THE BERENSTEIN–ZELEVINSKY QUANTUM CLUSTER ALGEBRA CONJECTURE

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Abstract. We prove the Berenstein–Zelevinsky conjecture that the quantized coordinate rings of the double Bruhat cells of all finite dimensional simple algebraic groups admit quantum cluster algebra structures with initial seeds as specified by [4]. We furthermore prove that the corresponding upper quantum cluster algebras coincide with the constructed quantum cluster algebras and exhibit a large number of explicit quantum seeds. Along the way a detailed study of the properties of quantum double Bruhat cells from the viewpoint of noncommutative UFDs is carried out and a quantum analog of the Fomin–Zelevinsky twist map is constructed and investigated for all double Bruhat cells. The results are valid over base fields of arbitrary characteristic and the deformation parameter is only assumed to be a non-root of unity.

1. Introduction

Cluster Algebras are an axiomatic class of algebras which was introduced by Fomin and Zelevinsky [10]. In the last 10 years they have played a fundamental role in many different contexts in combinatorics, representation theory, geometry and mathematical physics [28, 38].

One of the major applications of cluster algebras is to representation theory. Fomin and Zelevinsky [10] set up the program of (1) constructing cluster algebra structures on quantum algebras and coordinate rings of algebraic varieties, and (2) capturing a large part of the (dual) canonical basis of Kashiwara and Lusztig via the algorithmically constructed set of cluster monomials [7]. The first step has been carried out in several settings. Berenstein, Fomin and Zelevinsky constructed upper cluster algebra structures on the double Bruhat cells of all complex simple Lie groups [2]. Geiss, Leclerc and Schröer constructed cluster structures on the coordinate rings of Schubert cells for symmetric Kac–Moody algebras [15]. Scott and the last group of authors constructed cluster structures on the homogeneous coordinate rings of flag varieties [14, 37].

Let G be a connected, simply connected, complex simple algebraic group with a fixed pair of opposite Borel subgroups $B^{\pm}$ and a Weyl group W. The double Bruhat cells [9] of G are the subvarieties

$$G^{u,w} := B^+uB^+ \cap B^-wB^-, \quad u, w \in W.$$

Berenstein and Zelevinsky introduced the notion of a quantum cluster algebra and conjectured that the quantized coordinate rings $R_q[G^{u,w}]$ of all double Bruhat cells admit quantum cluster algebras structures with an explicit initial seed which they constructed.

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In a certain sense this is the most general situation and the cases above arise via reductions from it. The conjecture has been proved only in very special cases. Geiss, Leclerc and Schröer proved it in the case when $u = 1$ and G is a symmetric Kac–Moody group [16], and in a recent breakthrough Kang–Kashiwara–Kim–Oh [25, 26] proved that all cluster monomials belong to the dual canonical basis, thus completing the Fomin–Zelevinsky program in this important case. The case of the conjecture for $u = 1$ and general simple G was established in [3, 19]. These cases are simpler and none of the existing methods allows for an extension to the situation when both $w, u \neq 1$, even for special G.

In this paper we prove the Berenstein–Zelevinsky conjecture in full generality and deduce a number of desired properties of the quantum cluster algebras in question. In order to formulate these results, we recall that for all fundamental weights ϖ_i and pairs $u', w' \in W$, one defines the quantum minors $\Delta_{u'\varpi_i, w'\varpi_i} \in R_q[G^{u,w}]$. Fix reduced expressions

$$w = s_{i_1} \dots s_{i_N}, \quad u = s_{i'_1} \dots s_{i'_M}$$

and set

$$w_{<k}^{-1} := s_{i_N} \dots s_{i_{k+1}}, \;\; 1 \leq k \leq N, \quad u_{\leq j}^{-1} := s'_{i_M} \dots s'_{i_j}, \;\; 1 \leq j \leq M.$$

Main Theorem. *For all connected, simply connected, complex simple Lie groups G, Weyl group elements $u, w \in W$, infinite base fields $\mathbb{K}$ of arbitrary characteristic, and non-roots of unity $q \in \mathbb{K}^*$ such that $\sqrt{q} \in \mathbb{K}$, the quantized coordinate ring $R_q[G^{u,w}]$ of the double Bruhat cell $G^{u,w}$ is isomorphic to the quantum cluster algebra with an initial seed having*

(1) *cluster variables $\Delta_{\varpi_i, w^{-1}\varpi_i}$, $1 \leq i \leq r$, $\Delta_{\varpi_{i_k}, w_{<k}^{-1}\varpi_{i_k}}$, $1 \leq k \leq N$, $\Delta_{u_{\leq j}\varpi_{i'_j}, \varpi_{i'_j}}$, $1 \leq j \leq M$,*

(2) *exchange matrix equal to the Berenstein–Fomin–Zelevinsky matrix* (7.7) *and*

(3) *set of mutable variables given* (7.6).

Furthermore, the corresponding upper quantum cluster algebra coincides with the quantum cluster algebra, and thus is isomorphic to $R_q[G^{u,w}]$.

In addition, we construct large sets of explicit seeds of these quantum cluster algebras. The seeds are described in Theorem 6.2 (a)–(b). (Theorem 9.1 establishes a needed isomorphism between two settings.) There is one such seed for each element of the following subset of the symmetric group S_{M+N}

$$\Xi_{M+N} := \{\sigma \in S_{M+N} \mid \sigma([1,k]) \text{ is an interval for all } k \in [2, M+N]\}. \tag{1.1}$$

The second property in the theorem is a very desired property of a cluster algebra and enters as an assumption in various results such as the Gross–Hacking–Keel–Kontsevich proof [20] of the Fock-Goncharov conjecture [12] on canonical bases of cluster algebras. In a companion paper we prove that the equality of cluster and upper cluster algebras for all double Bruhat cells also holds at the classical level (which is a somewhat easier result) and that one can construct (many) maximal green sequences in the sense of Keller [27] using the explicit clusters in Theorem 6.2 indexed by the set (1.1). Using these two results, the main theorem of [20] implies the validity of the Fock-Goncharov conjecture [12] for all Berenstein–Fomin–Zelevinsky cluster algebras for double Bruhat cells $G^{u,w}$. We expect that the constructed large sets of explicit clusters of $R_q[G^{u,w}]$ will have many additional applications, just like Scott's clusters [37] for Grassmannians coming from Postnikov diagrams which are used in a wide array of situations, such as integrable systems [29], mirror symmetry [33, 36], scattering amplitudes [1], local acyclicity [35] and many others.

We carry out the proof of the Main Theorem in two steps:

(I) Using our results on quantum cluster algebra structures on quantum nilpotent algebras [19], we construct a quantum cluster algebra structure on each $R_q[G^{u^{-1},w^{-1}}]$ and prove that it has the additional property that it coincides with the corresponding upper quantum cluster algebra.

(II) In the second step we construct a quantum analog of the Fomin–Zelevinsky twist map which is an algebra antiisomorphism $\zeta_{u,w} : R_q[G^{u^{-1},w^{-1}}] \to R_q[G^{u,w}]$ and show that it transforms the quantum cluster algebra from step I to the one conjectured by Berenstein and Zelevinsky [4]. (Additional minor details require an interchange of the Weyl group elements w and u, realized via another antiismorphism involving the antipode of $R_q[G]$.)

In the remaining part of the introduction we briefly indicate the nature of the proof. For every nilpotent Lie algebra $\mathfrak{n}$, the universal enveloping algebra $U(\mathfrak{n})$ can be realized as an iterated skew polynomial extension

$$U(\mathfrak{n}) = \mathbb{K}[x_1][x_2; \mathrm{id}, \delta_2] \dots [x_N; \mathrm{id}, \delta_N] \tag{1.2}$$

where all derivations δ_k are locally nilpotent. Quantum nilpotent algebras are similar kinds of skew polynomial extensions

$$R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \dots [x_N; \sigma_N, \delta_N] \tag{1.3}$$

equipped with a rational action of a torus $\mathcal{H}$ for which the automorphisms σ_k come from the $\mathcal{H}$-action, the skew derivations δ_k are locally nilpotent, and certain diagonal eigenvalues of the $\mathcal{H}$-action are non-roots of unity (Definition 2.2). The algebras (1.2) have all but the last property; their corresponding eigenvalues are all equal to 1. In [19] we proved that all quantum nilpotent algebras (under mild conditions) possess quantum cluster algebra structures where the initial cluster variables are sequences of the form $p_1, \dots, p_N$ such that p_k is the unique homogeneous prime element in R_k that is not in R_{k-1} (for the intermediate algebras R_k generated by $x_1, \dots, x_k$). These arguments are based on the theory of noncommutative UFDs.

To carry out step I, we first realize each quantum double Bruhat cell $R_q[G^{u,w}]$ in terms of a bicrossed product

$$\mathcal{U}^-[u]_{\mathrm{op}} \bowtie \mathcal{U}^+[w]. \tag{1.4}$$

The algebras $\mathcal{U}^{\pm}[w]$ are the *quantum Schubert cell algebras* (also called *quantum unipotent cells*) introduced by De Concini–Kac–Procesi [8] and Lusztig [32]. They are deformations of the universal enveloping algebras $\mathcal{U}(\mathfrak{n}_{\pm} \cap w(\mathfrak{n}_{\mp}))$ where $\mathfrak{n}_{\pm}$ are the unipotent radicals of the Borel subalgebras $\mathrm{Lie}\,(B_{\pm})$. Here $(.)_{\mathrm{op}}$ refers to the opposite algebra structure. Part I of the proof proceeds in the following steps:

(A) Proving that the algebras $\mathcal{U}^-[u]_{\mathrm{op}} \bowtie \mathcal{U}^+[w]$ are quantum nilpotent algebras with respect to a particular ordering of the Lusztig root vectors of $\mathcal{U}^-[u]$ and $\mathcal{U}^+[w]$.

(B) Showing that for different pairs of Weyl group elements $w, u \in W$, the algebras (1.4) embed in each other appropriately and that the intermediate algebras R_k in the extension (1.3) are algebras of the same type for particular subwords of w and u.

(C) Classifying the prime elements of the algebras $\mathcal{U}^-[u]_{\mathrm{op}} \bowtie \mathcal{U}^+[w]$ using techniques from the study of the spectra of quantum groups [22, 24, 41].

The Fomin–Zelevinsky twist map [9] is a birational isomorphism $\zeta_{u,w} : G^{u,w} \to G^{u^{-1},w^{-1}}$ which, in addition to cluster algebras, plays a major role in many other constructions. Part II of the proof of the Main Theorem relies on the construction of a quantum version of this map and establishing compatibility properties of it with respect to two different kinds of quantum cluster algebras (the ones that are the result of step I and the conjectured ones by Berenstein and Zelevinsky). The original twist map is defined in terms of set theoretic factorizations in G, not in terms of maps on coordinate rings that are manifestly homomorphisms of algebras. This makes the construction of a quantum twist map rather delicate. For the construction of the map we employ the open embedding of the reduced double Bruhat cell

$$G^{u,w}/H \hookrightarrow (B^+uB^+)/B^+ \times (B^-wB^-)/B^- \subset G/B^+ \times G/B^-$$

obtained from the projections of G to the flag varieties $G/B^\pm$ (here $H := B^+ \cap B^-$). We express the restriction of the action of the classical twist map to the pullbacks of the two projections in geometric terms and then quantize that picture. The fact that the quantum twist map is an algebra antiisomorphism $R_q[G^{u,w}] \cong R_q[G^{u^{-1},w^{-1}}]$ is established by first showing that this happens at the level of division rings of fractions and then going back to the original algebras.

The paper is organized as follows. Sect. 2 contains necessary background on quantum cluster algebras and a brief description of our construction of cluster structures on quantum nilpotent algebras [19]. Sect. 4 contains background on quantum groups and the realization of the quantum double Bruhat cell algebras $R_q[G^{w,u}]$ in terms of bicrossed products $\mathcal{U}^-[u]_{\mathrm{op}} \bowtie \mathcal{U}^+[w]$. Sect. 4–5 carry out steps (A), (B) and (C) above. Sect. 6 constructs a cluster algebra structure on each of the algebras $\mathcal{U}^-[u]_{\mathrm{op}} \bowtie \mathcal{U}^+[w]$, proves that the corresponding upper quantum cluster algebra equals the quantum cluster algebra and constructs seeds associated to the elements of the set (1.1). Sect. 8 constructs the quantum twist map. Sect. 7 and 9 describe the connection between the cluster algebra structure on $\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u]$ and the conjectured one on $R_q[G^{u,w}]$ and contain the proof of the Berenstein–Zelevinsky conjecture, Theorem 9.3. To establish this connection, a second isomorphism $R_q[G^{u,w}] \to R_q[G^{w^{-1},u^{-1}}]$ is used in Sect. 7, which is the product of the antipode and Cartan involution of $R_q[G]$. This one is simpler than the quantum twist map and only plays an auxiliary role. However, as a result of it, in the body of the paper the roles of u and w are interchanged, compared to the introduction.

Throughout the paper we work over an infinite field $\mathbb{K}$ of arbitrary characteristic. For a $\mathbb{K}$-algebra A and elements $a_1, \ldots, a_k \in A$, we denote by $\mathbb{K}\langle a_1, \ldots, a_k\rangle$ the unital subalgebra of A generated by $a_1, \ldots, a_k$. For two integers $m \le n$, denote $[m, n] := \{m, m+1, \ldots, n\}$.

2. Quantum cluster algebra structures on quantum nilpotent algebras

In this section we review quantum cluster algebras and noncommutative unique factorization domains. We describe the key parts of our construction of quantum cluster algebra structures on all symmetric quantum nilpotent algebras (alias CGL extensions) [19], which plays a key role in the paper. We also derive some additional general results to those in [19].

2.1. **Quantum cluster algebras.** Fix a positive integer N, and denote by $\{e_1, \ldots, e_N\}$ the standard basis of $\mathbb{Z}^N$. Write elements of $\mathbb{Z}^N$ as column vectors.

A matrix $\mathbf{r} \in M_N(\mathbb{K}^*)$ is called *multiplicatively skew-symmetric* if $r_{kk} = 1$ and $r_{jk} = r_{kj}^{-1}$, for all $j, k \in [1, N]$. Corresponding to any such matrix is a *skew-symmetric bicharacter*

$$\Omega_{\mathbf{r}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^*, \quad \text{given by} \quad \Omega_{\mathbf{r}}(e_k, e_j) = r_{kj}, \ \ \forall\, j, k \in [1, N]. \tag{2.1}$$

Let $\mathcal{F}$ be a division algebra over $\mathbb{K}$. A *toric frame* for $\mathcal{F}$ is a mapping

$$M : \mathbb{Z}^N \to \mathcal{F}$$

such that

$$M(f)M(g) = \Omega_{\mathbf{r}}(f,g)M(f+g), \quad \forall\, f,g \in \mathbb{Z}^N, \tag{2.2}$$

for some multiplicatively skew-symmetric bicharacter $\Omega_{\mathbf{r}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^*$, and such that the elements in the image of M are linearly independent over $\mathbb{K}$ and $\operatorname{Fract} \mathbb{K}\langle M(\mathbb{Z}^N)\rangle = \mathcal{F}$, [4, 19]. The matrix $\mathbf{r}$ is uniquely reconstructed from the toric frame M, and will be denoted by $\mathbf{r}(M)$. The elements $M(e_1), \dots M(e_N)$ are called *cluster variables*. Fix a subset $\mathbf{ex} \subset [1,N]$, to be called the set of *exchangeable indexes*; the indices in $[1,N]\backslash\mathbf{ex}$ will be called *frozen*. An integral $N \times \mathbf{ex}$ matrix $\widetilde{B}$ will be called an *exchange matrix* if its principal part (the $\mathbf{ex} \times \mathbf{ex}$ submatrix) is skew-symmetrizable. A *quantum seed* for $\mathcal{F}$ is a pair $(M, \widetilde{B})$ consisting of a toric frame for $\mathcal{F}$ and an exchange matrix such that

$$\Omega_{\mathbf{r}}(b^k, e_j) = 1, \ \ \forall\, k \in \mathbf{ex}, \ j \in [1,N], \ k \neq j \quad \text{and}$$
$$\Omega_{\mathbf{r}}(b^k, e_k) \ \text{ are not roots of unity}, \ \ \forall\, k \in \mathbf{ex}$$

where b^k denotes the k-th column of $\widetilde{B}$.

One defines mutations of quantum seeds with respect to all exchangeable indices and iterates them infinitely many times.

Fix a subset $\mathbf{inv}$ of the set $[1,N]\backslash\mathbf{ex}$ of frozen indices – the corresponding cluster variables will be inverted. The *quantum cluster algebra* $\mathcal{A}(M, \widetilde{B}, \mathbf{inv})$ is the unital subalgebra of $\mathcal{F}$ generated by the cluster variables of all seeds obtained from $(M, \widetilde{B})$ by mutation and by $\{M(e_k)^{-1} \mid k \in \mathbf{inv}\}$. To each quantum seed $(M, \widetilde{B})$ one associates the mixed quantum torus/quantum affine space algebra

$$\mathcal{T}_{(M,\widetilde{B})} := \mathbb{K}\langle M(e_k)^{\pm 1}, \ M(e_j) \mid k \in \mathbf{ex} \cup \mathbf{inv}, \ j \in [1,N]\backslash(\mathbf{ex} \cup \mathbf{inv})\rangle \subset \mathcal{F}.$$

The intersection of all such subalgebras of $\mathcal{F}$ associated to all seeds that are obtained by mutation from the seed $(M, \widetilde{B})$ is called the *upper quantum cluster algebra* of $(M, \widetilde{B})$ and is denoted by $\mathcal{U}(M, \widetilde{B}, \mathbf{inv})$.

2.2. **Equivariant noncommutative unique factorization domains.** Recall that a *prime element* of a domain R is any nonzero normal element $p \in R$ (*normality* meaning that $Rp = pR$) such that Rp is a completely prime ideal, i.e., R/Rp is a domain. Assume that in addition R is a $\mathbb{K}$-algebra and $\mathcal{H}$ a group acting on R by $\mathbb{K}$-algebra automorphisms. An *$\mathcal{H}$-prime ideal* of R is any proper $\mathcal{H}$-stable ideal P of R such that ($IJ \subseteq P \implies I \subseteq P$ or $J \subseteq P$) for all $\mathcal{H}$-stable ideals I and J of R. One says that R is an *$\mathcal{H}$-UFD* if each nonzero $\mathcal{H}$-prime ideal of R contains a prime $\mathcal{H}$-eigenvector. This is an equivariant version of Chatters' notion [5] of noncommutative unique factorization domain given in [30, Definition 2.7].

The following fact is an equivariant version of results of Chatters and Jordan [5, Proposition 2.1], [6, p. 24], see [18, Proposition 2.2] and [41, Proposition 6.18 (ii)].

Proposition 2.1. *Let R be a noetherian $\mathcal{H}$-UFD. Every normal $\mathcal{H}$-eigenvector in R is either a unit or a product of prime $\mathcal{H}$-eigenvectors. The factors are unique up to reordering and taking associates.*

2.3. **CGL extensions.** Let S be a unital $\mathbb{K}$-algebra. We use the standard notation $S[x;\theta,\delta]$ for a *skew polynomial algebra*, or *skew polynomial extension*; it denotes a $\mathbb{K}$-algebra generated by S and an element x satisfying $xs = \theta(s)x + \delta(s)$ for all $s \in S$, where θ is an algebra endomorphism of S and δ is a (left) θ-derivation of S. The algebra $S[x;\theta,\delta]$ is a free left S-module, with the nonnegative powers of x forming a basis.

We focus on iterated skew polynomial extensions

$$R := \mathbb{K}[x_1][x_2; \theta_2, \delta_2] \cdots [x_N; \theta_N, \delta_N], \tag{2.3}$$

where it is taken as implied that $\mathbb{K}[x_1] = \mathbb{K}[x_1; \theta_1, \delta_1]$ with $\theta_1 = \mathrm{id}_{\mathbb{K}}$ and $\delta_1 = 0$. For $k \in [0, N]$, set

$$R_k := \mathbb{K}\langle x_1, \ldots, x_k \rangle = \mathbb{K}[x_1][x_2; \theta_2, \delta_2] \cdots [x_k; \theta_k, \delta_k].$$

In particular, $R_0 = \mathbb{K}$.

Definition 2.2. An iterated skew polynomial extension (2.3) is called a *CGL extension* [30, Definition 3.1] or a *quantum nilpotent algebra* if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(i) The elements $x_1, \ldots, x_N$ are $\mathcal{H}$-eigenvectors.
(ii) For every $k \in [2, N]$, δ_k is a locally nilpotent θ_k-derivation of the algebra R_{k-1}.
(iii) For every $k \in [1, N]$, there exists $h_k \in \mathcal{H}$ such that $\theta_k = (h_k \cdot)|_{R_{k-1}}$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

Conditions (i) and (iii) imply that

$$\theta_k(x_j) = \lambda_{kj} x_j \ \text{ for some } \lambda_{kj} \in \mathbb{K}^*, \ \ \forall\, 1 \le j < k \le N.$$

We then set $\lambda_{kk} := 1$ and $\lambda_{jk} := \lambda_{kj}^{-1}$ for $j < k$. This gives rise to a multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} := (\lambda_{kj}) \in M_N(\mathbb{K}^*)$ and the corresponding skew-symmetric bicharacter $\Omega_{\boldsymbol{\lambda}}$ from (2.1).

Define the *length* of R to be N and the *rank* of R by

$$\mathrm{rk}(R) := \{k \in [1, N] \mid \delta_k = 0\} \in \mathbb{Z}_{>0} \tag{2.4}$$

(cf. [18, Eq. (4.3)]). Denote the character group of the torus $\mathcal{H}$ by $X(\mathcal{H})$. The action of $\mathcal{H}$ on R gives rise to an $X(\mathcal{H})$-grading of R, and the $\mathcal{H}$-eigenvectors are precisely the nonzero homogeneous elements with respect to this grading. The $\mathcal{H}$-eigenvalue of a nonzero homogeneous element $u \in R$ will be denoted by χ_u. In other words, $\chi_u = X(\mathcal{H})\text{-}\deg(u)$ in terms of the $X(\mathcal{H})$-grading.

