(t^{-1}\mathfrak{g}[t^{-1}])^{-p+k-\varphi(i)}$, $b_i \in U(\mathfrak{g})$, $c_i = 1$ if $\varphi(i) = 0$ and $c_i \in U(t\mathfrak{g}[t])^{\varphi(i)}$ if $\varphi(i) > 0$. Here φ is a map from the index set in the RHS of (5.16) to $\mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}\big(\sigma^*(xt^{-1})\eta\big)(u \otimes m) &= \sum_i \eta(a_i \otimes \sigma(b_i c_i) m) \\ &= \delta_{p,k+j} \sum_{i\,:\varphi(i)=0} \eta(a_i \otimes \pi(b_i) m).\end{aligned}$$

The second equality follows from $\eta \in (\mathcal{M}^d)^{-j}$ and $p - k + \varphi(i) > j$ unless $p = k+j$ and $\varphi(i) = 0$.

The same formula holds for $\overline{\sigma}^*$ with π substituted in the RHS by $\overline{\pi}$. The compatibility of π and $\overline{\pi}$ on $\mathfrak{g} \hookrightarrow \mathfrak{g}[t]$ implies (5.15).

According to Proposition 4.1 $D(\mathrm{I}(\overline{M})_\kappa) \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$. From Corollary 3.6 we get that $D(\mathrm{I}(\overline{M})_\kappa)$ is finitely generated as a $U(\mathfrak{g}[t^{-1}])$ module. We can assume that it is generated by some homogeneous elements

$$\eta_i \in (\mathcal{M}^d)^{-j_i},\ i = 1, \ldots, n.$$

Then by induction (5.14) easily gives that $D(\mathrm{I}(M)_\kappa)$ is generated as a $U(\mathfrak{g}[t^{-1}])$ module by the same set $\{\eta_1, \ldots, \eta_n\}$. Thus $D(\mathrm{I}(M)_\kappa)$ is finitely generated as a $U(\mathfrak{g})$ module and

$$D(\mathrm{I}(M)_\kappa)(N) \in \mathcal{C}_{(\mathfrak{g},\mathfrak{f})}$$

because of (5.13) which shows that $D(\mathrm{I}(M)_\kappa) \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$. □

Now as in [13] Theorem 5.3 easily implies Theorem 1.6 and the following

Proposition 5.8. *In the setting of Theorem 1.6 and Theorem 5.3 if $U \in \mathcal{O}_\kappa$ and $V \in \mathcal{AFF}(\mathcal{C}_{(\mathfrak{g},\mathfrak{f})})_\kappa$ then $U \dot{\otimes} V$ and $D(T'(U,V))$ are naturally isomorphic.*

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