By [30, Proposition 3.2, Theorem 3.7], every CGL extension is an $\mathcal{H}$-UFD. A recursive description of the sets of homogeneous prime elements of the intermediate algebras R_k of a CGL extension R was obtained in [18]. The statement of the result involves the standard predecessor and successor functions, $p = p_\eta$ and $s = s_\eta$, of a function $\eta : [1, N] \to \mathbb{Z}$, defined as follows:

$$\begin{aligned} p(k) &= \max\{j < k \mid \eta(j) = \eta(k)\}, \\ s(k) &= \min\{j > k \mid \eta(j) = \eta(k)\}, \end{aligned} \tag{2.5}$$

where $\max \varnothing = -\infty$ and $\min \varnothing = +\infty$. Define corresponding order functions $O_\pm : [1, N] \to \mathbb{Z}_{\ge 0}$ by

$$\begin{aligned} O_-(k) &:= \max\{m \in \mathbb{Z}_{\ge 0} \mid p^m(k) \ne -\infty\}, \\ O_+(k) &:= \max\{m \in \mathbb{Z}_{\ge 0} \mid s^m(k) \ne +\infty\}. \end{aligned} \tag{2.6}$$

Theorem 2.3. [18, Theorem 4.3] *Let R be a CGL extension of length N and rank* $\mathrm{rk}(R)$ *as in* (2.3). *There exist a function $\eta : [1, N] \to \mathbb{Z}$ whose range has cardinality* $\mathrm{rk}(R)$ *and elements*

$$c_k \in R_{k-1} \ \textit{ for all } \ k \in [2, N] \ \textit{ with } \ p(k) \ne -\infty$$

such that the elements $y_1, \dots, y_N \in R$, recursively defined by

$$y_k := \begin{cases} y_{p(k)} x_k - c_k, & \text{if } p(k) \neq -\infty \\ x_k, & \text{if } p(k) = -\infty, \end{cases} \tag{2.7}$$

are homogeneous and have the property that for every $k \in [1, N]$,

$$\{y_j \mid j \in [1, k],\, s(j) > k\} \tag{2.8}$$

is a list of the homogeneous prime elements of R_k up to scalar multiples.

The elements $y_1, \dots, y_N \in R$ with these properties are unique. The function η satisfying the above conditions is not unique, but the partition of $[1, N]$ into a disjoint union of the level sets of η is uniquely determined by R, as are the predecessor and successor functions p and s. The function p has the property that $p(k) = -\infty$ if and only if $\delta_k = 0$.

The uniqueness of the level sets of η was not stated in [18, Theorem 4.3], but it follows at once from [18, Theorem 4.2]. This uniqueness immediately implies the uniqueness of p and s. In the setting of the theorem, the rank of R is also given by

$$\mathrm{rk}(R) = |\{j \in [1, N] \mid s(j) > N\}| \tag{2.9}$$

[18, Eq. (4.3)].

Definition 2.4. Denote by $\prec$ the reverse lexicographic order on $\mathbb{Z}^N_{\geq 0}$:

$$(m'_1, \dots, m'_N) \prec (m_1, \dots, m_N) \quad \text{if there exists} \\ n \in [1, N] \text{ with } m'_n < m_n,\, m'_{n+1} = m_{n+1},\, \dots,\, m'_N = m_N. \tag{2.10}$$

A CGL extension R as in (2.3) has the $\mathbb{K}$-basis

$$\{x^f := x_1^{m_1} \cdots x_N^{m_N} \mid f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N_{\geq 0}\}.$$

We say that a nonzero element $b \in R$ has *leading term* tx^f where $t \in \mathbb{K}^*$ and $f \in \mathbb{Z}^N_{\geq 0}$ if

$$b = tx^f + \sum_{g \in \mathbb{Z}^N_{\geq 0},\, g \prec f} t_g x^g$$

for some $t_g \in \mathbb{K}$, and we set $\mathrm{lt}(b) := tx^f$.

The leading terms of the prime elements y_k in Theorem 2.3 are given by

$$\mathrm{lt}(y_k) = x_{p^{O_-(k)}(k)} \dots x_{p(k)} x_k, \ \ \forall\, k \in [1, N]. \tag{2.11}$$

The leading terms of reverse-order monomials $x_N^{m_N} \dots x_1^{m_1}$ involve symmetrization scalars in $\mathbb{K}^*$ defined by

$$\mathcal{S}_{\boldsymbol{\lambda}}(f) := \prod_{1 \leq j < k \leq N} \lambda_{jk}^{-m_j m_k}, \ \ \forall\, f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N. \tag{2.12}$$

Namely,

$$\mathrm{lt}(x_N^{m_N} \dots x_1^{m_1}) = \mathcal{S}_{\boldsymbol{\lambda}}((m_1, \dots, m_N)^T) x_1^{m_1} \dots x_N^{m_N}, \ \ \forall\, (m_1, \dots, m_N)^T \in \mathbb{Z}^N. \tag{2.13}$$

2.4. **Symmetric CGL extensions.** Given an iterated skew polynomial extension R as in (2.3), denote

$$R_{[j,k]} := \mathbb{K}\langle x_i \mid j \le i \le k \rangle, \ \ \forall\, j,k \in [1,N].$$

So, $R_{[j,k]} = \mathbb{K}$ if $j \not\le k$.

Definition 2.5. We call a CGL extension R of length N as in Definition 2.2 *symmetric* if the following two conditions hold:

(i) For all $1 \le j < k \le N$,

$$\delta_k(x_j) \in R_{[j+1,k-1]}.$$

(ii) For all $j \in [1,N]$, there exists $h_j^* \in \mathcal{H}$ such that

$$h_j^* \cdot x_k = \lambda_{kj}^{-1} x_k = \lambda_{jk} x_k, \ \ \forall\, k \in [j+1, N]$$

and $h_j^* \cdot x_j = \lambda_j^* x_j$ for some $\lambda_j^* \in \mathbb{K}^*$ which is not a root of unity.

The above conditions imply that R has a CGL extension presentation with the variables x_k in descending order:

$$R = \mathbb{K}[x_N][x_{N-1}; \theta_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \theta_1^*, \delta_1^*],$$

see [18, Corollary 6.4].

Proposition 2.6. [19, Proposition 5.8] *Let R be a symmetric CGL extension of length N. If $l \in [1,N]$ with $O_+(l) = m > 0$, then*

$$\lambda_l^* = \lambda_{s(l)}^* = \ldots = \lambda_{s^{m-1}(l)}^* = \lambda_{s(l)}^{-1} = \lambda_{s^2(l)}^{-1} = \ldots = \lambda_{s^m(l)}^{-1}. \tag{2.14}$$

Definition 2.7. Denote the following subset of the symmetric group S_N:

$$\begin{aligned} \Xi_N := \{\sigma \in S_N \mid \sigma(k) &= \max \sigma([1,k-1]) + 1 \ \text{ or} \\ \sigma(k) &= \min \sigma([1,k-1]) - 1, \ \ \forall\, k \in [2,N]\}. \end{aligned} \tag{2.15}$$

In other words, Ξ_N consists of those $\sigma \in S_N$ such that $\sigma([1,k])$ is an interval for all $k \in [2,N]$.

If R is a symmetric CGL extension of length N, then for each $\sigma \in \Xi_N$ we have a CGL extension presentation

$$R = \mathbb{K}[x_{\sigma(1)}][x_{\sigma(2)}; \theta''_{\sigma(2)}, \delta''_{\sigma(2)}] \cdots [x_{\sigma(N)}; \theta''_{\sigma(N)}, \delta''_{\sigma(N)}], \tag{2.16}$$

see [18, Remark 6.5], [19, Proposition 3.9].

Denote by Γ_N the following subset of Ξ_N:

$$\begin{aligned} &\Gamma_N := \{\sigma_{i,j} \mid 1 \le i \le j \le N\}, \ \text{ where} \\ &\sigma_{i,j} := [i+1, \ldots, j, i, j+1, \ldots, N, i-1, i-2, \ldots, 1]. \end{aligned} \tag{2.17}$$

If R is a symmetric CGL extension of length N and $1 \le i \le k \le N$, then the subalgebra $R_{[i,k]}$ of R is a symmetric CGL extension, and Theorem 2.3 may be applied to it. Moreover, for the case $k = s^m(i)$ we have

Proposition 2.8. [19, Theorem 5.1] *Assume that R is a symmetric CGL extension of length N, and $i \in [1,N]$ and $m \in \mathbb{Z}_{\ge 0}$ are such that $s^m(i) \in [1,N]$. Then there is a unique homogeneous prime element $y_{[i,s^m(i)]} \in R_{[i,s^m(i)]}$ such that*

(i) $y_{[i,s^m(i)]} \notin R_{[i,s^m(i)-1]}$ *and* $y_{[i,s^m(i)]} \notin R_{[i+1,s^m(i)]}$.

(ii) $\mathrm{lt}(y_{[i,s^m(i)]}) = x_i x_{s(i)} \cdots x_{s^m(i)}$.

Certain combinations of the homogeneous prime elements from the above proposition play an important role in the mutation formulas for quantum cluster variables of symmetric CGL extensions. They are given in the following theorem, where we denote

$$\begin{aligned} e_{[j,s^l(j)]} &:= e_j + e_{s(j)} + \cdots + e_{s^l(j)}, \\ &\forall\, j \in [1,N],\ l \in \mathbb{Z}_{\geq 0} \text{ such that } s^l(j) \in [1,N]. \end{aligned} \tag{2.18}$$

Theorem 2.9. [19, Corollary 5.10] *Assume that R is a symmetric CGL extension of length N, and $i \in [1,N]$ and $m \in \mathbb{Z}_{>0}$ are such that $s^m(i) \in [1,N]$. Then*

$$u_{[i,s^m(i)]} := y_{[i,s^{m-1}(i)]} y_{[s(i),s^m(i)]} - \Omega_{\boldsymbol{\lambda}}(e_i, e_{[s(i),s^{m-1}(i)]}) y_{[s(i),s^{m-1}(i)]} y_{[i,s^m(i)]} \tag{2.19}$$

is a nonzero homogeneous normal element of $R_{[i+1,s^m(i)-1]}$ which is not a multiple of $y_{[s(i),s^{m-1}(i)]}$ if $m \geq 2$. It normalizes the elements of $R_{[i+1,s^m(i)-1]}$ in exactly the same way as $y_{[s(i),s^{m-1}(i)]} y_{[i,s^m(i)]}$ does.

2.5. **Rescaling of generators.** Suppose R is a CGL extension of length N as in (2.3). Given any nonzero scalars $t_1, \ldots, t_N \in \mathbb{K}^*$, one can rescale the generators x_j of R in the fashion

$$x_j \longmapsto t_j x_j, \quad \forall\, j \in [1,N], \tag{2.20}$$

meaning that R can be viewed as an iterated skew polynomial extension with generators $t_j x_j$:

$$R := \mathbb{K}[t_1 x_1][t_2 x_2; \theta_2, t_2 \delta_2] \cdots [t_N x_N; \theta_N, t_N \delta_N]. \tag{2.21}$$

This is also a CGL extension presentation of R, and if (2.3) is a symmetric CGL extension, then so is (2.21).

A rescaling as in (2.21) does not affect the $\mathcal{H}$-action or the matrix $\boldsymbol{\lambda}$, but various elements computed in terms of the new generators are correspondingly rescaled, such as the homogeneous prime elements from Theorem 2.3 and Proposition 2.8, which transform according to the rules

$$y_k \longmapsto \Bigg(\prod_{l=0}^{O_-(k)} t_{p^l(k)} \Bigg) y_k \quad \text{and} \quad y_{[i,s^m(i)]} \longmapsto \Bigg(\prod_{l=0}^{m} t_{s^l(i)} \Bigg) y_{[i,s^m(i)]}. \tag{2.22}$$

Consequently, the homogeneous normal elements (2.19) transform as follows:

$$u_{[i,s^m(i)]} \longmapsto \big(t_i t_{s(i)}^2 \cdots t_{s^{m-1}(i)}^2 t_{s^m(i)} \big) u_{[i,s^m(i)]}. \tag{2.23}$$

2.6. **Normalization conditions.** We next describe some normalizations which are required in order for the homogeneous prime elements y_k from Theorem 2.3 to function as quantum cluster variables. Throughout this subsection, assume that R is a symmetric CGL extension of length N as in Definitions 2.2 and 2.5. Assume also that the following mild conditions on scalars are satisfied:

Condition (A). The base field $\mathbb{K}$ contains square roots $\nu_{kl} = \sqrt{\lambda_{kl}}$ of the scalars λ_{kl} for $1 \leq l < k \leq N$ such that the subgroup of $\mathbb{K}^*$ generated by all of them contains no elements of order 2. Then set $\nu_{kk} := 1$ and $\nu_{kl} := \nu_{lk}^{-1}$ for $k < l$, so that $\boldsymbol{\nu} := (\nu_{kl})$ is a multiplicatively skew-symmetric matrix.

Condition (B). There exist positive integers d_n, $n \in \eta([1,N])$, such that

$$\lambda_k^{d_{\eta(l)}} = \lambda_l^{d_{\eta(k)}}$$

for all $k, l \in [1, N]$ with $p(k), p(l) \neq -\infty$. In view of Proposition 2.6, this is equivalent to the condition

$$(\lambda_k^*)^{d_{\eta(l)}} = (\lambda_l^*)^{d_{\eta(k)}}, \ \ \forall\, k, l \in [1, N] \text{ with } s(k), s(l) \neq +\infty.$$

In parallel with (2.12), define

$$\mathcal{S}_{\boldsymbol{\nu}}(f) := \prod_{1 \le j < k \le N} \nu_{jk}^{-m_j m_k}, \ \ \forall\, f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N. \tag{2.24}$$

Then set

$$\overline{e}_j := e_j + e_{p(j)} + \dots + e_{p^{O_-(j)}(j)} \quad \text{and} \quad \overline{y}_j := \mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_j) y_j, \ \ \forall\, j \in [1, N]. \tag{2.25}$$

The homogeneous prime elements appearing in Proposition 2.8 are normalized analogously to (2.25):

$$\begin{aligned} \overline{y}_{[i,s^m(i)]} &:= \mathcal{S}_{\boldsymbol{\nu}}(e_{[i,s^m(i)]}) y_{[i,s^m(i)]}, \\ &\qquad \forall\, i \in [1, N], \ m \in \mathbb{Z}_{\ge 0} \ \text{such that} \ \ s^m(i) \in [1, N]. \end{aligned} \tag{2.26}$$

One additional normalization, for the leading coefficients of the homogeneous normal elements $u_{[i,s^m(i)]}$, is needed in order to establish mutation formulas for the quantum cluster variables $\overline{y}_k$. For $i \in [1, N]$ and $m \in \mathbb{Z}_{\ge 0}$ such that $s^m(i) \in [1, N]$, write

$$\begin{aligned} &\mathrm{lt}(u_{[i,s^m(i)]}) = \pi_{[i,s^m(i)]} x^{f_{[i,s^m(i)]}}, \\ &\qquad\qquad \pi_{[i,s^m(i)]} \in \mathbb{K}^*, \ \ f_{[i,s^m(i)]} \in \sum_{j=1+1}^{s^m(i)-1} \mathbb{Z}_{\ge 0}\, e_j \subset \mathbb{Z}_{\ge 0}^N. \end{aligned} \tag{2.27}$$

We will require the condition

$$\pi_{[i,s(i)]} = \mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s(i)]}), \ \ \forall\, i \in [1, N] \ \text{ such that } \ s(i) \neq +\infty. \tag{2.28}$$

This can always be satisfied after a suitable rescaling of the generators of R, as follows.

Proposition 2.10. [19, Propositions 6.3, 6.1] *Let R be a symmetric CGL extension of length N, satisfying condition* (A).

(i) *There exist N-tuples $(t_1, \dots, t_N) \in (\mathbb{K}^*)^N$ such that after the rescaling* (2.20), *condition* (2.28) *holds.*

(ii) *The set of N-tuples occurring in* (a) *is parametrized by $(\mathbb{K}^*)^{\mathrm{rk}(R)}$.*

(iii) *Once the x_j have been rescaled so that* (2.28) *holds, then*

$$\pi_{[i,s^m(i)]} = \ \mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})^{-2} \mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s^m(i)]})$$

for all $i \in [1, N]$, $m \in \mathbb{Z}_{\ge 0}$ with $s^m(i) \in [1, N]$.

2.7. **Main theorem.** In this subsection, we present the main theorem from [19], which gives quantum cluster algebra structures for symmetric CGL extensions.

Recall the notation on quantum cluster algebras from §2.1. There is a right action of S_N on the set of toric frames for a division algebra $\mathcal{F}$, given by re-enumeration,

$$(M \cdot \tau)(e_k) := M(e_{\tau(j)}), \ \ \mathbf{r}(M \cdot \tau)_{jk} = \mathbf{r}(M)_{\tau(j), \tau(k)}, \quad \tau \in S_N, \ \ k \in [1, N]. \tag{2.29}$$

Fix a symmetric CGL extension R of length N such that Conditions (A) and (B) hold. Define the multiplicatively skew-symmetric matrix $\boldsymbol{\nu}$ as in Condition (A), with associated bicharacter $\Omega_{\boldsymbol{\nu}}$, and define a second multiplicatively skew-symmetric matrix $\mathbf{r} = (r_{kj})$ by

$$r_{kj} := \Omega_{\boldsymbol{\nu}}(\overline{e}_k, \overline{e}_j), \ \ \forall\, k, j \in [1, N]. \tag{2.30}$$

Let $\overline{y}_1, \ldots, \overline{y}_N$ be the sequence of normalized homogeneous prime elements given in (2.25). (We recall that generally each of those is a prime element of some of the subalgebras R_l, not of the full algebra $R = R_N$.) There is a unique toric frame $M : \mathbb{Z}^N \to \mathrm{Fract}(R)$ whose matrix is $\mathbf{r}(M) := \mathbf{r}$ and such that $M(e_k) := \overline{y}_k$, for all $k \in [1, N]$, [19, Proposition 4.6].

Next, consider an arbitrary element $\sigma \in \Xi_N \subset S_N$, recall (2.15). By the definition of Ξ_N, for all $k \in [1, N]$,

$$\eta^{-1}\eta\sigma(k) \cap \sigma([1,k]) = \begin{cases} \{p^n(\sigma(k)), \ldots, p(\sigma(k)), \sigma(k)\}, & \text{if } \sigma(1) \le \sigma(k) \\ \{\sigma(k), s(\sigma(k)), \ldots, s^n(\sigma(k))\}, & \text{if } \sigma(1) \ge \sigma(k) \end{cases} \tag{2.31}$$

for some $n \in \mathbb{Z}_{\ge 0}$. Corresponding to σ, we have the CGL extension presentation (2.16), whose $\boldsymbol{\lambda}$-matrix is the matrix $\boldsymbol{\lambda}_\sigma$ with entries $(\boldsymbol{\lambda}_\sigma)_{ij} := \boldsymbol{\lambda}_{\sigma(i)\sigma(j)}$. Analogously we define the matrix $\boldsymbol{\nu}_\sigma$, and denote by $\mathbf{r}_\sigma$ the corresponding multiplicatively skew-symmetric matrix derived from $\boldsymbol{\nu}_\sigma$ by applying (2.30) to the presentation (2.16). It is explicitly given by

$$(\mathbf{r}_\sigma)_{kj} = \prod \{\nu_{il} \mid i \in \sigma([1,k]),\ \eta(i) = \eta\sigma(k),\ l \in \sigma([1,j]),\ \eta(l) = \eta\sigma(j)\}, \tag{2.32}$$

cf. (2.31). Let $\overline{y}_{\sigma,1}, \ldots, \overline{y}_{\sigma,N}$ be the sequence of normalized prime elements given by (2.25) applied to the presentation (2.16). By [19, Proposition 4.6], there is a unique toric frame $M_\sigma : \mathbb{Z}^N \to \mathrm{Fract}(R)$ whose matrix is $\mathbf{r}(M_\sigma) := \mathbf{r}_\sigma$ and such that for all $k \in [1, N]$

$$M_\sigma(e_k) := \overline{y}_{\sigma,k} = \begin{cases} \overline{y}_{[p^n(\sigma(k)), \sigma(k)]}, \\ \overline{y}_{[\sigma(k), s^n(\sigma(k))]} \end{cases} \tag{2.33}$$

in the two cases of (2.31), respectively. The last equality is proved in [19, Theorem 5.2].

Recall that $P(N) := \{k \in [1, N] \mid s(k) = +\infty\}$ parametrizes the set of homogeneous prime elements of R, i.e.,

$$\{y_k \mid k \in P(N)\} \text{ is a list of the homogeneous prime elements of } R$$

up to scalar multiples (Theorem 2.3). Define

$$\mathbf{ex} := [1, N] \setminus P(N) = \{l \in [1, N] \mid s(l) \neq +\infty\}.$$

Since $|P(N)| = \mathrm{rk}(R)$, the cardinality of this set is $|\mathbf{ex}| = N - \mathrm{rk}(R)$. More generally, for $\sigma \in \Xi_N$, define the set

$$\mathbf{ex}_\sigma = \{l \in [1, N] \mid \exists\, k > l \text{ with } \eta\sigma(k) = \eta\sigma(l)\}$$

of the some cardinality. Finally, recall that for a homogeneous element $u \in R$, $\chi_u \in X(\mathcal{H})$ denotes its $\mathcal{H}$-eigenvalue.

In [19, Theorem 8.2] we re-indexed all toric frames M_σ in such a way that the right action in Theorem 2.11 (c) was trivialized and the exchangeable variables in all such seeds were parametrized just by $\mathbf{ex}$, rather than by $\mathbf{ex}_\sigma$. We will not do the re-indexing here to simply the exposition.

Theorem 2.11. [19, Theorem 8.2] *Let R be a symmetric CGL extension of length N and rank* $\mathrm{rk}(R)$ *as in Definitions* 2.2, 2.5. *Assume that Conditions* (A), (B) *hold, and that the sequence of generators $x_1, \ldots, x_N$ of R is normalized* (*rescaled*) *so that condition* (2.28) *is satisfied. Then the following hold:*

(a) *For all $\sigma \in \Xi_N$* (*see* (2.15)) *and $l \in \mathbf{ex}_\sigma$, there exists a unique vector $b^l_\sigma \in \mathbb{Z}^N$ such that $\chi_{M_\sigma(b^l_\sigma)} = 1$ and*

$$\Omega_{\mathbf{r}_\sigma}(b^l_\sigma, e_j) = 1, \ \ \forall\, j \in [1, N],\ j \neq l \quad \textit{and} \quad \Omega_{\mathbf{r}_\sigma}(b^l_\sigma, e_l)^2 = \lambda^*_{\min \eta^{-1}\eta(\sigma(l))}. \tag{2.34}$$

Denote by $\widetilde{B}_\sigma \in M_{N\times|\mathbf{ex}|}(\mathbb{Z})$ *the matrix with columns* b^l_σ, $l \in \mathbf{ex}_\sigma$. *Let* $\widetilde{B} := \widetilde{B}_{\mathrm{id}}$.

(b) *For all* $\sigma \in \Xi_N$, *the pair* $(M_\sigma, \widetilde{B}_\sigma)$ *is a quantum seed for* $\mathrm{Fract}(R)$. *The principal part of* $\widetilde{B}_\sigma$ *is skew-symmetrizable via the integers* $d_{\eta(k)}$, $k \in \mathbf{ex}_\sigma$ *from Condition* (B).

(c) *All such quantum seeds are mutation-equivalent to each other up to the* S_N *action. They are linked by the following one-step mutations. Let* $\sigma, \sigma' \in \Xi_N$ *be such that*

$$\sigma' = (\sigma(k), \sigma(k+1))\sigma = \sigma(k, k+1)$$

for some $k \in [1, N-1]$. *If* $\eta(\sigma(k)) \neq \eta(\sigma(k+1))$, *then* $M_{\sigma'} = M_\sigma \cdot (k, k+1)$ *in terms of the action* (2.29). *If* $\eta(\sigma(k)) = \eta(\sigma(k+1))$, *then* $M_{\sigma'} = \mu_k(M_\sigma) \cdot (k, k+1)$.

(d) *We have the following equality between the CGL extension* R *and the quantum cluster and upper cluster algebras associated to* M, $\widetilde{B}$, $\varnothing$:

$$R = \mathcal{A}(M, \widetilde{B}, \varnothing) = \mathcal{U}(M, \widetilde{B}, \varnothing).$$

In particular, $\mathcal{A}(M, \widetilde{B}, \varnothing)$ *and* $\mathcal{U}(M, \widetilde{B}, \varnothing)$ *are affine and noetherian, and more precisely* $\mathcal{A}(M, \widetilde{B}, \varnothing)$ *is generated by the cluster variables in the seeds parametrized by the finite subset* Γ_N *of* Ξ_N, *recall* (2.17).

(e) *Let* $\mathbf{inv}$ *be any subset of the set* $P(N)$ *of frozen variables. Then*

$$R[y_k^{-1} \mid k \in \mathbf{inv}] = \mathcal{A}(M, \widetilde{B}, \mathbf{inv}) = \mathcal{U}(M, \widetilde{B}, \mathbf{inv}).$$

The next theorem gives an explicit formula for each of the exchange matrices $\widetilde{B}_\sigma$ in terms of the original exchange matrix $\widetilde{B} := \widetilde{B}_{\mathrm{id}}$. This implies that the only input needed to describe the quantum seeds $(M_\sigma, \widetilde{B}_\sigma)$ for a given algebra in Theorem 2.11 are the normalized interval prime elements (used in (2.32)–(2.33)) and the explicit form of the exchange matrix $\widetilde{B}$. Define the matrices

$$Z = (z_{jk}), \quad Z_\sigma = (z^\sigma_{jk}) \in M_N(\mathbb{Z}), \quad \sigma \in \Xi_N,$$

by $z_{jk} = 1$ if $j = p^m(k)$ for some $m \in \mathbb{Z}_{\geq 0}$ and $z_{jk} = 0$ otherwise, and $z^\sigma_{jk} = 1$ if $j \in \sigma([1,k]) \cap \eta^{-1}\eta(\sigma(k))$ and $z_{jk} = 0$ otherwise.

Theorem 2.12. *For each* $\sigma \in \Xi_N$, *the columns of the exchange matrix* $\widetilde{B}_\sigma$ *are explicitly given in terms of the columns of the original exchange matrix* $\widetilde{B}$ *as follows. For each* $l \in \mathbf{ex}_\sigma$, *there exists* $k > l$ *such that* $\eta\sigma(k) = \eta\sigma(l)$; *set* $k := \min\{j > l \mid \eta\sigma(j) = \eta\sigma(l)\}$. *Then either* $\sigma(k) = s^m\sigma(l)$ *or* $\sigma(k) = p^m\sigma(l)$ *for some* $m \in \mathbb{Z}_{>0}$, *and*

$$b^l_\sigma = \begin{cases} Z_\sigma^{-1} Z (b^{\sigma(l)} + \cdots + b^{s^{m-1}\sigma(l)}), \\ -Z_\sigma^{-1} Z (b^{p^m\sigma(l)} + \cdots + b^{p\sigma(l)}), \end{cases} \tag{2.35}$$

in the two cases respectively.

Proof. For $j \in [1, N]$, set $\mu(j) := \min \eta^{-1}\eta(j)$ and

$$\overline{e}_{\sigma,j} := \sum \{e_i \mid i \in \sigma([1,j]) \cap \eta^{-1}\eta(\sigma(j))\} \in \mathbb{Z}^N.$$

Thus, in the two cases of (2.31), $\overline{e}_{\sigma,j} = e_{[p^n(\sigma(j)),\sigma(j)]}$ and $\overline{e}_{\sigma,j} = e_{[\sigma(j),s^n(\sigma(j))]}$, respectively (using the notation (2.18)). In particular, $\overline{e}_j = \overline{e}_{\mathrm{id},j}$. Define the homomorphism $\chi : \mathbb{Z}^N \to X(\mathcal{H})$ by $\chi(\sum_j m_j e_j) := \prod_j \chi_{x_j}^{m_j}$. Thus, $\chi(\overline{e}_{\sigma,j}) = \chi_{M_\sigma(e_j)}$. For $\sigma \in \Xi_N$ and $l \in \mathbf{ex}_\sigma$, define the vectors

$$v^l_\sigma := \sum_{j=1}^N (\widetilde{B}_\sigma)_{jl} \overline{e}_{\sigma,j} \in \mathbb{Z}^N, \quad v^l := v^l_{\mathrm{id}}.$$

The sets $\{e_j\}$, $\{\overline{e}_j\}$ and $\{\overline{e}_{\sigma,j}\}$ are bases of $\mathbb{Z}^N$ and $b^l_\sigma = [v^l_\sigma]_{\{\overline{e}_{\sigma,j}\}}$ where $[v]_{\{\bullet\}}$ denotes the vector of coordinates of $v \in \mathbb{Z}^N$ with respect to a given basis.

By [19, Lemma 8.14], the statement of Theorem 2.11 (a) is valid if the first condition in (2.34) is replaced by

$$\Omega_{\mathbf{r}_\sigma}(b^l_\sigma, e_j)^2 = (\lambda^*_{\mu(\sigma(l))})^{\delta_{jl}}, \quad \forall\, l \in \mathbf{ex}_\sigma, \;\; j \in [1, N].$$

Therefore, for $\sigma \in \Xi_N$ and $l \in \mathbf{ex}_\sigma$, the vector $v^l_\sigma \in \mathbb{Z}^N$ is the unique solution of

$$\chi(v^l_\sigma) = 1 \quad \text{and} \quad \Omega_{\boldsymbol{\nu}}(v^l_\sigma, \overline{e}_{\sigma,j})^2 = (\lambda^*_{\mu(\sigma(l))})^{\delta_{jl}}, \quad \forall\, j \in [1, N]. \tag{2.36}$$

In the case $\sigma = \mathrm{id}$, the second condition is equivalent to $\Omega_{\boldsymbol{\nu}}(v^l, e_j)^2 = (\lambda^*_{\mu(l)})^{a_{jl}}$ where $a_{jl} = 1$ if $j = l$, $a_{jl} = -1$ if $j = s(l)$ and $a_{jl} = 0$ for $j \in [1, N] \backslash \{j, s(j)\}$. From the uniqueness statement in (2.36) it follows that

$$v^l_\sigma = \begin{cases} v^{\sigma(l)} + \cdots + v^{s^{m-1}\sigma(l)}, \\ -v^{p^m\sigma(l)} - \cdots - v^{p\sigma(l)}, \end{cases}$$

in the respective cases $\sigma(k) = s^m\sigma(l)$ and $\sigma(k) = p^m\sigma(l)$. The theorem follows from this by taking into account that Z and Z_σ are change of bases matrices

$$[v]_{\{e_j\}} = Z[v]_{\{\overline{e}_j\}}, \quad [v]_{\{e_j\}} = Z_\sigma[v]_{\{\overline{e}_{\sigma,j}\}}, \quad \forall\, v \in \mathbb{Z}^N, \;\; j \in [1, N]$$

and the fact that $b^l_\sigma = [v^l_\sigma]_{\{\overline{e}_{\sigma,j}\}}$. □

3. Quantum groups

In this section we gather some facts on quantum groups, quantized coordinate rings of double Bruhat cells and quantum Schubert cell algebras. We describe the connections between these classes of algebras, as well as their relation to the axiomatic class of quantum nilpotent algebras from the previous section.

3.1. **Quantized universal enveloping algebras.** Throughout $\mathfrak{g}$ will denote an arbitrary finite dimensional complex simple Lie algebra with set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ and Weyl group W. Let $\{s_i\}$, $\{\alpha_i^\vee\}$ and $\{\varpi_i\}$ be the sets of simple reflections, coroots and fundamental weights of $\mathfrak{g}$. The the weight and root lattices of $\mathfrak{g}$ will be denoted by $\mathcal{P}$ and $\mathcal{Q}$. Let $\mathcal{P}^+$ be the set of dominant integral weights of $\mathfrak{g}$ and $\mathcal{Q}^+ := \sum_i \mathbb{Z}_{\geq 0}\alpha_i$. Denote by $\langle ., . \rangle$ the invariant bilinear form on $\mathbb{R}\Pi$ normalized by $\langle \alpha_i, \alpha_i \rangle = 2$ for short roots α_i. The Cartan matrix of $\mathfrak{g}$ is given by

$$(c_{ij}) := (\langle \alpha_i^\vee, \alpha_j \rangle) \in M_r(\mathbb{Z}). \tag{3.1}$$

For $\gamma \in \mathbb{R}\Pi$, set

$$\|\gamma\|^2 = \langle \gamma, \gamma \rangle.$$

For a set of Weyl group elements $w_1, \ldots, w_m \in W$ denote the union of their supports

$$\mathcal{S}(w_1, \ldots, w_m) := \{i \in [1, r] \mid s_i \text{ appears in a reduced expression of some } w_j\}.$$

For an arbitrary (infinite) field $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$, denote by $\mathcal{U}_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{K}$ with deformation parameter q. We will follow the conventions of Jantzen's book [23] with the exception that we will denote the standard generators of $\mathcal{U}_q(\mathfrak{g})$ by $K_i^{\pm 1}$, X_i^+ and X_i^- instead of $K_{\pm\alpha_i}$, E_{α_i} and F_{α_i} for the purpose of not repeating formulas for the positive and negative parts of $\mathcal{U}_q(\mathfrak{g})$. In particular we will use the form of the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ with relations, coproduct, etc. as in [23, §4.3, 4.8].

The subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_i^\pm\}$ and $\{K_i\}$ will be denoted by $\mathcal{U}^\pm$ and $\mathcal{U}^0$, respectively. The algebra $\mathcal{U}_q(\mathfrak{g})$ is $\mathcal{Q}$-graded by

$$\deg K_i = 0, \ \ \deg X_i^\pm = \pm\alpha_i, \quad i \in [1, r].$$

Its graded components will be denoted by $\mathcal{U}_q(\mathfrak{g})_\gamma$, $\gamma \in \mathcal{Q}$.

We will use the standard notation for q-factorials and binomial coefficients based on setting $[n]_q := (q^n - q^{-n})/(q - q^{-1})$ for q-integers. Let $q_i := q^{\|\alpha_i\|^2/2}$.

3.2. **Quantum groups.** A (left) $\mathcal{U}_q(\mathfrak{g})$-module V is called a *type one* module if it is a direct sum of its weight spaces defined by

$$V_\mu := \{v \in V \mid K_i v = q^{\langle \mu, \alpha_i \rangle} v, \ \forall\, i \in [1, r]\}, \quad \mu \in \mathcal{P}.$$

The category of finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules is semisimple (see [23, Theorem 5.17] and the remark on p. 85 of [23] for the case of general fields $\mathbb{K}$) and is closed under taking tensor products and duals. The irreducible modules in this category are parametrized by the dominant integral weights $\mathcal{P}^+$, [23, Theorem 5.10]. For $\mu \in \mathcal{P}^+$ we will denote by $V(\mu)$ the corresponding irreducible module and we will fix a highest weight vector v_μ of $V(\mu)$.

Denote by G the connected, simply connected, complex simple algebraic group with Lie algebra $\mathfrak{g}$. The quantum function algebra $R_q[G]$ is defined as the Hopf subalgebra of the restricted dual $\mathcal{U}_q(\mathfrak{g})^\circ$ consisting of the matrix coefficients of all finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules; that is, $R_q[G]$ is spanned by the matrix coefficients $c_{\xi,v}$ of the modules $V(\mu)$, $\mu \in \mathcal{P}^+$, given by

$$c_{\xi,v}(x) := \xi(xv) \quad \text{for all } \ x \in \mathcal{U}_q(\mathfrak{g}) \tag{3.2}$$

where $v \in V(\mu)$, $\xi \in V(\mu)^*$. The algebra $R_q[G]$ is $\mathcal{P} \times \mathcal{P}$-graded by

$$R_q[G]_{\nu,\nu'} = \{c_{\xi,v} \mid \mu \in \mathcal{P}^+, \ \xi \in (V(\mu)^*)_\nu, \ v \in V(\mu)_{\nu'}\}, \quad \nu, \nu' \in \mathcal{P}. \tag{3.3}$$

Throughout the paper, we will make use of the compatible Lusztig actions of the braid group $\mathcal{B}_\mathfrak{g}$ on the finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules [23, §8.6] and on $\mathcal{U}_q(\mathfrak{g})$ [23, §8.14]. For $\mu \in \mathcal{P}^+$ and $w \in W$, set

$$v_{w\mu} := T_{w^{-1}}^{-1} v_\mu \in V(\mu)_{w\mu} \tag{3.4}$$

which by [32, Proposition 39.3.7] only depends on $w\mu$. Let $\xi_{w\mu} \in (V(\mu)^*)_{-w\mu}$ be the unique vector such that $\langle \xi_{w\mu}, T_{w^{-1}}^{-1} v_\mu \rangle = 1$. For $w, u \in W$ and $\mu \in \mathcal{P}^+$ define the (generalized) quantum minors

$$\Delta_{w\mu,u\mu} := c_{\xi_{w\mu}, v_{u\mu}} \in R_q[G]_{-w\mu,u\mu}.$$

They satisfy

$$\Delta_{w\mu,u\mu}(x) = \Delta_{\mu,\mu}(T_{w^{-1}} \cdot x \cdot T_{u^{-1}}^{-1}), \quad \forall\, x \in \mathcal{U}_q(\mathfrak{g}) \tag{3.5}$$

and are equal [19, §9.3] to the Berenstein–Zelevinsky quantum minors [4, Eq. (9.10)].

Remark 3.1. The use of $T_{w^{-1}}^{-1}$ (vs. T_w) is motivated by the fact that in the specialization $q = 1$, the action of $T_{w^{-1}}^{-1}$ on $\mathcal{U}_q(\mathfrak{g})$-modules specializes to the action of the Weyl group representative $\overline{w}$ of Fomin and Zelevinsky [9], while T_w spezializes to $\overline{\overline{w}}$.

The quantum minors satisfy

$$\Delta_{w\mu,u\mu}\Delta_{w\mu',u\mu'} = \Delta_{w(\mu+\mu'),u(\mu+\mu')} \quad \text{for } \ \mu, \mu' \in \mathcal{P}^+. \tag{3.6}$$

3.3. **Quantized coordinate rings of double Bruhat cells.** Define the subalgebras of $R_q[G]$

$$R^+ = \mathrm{Span}\{c_{\xi,v} \mid \mu \in \mathcal{P}^+,\ v \in V(\mu)_\mu,\ \xi \in V(\mu)^*\},$$
$$R^- = \mathrm{Span}\{c_{\xi,v} \mid \mu \in \mathcal{P}^+,\ v \in V(\mu)_{w_0\mu},\ \xi \in V(\mu)^*\}$$

where w_0 denotes the longest element of W. Joseph proved [24, Proposition 9.2.2] that $R_q[G] = R^+R^- = R^-R^+$. A special case of the notation (3.4) gives the lowest weight vectors

$$v_{-\mu} = T_{w_0}^{-1} v_{-w_0\mu} \in V(-w_0\mu)_{-\mu}, \quad \mu \in \mathcal{P}^+ \tag{3.7}$$

in terms of which we have

$$v_{-w\mu} = v_{ww_\circ(-w_\circ\mu)} = T_w v_{-\mu}$$

(the point here is that $-w_\circ\mu \in \mathcal{P}^+$). The latter enter in the definition of the quantum minors $\Delta_{-w\mu,-u\mu} \in R_q[G]$, $\mu \in \mathcal{P}$ which are another way of writing those in (3.5). For $\xi \in V(\mu)^*$ and $\xi' \in V(-w_0\mu)^*$, denote for brevity

$$c^+_\xi := c_{\xi,v_\mu} \ \text{ and } \ c^-_{\xi'} := c_{\xi',v_{-\mu}}. \tag{3.8}$$

Consider the Demazure modules

$$V^+_w(\mu) = \mathcal{U}^+V(\mu)_{w\mu} \subseteq V(\mu) \quad \text{and} \quad V^-_u(\mu) = \mathcal{U}^-V(-w_0\mu)_{-u\mu} \subseteq V(-w_0\mu). \tag{3.9}$$

The ideals

$$I^\pm_w := \mathrm{Span}\{c^\pm_\xi \mid \xi \in V^\pm_w(\mu)^\perp,\ \mu \in \mathcal{P}^+\} \subset R^\pm,$$
$$I_{w,u} := I^+_w R^- + R^+ I^-_u \subset R_q[G]$$

are completely prime and homogeneous with respect to the $\mathcal{P} \times \mathcal{P}$-grading, [24, Propositions 10.1.8, 10.3.5]. For all $\mu \in \mathcal{P}^+$, $w \in W$, $\nu, \nu' \in \mathcal{P}$ and $a \in R_q[G]_{-\nu,\nu'}$

$$\Delta_{w\mu,\mu} a = q^{\langle w\mu,\nu\rangle - \langle \mu,\nu'\rangle} a \Delta_{w\mu,\mu} \mod I^+_w R^-, \tag{3.10}$$

$$\Delta_{-w\mu,-\mu} a = q^{\langle w\mu,\nu\rangle - \langle \mu,\nu'\rangle} a \Delta_{-w\mu,-\mu} \mod R^+ I^-_w, \tag{3.11}$$

see [41, (2.22), (2.23)]. Denote $E^\pm_w := \{\Delta_{\pm w\mu,\pm\mu} \mid \mu \in \mathcal{P}^+\}$ and $E_{w,u} := q^{\mathbb{Z}} E^+_w E^-_u \subset R_q[G]$. By abuse of notation we will denote by the same symbols the images of elements and subsets in factor rings. The set $E_{w,u}$ is a multiplicative subset of normal elements of $R_q[G]/I_{w,u}$ and $E^\pm_w$ are multiplicative subsets of normal elements of $R^\pm/I^\pm_w$ (by (3.6), (3.10)-(3.11)). In particular,

$$E_{w,u} \cap I_{w,u} = \varnothing \quad \text{and} \quad E^\pm_w \cap I^\pm_w = \varnothing. \tag{3.12}$$

The localization

$$R_q[G^{w,u}] := (R_q[G]/I_{w,u})[E^{-1}_{w,u}]$$

is called the *quantized coordinate ring of the double Bruhat cell* $G^{w,u}$.

3.4. **The algebras $S^\pm_w$ and a presentation of the quantum double Bruhat cell algebras.** The localizations $R^\pm_w := (R^\pm/I^\pm_w)[(E^\pm_w)^{-1}]$ are $\mathcal{P} \times \mathcal{P}$-graded. Their subalgebras consisting of elements in degrees $\mathcal{P} \times \{0\}$ will be denoted by $S^\pm_w$. A simple computation shows that they are effectively $\mathcal{Q} \cong \mathcal{Q} \times \{0\}$–graded. The graded components of $S^\pm_w$ will be denoted by $(S^\pm_w)_\gamma = (S^\pm_w)_{\gamma,0}$, $\gamma \in \mathcal{Q}$. Each element of S^+_w (resp. S^-_u) can be represented in the form $c^+_\xi \Delta^{-1}_{w\mu,\mu}$ (resp. $\Delta^{-1}_{-w\mu,-\mu} c^-_{\xi'}$) for some $\mu \in \mathcal{P}^+$ and $\xi \in V(\mu)^*$ (resp. $\xi' \in V(-w_0\mu)^*$), see [24, §10.3.1]. Here and below the coset notations for factor rings are omitted.

For $\gamma \in \mathcal{Q}^+$ set $m_w(\gamma) := \dim \mathcal{U}^+_\gamma = \dim \mathcal{U}^-_{-\gamma}$. Let $\{x_{\gamma,n}\}_{n=1}^{m_w(\gamma)}$ and $\{x_{-\gamma,n}\}_{n=1}^{m_w(\gamma)}$ be dual bases of $\mathcal{U}^+_\gamma$ and $\mathcal{U}^-_{-\gamma}$ with respect to the Rosso–Tanisaki form, cf. [23, Ch. 6].

Joseph proved [24, §10.3.2] that there exists a unique $\mathcal{Q} \cong \mathcal{Q} \times \{0\}$-graded $\mathbb{K}$-algebra $S^+_w \bowtie S^-_u$ which satisfies the following properties:

(i) $S^+_w \bowtie S^-_u \cong S^+_w \otimes_{\mathbb{K}} S^-_u$ as $\mathbb{K}$-vector spaces and the canonical embeddings of S^+_w and S^-_u in it are graded algebra embeddings,

(ii) the elements of S^+_w and S^-_u satisfy the commutation relations

$$\begin{aligned}
(3.13) \qquad & \left[\Delta^{-1}_{-u\mu',-\mu'} c^-_{\xi'}\right]\left[c^+_\xi \Delta^{-1}_{w\mu,\mu}\right] = q^{-\langle \nu'+u\mu', \nu - w\mu\rangle} \left[c^+_\xi \Delta^{-1}_{w\mu,\mu}\right]\left[\Delta^{-1}_{-u\mu',-\mu'} c^-_{\xi'}\right] \\
& + \sum_{\gamma \in \mathcal{Q}^+, \gamma \neq 0} \sum_{i=1}^{m(\gamma)} q^{-\langle \nu'+\gamma+u\mu', \nu-\gamma-w\mu\rangle} \left[c^+_{S^{-1}(x_{\gamma,i})\xi} \Delta^{-1}_{w\mu,\mu}\right].\left[\Delta^{-1}_{-u\mu',-\mu'} c^-_{S^{-1}(x_{-\gamma,i})\xi'}\right],
\end{aligned}$$

for all $\mu, \mu' \in \mathcal{P}^+$, $\xi \in (V(\mu)^*)_{-\nu}$, $\xi' \in (V(-w_0\mu')^*)_{-\nu'}$. The commutation relation (3.13) is similar to a bicrossed product of Hopf algebras. The multiplication in the algebra induces the linear isomorphism

$$(3.14) \qquad S^+_w \bowtie S^-_u \cong S^-_u \otimes_{\mathbb{K}} S^+_w.$$

Furthermore, Joseph proved [24, §10.3.2] that there exists a graded algebra embedding

$$(3.15) \qquad S^+_w \bowtie S^-_u \hookrightarrow R_q[G^{w,u}]$$

whose restrictions to the subalgebras S^+_w and S^-_u are the compositions

$$\begin{aligned}
& S^+_w \hookrightarrow (R^+/I^+_w)[(E^+_w)^{-1}] \hookrightarrow (R_q[G]/I_{w,u})[E^{-1}_{w,u}] = R_q[G^{w,u}] \quad \text{and} \\
& S^-_u \hookrightarrow (R^-/I^-_u)[(E^-_u)^{-1}] \hookrightarrow (R_q[G]/I_{w,u})[E^{-1}_{w,u}] = R_q[G^{w,u}].
\end{aligned}$$

We will identify $S^+_w \bowtie S^-_u$ with its image. Consider the identification of $\mathcal{U}_q(\mathfrak{g})$-modules

$$(3.16) \qquad V(\varpi_i)^* \cong V(-w_0\varpi_i) \quad \text{normalized by} \quad \xi_{\varpi_i} \mapsto v_{-\varpi_i}.$$

Let $\{\xi_{i,j}\}$ and $\{\xi'_{i,j}\}$ be a pair of dual bases of $V(\varpi_i)^* \cong V(-w_0\varpi_i)$ and $V(-w_0\varpi_i)^*$ with respect to the corresponding dual pairing. Define the elements

$$(3.17) \qquad p_i := \sum_j \left[\Delta^{-1}_{-u\varpi_i,-\varpi_i} c^-_{\xi'_{i,j}}\right]\left[c^+_{\xi_{i,j}} \Delta^{-1}_{w\varpi_i,\varpi_i}\right] \in (S^+_w \bowtie S^-_u)_{(w-u)\varpi_i}.$$

The element p_i is not a scalar if and only if $i \in \mathcal{S}(w,u)$, [41, Proposition 7.6]. We have $\sum_j c^-_{\xi'_{i,j}} c^+_{\xi_{i,j}} = 1$ because

$$\begin{aligned}
(3.18) \qquad \sum_j c^-_{\xi'_{i,j}}(x) c^+_{\xi_{i,j}}(y) &= \sum_j \langle \xi'_{i,j}, x v_{-\varpi_i}\rangle \langle \xi_{i,j}, y v_{\varpi_i}\rangle \\
&= \langle x\xi_{\varpi_i}, y v_{\varpi_i}\rangle = \langle \xi_{\varpi_i}, S(x) y v_{\varpi_i}\rangle, \quad \forall\, x, y \in \mathcal{U}_q(\mathfrak{g}).
\end{aligned}$$

Therefore, in the embedding (3.15),

$$(3.19) \qquad p_i = \Delta^{-1}_{-u\varpi_i,-\varpi_i} \Delta^{-1}_{w\varpi_i,\varpi_i}.$$

By (3.10)–(3.11), all p_i are normal elements of $S^+_w \bowtie S^-_u$. Extend (3.15) to the embedding

$$(3.20) \qquad (S^+_w \bowtie S^-_u)[p_i^{-1}, i \in \mathcal{S}(w,u)] \hookrightarrow R_q[G^{w,u}].$$

Denote by $\mathcal{L}^+_w$ and $\mathcal{L}^-_u$ the respective subalgebras of $R_q[G^{w,u}]$ generated by the elements of $E^+_w \cup (E^+_w)^{-1}$ and the elements of $E^-_u \cup (E^-_u)^{-1}$. It follows from (3.6) and (3.12) that $\mathcal{L}^+_w$ and $\mathcal{L}^-_u$ are Laurent polynomial rings

$$(3.21) \qquad \mathcal{L}^+_w,\ \mathcal{L}^-_u \cong \mathbb{K}[z_1^{\pm 1}, \dots, z_r^{\pm 1}] \quad \text{by} \quad \Delta^{\pm 1}_{w\varpi_i,\varpi_i},\ \Delta^{\pm 1}_{-u\varpi_i,-\varpi_i} \mapsto z_i^{\pm 1}.$$

Finally, Joseph proved [24, 10.3.2(5)] that

$$R_q[G^{w,u}] \cong \big((S_w^+ \bowtie S_u^-)[p_i^{-1}, i \in \mathcal{S}(w,u)]\big)\#\mathcal{L}_w^+ \tag{3.22}$$
$$\cong \big((S_w^+ \bowtie S_u^-)[p_i^{-1}, i \in \mathcal{S}(w,u)]\big)\#\mathcal{L}_u^-$$

where the smash products are defined via the commutation relations (3.10)-(3.11). Since $\Delta_{w\mu,\mu}$ is a nonzero element of $R_q[G^{w,u}]_{-w\mu,\mu}$ for $\mu \in \mathcal{P}^+$, (3.20) induces the isomorphism

$$(S_w^+ \bowtie S_u^-)[p_i^{-1}, i \in \mathcal{S}(w,u)] \xrightarrow{\cong} \bigoplus_{\nu \in \mathcal{Q}} R_q[G^{w,u}]_{\nu,0} = R_q[G^{w,u}/H]. \tag{3.23}$$

The algebra on the right is called the *quantized coordinate ring of the reduced double Bruhat cell* $G^{w,u}/H$ where $H = B^+ \cap B^-$.

Some of the quoted results of Joseph were stated in [24] for base fields of characteristic 0 and $q \in \mathbb{K}$ which is transcendental over $\mathbb{Q}$. However all hold in full generality [40, 41], see in particular [41, §3.4] for the above set of embeddings and isomorphisms.

3.5. **Quantum Schubert cell algebras.** We will identify the rational character lattice $X(\mathcal{H})$ of the torus $\mathcal{H} := (\mathbb{K}^*)^r$ with $\mathcal{Q}$ by mapping $\gamma \in \mathcal{Q}$ to the character

$$h = (t_1, \dots, t_r) \in \mathcal{H} \longmapsto h^\gamma = \prod t_i^{\langle \gamma, \varpi_i^\vee \rangle}. \tag{3.24}$$

A $\mathcal{Q}$-graded algebra R (e.g., $\mathcal{U}_q(\mathfrak{g})$) has a canonical rational $\mathcal{H}$-action by algebra automorphisms where $h \cdot x = h^\gamma x$ for $x \in R_\gamma$, $h \in \mathcal{H}$. Define the embedding $\mathcal{P}^\vee \hookrightarrow \mathcal{H}$ by taking $\nu \in \mathcal{P}^\vee$ to the unique element $h(\nu) \in \mathcal{H}$ such that $h(\nu)^\gamma = q^{\langle \gamma, \nu \rangle}$ for all $\gamma \in \mathcal{Q}$.

Consider a Weyl group element w and a reduced expression $w = s_{i_1} \dots s_{i_N}$. Define the roots

$$\beta_k := s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}), \quad \forall\, k \in [1, N]. \tag{3.25}$$

The *quantum Schubert cell algebras* $\mathcal{U}^\pm[w]$, $w \in W$, were defined by De Concini, Kac, and Procesi [8], and Lusztig [32, §40.2] as the subalgebras of $\mathcal{U}^\pm$ generated by

$$X_{\beta_k}^\pm := T_{i_1} \dots T_{i_{k-1}}(X_{i_k}^\pm), \quad k \in [1, N] \tag{3.26}$$

see [32, §39.3]. These algebras do not depend on the choice of a reduced expression for w. They are $\mathcal{Q}$-graded subalgebras of $\mathcal{U}_q(\mathfrak{g})$ and are thus stable under the $\mathcal{H}$-action.

A reduced expression of w gives rise to presentations of the algebras $\mathcal{U}^\pm[w]$ as symmetric CGL extensions of the form

$$\mathcal{U}^\pm[w] = \mathbb{K}[X_{\beta_1}^\pm][X_{\beta_2}^\pm; (h(\mp\beta_2)\cdot), \delta_2)] \dots [X_{\beta_N}^\pm; (h(\mp\beta_N)\cdot), \delta_N] \tag{3.27}$$

[19, Lemma 9.1]. The reverse presentations have the forms

$$\mathcal{U}^\pm[w] = \mathbb{K}[X_{\beta_N}^\pm][X_{\beta_{N-1}}^\pm; (h(\pm\beta_{N-1})\cdot), \delta_{N-1}^*)] \dots [X_{\beta_1}^\pm; (h(\pm\beta_1)\cdot), \delta_1^*]. \tag{3.28}$$

This is derived from the Levendorskii–Soibelman straightening law

$$X_{\beta_k}^\pm X_{\beta_j}^\pm - q^{-\langle \beta_k, \beta_j \rangle} X_{\beta_j}^\pm X_{\beta_k}^\pm \in \mathbb{K}\langle X_{\beta_{j+1}}^\pm, \dots, X_{\beta_{k-1}}^\pm \rangle, \quad \forall\, 1 \le j < k \le N. \tag{3.29}$$

The interval subalgebras for the algebras $\mathcal{U}^\pm[w]$ are given by

$$\mathcal{U}^\pm[w]_{[j,k]} = T_{i_1} \dots T_{i_{j-1}}(\mathcal{U}^\pm[s_{i_j} \dots s_{i_k}]) \tag{3.30}$$

for all $1 \le j \le k \le N$.

There is an algebra isomorphism $\omega : \mathcal{U}^+[w] \to \mathcal{U}^-[w]$ obtained by restricting the algebra automorphism ω of $\mathcal{U}_q(\mathfrak{g})$ given by

$$\omega(K_i) = K_i^{-1}, \;\; \omega(X_i^\pm) = X_i^\mp, \quad \forall\, i \in [1, r].$$

It is graded if the grading of one of the algebras $\mathcal{U}^{\pm}[w]$ is reversed (i.e., $\gamma \mapsto -\gamma$).

3.6. **Quantum function algebras and isomorphisms.** We will make extensive use of the fact that the quantum Schubert cell algebras $\mathcal{U}^{\pm}[w]$ are (anti)isomorphic to the algebras $S^{\mp}_w$. To describe these (anti)isomorphisms we will need the quantum R-matrix $\mathcal{R}^w$ corresponding to w. It equals the sum of the tensor products of the elements in any two dual bases of $\mathcal{U}^+[w]$ and $\mathcal{U}^-[w]$ with respect to the Rosso-Tanisaki form and is explicitly given by

$$\mathcal{R}^w := \sum_{m_1,\dots,m_N \in \mathbb{Z}_{\geq 0}} \left(\prod_{j=1}^N \frac{(q_{i_j}^{-1} - q_{i_j})^{m_j}}{q_{i_j}^{m_j(m_j-1)/2}[m_j]_{q_{i_j}}!} \right) (X^+_{\beta_N})^{m_N} \dots (X^+_{\beta_1})^{m_1} \otimes (X^-_{\beta_N})^{m_N} \dots (X^-_{\beta_1})^{m_1}. \tag{3.31}$$

It belongs to the completion $\mathcal{U}^+[w] \widehat{\otimes} \mathcal{U}^-[w]$ of $\mathcal{U}^+[w] \otimes \mathcal{U}^-[w]$ with respect to the descending filtration [32, §4.1.1]. We will denote the flip of the two components of $\mathcal{R}^w$ by

$$\mathcal{R}^w_{\mathrm{op}} \quad \text{and also set} \quad \mathcal{R} := \mathcal{R}^{w_\circ}, \;\; \mathcal{R}_{\mathrm{op}} = \mathcal{R}^{w_\circ}_{\mathrm{op}}. \tag{3.32}$$

We will also need the unique graded algebra antiautomorphism τ of $\mathcal{U}_q(\mathfrak{g})$ given by

$$\tau(X^{\pm}_i) = X^{\pm}_i, \;\; \tau(K_i) = K_i^{-1}, \quad \forall\, i \in [1, r], \tag{3.33}$$

see [23, Lemma 4.6(b)]. It is compatible with the braid group action:

$$\tau(T_w x) = T^{-1}_{w^{-1}}(\tau(x)), \quad \forall\, x \in \mathcal{U}_q(\mathfrak{g}), \; w \in W, \tag{3.34}$$

see [23, Eq. 8.18(6)]. For a linear map $\theta : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$, denote

$${}^{\theta \times 1}\mathcal{R} := (\theta \otimes \mathrm{id})\mathcal{R}. \tag{3.35}$$

Analogously, we define ${}^{\theta\times 1}\mathcal{R}^w$, ${}^{\theta\times 1}\mathcal{R}_{\mathrm{op}}$, ${}^{1\times\theta}\mathcal{R}$, etc.

Theorem 3.2. [41, Theorem 2.6] *For all finite dimensional simple Lie algebras $\mathfrak{g}$ and Weyl group elements $w \in W$, the maps $\varphi^{\pm}_w : S^{\pm}_w \to \mathcal{U}^{\mp}[w]$ given by*

$$\varphi^+_w\big(c^+_\xi \Delta^{-1}_{w\mu,\mu}\big) := \big(c_{\xi, v_{w\mu}} \otimes \mathrm{id}\big)^{\tau\times 1}\mathcal{R}^w, \;\; \forall\, \mu \in \mathcal{P}^+, \; \xi \in V(\mu)^*$$
$$\varphi^-_w\big(\Delta^{-1}_{-w\mu,-\mu} c^-_\xi\big) := \big(c_{\xi, v_{-w\mu}} \otimes \mathrm{id}\big)^{1\times\tau}\mathcal{R}^w_{\mathrm{op}}, \;\; \forall\, \mu \in \mathcal{P}^+, \; \xi \in V(-w_0\mu)^*$$

are well defined $\mathcal{Q}$-graded algebra antiisomorphisms and isomorphisms, respectively. In the definition of $\varphi^{\pm}_w$ one can replace $\mathcal{R}^w$ with $\mathcal{R}$.

4. Quantum double Bruhat cells and symmetric CGL extensions

In this section we prove that all algebras $S^+_w \bowtie S^-_u$ are symmetric CGL extensions. Their interval subalgebras are identified with algebras of the same kind. We also describe an explicit model for the isomorphic bicrossed products $\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^-[u]$.

4.1. **Statements of the results.** Throughout the section we fix reduced expressions of the Weyl group elements $w, u \in W$,

$$w = s_{i_1} \dots s_{i_N} \quad \text{and} \quad u = s_{i'_1} \dots s_{i'_M}. \tag{4.1}$$

Consider the root vectors (3.25), (3.26) and

$$\beta'_k := s_{i'_1} \dots s_{i'_{k-1}}(\alpha_{i'_k}), \quad X^{\pm}_{\beta'_k} := T_{i'_1} \cdots T_{i'_{k-1}}(X^{\pm}_{i'_k}), \quad k \in [1, M]. \tag{4.2}$$

Define

$$w_{\leq k} := s_{i_1} \dots s_{i_k}$$

and

$$w_{[j,k]} := \begin{cases} s_{i_j} \dots s_{i_k}, & \text{if } j \leq k \\ 1, & \text{if } j > k. \end{cases}$$

(For the simplicity of notation we will not explicitly show the dependence of this notation on the choice of reduced decomposition.) Define in a similar fashion $u_{\leq k}$ and $u_{[j,k]}$.

Let

$$C^+_{\beta_k} := \left(\varphi^+_w\right)^{-1}(X^-_{\beta_k}) \in (S^+_w)_{-\beta_k} \quad \text{and} \quad C^-_{\beta'_l} := \left(\varphi^-_u\right)^{-1}(X^+_{\beta'_l}) \in (S^-_u)_{\beta'_l} \tag{4.3}$$

for $k \in [1, N]$, $l \in [1, M]$.

Theorem 4.1. *Let $\mathbb{K}$ be an arbitrary base field and $q \in \mathbb{K}^*$ a non-root of unity. For all finite dimensional simple Lie algebras $\mathfrak{g}$ and pairs of Weyl group elements (w, u), the algebra $S^+_w \bowtie S^-_u$ is a symmetric CGL extension. It has the presentation*

$$\begin{aligned} S^+_w \bowtie S^-_u = \mathbb{K}[C^+_{\beta_N}][C^+_{\beta_{N-1}}, (h(\beta_{N-1})\cdot), \delta_{N-1}] \dots [C^+_{\beta_1}, (h(\beta_1)\cdot), \delta_1] \\ [C^-_{\beta'_1}, (h(-\beta'_1)\cdot), \partial_1] \dots [C^-_{\beta'_M}, (h(-\beta'_M)\cdot), \partial_M] \end{aligned} \tag{4.4}$$

and the reverse presentation

$$\begin{aligned} S^+_w \bowtie S^-_u = \mathbb{K}[C^-_{\beta'_M}][C^-_{\beta'_{M-1}}, (h(\beta'_{M-1})\cdot), \partial^*_{M-1}] \dots [C^-_{\beta'_1}, (h(\beta'_1)\cdot), \partial^*_1] \\ [C^+_{\beta_1}, (h(-\beta_1)\cdot), \delta^*_1] \dots [C^+_{\beta_N}, (h(-\beta_N)\cdot), \delta^*_N] \end{aligned} \tag{4.5}$$

where $\delta_k, \delta^*_k, \partial_l, \partial^*_l$ are locally nilpotent skew derivations.

Denote for brevity

$$|k| := \begin{cases} N - k + 1, & \text{if } k \in [1, N] \\ k - N, & \text{if } k \in [N+1, N+M]. \end{cases} \tag{4.6}$$

Corollary 4.2. *The scalars λ_k, λ^*_k (from Definition 2.2 (iii)) for the symmetric CGL extension $S^+_w \bowtie S^-_u$ are given by*

$$\lambda^*_k = \lambda^{-1}_k = \begin{cases} q^2_{i_{|k|}}, & \textit{if } 1 \leq k \leq N \\ q^2_{i'_{|k|}}, & \textit{if } N < k \leq M + N. \end{cases}$$

Proof. By Theorem 4.1

$$\lambda_k C^+_{\beta_{|k|}} = h(\beta_{|k|}) \cdot C^+_{\beta_{|k|}} = q^{-\|\beta_{|k|}\|^2} C^+_{\beta_{|k|}} = q^{-\|\alpha_{i_{|k|}}\|^2} C^+_{\beta_{|k|}} = q^{-2}_{i_{|k|}} C^+_{\beta_{|k|}}$$

for $1 \leq k \leq N$. The other formulas and cases are analogous. □

Theorem 4.3. *In the setting of Theorem 4.1 the interval subalgebras of $S^+_w \bowtie S^-_u$ satisfy*

$$\begin{aligned} \left(S^+_w \bowtie S^-_u\right)_{[j,k]} &\cong \mathcal{U}^-[w_{[|k|,|j|]}]^{\mathrm{op}} \cong S^+_{w_{[|k|,|j|]}} \quad \textit{for} \quad 1 \leq j \leq k \leq N, \\ \left(S^+_w \bowtie S^-_u\right)_{[j,k]} &\cong \mathcal{U}^+[u_{[|j|,|k|]}] \cong S^-_{u_{[|j|,|k|]}} \quad \textit{for} \quad N < j \leq k \leq N + M, \\ \left(S^+_w \bowtie S^-_u\right)_{[j,k]} &\cong S^+_{w_{\leq |j|}} \bowtie S^-_{u_{\leq |k|}} \quad \textit{for} \quad 1 \leq j \leq N < k \leq N + M. \end{aligned}$$

The first two sets of isomorphisms are obtained from the ones in Theorem 3.2. For example, for $1 \leq j \leq k \leq N$, the isomorphism

$$\big(S_w^+ \bowtie S_u^-\big)_{[j,k]} \cong \mathcal{U}^-[w_{[N-k+1,N-j+1]}]^{\mathrm{op}}$$

is the restriction of φ_w^+ to $\mathbb{K}\langle C^-_{N-k+1}, \dots, C^-_{N-j+1}\rangle$. The isomorphism

$$\mathcal{U}^-[w_{[N-k+1,N-j+1]}]^{\mathrm{op}} \cong S^+_{w_{[N-k+1,N-j+1]}}$$

is given by $(\varphi^-_{w_{[N-k+1,N-j+1]}})^{-1}$. The last isomorphism is constructed in §4.3.

4.2. **Proof of Theorem 4.1.** Theorem 3.2 and Eq. (3.29) imply that

$$C^+_{\beta_k} C^+_{\beta_j} - q^{\langle \beta_k, \beta_j\rangle} C^+_{\beta_j} C^+_{\beta_k} \in \mathbb{K}\langle C^+_{\beta_{j+1}}, \dots, C^+_{\beta_{k-1}}\rangle \quad \text{for } 1 \leq j \leq k \leq N, \tag{4.7}$$

$$C^-_{\beta'_k} C^-_{\beta'_j} - q^{-\langle \beta'_k, \beta'_j\rangle} C^-_{\beta_j} C^-_{\beta_k} \in \mathbb{K}\langle C^-_{\beta'_{j+1}}, \dots, C^-_{\beta'_{k-1}}\rangle \quad \text{for } 1 \leq j \leq k \leq M. \tag{4.8}$$

Theorem 4.1 easily follows by induction on $j = N, \dots, 1$, $k = N+1, \dots, N+M$ from the following proposition, these two identities, and the definition of the elements $h(\gamma)$, $\gamma \in \mathcal{Q}^\vee$. We leave the details to the reader.

Proposition 4.4. *In the setting of Theorem 4.1, for all $j \in [1,N]$ and $k \in [1,M]$,*

$$C^-_{\beta'_k} C^+_{\beta_j} - q^{\langle \beta'_k, \beta_j\rangle} C^+_{\beta_j} C^-_{\beta'_k} \in \mathbb{K}\langle C^+_{\beta_{j-1}}, \dots, C^+_{\beta_1}\rangle . \mathbb{K}\langle C^-_{\beta'_1}, \dots, C^-_{\beta'_{k-1}}\rangle.$$

Before we proceed with the proof of the proposition we obtain some auxiliary results. Extend the reduced expression of w to a reduced expression of the longest element of W

$$w_0 = s_{i_1} \dots s_{i_N} \dots s_{i_l}.$$

Define the roots β_k and the root vectors $X^\pm_{\beta_k}$, $k \in [1,l]$ by extending (3.25) and (3.26) to $k \in [1,l]$. For $\mathbf{m} = (m_1, \dots, m_l) \in \mathbb{Z}^l_{\geq 0}$ set

$$X^{\mathbf{m},\pm} = (X^\pm_{\beta_l})^{m_l} \dots (X^\pm_{\beta_1})^{m_1}.$$

The Levendorskii–Soibelman straightening law (3.29) and the fact that $\mathcal{U}^\pm$ are $\mathcal{Q}^\pm$-graded imply that

$$X^{\mathbf{m}',\pm} X^{\mathbf{m},\pm} \in \mathrm{Span}\{X^{\mathbf{m}'',\pm} \mid \mathbf{m}'' \in \mathbb{Z}^l_{\geq 0},\ \mathbf{m}' \preceq \mathbf{m}''\}, \quad \forall\, \mathbf{m}, \mathbf{m}' \in \mathbb{Z}^l_{\geq 0} \tag{4.9}$$

with respect to the reverse lexicographic order (2.10) on $\mathbb{Z}^l_{\geq 0}$.

Lemma 4.5. *If $j \in [1,N]$, $\mu \in \mathcal{P}^+$, and $\xi \in V(\mu)^*$ are such that*

$$C^+_{\beta_j} = c^+_\xi \Delta^{-1}_{w\mu,\mu},$$

then

$$\varphi^+_w(c^+_{S^{-1}(\tau(X^{\mathbf{m},+}))\xi} \Delta^{-1}_{w\mu,\mu}) \in \mathcal{U}^-[w_{\leq j-1}], \quad \forall\, \mathbf{m} \in \mathbb{Z}^l_{\geq 0},\ \mathbf{m} \neq 0.$$

Proof. The last statement in Theorem 3.2 implies that

$$\langle \xi, \tau(X^{\mathbf{m}',+}) v_{w\mu}\rangle = 0 \quad \forall \mathbf{m}' \in \mathbb{Z}^l_{\geq 0}, \ \ \mathbf{m}' \neq e_j \tag{4.10}$$

where $e_1, \dots, e_l$ is the standard basis of $\mathbb{Z}^l_{\geq 0}$. Assume that $\mathbf{m}, \mathbf{m}' = (m'_1, \dots, m'_l) \in \mathbb{Z}^l_{\geq 0}$ are such that $\mathbf{m} \neq 0$ and $m'_j + \dots + m'_N > 0$. By (4.9),

$$X^{\mathbf{m}',+} X^{\mathbf{m},+} \in \mathrm{Span}\{X^{\mathbf{m}'',+} \mid \mathbf{m}'' \in \mathbb{Z}^l_{\geq 0},\ \mathbf{m}' \prec \mathbf{m}''\}.$$

Combining this with (4.10) leads to

$$\langle S^{-1}(\tau(X^{\mathbf{m},+}))\xi, \tau(X^{\mathbf{m}',+}) v_{w\mu}\rangle = \langle \xi, \tau(X^{\mathbf{m}',+} X^{\mathbf{m},+}) v_{w\mu}\rangle = 0.$$

This implies the statement of the lemma in view of the definition of the map φ^+_w. □

Proof of Proposition 4.4. The fact that $S^w_+ \bowtie S^u_- \cong S^w_+ \otimes S^u_-$ as $\mathbb{K}$-vector spaces, Theorem 3.2 and the iterated skew polynomial extension presentations of $\mathcal{U}^\pm[w]$ from §3.5 imply

$$\begin{aligned}(4.11)\quad \mathbb{K}\langle C^+_{\beta_{j-1}}, \dots, C^+_{\beta_1}\rangle . S^-_u \cap S^+_w . \mathbb{K}\langle C^-_{\beta'_1}, \dots, C^-_{\beta'_{k-1}}\rangle \\ = \mathbb{K}\langle C^+_{\beta_{j-1}}, \dots, C^+_{\beta_1}\rangle . \mathbb{K}\langle C^-_{\beta'_1}, \dots, C^-_{\beta'_{k-1}}\rangle .\end{aligned}$$

We will prove that

$$C^-_{\beta'_k} C^+_{\beta_j} - q^{\langle \beta'_k, \beta_j\rangle} C^+_{\beta_j} C^-_{\beta'_k} \in \mathbb{K}\langle C^+_{\beta_{j-1}}, \dots, C^+_{\beta_1}\rangle . S^-_u \tag{4.12}$$

Analogously one proves

$$C^-_{\beta'_k} C^+_{\beta_j} - q^{\langle \beta'_k, \beta_j\rangle} C^+_{\beta_j} C^-_{\beta'_k} \in S^+_w . \mathbb{K}\langle C^-_{\beta'_1}, \dots, C^-_{\beta'_{k-1}}\rangle . \tag{4.13}$$

The proposition follows from the combination of Eqs. (4.11), (4.12) and (4.13).

Let $\mu, \mu' \in \mathcal{P}^+$, $\xi \in V(\mu)_{-\nu}$, $\xi' \in V(-w_0\mu')_{-\nu'}$ be such that

$$C^+_{\beta_j} = c^+_\xi \Delta^{-1}_{w\mu,\mu} \quad \text{and} \quad C^-_{\beta'_k} = \Delta^{-1}_{-u\mu',-\mu'} c^-_{\xi'} .$$

Since φ^+_w, φ^-_u are graded maps,

$$-\beta_j = -\nu + w\mu \quad \text{and} \quad \beta'_k = -\nu' - u\mu' . \tag{4.14}$$

The sets

$$\{X^{\mathbf{m},+} \mid \mathbf{m} \in \mathbb{Z}^l_{\geq 0},\ \mathbf{m} \neq 0\} \quad \text{and} \quad \{t_{\mathbf{m}} X^{\mathbf{m},-} \mid \mathbf{m} \in \mathbb{Z}^l_{\geq 0},\ \mathbf{m} \neq 0\} \tag{4.15}$$

are dual bases of $\oplus_{\gamma \in \mathcal{Q}^+, \gamma \neq 0} \mathcal{U}^+_\gamma$ and $\oplus_{\gamma \in \mathcal{Q}^+, \gamma \neq 0} \mathcal{U}^-_{-\gamma}$ with respect to the Rosso–Tanisaki form for some scalars $t_{\mathbf{m}} \in \mathbb{K}^*$ (equal to those in (3.31)). By [23, Lemma 6.16], the antiautomorphism τ of $\mathcal{U}_q(\mathfrak{g})$ satisfies

$$\langle \tau(x_+), \tau(x_-)\rangle = \langle x_+, x_-\rangle ,$$

for all $x_\pm \in \mathcal{U}^\pm$. So applying τ to (4.15) produces another set of dual bases. Invoking the commutation relation (3.13) with the use of the last pair of dual bases, Lemma 4.5 and Eq. (4.14) gives

$$C^-_{\beta'_k} C^+_{\beta_j} - q^{\langle \beta'_k, \beta_j\rangle} C^+_{\beta_j} C^-_{\beta'_k} \in \mathbb{K}\langle C^+_{\beta_{j-1}}, \dots, C^+_{\beta_1}\rangle S^u_- .$$

This proves (4.12) which completes the proof of the proposition. □

4.3. **Proof of Theorem 4.3.** The first two sets of isomorphisms in Theorem 4.3 were defined and proved in §4.1. We proceed with the last isomorphism. Fix two integers $j \in [1, N]$ and $k \in [N+1, N+M]$. Set for brevity

$$A^+ := \mathbb{K}\langle C^+_{\beta_{N-j+1}}, \dots, C^+_{\beta_1}\rangle, \quad A^- := \mathbb{K}\langle C^-_{\beta'_1}, \dots, C^-_{\beta'_{k-N}}\rangle .$$

Identify

$$(S^+_w \bowtie S^-_u)_{[j,k]} = \mathbb{K}\langle C^+_{\beta_{N-j+1}}, \dots, C^+_{\beta_1}, C^-_{\beta'_1}, \dots, C^-_{\beta'_{k-N}}\rangle \cong A^+ \otimes_{\mathbb{K}} A^-$$

as $\mathbb{K}$-vector spaces. Define the $\mathbb{K}$-linear isomorphism

$$\pi_{[j,k]} : (S^+_w \bowtie S^-_u)_{[j,k]} \to S^+_{w_{\leq N-j+1}} \bowtie S^-_{u_{\leq k-N}}$$

by

$$\pi_{[j,k]}|_{A^+} := \big(\varphi^+_{w_{\leq N-j+1}}\big)^{-1} \varphi^+_w|_{A^+} \quad \text{and} \quad \pi_{[j,k]}|_{A^-} := \big(\varphi^-_{u_{\leq k-N}}\big)^{-1} \varphi^-_u .$$

It provides the last isomorphism in Theorem 4.3 by the following result.

Theorem 4.6. *The map*

$$\pi_{[j,k]} : (S_w^+ \bowtie S_u^-)_{[j,k]} \to S^+_{w_{\leq N-j+1}} \bowtie S^-_{u_{\leq k-N}}$$

is a $\mathcal{Q}$-graded algebra isomorphism for all $1 \leq j \leq N < k \leq N+M$.

Proof. By Theorem 3.2, the restrictions of $\pi_{[j,k]}$ to $A^\pm$ are $\mathcal{Q}$-graded algebra isomorphisms. The statement of the theorem amounts to verifying that the commutation rule (3.13) is transformed appropriately under $\pi_{[j,k]}$.

It is sufficient to establish the theorem in the two cases $j \in [1,N]$, $k = N+M$ and $j = 1$, $k \in [N+1, N+M]$, and apply a composition of the maps π in the general case. We will restrict to the first case, the second being analogous. Thus, from now on $k = N+M$.

Fix $m \in [1, N-j+1]$. Let $\mu \in \mathcal{P}^+$, $\xi \in V(\mu)^*$, $\overline{\xi} \in V(\mu)^*$ be such that $C^+_{\beta_m} = c^+_\xi \Delta^{-1}_{w\mu,\mu}$ and $\pi_{[j,N+M]}(C^+_{\beta_m}) = \big(\varphi^+_{w_{\leq N-j+1}}\big)^{-1}(X^-_{\beta_m}) = c^+_{\overline{\xi}} \Delta^{-1}_{w_{\leq N-j+1}\mu,\mu}$. Consider the $\mathbb{K}$-basis $\{X^{\mathbf{m},+} \mid \mathbf{m} \in \mathbb{Z}^l_{\geq 0}\}$ of $\mathcal{U}^+$ from §4.2. As in §4.2, the definition of φ^+_w implies that

$$\langle \xi, \tau(X^{\mathbf{m}}) v_{w\mu} \rangle = \langle \overline{\xi}, \tau(X^{\mathbf{m}}) v_{w_{\leq N-k+1}(\mu)} \rangle = \delta_{\mathbf{m}, e_m}$$

for all $\mathbf{m} \in \mathbb{Z}^l_{\geq 0}$. Therefore

$$\langle \xi, x v_{w\mu} \rangle = \langle \overline{\xi}, x V_{w_{\leq N-k+1}(\mu)} \rangle, \quad \forall\, x \in \mathcal{U}^+$$

and

$$\varphi^+_w \left(c^+_{S^{-1}(x)\xi} \Delta^{-1}_{w\mu,\mu} \right) = \varphi^+_{w_{\leq N-j+1}} \left(c^+_{S^{-1}(x)\overline{\xi}} \Delta^{-1}_{w_{\leq N-j+1}\mu,\mu} \right), \quad \forall\, x \in \mathcal{U}^+.$$

This implies that $\pi_{[j,N+M]}$ transforms appropriately the commutation relation (3.13) in the case when the first term is taken to be an arbitrary element of S^-_u and the second term is taken to be $C^+_{\beta_m}$, $m \in [1, N-j+1]$. Hence, $\pi_{[j,N+M]}$ is a $\mathcal{Q}$-graded algebra isomorphism. □

4.4. **The algebras $\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u]$.** We use the linear isomorphism $\varphi^+_w \otimes \varphi^-_u$ from $S^+_w \bowtie S^-_u \cong S^+_w \otimes S^-_u$ to $\mathcal{U}^-[w]_{\mathrm{op}} \otimes \mathcal{U}^+[u]$ to transfer the algebra structure on $S^+_w \bowtie S^-_u$ to an algebra structure on $\mathcal{U}^-[w]_{\mathrm{op}} \otimes \mathcal{U}^+[u]$ which will be denoted by $\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^-[u]$. Theorem 3.2 implies that

$$\varphi_{w,u} := \varphi^+_w \otimes \varphi^-_u : S^+_w \bowtie S^-_u \cong S^+_w \otimes S^-_u \to \mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u] \tag{4.16}$$

is an algebra isomorphism and that $\mathcal{U}^-[w]_{\mathrm{op}}, \mathcal{U}^+[u] \hookrightarrow \mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u]$ are algebra embeddings. By way of definition $\varphi_{w,u}|_{S^+_w} = \varphi^+_w$ and $\varphi_{w,u}|_{S^-_u} = \varphi^-_u$. By Theorem 4.6 we have the algebra embeddings

$$\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u] \hookrightarrow \mathcal{U}^-_{\mathrm{op}} \bowtie \mathcal{U}^+ = \mathcal{U}^-[w_0]_{\mathrm{op}} \bowtie \mathcal{U}^+[w_0]$$

induced by the canonical embeddings $\mathcal{U}^-[w] \hookrightarrow \mathcal{U}^-$ and $\mathcal{U}^+[u] \hookrightarrow \mathcal{U}^+$. This gives the following explicit description of the algebras $\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u] \cong S^+_w \bowtie S^-_u$.

Theorem 4.7. (i) *For any simple Lie algebra $\mathfrak{g}$, the algebra $\mathcal{U}^-_{\mathrm{op}} \bowtie \mathcal{U}^+$ is isomorphic to the tensor product $\mathcal{U}^-_{\mathrm{op}} \otimes \mathcal{U}^+$ in which $\mathcal{U}^-_{\mathrm{op}}$ and $\mathcal{U}^+$ sit as subalgebras and commute as follows:*

$$X^+_i X^-_j = q^{\langle \alpha_i, \alpha_j \rangle} X^-_j X^+_i + \delta_{i,j} (q_i^{-1} - q_i)^2, \quad \forall\, i,j \in [1,r]. \tag{4.17}$$

In other words, $\mathcal{U}^-_{\mathrm{op}} \bowtie \mathcal{U}^+$ is the algebra with generators $X^\pm_i$ subject to the quantum Serre relations for X^+_i, the opposite quantum Serre relations for X^-_i and the above mixed relations.

(ii) *For all Weyl group elements w, u, the algebra $\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u]$ is a subalgebra of $\mathcal{U}^-_{\mathrm{op}} \bowtie \mathcal{U}^+$ equal to the subspace $\mathcal{U}^-[w]_{\mathrm{op}} \otimes \mathcal{U}^+[u]$. In other words, it is the subalgebra generated by $X^-_{\beta_1}, \dots X^-_{\beta_N}$, $X^+_{\beta'_1}, \dots, X^+_{\beta'_M}$ in the notation of* §4.1.

The scalars in the right hand side of (4.17) can be rescaled to any collection in $\mathbb{K}^*$ by rescaling $X^{\pm}_1, \dots, X^{\pm}_r$, which does not change the Serre relations.

Proof. For given $i, j \in [1, r]$ choose two reduced expression of w_0 that start with s_i and s_j respectively. Then in the notation from §4.1 for $w = u = w_0$, $X^-_{\beta_1} = X^-_j$ and $X^+_{\beta'_1} = X^+_i$. Theorem 4.1 implies that

$$X^+_i X^-_j = q^{\langle \alpha_i, \alpha_j \rangle} X^-_j X^+_i + t_{i,j}$$

for some $t_{i,j} \in \mathbb{K}$. For degree reasons $t_{i,j} = 0$ for $i \neq j$. The scalars $t_{i,i}$ are evaluated directly by applying (3.13). The second part of the theorem follows from Theorem 4.6 □

Using Theorem 3.2 and a direct calculation, one shows that the elements p_i from (3.17) are given by

$$p_i = t \varphi^{-1}_{w,u} \left((\Delta_{u\varpi_i, w\varpi_i} \otimes \mathrm{id}) \left({}^{S\tau \times 1}\mathcal{R}^u_{\mathrm{op}} \, {}^{\tau \times 1}\mathcal{R}^w \right) \right) \tag{4.18}$$

for some $t \in \mathbb{K}^*$, recall the notation (3.32) and (3.35). Theorem 4.7 gives an explicit model for the quantized coordinate rings of all reduced double Bruhat cells (cf. §3.4),

$$R_q[G^{w,u}/H] \cong (\mathcal{U}^-[w]_{\mathrm{op}} \bowtie \mathcal{U}^+[u]) \big[\big((\Delta_{u\varpi_i, w\varpi_i} \otimes \mathrm{id}) ({}^{S\tau \times 1}\mathcal{R}^u_{\mathrm{op}} \, {}^{\tau \times 1}\mathcal{R}^w) \big)^{-1}, i \in \mathcal{S}(w,u) \big].$$

5. Classification of the homogeneous prime elements of $S^+_w \bowtie S^-_u$

In this section we classify the homogeneous prime elements of the algebras $S^+_w \bowtie S^-_u$. This determines the ranks of these algebras as CGL extensions. As other corollaries of the main result we obtain classifications of all normal elements of the algebras $S^+_w \bowtie S^-_u$ and $R_q[G^{w,u}]$.

5.1. **Statements of the main result and its corollaries.** As in the previous sections, all of the following results are for an arbitrary base field $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$.

Recall the definition of the normal elements $p_i \in (S^+_w \bowtie S^-_u)_{(w-u)\varpi_i}$ from §3.4.

Theorem 5.1. *For all finite dimensional simple Lie algebras $\mathfrak{g}$ and $w, u \in W$, the homogeneous prime elements of $S^+_w \bowtie S^-_u$ are the nonzero scalar multiples of the elements $p_i \in (S^+_w \bowtie S^-_u)_{(w-u)\varpi_i}$, $i \in \mathcal{S}(w,u)$, defined in* (3.17), *and those elements are not associates of each other. In particular, the rank of the algebra $S^+_w \bowtie S^-_u$ equals $|\mathcal{S}(w,u)|$.*

The theorem is proved in §5.4 and §§5.2–5.3 contain preparatory material.

The elements p_i quasicommute and the monomials in those elements are homogeneous normal elements of $S^+_w \bowtie S^-_u$. By [18, Theorem 1.2] for every CGL extension R, the monomials in the homogeneous prime elements of R with different collections of exponents have different weights with respect to the $X(\mathcal{H})$-grading. The normalizing automorphisms of the p_i are determined from

$$p_i a = q^{-\langle (w+u)\varpi_i, \gamma \rangle} a p_i, \quad \forall \, a \in (S^+_w \bowtie S^-_u)_\gamma, \ \gamma \in \mathcal{Q} \tag{5.1}$$

which follows from (3.10)–(3.11) and (3.19).

Corollary 5.2. *The homogeneous normal elements of $S^+_w \bowtie S^-_u$ are precisely the monomials in the elements p_i, $i \in \mathcal{S}(w,u)$.*

The normal elements of $S^+_w \bowtie S^-_u$ are the sums of monomials in the p_i that have the same normalizing automorphism (*which is explicitly computed from* (5.1)).

The corollary follows from Theorem 5.1 and [18, Proposition 2.6] which states that the normal elements of an $\mathcal{H}$-UFD R (for a torus $\mathcal{H}$) are all sums of monomials in the homogeneous prime elements of R that have the same normalizing automorphism.

The elements $\Delta_{w\varpi_i,\varpi_i}$, $i \in \mathcal{S}(w,u)$, and $\Delta_{-u\varpi_i,-\varpi_i}$, $i \in [1,r]$, are normal elements of $R_q[G^{w,u}]$ and their normalizing automorphisms are given by (3.10)–(3.11). By the above discussion and Eq. (3.19) the Laurent polynomials in those elements with different sets of exponents are linearly independent. On the other hand, for all $i \notin \mathcal{S}(w,u)$, $\Delta_{w\varpi_i,\varpi_i} = \Delta^{-1}_{-u\varpi_i,-\varpi_i}$ (as elements of $R_q[G^{w,u}]$).

Corollary 5.3. *The normal elements of $R_q[G^{w,u}]$ are the sums of Laurent monomials in $\Delta_{w\varpi_i,\varpi_i}$, $i \in \mathcal{S}(w,u)$, and $\Delta_{-u\varpi_i,-\varpi_i}$, $i \in [1,r]$, that have the same normalizing automorphism.*

The homogeneous normal elements of $R_q[G^{w,u}]$ with respect to the $\mathcal{P} \times \mathcal{P}$-grading are the Laurent monomials in $\Delta_{w\varpi_i,\varpi_i}$, $i \in \mathcal{S}(w,u)$, and $\Delta_{-u\varpi_i,-\varpi_i}$, $i \in [1,r]$

Proof. Each element x of $R_q[G^{w,u}]$ has the form $\sum_j x_j \# m_j$ where $x_j \in S_w^+ \bowtie S_u^-$ and m_j are linearly independent Laurent monomials in $\Delta_{-u\varpi_i,-\varpi_i}$, $i \in [1,N]$. Such an element is homogeneous with respect to the $\mathcal{P} \times \mathcal{P}$-grading if and only if the sum has only one term and x_1 is a homogeneous element of $S_w^+ \bowtie S_u^-$. Under those conditions, if x is a normal element of $R_q[G^{w,u}]$, then x_1 is a normal element of $S_w^+ \bowtie S_u^-$. Applying Corollary 5.2 and Eq. (3.19) completes the proof of the second part.

The first part follows from the fact that the normal elements of a $\mathbb{Z}^l$-graded ring R are the linear combinations of homogeneous normal elements of R that have the same normalizing automorphism. □

Corollary 5.3 strengthens [41, Theorem 1.1] which classified the centers of $R_q[G^{w,u}]$ by investigating certain sets of q-normal elements of $R_q[G^{w,u}]$. The results in Corollaries 5.2 and 5.3 are of independent interest due to the relationship of those sets of normal elements to prime spectra of quantum groups and related algebras [24].

5.2. **Factors of the elements p_i.**

Lemma 5.4. *If p is a homogeneous prime element of $S_w^+ \bowtie S_u^-$, then $p|p_i$ for some $i \in \mathcal{S}(w,u)$.*

Proof. Assume that this is not the case. It follows from (3.22) that p is a homogeneous prime element of $R_q[G^{w,u}] \cong (S_w^+ \bowtie S_u^-)\#\mathcal{L}_u$. This contradicts the fact that $R_q[G^{w,u}]$ has no nontrivial homogeneous prime ideals [24, Theorem 10.3.4]. □

5.3. **Maximal terms of elements of $S_w^+ \bowtie S_u^-$.** The $\mathcal{Q}$-grading of the algebras S_w^+ and S_u^- is supported in $-\mathcal{Q}^+$ and $\mathcal{Q}^+$, respectively. Consider the vector space decomposition

$$S_w^+ \bowtie S_u^- = \bigoplus_{\nu \in -\mathcal{Q}^+,\, \nu' \in \mathcal{Q}^+} (S_w^+)_\nu \otimes (S_u^-)_{\nu'}.$$

For an element $a \in S_w^+ \bowtie S_u^-$ denote by $[a]_{\nu,\nu'}$ its component in $(S_w^+)_\nu \otimes (S_u^-)_{\nu'}$. Denote its support by

$$\operatorname{supp}(a) := \{(\nu,\nu') \in (-\mathcal{Q}^+) \times \mathcal{Q}^+ \mid [a]_{\nu,\nu'} \neq 0\}.$$

Define the partial order $\geq$ on $(-\mathcal{Q}^+) \times \mathcal{Q}^+$ by

$$(\nu_1,\nu_1') \geq (\nu_2,\nu_2') \quad \text{if and only if there exist } \gamma,\gamma' \in \mathcal{Q}^+ \\ \text{such that } \nu_1 = \nu_2 - \gamma \text{ and } \nu_1' = \nu_2' + \gamma'.$$

Denote by $\mathrm{msupp}(a)$ the maximal elements in the support of an element $a \in S_w^+ \bowtie S_u^-$. For $a \in S_w^+ \bowtie S_u^-$, $a \neq 0$,

$$a \in \mathbb{K}^* \Leftrightarrow \mathrm{msupp}(a) = \{(0,0)\}.$$

Call the components $[a]_{\nu,\nu'}$ for $(\nu,\nu') \in \mathrm{msupp}(a)$ the *maximal terms* of a. By (3.13),

$$\mathrm{msupp}(xy) = \{\text{maximal elements of } \mathrm{msupp}(x) + \mathrm{msupp}(y)\}, \quad \forall x, y \in S_w^+ \bowtie S_u^-. \tag{5.2}$$

For a subset $I \subseteq [1,r]$, set

$$\mathcal{P}_I^+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i.$$

Define the elements

$$d_{w,\mu}^{\pm} := \Delta_{\pm\mu,\pm\mu} \Delta_{\pm w\mu,\pm\mu}^{-1} \in (S_w^{\pm})_{\pm(w-1)\mu}, \quad \forall \mu \in \mathcal{P}_{\mathcal{S}(w)}^+.$$

(For the rest of this section the previous convention for right/left denominators for $S_w^{\pm}$ will not be needed.) Their scalar multiples exhaust all homogeneous normal elements of $S_w^{\pm}$ by [41, Theorem 6.1 (i)] and

$$d_{w,\mu}^{\pm} a = q^{\langle (w+1)\mu, \gamma \rangle} a d_w^{\pm}, \quad \forall\, a \in (S_w^{\pm})_\gamma,\ \gamma \in \mathcal{Q},$$

cf. [41, Eq. (3.30)].

Lemma 5.5. *The maximal terms of each homogeneous normal element of $S_w^+ \bowtie S_u^-$ are of the form*

$$t d_{w,\mu}^+ d_{u,\mu'}^-$$

for some $\mu \in \mathcal{P}_{\mathcal{S}(w)}^+$, $\mu' \in \mathcal{P}_{\mathcal{S}(u)}^+$ and $t \in \mathbb{K}^*$.

Proof. Each maximal term of a normal element $c \in S_w^+ \bowtie S_u^-$ can be written in the form

$$a_1 b_1 + \cdots + a_m b_m$$

where $a_1, \ldots, a_m \in (S_w^+)_\nu$ and $b_1, \ldots, b_m$ are linearly independent elements of $(S_u^-)_{\nu'}$ for some $\nu \in -\mathcal{Q}^+$, $\nu' \in \mathcal{Q}^+$. The normality of c, linear independence of $b_1, \ldots, b_m$ and Eq. (3.13) easily imply that $a_1, \ldots, a_m$ are normal elements. By the above mentioned [41, Theorem 6.1 (i)] all of them must be scalar multiples of $d_{w,\mu}^+$ where $\mu \in \mathcal{P}_{\mathcal{S}(w)}^+$ is the unique element such that $\nu = (w-1)\mu$ (if ν can be written in this form). Reversing the roles of S_w^+ and S_u^- leads to the desired result. □

5.4. **Proof of the main result.** The definition of the element p_i implies that it has one maximal term which equals

$$\overline{p}_i := \begin{cases} t d_{w,\varpi_i}^+ d_{u,\varpi_i}^-, & \text{if } i \in \mathcal{S}(w) \cap \mathcal{S}(u) \\ t d_{w,\varpi_i}^+, & \text{if } i \in \mathcal{S}(w) \backslash \mathcal{S}(u) \\ t d_{u,\varpi_i}^-, & \text{if } i \in \mathcal{S}(u) \backslash \mathcal{S}(w) \end{cases}$$

for some $t \in \mathbb{K}^*$.

Proof of Theorem 5.1. Because of Lemma 5.4 and [18, Proposition 2.2 (b)], all we need to prove is that p_i are irreducible elements of $S_w^+ \bowtie S_u^-$ for $i \in \mathcal{S}(w,u)$. Let $p_i = ab$ for some $a, b \in S_w^+ \bowtie S_u^-$. The fact that p_i has only one maximal term and Eqs. (3.13) and (5.2) imply that a and b also have one maximal term and those maximal terms (to be denoted by $\overline{a}, \overline{b}$) satisfy

$$\overline{a}\overline{b} = \overline{p}_i.$$

Case 1: $i \in \mathcal{S}(w) \backslash \mathcal{S}(u)$. The tensor product decomposition of $S_w^+ \bowtie S_u^-$ into S_w^+ and S_u^-, and the fact that d_{w,ϖ_i}^+ are prime elements of S_w^+ ([41, Theorem 6.1 (ii)]) imply that one of the factors $\overline{a}, \overline{b}$ is a scalar. This means that either $a \in \mathbb{K}^*$ or $b \in \mathbb{K}^*$, so p_i is irreducible.

Case 2: $i \in \mathcal{S}(u) \backslash \mathcal{S}(w)$. This case is analogous to the previous one.

Case 3: $i \in \mathcal{S}(w) \cap \mathcal{S}(u)$. In this case

$$\overline{a}\overline{b} = t d^+_{w,\varpi_i} d^-_{u,\varpi_i}, \quad t \in \mathbb{K}^*$$

If $\overline{a}$ or $\overline{b}$ is a scalar, we complete the proof as in case 1. Otherwise, the primality of $d^+_{w,\varpi_i} \in S^+_w$ and $d^-_{u,\varpi_i} \in S^-_u$ and (3.13) imply that either

$$\overline{a} = t_1 d^+_{w,\varpi_i},\ \overline{b} = t_2 d^-_{u,\varpi_i} \quad \text{or} \quad \overline{a} = t_1 d^-_{u,\varpi_i},\ \overline{b} = t_2 d^-_{w,\varpi_i}$$

for some $t_1, t_2 \in \mathbb{K}^*$. We will restrict to the first case, leaving the analogous second case to the reader. The definition of maximal support implies that $a \in S^+_w$ and $b \in S^-_u$. Furthermore the tensor product decomposition of $S^+_w \bowtie S^-_u$ into S^+_w and S^-_u and the homogeneity of p_i imply that a and b also need to be homogeneous. Thus $a = \overline{a} = t_1 d^+_{w,\varpi_i}$, $b = \overline{b} = t_2 d^-_{u,\varpi_i}$ and

$$p_i = t d^+_{w,\varpi_i} d^-_{u,\varpi_i}. \tag{5.3}$$

Since $i \in \mathcal{S}(w) \cap \mathcal{S}(u)$, by [24, Corollary 4.4.6], $c^+_{\xi_{s_i\varpi_i}} = \Delta_{s_i\varpi_i,\varpi_i} \notin I^+_w$ and $c^-_{\xi_{-s_i\varpi_i}} = \Delta_{-s_i\varpi_i,-\varpi_i} \notin I^-_u$. Therefore, as elements of S^+_w and S^-_u,

$$c^+_{\xi_{s_i\varpi_i}} \Delta^{-1}_{w\varpi_i,\varpi_i} \neq 0 \quad \text{and} \quad \Delta^{-1}_{-u\varpi_i,-\varpi_i} c^-_{\xi_{-s_i\varpi_i}} \neq 0.$$

The definition of p_i implies that $[p_i]_{(w-s_i)\varpi_i,(s_i-u)\varpi_i} \neq 0$ which contradicts (5.3). This completes the proof. □

6. Quantum cluster algebra structure on $S^+_w \bowtie S^-_u$ and reduced double Bruhat cells

The section describes the interval prime elements of the CGL extensions $S^+_w \bowtie S^-_u$. We construct quantum cluster algebra structures on $S^+_w \bowtie S^-_u$ and on the reduced double Bruhat cells $R_q[G^{w,u}/H]$, exhibit a large number of explicit seeds for them, and prove that they coincide with the corresponding upper quantum cluster algebras.

6.1. Classification of the prime elements of the interval subalgebras of $S^+_w \bowtie S^-_u$. As in Section 4 we fix reduced expressions (4.1) of the Weyl group elements w and u. Recall the notation $k \mapsto |k|$ from (4.6). Define the function $\eta : [1, N+M] \to \mathbb{Z}$ by

$$\eta(k) := \begin{cases} i_{|k|}, & \text{if } k \in [1, N] \\ i'_{|k|}, & \text{if } k \in [N+1, N+M]. \end{cases} \tag{6.1}$$

Consider the following rescaled generators of the CGL extension presentation of $S^+_w \bowtie S^-_u$ from Theorem 4.1:

$$x_k := \begin{cases} q^{1/2}_{i_{|k|}} (q^{-1}_{i_{|k|}} - q_{i_{|k|}}) C^+_{\beta_{|k|}}, & \text{if } k \in [1, N] \\ q^{1/2}_{i'_{|k|}} (q^{-1}_{i'_{|k|}} - q_{i'_{|k|}}) C^-_{\beta'_{|k|}}, & \text{if } k \in [N+1, N+M], \end{cases} \tag{6.2}$$

recall (4.3).

Theorem 6.1. *Let $\mathfrak{g}$ be an arbitrary finite dimensional simple Lie algebra and (w,u) a pair of Weyl group elements. Let $\mathbb{K}$ be a base field of arbitrary characteristic and $q \in \mathbb{K}^*$ a non-root of unity such that $\sqrt{q} \in \mathbb{K}$. Consider the CGL extension presentation* (4.4) *of $S^+_w \bowtie S^-_u$ with the rescaled generators* (6.2)*. Then the following hold:*

(a) *The function* η *from Theorem* 2.3 *can be chosen as the function* (6.1)*.*
(b) *The normalized interval prime elements* (2.26) *are given as follows.*
1) *If* $1 \le j < k \le N$ *and* $i_{|j|} = i_{|k|} (=: i)$, *then*

$$\overline{y}_{[j,k]} = \sqrt{q}^{\,\|(w_{\le |j|} - w_{<|k|})\varpi_i\|^2/2} (\varphi_w^+)^{-1} \left((\Delta_{w_{<|k|}\varpi_i, w_{\le |j|}\varpi_i} \otimes \mathrm{id})\, {}^{\tau \times 1}\mathcal{R}^w \right). \tag{6.3}$$

2) *If* $N < j < k \le N+M$ *and* $i'_{|j|} = i'_{|k|} (=: i)$, *then*

$$\overline{y}_{[j,k]} = \sqrt{q}^{\,\|(u_{\le |k|} - u_{<|j|})\varpi_i\|^2/2} \times \\ (\varphi_u^-)^{-1} \left((\Delta_{\varpi_i, \varpi_i} \otimes \mathrm{id}) \left((T^{-1}_{u_{\le |k|}} \otimes 1) \cdot {}^{S\tau \times 1}\mathcal{R}^u_{\mathrm{op}} \cdot (T_{u_{<|j|}} \otimes 1) \right) \right). \tag{6.4}$$

3) *If* $1 \le j \le N < k \le N+M$ *and* $i_{|j|} = i'_{|k|} (=: i)$, *then*

$$\overline{y}_{[j,k]} = \sqrt{q}^{\,\|(u_{\le |k|} - w_{\le |j|})\varpi_i\|^2/2} \times \\ \varphi_{w,u}^{-1} \left((\Delta_{\varpi_i, w_{\le |j|}\varpi_i} \otimes \mathrm{id}) \left((T^{-1}_{u_{\le |k|}} \otimes 1) \cdot {}^{S\tau \times 1}\mathcal{R}^u_{\mathrm{op}} \, {}^{\tau \times 1}\mathcal{R}^w \right) \right) \tag{6.5}$$

in terms of the algebra isomorphism (4.16)*.*
(c) *The condition* (2.28) *is satisfied.*
Furthermore, all formulas remain valid if $\mathcal{R}^w$ *and* $\mathcal{R}^u_{\mathrm{op}}$ *are replaced by* $\mathcal{R}$ *and* $\mathcal{R}_{\mathrm{op}}$, *respectively, cf.* (3.32)*.*

Note that the three braid group terms in (6.4)-(6.5) are the opposite to the ones defining the quantum minors (3.5). They specialize to the Fomin–Zelevinsky elements $\overline{\overline{w}}$.
Theorem 6.1 is proved in §6.2.

6.2. **Quantum cluster algebra structures on** $S_w^+ \bowtie S_u^-$ **and** $R_q[G^{w,u}/H]$**.** Recall the setting of §2.7 and the definitions (2.15)–(2.17) of the sets Ξ_N and Γ_N. We replace everywhere N with $N+M$ to account for the GK dimension of $S_w^+ \bowtie S_u^-$.

Let η be the function given by (6.1) and $\mathbf{ex} := \{k \in [1, N+M] \mid \eta(k) \neq +\infty\}$. Denote by w_N° the longest element of S_N, viewed as an element of S_{N+M}. Let $p', s' : [1, N+M] \to [1, N+M] \sqcup \{\pm\infty\}$ be the predecessor and successor functions for the function $\eta w_N^\circ \colon [1, N+M] \to \mathbb{Z}$. Define

$$\epsilon : [1, N+M] \to \{\pm 1\}, \;\; \epsilon(k) = 1 \;\text{ for }\; k > N, \;\; \epsilon(k) = -1 \;\text{ for }\; k \le N.$$

We will use the notation $\mathbf{ex}_\sigma$, Z and Z_σ for $\sigma \in \Xi_{N+M}$ from §2.7. The Berenstein–Fomin–Zelevinsky exchange matrix $\overline{B}$ for the double word $-i_1, \ldots, -i_N, i'_1, \ldots, i'_M$ is given by [2, Definition 2.3],

$$\overline{b}_{jk} := \begin{cases} -\epsilon(k), & \text{if } \; j = p'(k) \\ -\epsilon(k) c_{i_j, i_k}, & \text{if } \; j < k < s'(j) < s'(k), \; \epsilon(k) = \epsilon(s'(j)) \\ & \text{or } \; j < k < s'(k) < s'(j), \; \epsilon(k) = -\epsilon(s'(k)) \\ \epsilon(j) c_{i_j, i_k}, & \text{if } \; k < j < s'(k) < s'(j), \; \epsilon(j) = \epsilon(s'(k)) \\ & \text{or } \; k < j < s'(j) < s'(k), \; \epsilon(j) = -\epsilon(s(j)) \\ \epsilon(j), & \text{if } \; j = s'(k) \\ 0, & \text{otherwise,} \end{cases} \tag{6.6}$$

where $j \in [1, N+M]$, $k \in \mathbf{ex}_{w_N^\circ}$. Denote its columns by $\overline{b}^k$. Define the exchange matrix $\widetilde{B}$ of size $(N+M) \times \mathbf{ex}$ with columns b^l by

$$b^l = Z^{-1} Z_{w_N^\circ} \cdot \begin{cases} \overline{b}^l, & \text{if } \ l > N \\ \overline{b}^{N+1-s(l)}, & \text{if } \ l \leq N \ \text{ and } \ s(l) \leq N \\ -\overline{b}^{N+1-p(l)} - \cdots - \overline{b}^{N+1-p^m(l)}, & \text{if } \ l \leq N \ \text{ and } \ s(l) > N, \end{cases}$$

where $m := \max\{j \mid p^j(l) \neq -\infty\}$ in the last case. For $\sigma \in \Xi_{N+M}$, define the exchange matrix $\widetilde{B}_\sigma$ of size $(N+M) \times \mathbf{ex}_\sigma$ with columns b^l_σ using (2.35) in Theorem 2.12. By way of definition, $\widetilde{B}_{w_N^\circ} = \overline{B}$.

Define toric frames M_σ of $\mathrm{Fract}(S_w^+ \bowtie S_u^-)$ for $\sigma \in \Xi_{N+M}$ as follows. As in (2.33), for $k \in [1, N+M]$, set

(6.7)
$$M_\sigma(e_k) = \begin{cases} \overline{y}_{[p^m(\sigma(k)), \sigma(k)]}, & \text{if } \ \sigma(1) \leq \sigma(k), \ \text{where}\, m := \max\{l \mid p^l(\sigma(k)) \geq \sigma(1)\} \\ \overline{y}_{[\sigma(k), s^m(\sigma(k))]}, & \text{if } \ \sigma(1) > \sigma(k), \ \text{where}\, m := \max\{l \mid s^l(\sigma(k)) \leq \sigma(1)\}. \end{cases}$$

The matrix $\mathbf{r}_\sigma$ of the toric frame M_σ is defined as follows. Let a and b be the maximum and minimum of the set $\{|\sigma(1)|, |\sigma(k)|\}$. Set $i := i_{|\sigma(k)|}$, if $\sigma(k) \leq N$ and $i := i'_{|\sigma(k)|}$, otherwise. Define

$$\chi_k^\pm := \begin{cases} (w_{\leq a} \pm w_{<b})\varpi_i, & \text{if } \ \sigma(1), \sigma(k) \leq N \\ (u_{<b} \pm u_{\leq a})\varpi_i, & \text{if } \ \sigma(1), \sigma(k) > N \\ (u_{\leq a} \pm w_{\leq b})\varpi_i, & \text{otherwise.} \end{cases}$$

Define the skewsymmetric matrix $(\psi^\sigma_{kj}) \in M_{N+M}(\mathbb{Z})$ by $\psi^\sigma_{kj} := \langle \chi_k^+, \chi_j^- \rangle / 2 \in \mathbb{Z}$ if $k > j$, $\psi^\sigma_{kj} := -\psi^\sigma_{jk}$ if $k < j$, and $\psi^\sigma_{kk} = 0$, and set

$$(\mathbf{r}_\sigma)_{kj} := q^{\psi^\sigma_{kj}}, \quad \forall k, j \in [1, N+M].$$

The fact that $M_\sigma(e_k) M_\sigma(e_j) = (\mathbf{r}_\sigma)^2_{kj} M_\sigma(e_j) M_\sigma(e_k)$ for $k, j \in [1, N+M]$ is verified by applying (3.10)-(3.11) and Theorem 4.6, the point being that $\overline{y}_{lk}$ are normal elements of $(S_w^+ \bowtie S_u^-)_{[l,k]}$. In the case of the longest element of S_N, $\sigma = w_N^\circ$, the quantum seed $(M_{w_N^\circ}, \mathbf{r}_{w_N^\circ})$ has a particularly simple form. We will denote it by $(\overline{M}, \overline{\mathbf{r}})$; it will play a special role in the rest of the paper. For $k \leq N$ and $k > N$, $\overline{M}(e_k)$ is given by

$$\overline{M}(e_k) = \sqrt{q}^{\|(w_{\leq k-1})\varpi_{i_k}\|^2/2} (\varphi_w^+)^{-1} \left((\Delta_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} \otimes \mathrm{id})^{\tau \times 1} \mathcal{R}^w \right) \quad \text{and} \tag{6.8}$$

$$\overline{M}(e_k) = \sqrt{q}^{\|(u_{\leq k-N} - w)\varpi_{i'_{k-N}}\|^2/2} \times \\ \varphi_{w,u}^{-1} \left((\Delta_{\varpi_{i'_{k-N}}, w\varpi_{i'_{k-N}}} \otimes \mathrm{id}) \left((T^{-1}_{u_{\leq k-N}} \otimes 1) \cdot {}^{S\tau \times 1}\mathcal{R}^u_{\mathrm{op}} \, {}^{\tau \times 1}\mathcal{R}^w \right) \right), \tag{6.9}$$

respectively. The corresponding matrices $(\overline{\psi}_{kj})$ and $\overline{\mathbf{r}}$ also simplify – for $\sigma = w_N^\circ$ we have $a = 1$, and $b = k$ if $k \leq N$, $b = k - N$ if $k > N$. Set $\overline{\mathbf{ex}} := \mathbf{ex}_{w_\circ^N}$.

Theorem 6.2. *Let $\mathfrak{g}$ be a complex simple Lie algebra, $w, u \in W$, $\mathbb{K}$ an arbitrary base field and $q \in \mathbb{K}$ a non-root of unity such that $\sqrt{q} \in \mathbb{K}$.*

(a) *For all $\sigma \in \Xi_N$, the pair $(M_\sigma, \widetilde{B}_\sigma)$ is a quantum seed for* $\mathrm{Fract}(S_w^+ \bowtie S_u^-)$. *The principal part of $\widetilde{B}_\sigma$ is skew-symmetrizable via the integers d_k, $k \in \mathbf{ex}_\sigma$, where $d_k = \|\alpha_{i_{|\sigma(k)|}}\|^2/2$ if $\sigma(k) \leq N$ and $d_k = \|\alpha_{i'_{|\sigma(k)|}}\|^2/2$ if $\sigma(k) > N$.*

(b) *These quantum seeds are mutation-equivalent to each other and are linked by the following mutations. Let* $\sigma, \sigma' \in \Xi_{N+M}$ *be such that*

$$\sigma' = (\sigma(k), \sigma(k+1))\sigma = \sigma(k, k+1)$$

for some $k \in [1, N+M-1]$. *If* $\eta(\sigma(k)) \neq \eta(\sigma(k+1))$, *then* $M_{\sigma'} = M_\sigma \cdot (k, k+1)$ *in terms of the action* (2.29). *If* $\eta(\sigma(k)) = \eta(\sigma(k+1))$, *then* $M_{\sigma'} = \mu_k(M_\sigma) \cdot (k, k+1)$.

(c) *We have the following equality between the algebra* $S_w^+ \bowtie S_u^-$ *and the quantum cluster and upper cluster algebras associated to the seed* $(\overline{M}, \overline{B})$ *with no inverted indices* (*i.e.,* $\mathbf{inv} := \varnothing$)*:*

$$S_w^+ \bowtie S_u^- = \mathcal{A}(\overline{M}, \overline{B}, \varnothing) = \mathcal{U}(\overline{M}, \overline{B}, \varnothing).$$

Each generator x_k *of* $S_w^+ \bowtie S_u^-$ *is a cluster variable in one of the seeds corresponding to* $\sigma \in \Gamma_N$. *In particular, the cluster variables in those seeds generate* $S_w^+ \bowtie S_u^-$.

(d) *The quantized coordinate ring* $R_q[G^{w,u}/H]$ *of the reduced double Bruhat cell* $G^{w,u}/H$ *equals the corresponding quantum and upper quantum cluster algebras with all frozen variables inverted:*

$$R_q[G^{w,u}/H] = \mathcal{A}(\overline{M}, \overline{B}, [1, N+M]\backslash\mathbf{ex}) = \mathcal{U}(\overline{M}, \overline{B}, [1, N+M]\backslash\mathbf{ex}).$$

Proof. Theorem 4.1 implies that $S_w^+ \bowtie S_u^-$ is a symmetric CGL extension with respect to the presentation (4.4). By Corollary 4.2 the conditions (A) and (B) in §2.6 are satisfied. It follows from Theorem 6.1 that the condition (2.28) holds for the sequence of rescaled generators $x_1, \ldots, x_{N+M}$ from (6.2). Thus, we can apply Theorem 2.11.

Set $\overline{\Psi} := (\overline{\psi}_{kj}) \in M_{M+N}(\mathbb{Z})$. Showing that $(\overline{M}, \overline{\mathbf{r}})$ is a quantum seed amounts to showing that

$$(\overline{\Psi}\,\overline{B})_{kj} = -\delta_{kj} d_k$$

where $d_k = \|\alpha_{i_k}\|^2/2$ if $k \leq N$ and $d_k = \|\alpha_{i'_k}\|^2/2$ if $k > N$. A compatibility of this type was proved by Berenstein–Zelevinsky in [4, Theorem 8.3] where the matrix $\overline{B}$ was replaced with one that has r more rows and the matrix in place of $\overline{\Psi}$ is slightly different. However, if one subtracts the entries in the above compatibility relation from the ones in [4] then the result is easily verified to be 0 as a consequence of the fact that the mutations of the seeds in [4] are graded with respect to the $\mathcal{P} \times \mathcal{P}$ grading of $\mathbb{C}[G]$.

The above compatibility and Theorem 6.1 imply that the quantum seed of $S_w^+ \bowtie S_u^-$ obtained by applying Theorem 2.11 (b) to this algebra and $\tau = w_N^\circ$ is precisely $(\overline{M}, \overline{B})$. Now parts (a) and (b) of the theorem follow from Theorem 2.12, Theorem 6.1, and Theorem 2.11 (b)-(c).

Part (c) of the theorem follows from Theorem 2.11 (d). Part (d) folllows from the isomorphism (3.23), the fact that the prime elements p_i, $i \in \mathcal{S}(w, u)$ are precisely the frozen variables of the toric frame $\overline{M}$ (up to non-zero scalars), and Theorem 2.11 (e). □

6.3. **Proof of Theorem 6.1.** The validity of (6.3) and the fact that the restriction to $[1, N]$ of the function η from (6.1) satisfies the conditions of Theorem 2.3 was established in [19, Theorem 10.1].

We proceed with (6.5). It follows from (4.18) that (6.5) is valid up to a scalar. Denote by z the term on the second line of (6.5). Recursively using the $\mathfrak{sl}_2$ relations

$$X_i^+ T_i^{-1} v_{\varpi_i} = v_{\varpi_i} = T_i^{-1}((S X_i^-) v_{\varpi_i}) \tag{6.10}$$

one obtains that

$$\text{the leading term of } z \text{ is } \sqrt{q}^{\,(n+1)\|\alpha_i\|^2/2} x_k x_{p(k)} \ldots x_j \tag{6.11}$$

where $n \in \mathbb{Z}_{>0}$ is such that $s^n(j) = k$. The above is an ordered product of generators of S_u^- and then generators of S_w^+. Lemma 10.3(a) of [19] and Proposition 4.4 imply

$$\overline{y}_{[j,k]} = \prod_{l \le |j|,\, m \le |k|,\, i_l = i_m = i} \sqrt{q}^{\;-\langle \beta_l, \beta'_m \rangle} \sqrt{q}^{\,(\|(1-w_{\le |j|})\varpi_i\|^2 + \|(1-u_{\le |k|})\varpi_i\|^2)/2} z.$$

The first product equals $\sqrt{q}^{\;-\langle (1-w_{\le |j|})\varpi_i, (1-u_{\le |k|})\varpi_i \rangle}$ since

$$\sum_{l \le |j|,\, i_l = i} \beta_l = (1 - w_{\le |j|})\varpi_i \quad \text{and} \quad \sum_{m \le |k|,\, i_m = i} \beta'_m = (1 - u_{\le |k|})\varpi_i.$$

This proves (6.5). Eq. (6.4) is proved analogously only the second relation in (6.10). The rest of part (a) of the theorem concerning the restriction of η to $[N+1, N+M]$ follows from the combination of (6.11) and Theorem 2.3.

Part (c) of Theorem 6.1: It follows from [19, Lemma 10.7] that the condition (2.28) is satisfied when $1 \le j < s(j) \le N$. The fact that the condition (2.28) is satisfied when $N < j < s(j) \le N + M$ follows from a property of the isomorphism $\omega : \mathcal{U}^-[u] \xrightarrow{\cong} \mathcal{U}^+[u]$:

$$\omega(X^-_{\beta'_k}) = (-q_{i'_k}^{-1})(-1)^{\langle \beta'_k, \rho^\vee \rangle} q^{\langle \beta'_k, \rho \rangle} X^+_{\beta'_k},$$

where ρ, $\rho^\vee$ are the sums of the fundamental weights (resp. coweights) of $\mathfrak{g}$, see [13, Eq. (2.10)]). Last, we consider the case $1 \le j \le N < s(j) \le N + M$. The third statement in Theorem 4.3 implies that it is sufficient to verify the condition (2.28) when $j = 1$, $k = N + M$ and $i_1 = i'_M$, $i_l \ne i_1$ for $l \in [2, N]$, $i'_m \ne i_M$ for $m \in [1, M-1]$. This is done analogously to the proof of [19, Lemma 10.7]. $\square$

7. Modified Berenstein–Zelevinsky quantum seeds

In this section we recall the definition of the Berenstein–Zelevinsky seeds for quantum double Bruhat cells, study the automorphism ωS on $R_q[G]$, and describe a modification of the BZ seeds via ωS. Throughout the section we assume that $\sqrt{q} \in \mathbb{K}$.

7.1. **The Berenstein–Zelevinsky seeds.** Fix two Weyl group elements w and u and reduced expressions of them as in (4.1). Consider the reversed reduced expressions

$$w^{-1} = s_{i_N} \dots s_{i_1} \quad \text{and} \quad u^{-1} = s_{i'_M} \dots s_{i'_1}, \tag{7.1}$$

and set

$$w^{-1}_{<k} := s_{i_N} \dots s_{i_{k+1}}, \;\; \forall k \in [1, N], \quad \text{and} \quad u^{-1}_{\le k} := s'_{i_M} \dots s'_{i_k}, \;\; \forall k \in [1, M]. \tag{7.2}$$

Consider the reduced double word

$$1, \dots, r, i_1, \dots, i_N, -i'_1, \dots, -i'_M \tag{7.3}$$

in the terminology of [4]. The last two parts form a reduced word for the Weyl group element $(u, w) \in W \times W$ where the generators of the first copy of W are assigned negative indices.

The Berenstein–Zelevinsky toric frame [4] of $\mathrm{Fract}(R_q[G^{u,w}])$ associated to (7.3) has *cluster variables*

$$\begin{aligned} M_{BZ}(e_i) &:= \Delta_{\varpi_i, w^{-1}\varpi_i}, & i \in [1, r], \\ M_{BZ}(e_{r+k}) &:= \Delta_{\varpi_{i_k}, w^{-1}_{<k}\varpi_{i_k}}, & k \in [1, N], \\ M_{BZ}(e_{r+N+k}) &:= \Delta_{u_{\le k}\varpi_{i'_k}, \varpi_{i'_k}}, & k \in [1, M] \end{aligned} \tag{7.4}$$

in the notation (7.2). The *matrix of the toric frame* is $\mathbf{r}_{BZ} := (q^{\mu_{jk}/2})$ where

$$\mu_{jk} := \langle \gamma_j, \gamma_k \rangle - \langle \delta_j, \delta_k \rangle \text{ for } j > k, \ \ \mu_{jk} := \mu_{kj} \text{ for } j < k, \ \ \mu_{jj} = 0, \tag{7.5}$$

and γ_k, δ_k are the first and second weight of the quantum minor defining $M(e_k)$, cf. [4, Eqs. (8.5), (10.13)], We interchange the roles of u and w ($R_q[G^{u,w}]$ vs. $R_q[G^{w,u}]$) to match the Berenstein–Zelevinsky picture to ours in Sect. 4. Consider the function $\eta\colon [1, r+N+M] \to \mathbb{Z}$ defined by taking the absolute values of the numbers in the list (7.3). Denote the corresponding predecessor/successor functions $p, s : [1, r+N+M] \to [1, r+N+M] \cup \{\pm\infty\}$ by (2.5). Define the set of *exchangeable indices* to be

$$\mathbf{ex} := \{k \in [r+1, r+N+M] \mid s(k) \neq +\infty\}. \tag{7.6}$$

The *BZ quantum seed* [4, Eq. (8.7)] is the pair $(M_{BZ}, \widetilde{B}_{BZ})$ for the *exchange matrix* $\widetilde{B}_{BZ}$ with entries b_{jk} given by

$$b_{jk} := \begin{cases} -\epsilon(k), & \text{if } \ j = p(k) \\ -\epsilon(k) c_{\eta(j)\eta(k)}, & \text{if } \ j < k < s(j) < s(k) \text{ and } \epsilon(k) = \epsilon(s(j)) \\ & \text{or } \ j < k \leq r+M < s(k) < s(j) \\ \epsilon(j) c_{\eta(j)\eta(k)}, & \text{if } \ k < j < s(k) < s(j) \text{ and } \epsilon(j) = \epsilon(s(k)) \\ & \text{or } \ k < j \leq r+M < s(j) < s(k) \\ \epsilon(j), & \text{if } \ j = s(k) \\ 0, & \text{otherwise} \end{cases} \tag{7.7}$$

for $j \in [1, r+N+M]$, $k \in \mathbf{ex}$, where $\epsilon(k) := +1$ for $k \leq r+N$ and $\epsilon(k) = -1$ otherwise.

7.2. **The action of ωS on quantum minors.** Recall the definition of the automorphism ω of $\mathcal{U}_q(\mathfrak{g})$ from §3.5. It is a coalgebra antiautomorphism, so, $(\omega S)^* \in \operatorname{Aut} R_q[G]$. For simplicity we will denote $\omega^* : R_q[G] \to R_q[G]$ by ω, just like the antipode S^* of $R_q[G]$ is denoted by S. Since $\omega S = S\omega$ on $\mathcal{U}_q(\mathfrak{g})$, the same holds on $R_q[G]$.

Proposition 7.1. (i) *For all $u, w \in W$, $\mu \in \mathcal{P}^+$, and $\iota_1, \iota_2 = \pm 1$,*

$$\Delta_{\mu,\mu}(T_w^{\iota_1} \cdot \omega S(x) \cdot T_u^{\iota_2}) = \Delta_{\mu,\mu}(T_{u^{-1}}^{\iota_2} \cdot x \cdot T_{w^{-1}}^{\iota_1}), \quad \forall\, x \in \mathcal{U}_q(\mathfrak{g}).$$

(ii) *For $u, w \in W$ and $\mu \in \mathcal{P}^+$,*

$$\omega S(\Delta_{u\mu, w\mu}) = (-1)^{\langle \rho^\vee, (w-u)\mu \rangle} q^{\langle \rho, (w-u)\mu \rangle} \Delta_{w\mu, u\mu}$$

where ρ, $\rho^\vee$ are the sums of fundamental weights (resp. coweights) of $\mathfrak{g}$.

Proof. Part (i). Using $\omega S(K_i) = K_i$, $\omega S(X_i^+) = -K_i X_i^-$, $\omega S(X_i^-) = -X_i^+ K_i^{-1}$, we obtain that for any $x = x_- x_0 x_+ \in \mathcal{U}_q(\mathfrak{g})$ where $x_\pm \in \mathcal{U}^\pm$, $x_0 \in \mathcal{U}^0$,

$$\Delta_{\mu,\mu}(x) = \varepsilon(x_-)\varepsilon(x_+)\Delta_{\mu,\mu}(x_0) = \Delta_{\mu,\mu}(\omega S(x)),$$

because $\omega S(x_0) = x_0$ (ε denotes the counit of $\mathcal{U}_q(\mathfrak{g})$). This proves the special case $w = u = 1$. For the general case, define the operators $\omega S(T_w^{-1})\colon V(\mu) \to V(\mu)$ by

$$\omega S(T_w) := \omega S(T_{i_N}) \dots \omega S(T_{i_1})$$

and by setting

$$(\omega S(T_i))v := \sum_{j,k,l \geq, j-k+l = \langle \lambda, \alpha_i^\vee \rangle} \frac{(-1)^k q_i^{k-jl}}{[j]_{q_i}! [k]_{q_i}! [l]_{q_i}!} (-K_i X_i^-)^j (-X_i^+ K_i^{-1})^k (-K_i X_i^-)^l$$

on $v \in V(\mu)_\lambda$. The latter are obtained by the direct application of ωS to the element of $\mathcal{U}_q(g)$ defining T_w, cf. [23, Eq. 8.6 (2)]. By commuting the K_i elements and using [23, Eq. 8.6 (7)] one shows that $\omega S(T_w) = T_{w^{-1}}$. Now we can apply the special case,

$$\begin{aligned}\Delta_{\mu,\mu}(T_u \cdot \omega S(x) \cdot T_w) &= \Delta_{\mu,\mu}((\omega S)T_{u^{-1}}(\omega S)(x)\omega S(T_{w^{-1}})) \\ &= \Delta_{\mu,\mu}(\omega S(T_{w^{-1}} \cdot x \cdot T_{u^{-1}})) = \Delta_{\mu,\mu}(T_{w^{-1}} \cdot x \cdot T_{u^{-1}})), \quad \forall\, x \in \mathcal{U}_q(\mathfrak{g}),\end{aligned}$$

the point being that in the last expression $T_{w^{-1}} \cdot x \cdot T_{u^{-1}}$ acts on v_μ by an element of $\mathcal{U}_q(\mathfrak{g})$ and the previous term is obtained by applying ωS to it. The other cases for $\iota_1, \iota_2 = \pm 1$ are analogous.

Part (ii). One checks that

$$T_w v_\mu = (-1)^{\langle \rho^\vee - w^{-1}\rho^\vee, \mu\rangle} q^{\langle \rho - w^{-1}\rho, \mu\rangle} T_{w^{-1}}^{-1} v_\mu$$

by iterating the special case of this formula for $\mathcal{U}_q(\mathfrak{sl}_2)$. This, (3.5), and part (i) give

$$\begin{aligned}(\omega S(\Delta_{u\mu,w\mu}))(x) &= \Delta_{\mu,\mu}(T_{u^{-1}} \cdot \omega S(x) \cdot T_{w^{-1}}^{-1}) = \\ &= \Delta_{\mu,\mu}(T_w^{-1} \cdot x \cdot T_u) = a\Delta_{\mu,\mu}(T_{w^{-1}} \cdot \omega S(x) \cdot T_{u^{-1}}^{-1}) = a\Delta_{w\mu,u\mu}(x)\end{aligned}$$

where $a = (-1)^{\langle (w^{-1}-u^{-1})\rho^\vee, \mu\rangle} q^{\langle (w^{-1}-u^{-1})\rho, \mu\rangle} = (-1)^{\langle \rho^\vee, (w-u)\mu\rangle} q^{\langle \rho, (w-u)\mu\rangle}$. □

7.3. **The action of ωS on quantum double Bruhat cells.**

Proposition 7.2. *Let $u, w \in W$.*

(i) $S^{\pm 1}(I_{u,w}) = I_{u^{-1},w^{-1}}$ *and* $S^{\pm 1}$ *induce antiisomorphisms between* $R_q[G^{u,w}]$ *and* $R_q[G^{u^{-1},w^{-1}}]$.

(ii) $\omega^{\pm 1}S^{\pm 1}(I_{u,w}) = I_{w^{-1},u^{-1}}$ *and* $\omega^{\pm 1}S^{\pm 1}$ *induce isomorphisms between* $R_q[G^{u,w}]$ *and* $R_q[G^{w^{-1},u^{-1}}]$.

Proof. Part (i). For $i \in [1, r]$ denote by $\mathcal{B}_i^\pm$ the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $X_i^\pm$ and $K_1^{\pm 1}, \dots, K_r^{\pm 1}$. It follows from [24, Corollary 4.4.6] and [4, Proposition 9.2] that the ideals $I_{u,w}$ are given by

$$I_{u,w} = (\mathcal{B}_{i'_1}^- \dots \mathcal{B}_{i'_M}^- \mathcal{B}_{i_1}^+ \dots \mathcal{B}_{i_N}^+)^\perp = (\mathcal{B}_{i_1}^+ \dots \mathcal{B}_{i_N}^+ \mathcal{B}_{i'_1}^- \dots \mathcal{B}_{i'_M}^-)^\perp$$

in terms of the reduced expressions (4.1), where the orthogonal complements are computed with respect to the natural pairing between $R_q[G]$ and $\mathcal{U}_q(\mathfrak{g})$. The first part of the proposition follows from this fact by using that $S(\mathcal{B}_i^\pm) = \mathcal{B}_i^\pm$ for all $i \in [1, r]$. Therefore $S^{\pm 1}$ induce antiisomorphisms between

$$R_q[G]/I_{u,w} \quad \text{and} \quad R_q[G]/I_{u^{-1},w^{-1}}.$$

Denote by $M_{w,u}$ the multiplicative subset of $R_q[G^{w,u}]$ generated by $\mathbb{K}^*$ and the elements $\Delta^\pm_{w\varpi_i,\varpi_i}$, $\Delta^\pm_{-u\varpi_i,-\varpi_i}$ for $i \in [1, r]$. The elements $S^{\pm 1}(\Delta_{u\varpi_i,\varpi_i})$ and $S^{\pm 1}(\Delta_{-w\varpi_i,-\varpi_i})$ are normal in $R_q[G^{u^{-1},w^{-1}}]$ and homogeneous with respect to the $\mathcal{Q} \times \mathcal{Q}$ grading. By Corollary 5.3 all of them belong to $M_{w^{-1},u^{-1}}$, so $S^{\pm 1}(M_{u,w}) \subseteq M_{u^{-1},w^{-1}}$. Using that S^{-1} is the inverse of S gives the inverse inclusion, and thus

$$S^{\pm 1}(M_{u,w}) = M_{u^{-1},w^{-1}} \quad \text{and} \quad R_q[G^{u,w}] = (R_q[G]/I_{u,w})[(S^{\pm 1}(E_{u^{-1},w^{-1}}))^{-1}]$$

which implies the second part of (i). Part (ii) is proved analogously. □

7.4. **A modification of the Berenstein–Zelevinsky seeds via ωS.** The isomorphism $\omega S : R_q[G^{u,w}] \to R_q[G^{w^{-1},u^{-1}}]$ from Proposition 7.2 and the relations in Proposition 7.1 (ii) give a quantum seed of $\mathrm{Fract}(R_q[G^{w^{-1},u^{-1}}])$ which we will call the *modified Berenstein–Zelevinsky quantum seed*. The *cluster variables* of the *toric frame* are given by

$$
\begin{aligned}
M_{mBZ}(e_i) &:= \Delta_{w^{-1}\varpi_i,\varpi_i}, && i \in [1,r],\\
M_{mBZ}(e_{r+k}) &:= \Delta_{w^{-1}_{<k}\varpi_{i_k},\varpi_{i_k}}, && k \in [1,N],\\
M_{mBZ}(e_{r+N+k}) &:= \Delta_{\varpi_{i'_k},u_{\leq k}\varpi_{i'_k}}, && k \in [1,M].
\end{aligned}
\tag{7.8}
$$

The *matrix of the toric frame* is given by (7.5) and *exchangeable indices* by (7.6). Define the modified BZ quantum seed to be the pair $(M_{mBZ}, \widetilde{B}_{BZ})$.

Define an action of $\mathcal{H} = (\mathbb{K}^*)^r$ on $R_q[G]$ by

$$h \cdot c = h^{\nu'+\nu} c \quad \text{for} \quad c \in R_q[G]_{\nu,\nu'}, h \in \mathcal{H},$$

recall the identification $X(\mathcal{H}) \cong \mathcal{Q}$ in (3.24). There exists a unique $h_* \in \mathcal{H}$ such that

$$h_*^{\gamma} = (-1)^{\langle \rho^\vee,\gamma\rangle} q^{\langle \rho,\gamma\rangle}, \quad \forall\, \gamma \in \mathcal{Q}, \tag{7.9}$$

recall that $\mathcal{P}^\vee \subseteq \mathcal{P}$. Taking into account that $\Delta_{w\mu,u\mu} \in R_q[G]_{-w\mu,u\mu}$ and Proposition 7.1 (ii) we obtain,

Proposition 7.3. *For all $w, u \in W$, the ωS-image of the Berenstein–Zelevinsky quantum seed and the modified BZ quantum seed are related by an automorphism of $R_q[G^{w^{-1},u^{-1}}]$,*

$$M_{mBZ}(f) = h_* \cdot (\omega S) M_{BZ}(f), \quad \forall\, f \in \mathbb{Z}_{r+N+M}.$$

8. A quantum twist map for double Bruhat cells

In this section we develop a quantum analog of the Fomin–Zelevinsky twist map [9] for all double Bruhat cells. We use it to relate the cluster variables in the quantum seed in Theorem 6.2 for $\mathrm{Fract}(S_w^+ \bowtie S_u^-)$ corresponding to the element of $\sigma = w_N^\circ \in \Xi_{N+M}$ to the cluster variables in the modified Berenstein–Zelevinsky quantum seed [4] for $\mathrm{Fract}(R_q[G^{w^{-1},u^{-1}}])$ described in the previous section. Throughout the section, $\mathfrak{g}$ is an arbitrary simple Lie algebra, w, u are Weyl group elements with fixed reduced expressions as in (4.1), and reversed expressions (7.1) are used for the elements w^{-1}, u^{-1}.

8.1. **First half of the construction: a twist map for quantum Schubert cells.** For $w \in W$ define the algebra antiisomorphism

$$\zeta_{w^{-1}} := \tau S^{-1} \tau \omega T_w^{-1} \colon \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g}).$$

The double appearance of τ is needed because of the nature of the isomorphisms in Theorem 3.2 and the fact that S and τ do not commute. Since

$$\tau\omega = \omega\tau \quad \text{and} \quad \omega S = S\omega, \tag{8.1}$$

we also have $\zeta_{w^{-1}} := \tau\omega S^{-1} \tau T_w^{-1}$.

Proposition 8.1. *For all simple Lie algebras $\mathfrak{g}$ and $w \in W$, the map $\zeta_{w^{-1}} : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$ restricts to antiisomorphisms*

$$\zeta_{w^{-1}} \colon \mathcal{U}^{\pm}[w] \to \mathcal{U}^{\pm}[w^{-1}]$$

and satisfies

$$\zeta_{w^{-1}}(T_{i_1} \dots T_{i_{k-1}}(X_{i_k}^{\pm})) = t_{w,k}^{\pm} T_{i_N} \dots T_{i_{k+1}}(X_{i_k}^{\pm}), \quad \forall\, k \in [1,N]$$

for some $t^{\pm}_{w,k} \in \mathbb{K}^*$, *in terms of* (4.1), (7.1). *Moreover,* $t^+_{w,k} t^-_{w,k} = 1$ *for* $k \in [1, N]$, *so,*

$$(\zeta_{w^{-1}} \otimes \zeta_{w^{-1}}) \mathcal{R}^w = \mathcal{R}^{w^{-1}}.$$

The map $\zeta_{w^{-1}}$ is inverse to the twist antiisomorphism

$$\Theta_w := T_w \omega \tau S \tau : \mathcal{U}^{\pm}[w^{-1}] \to \mathcal{U}^{\pm}[w]$$

studied in [31]. The first two statements of the proposition follow from [31, Proposition 6.1]. The property that $t^+_{w,k} t^-_{w,k} = 1$ follows from the fact that

$$T_w \omega(x) = t_\gamma \omega T_w(x), \quad \forall\, x \in \mathcal{U}_q(\mathfrak{g})_\gamma,\ \gamma \in \mathcal{Q}$$

for certain $t_\gamma \in \mathbb{K}^*$ satisfying $t_\gamma t_{-\gamma} = 1$ (see [23, Eq. 8.18 (5)]) and from (8.1).

We will denote by the same notation the antiisimorphisms

$$\zeta_{w^{-1}} \colon S^w_{\pm} \to S^{w^{-1}}_{\pm} \tag{8.2}$$

obtained from ζ_w by composing it with the (anti-)isomorphisms $\varphi^{\pm}_w : S^w_{\pm} \to \mathcal{U}^{\mp}[w]$ from Theorem 3.2 and their inverses.

Lemma 8.2. *For all* $w \in W$, *the twist antiismorphism* $\zeta_{w^{-1}} \colon S^w_+ \to S^{w^{-1}}_+$ *is given by*

$$\zeta_{w^{-1}}\big(c^+_\xi \Delta^{-1}_{w\mu,\mu}\big) = (\varphi^+_{w^{-1}})^{-1}\Big((c^+_\xi \otimes \mathrm{id})\big((T^{-1}_{w^{-1}} \otimes 1) \cdot {}^{\omega S \tau \times 1}\mathcal{R}^{w^{-1}}\big)\Big)$$

for $\mu \in \mathcal{P}^+$, $\xi \in V(\mu)^*$ *(recall* (3.35)*), and the equality holds with* $\mathcal{R}^{w^{-1}}$ *replaced by* $\mathcal{R}$.

Proof. Using Propositions 7.1 (i) and 8.1, and (3.34), we obtain

$$\begin{aligned}
\varphi^+_{w^{-1}}\big(\zeta_{w^{-1}}(c^+_\xi \Delta^{-1}_{w\mu,\mu})\big) &= (c_{\xi, v_{w\mu}} \otimes \zeta_{w^{-1}})\ {}^{\tau \times 1}\mathcal{R}^w \\
&= (c_{\xi, v_{w\mu}} \otimes \mathrm{id})\ {}^{\tau \zeta^{-1}_{w^{-1}} \times 1}\mathcal{R}^{w^{-1}} = (c_{\xi, v_{w\mu}} \otimes \mathrm{id})\ {}^{T^{-1}_{w^{-1}} \omega S \tau \times 1}\mathcal{R}^{w^{-1}} \\
&= (c^+_\xi \otimes \mathrm{id})\big((T^{-1}_{w^{-1}} \otimes 1) \cdot\ {}^{\omega S \tau \times 1}\mathcal{R}^{w^{-1}}\big).
\end{aligned}$$

At the end one can replace $\mathcal{R}^{w^{-1}}$ by $\mathcal{R}$ because $c^+_\xi = 0$ in $R_q[G^{w,u}]$ for $\xi \in (V(\mu)^+_{w\mu})^{\perp}$. $\square$

8.2. **Quantum Bruhat cells.** In the setting of §3.3, $I^-_{w_0} = 0$ and $I_{w,w_0} = I^+_w R^-$. The quantized coordinate ring of the Bruhat cell $B^+ w B^+$ is defined to be

$$R_q[B^+ w B^+] := (R_q[G]/I^+_w R^-)[(E^+_w)^{-1}].$$

Since $I^+_w R^- \subseteq I_{w,u}$ and $E^+_w \subseteq E_{w,u}$ for $u \in W$, we have the canonical projection

$$\kappa : R_q[B^+ w B^+] \to R_q[G^{w,u}] \tag{8.3}$$

Keeping in mind (3.6), define

$$\Delta_{w\mu,\mu} \in R_q[B^+ w B^+] \quad \text{for} \ \ \mu \in \mathcal{P} \tag{8.4}$$

by $\Delta_{w\mu,\mu} := \Delta_{w\mu_1,\mu_1} \Delta^{-1}_{w\mu_2,\mu_2}$ for any $\mu_i \in \mathcal{P}^+$ such that $\mu = \mu_1 - \mu_2$.

Proposition 8.3. *The set*

$$\overline{S}^+_w := \{c_{\xi_{w\mu}, v} \Delta^{-1}_{w\nu,\nu} \mid v \in V(\mu)_\nu,\ \mu \in \mathcal{P}^+,\ \nu \in \mathcal{P}\}$$

is a subalgebra of $R_q[B^+ w B^+]$ *and the map* $\psi^+_w : \overline{S}^+_w \to \mathcal{U}^-$ *given by*

$$\psi^+_w\big(c_{\xi_{w\mu}, v} \Delta^{-1}_{w\nu,\nu}\big) := (c_{\xi_\mu, v} \otimes \mathrm{id})({}^{\tau \times 1}\mathcal{R})$$

is a well defined algebra isomorphism.

Proof. The fact that the set $\overline{S}_w^+$ is a subalgebra of $R_q[B^+wB^+]$ follows from the identity $(T_{w^{-1}}^{-1}v_\lambda)\otimes(T_{w^{-1}}^{-1}v_\mu) = T_{w^{-1}}^{-1}(v_\lambda\otimes v_\mu)$ for $\lambda,\mu\in\mathcal{P}^+$ (see e.g. [41, (2.19)-(2.20)]). The second isomorphism is completely analogous to Theorem 3.2 proved in [41, Theorem 2.6]. $\square$

8.3. **Definition of the quantum twist map for double Bruhat cells.** Similarly to the previous subsection, define

$$\Delta_{-u\mu,-\mu}\in R_q[G^{w,u}],\ \ \Delta_{w^{-1}\mu,\mu}\in R_q[G^{w^{-1},u^{-1}}]\quad\text{for}\ \ \mu\in\mathcal{P}$$

by

$$\Delta_{-u\mu,-\mu}=\Delta_{-u\mu_1,-\mu_1}\Delta_{-u\mu_2,-\mu_2}^{-1},\quad \Delta_{w^{-1}\mu,\mu}=\Delta_{w^{-1}\mu_1,\mu_1}\Delta_{w^{-1}\mu_2,\mu_2}^{-1} \tag{8.5}$$

for any $\mu_i\in\mathcal{P}^+$ such that $\mu=\mu_1-\mu_2$. Recall the algebra isomorphisms (4.16). We define a linear map

$$\zeta_{w^{-1},u^{-1}}:(S_w^+\bowtie S_u^-)\#\mathcal{L}_u^-\to R_q[G^{w^{-1},u^{-1}}]\cong\big((S_{w^{-1}}^+\bowtie S_{u^{-1}}^-)[p_i^{-1},\ i\in\mathcal{S}(w,u)]\big)\#\mathcal{L}_{w^{-1}}^+$$

by the following formulas on the three components of the left hand side identified (as a vector space) with the tensor product $S_w^+\otimes S_u^-\otimes\mathcal{L}_u^-$. Firstly,

$$\zeta_{w^{-1},u^{-1}}|_{S_w^+}:=\zeta_{w^{-1}}\,. \tag{8.6}$$

Secondly, we define $\zeta_{w^{-1},u^{-1}}|_{S_u^-}$ as the composition

$$S_u^-\xrightarrow{\varphi_u^-}\mathcal{U}^+[u]\xrightarrow{\omega}\mathcal{U}^-[u]\hookrightarrow\mathcal{U}^-\xrightarrow{(\psi_{w^{-1}}^+)^{-1}}\overline{S}_{w^{-1}}^+\xrightarrow{\kappa}R_q[G^{w^{-1},u^{-1}}]. \tag{8.7}$$

The map $\zeta_{w^{-1},u^{-1}}|_{S_u^-}$ is an antihomomorphism because $\psi_{w^{-1}}^+$ is an antiisomorphism, φ_u^- and ω are isomorphisms and κ is a homomorphism.

Finally, we define $\zeta_{w^{-1},u^{-1}}|_{\mathcal{L}_u^-}$ by

$$\zeta_{w^{-1},u^{-1}}(\Delta_{-u\mu,-\mu}):=\Delta_{w^{-1}(u\mu),u\mu},\quad\forall\,\mu\in\mathcal{P} \tag{8.8}$$

where the right hand side uses the notation (8.5) for the weight $u\mu\in\mathcal{P}$. By (3.6), $\zeta_{w^{-1},u^{-1}}|_{\mathcal{L}_u^-}$ is an (anti)homomorphism of commutative algebras. It is straightforward to check that

$$\zeta_{w^{-1},u^{-1}}((S_w^+\bowtie S_u^-)\#\mathcal{L}_u^-)_{\mu,\nu})\subseteq R_q[G^{w^{-1},u^{-1}}]_{-w^{-1}\mu,-u\nu},\quad \mu,\nu\in\mathcal{P}. \tag{8.9}$$

Theorem 8.4. *For all simple Lie algebras $\mathfrak{g}$ and $w,u\in W$, the following hold:*

(i) *The map $\zeta_{w^{-1},u^{-1}}$ extends to an antiisomorphism $R_q[G^{w,u}]\to R_q[G^{w^{-1},u^{-1}}]$ whose inverse is $\zeta_{w,u}$.*

(ii) *The antiisomorphism $\zeta_{w^{-1},u^{-1}}$ links the modified BZ cluster variables to the cluster variables in the toric frame $\overline{M}$ in Theorem 6.2 as follows:*

$$\zeta_{w^{-1},u^{-1}}\big(\overline{M}(e_k)\big)=\sqrt{q}^{\,-\|(w_{\le k}-1)\varpi_{i_k}\|^2/2}\Delta_{w_{<k}^{-1}\varpi_{i_k},\varpi_{i_k}}\Delta_{w^{-1}\varpi_{i_k},\varpi_{i_k}}^{-1}, \tag{8.10}$$

$$\zeta_{w^{-1},u^{-1}}\big(\overline{M}(e_{N+j})\big)=\sqrt{q}^{\,-\|(u_{\le j}-w)\varpi_{i'_j}\|^2/2}\Delta_{\varpi_{i'_j},u_{\le j}\varpi_{i'_j}}\Delta_{w^{-1}(u_{\le j}\varpi_{i'_j}),u_{\le j}\varpi_{i'_j}}^{-1}, \tag{8.11}$$

for all $k\in[1,N]$, $j\in[1,M]$ in the notation (7.2) *and* (8.5).

We will call $\zeta_{w^{-1},u^{-1}}:R_q[G^{w,u}]\to R_q[G^{w^{-1},u^{-1}}]$ the *quantum twist map*. In §8.4 we describe its relation to the classical twist map of Fomin–Zelevinsky [9]. In §8.5 we prove that the map $\zeta_{w^{-1},u^{-1}}$ is an antihomomorphism. In §8.6 and 8.7 we prove parts (ii) and (i) of Theorem 8.4, respectively.

8.4. **Motivation for the quantum twist map.** Recall that G denotes a connected, simply connected complex simple Lie group with Lie algebra $\mathfrak{g}$, $B^{\pm}$ a pair of opposite Borel subgroups with unipotent radicals $U^{\pm}$, and $H = B^{+} \cap B^{-}$ the corresponding maximal torus of G. The set $G_0 = U_{-}HU_{+}$ is a Zariski open subset of G and we have the canonical projections

$$G_0 \to U^{-},\ H,\ U^{+}, \quad x \mapsto [x]_{-},\ [x]_0,\ [x]_{+}$$

where $x = [x]_{-}[x]_0[x]_{+}$ and $x \in G_0$, $[x]_{-} \in U^{-}$, $[x]_0 \in H$, $[x]_{+} \in U^{+}$. The elements $w \in W$ have two representatives $\overline{w}$, $\overline{\overline{w}}$ in the normalizer of H in G, defined in [9, §1.4]. It is easy to show that their action on integrable $\mathfrak{g}$-modules are the specializations of $T_{w^{-1}}^{-1}$ and T_w, respectively. The involution ω of $U_q(\mathfrak{g})$ specializes to the involution of $\mathfrak{g}$ given by $\omega(h_i) = -h_i$, $\omega(e_i) = f_i$, $\omega(f_i) = e_i$ in terms of the Chevalley generators of $\mathfrak{g}$. The latter involution integrates to an involution of G to be denoted by the same letter. The Fomin–Zelevinsky twist map [9] is the algebraic isomorphism

$$\zeta^{w,u} : G^{w,u} \to G^{w^{-1},u^{-1}}, \quad \zeta^{w,u}(x) := \omega\Big(\big[\overline{w}^{\,-1}x\big]_{-}^{-1}\, \overline{w}^{\,-1} x\, \overline{\overline{u}}^{\,-1} \Big[x\overline{\overline{u}}^{\,-1}\Big]_{+}^{-1} \Big).$$

Denote the canonical projections $\upsilon_{\pm} : G \to G/B^{\pm}$ and their restrictions $\upsilon_{+} : G^{w,u} \to (B^{+}wB^{+})/B^{+}$ and $\upsilon_{-} : G^{w,u} \to (B^{-}uB^{-})/B^{-}$. Following the specialization arguments in [39, Sect. 4], one can show that the pullback algebras

$$A_w^{+} := \upsilon_{+}^{*}\big(\mathbb{C}[(B^{+}wB^{+})/B^{+}]\big) \subseteq \mathbb{C}[G^{w,u}], \quad A_u^{-} := \upsilon_{-}^{*}\big(\mathbb{C}[(B^{-}wB^{-})/B^{-}]\big) \subseteq \mathbb{C}[G^{w,u}]$$

are specializations of $S_w^{+}, S_u^{-} \subseteq R_q[G^{w,u}]$ with respect to the nonrestricted integral form of $R_q[G]$. Each element $x \in G^{w,u}$ has a unique decomposition of the form

$$x = n_{+}\overline{w}m_{+}h = m_{-}h'\,\overline{\overline{u}}\,n_{-}$$

where $n_{+} \in U^{+} \cap wU^{-}w^{-1}$, $n_{-} \in U^{-} \cap u^{-1}U^{+}u$, $m_{\pm} \in U^{\pm}$ and $h, h' \in H$.

Lemma 8.5. *In the above setting, the following hold for* $x \in G^{w,u}$*:*

$$(\zeta^{w,u})^{*}(f)(x) = f(\omega(\overline{w}^{\,-1}n_{+}^{-1})), \quad \forall f \in A_{w^{-1}}^{+}, \tag{8.12}$$

$$(\zeta^{w,u})^{*}(g)(x) = g(\omega(m_{+}\overline{u}^{\,-1})), \quad \forall g \in A_{u^{-1}}^{-}. \tag{8.13}$$

Proof. Using that $\big[\overline{w}^{\,-1}x\big]_{-} = \overline{w}^{\,-1}n_{+}\overline{w}$ and $\Big[x\overline{\overline{u}}^{\,-1}\Big]_{+} = \overline{\overline{u}}n_{-}\overline{\overline{u}}^{\,-1}$, and the invariance properties of the functions in A_w^{+} and A_u^{-}, we obtain

$$(\zeta^{w,u})^{*}(f)(x) = f(\omega(\overline{w}^{\,-1}n_{+}^{-1}m_{-}h')) = f(\omega(\overline{w}^{\,-1}n_{+}^{-1}))$$

for $f \in A_{w^{-1}}^{+}$ and

$$(\zeta^{w,u})^{*}(g)(x) = g(\omega(m_{+}hn_{-}^{-1}\overline{\overline{u}}^{\,-1})) = g(\omega(m_{+}\overline{u}^{\,-1}))$$

for $g \in A_{u^{-1}}^{-}$. □

The defining formulas (8.6)–(8.8) for the quantum twist map were set up in such a way that they would quantize the geometric maps behind (8.12)–(8.13). In this way $\zeta_{w^{-1},u^{-1}} : R_q[G^{w,u}] \to R_q[G^{w^{-1},u^{-1}}]$ is a quantization of the pullback of the Fomin-Zelevinsky map $\big(\zeta^{w^{-1},u^{-1}}\big)^{*} : \mathbb{C}[G^{w,u}] \to \mathbb{C}[G^{w^{-1},u^{-1}}]$.

8.5. $\zeta_{w^{-1},u^{-1}}$ **is an antihomomorphism.** It was shown in §§8.1, 8.3 that the restrictions of $\zeta_{w^{-1},u^{-1}}$ to the three algebras S_w^+, S_u^- and $\mathcal{L}_u^-$ are antihomomorphisms. The algebra $\mathcal{L}_u^-$ is spanned by the localized minors $\Delta_{-u\mu,-\mu}$, $\mu \in \mathcal{P}$ which are normal elements of $R_q[G^{w^{-1},u^{-1}}]$. The corresponding commutation relations with the elements of $R_q[G^{w^{-1},u^{-1}}]$ are given by (3.11). It follows from (8.9) and (3.10) that $\zeta_{w^{-1},u^{-1}}$ anti-preserves those relations. It remains to prove that $\zeta_{w^{-1},u^{-1}}$ anti-preserves the commutation relation (3.13) between S_w^+ and S_u^-.

We will prove a stronger statement. It follows from the definitions of $S_{w^{-1}}^+$ and $R_q[B^+w^{-1}B^+]$ that the first algebra is canonically a subalgebra of the second. By Proposition 8.3, $\overline{S}_{w^{-1}}^+$ is also a subalgebra of $R_q[B^+w^{-1}B^+]$.

Lemma 8.6. *The map $S_w^+ \bowtie S_u^- \to R_q[B^+w^{-1}B^+]$ given by $\zeta_{w^{-1}}$ on the first term and by the composition*

$$S_u^- \xrightarrow{\varphi_u^-} \mathcal{U}^+[u] \xrightarrow{\omega} \mathcal{U}^-[u] \hookrightarrow \mathcal{U}^- \xrightarrow{(\psi_{w^{-1}}^+)^{-1}} \overline{S}_{w^{-1}}^+ \subset R_q[B^+w^{-1}B^+]. \tag{8.14}$$

on the second is an antihomomorphism.

The fact that $\zeta_{w^{-1},u^{-1}}$ is an antihomomorphism follows from the lemma and the fact that its restriction to $S_w^+ \bowtie S_u^-$ coincides with the composition of the map in the lemma and the the homomorphism $\kappa : R_q[B^+w^{-1}B^+] \to R_q[G^{w^{-1},u^{-1}}]$.

Proof. By Theorem 4.6, we have the embedding $S_w^+ \bowtie S_u^- \hookrightarrow S_w^+ \bowtie S_{w_\circ}^-$, so it is sufficient to prove the lemma for $u = w_\circ$. Furthermore, $S_{w_\circ}^-$ is generated by $(\varphi_{w_\circ}^-)^{-1}(X_i^+)$ for $i \in [1,r]$, and using the embeddings in Theorem 4.6 one more time, we see that it is sufficient to prove the lemma for $u = s_i$, $i \in [1,r]$. In this case

$$(\varphi_{s_i}^-)^{-1}(X_i^+) = (q_i^{-1} - q_i)^{-1} \Delta_{-s_i\varpi_i,-\varpi_i}^{-1} \Delta_{-\varpi_i,-\varpi_i}.$$

and the commutation relation (3.13) reduces to

$$\begin{aligned}&\big[\Delta_{-s_i\varpi_i,-\varpi_i}^{-1}\Delta_{-\varpi_i,-\varpi_i}\big]\big[c_\xi^+\Delta_{w^{-1}\mu,\mu}^{-1}\big]\\ &\quad = q^{\langle \alpha_i, \nu - w^{-1}\mu\rangle}\big[c_\xi^+\Delta_{w^{-1}\mu,\mu}^{-1}\big]\big[\Delta_{-s_i\varpi_i,-\varpi_i}^{-1}\Delta_{-\varpi_i,-\varpi_i}\big] + (q_i^{-1} - q_i)\big[c_{S^{-1}(X_i^+)\xi}^+\Delta_{w^{-1}\mu,\mu}^{-1}\big]\end{aligned}$$

for $\xi \in (V(\mu)^*)_{-\nu}$, $\mu \in \mathcal{P}^+$, $\nu \in \mathcal{P}$. It is straightforward to check that the composition map in the lemma anti-preserves this commutation relation by using Lemma 8.2 and computing directly the image of $\Delta_{-s_i\varpi_i,-\varpi_i}^{-1}\Delta_{-\varpi_i,-\varpi_i}$. The lemma now follows from the fact that the maps $\zeta_{w^{-1}}$ and (8.14) are antihomomorphisms. □

8.6. **Proof of Theorem 8.4 (ii).** First we prove (8.10). Similarly to the proof of Lemma 8.2 (using Propositions 7.1 (i) and 8.1 and (3.34)), we obtain

$$\begin{aligned}&\sqrt{q}^{\,\|(w_{\leq k}-1)\varpi_{i_k}\|^2/2}\varphi_{w^{-1}}^+\zeta_{w^{-1},u^{-1}}\big(\overline{M}(e_k)\big) = (\Delta_{\varpi_{i_k},w_{\leq k}\varpi_{i_k}} \otimes \mathrm{id})\big({}^{\tau\zeta_{w^{-1}}^{-1}\times 1}\mathcal{R}^{w^{-1}}\big)\\ &\quad = (\Delta_{\varpi_{i_k},\varpi_{i_k}} \otimes \mathrm{id})\big((T_{w^{-1}}^{-1} \otimes 1) \cdot {}^{\omega S\tau\times 1}\mathcal{R}^{w^{-1}} \cdot (T_{w_{<k}^{-1}} \otimes 1)\big)\\ &\quad = (\Delta_{\varpi_{i_k},\varpi_{i_k}} \otimes \mathrm{id})\big((T_{w_{<k}} \otimes 1) \cdot {}^{\tau\times 1}\mathcal{R}^{w^{-1}} \cdot (T_w^{-1} \otimes 1)\big) = \Delta_{w_{<k}^{-1}\varpi_{i_k},\varpi_{i_k}}\Delta_{w^{-1}\varpi_{i_k},\varpi_{i_k}}^{-1}.\end{aligned}$$

Next we proceed with (8.11). Because of the embeddings $S_w^+ \bowtie S_{u_{\leq j}}^- \hookrightarrow S_w^+ \bowtie S_u^-$ and the definition of $\zeta_{w^{-1},u^{-1}}$, it is sufficient to prove (8.11) for $j = M$. Set for brevity $i := i'_M$. We need to prove that

$$\sqrt{q}^{\,\|(u-w)\varpi_i\|^2/2}\zeta_{w^{-1},u^{-1}}\big(\overline{M}(e_{N+M})\big) \cdot \big(\Delta_{\varpi_i,u\varpi_i}\Delta_{w^{-1}(u\varpi_i),u\varpi_i}^{-1}\big)^{-1} = 1. \tag{8.15}$$

The frozen cluster variable $\overline{M}(e_{N+M})$ is a normal element of $R_q[G^{w,u}]$ by (3.10)–(3.11) and $\Delta_{\varpi_i,u\varpi_i}\Delta^{-1}_{w^{-1}(u\varpi_i),u\varpi_i^{-1}}$ is a normal element of $R_q[G^{w^{-1},u^{-1}}]$ by the same properties and Propositions 7.1 (ii) and 7.2 (i). Using that $\zeta_{w^{-1},u^{-1}}$ is an antiisomorphism gives that the element in the left hand side of (8.15) is in the center of $R_q[G^{w^{-1},u^{-1}}]$. It easily follows from (8.9) that this element is also in $R_q[G^{w^{-1},u^{-1}}]_{0,0}$. By [22, Theorem 4.14 (3)], $Z(R_q[G^{w^{-1},u^{-1}}])_{0,0} = \mathbb{K}$, thus

$$\text{(8.16)} \qquad \sqrt{q}^{\,\|(u-w)\varpi_i\|^2/2}\zeta_{w^{-1},u^{-1}}\big(\overline{M}(e_{N+M})\big)\cdot\big(\Delta_{\varpi_i,u\varpi_i}\Delta^{-1}_{w^{-1}(u\varpi_i),u\varpi_i}\big)^{-1} = t$$

for some $t \in \mathbb{K}$. For a set of elements $r_1,\dots r_n$ in an algebra R that generate a quantum torus, we will denote by $\mathcal{T}(r_1,\dots,r_n)$ this quantum torus. It follows from (3.17) that

$$\sqrt{q}^{\,\|(u-w)\varpi_i\|^2/2}\,\overline{M}(e_{N+M}) - \big(\Delta^{-1}_{-u\varpi_i,-\varpi_i}\Delta_{-\varpi_i,-\varpi_i}\big)\big(\Delta_{\varpi_i,\varpi_i}\Delta^{-1}_{w\varpi_i,\varpi_i}\big) \in \mathcal{T}(\{\overline{M}(e_k) \mid k \in [1, N+M-1]\}).$$

Applying $\zeta_{w^{-1},u^{-1}}$ to both sides and using (8.10) and (8.11) for $j < M$, leads to

$$\text{(8.17)} \quad \sqrt{q}^{\,\|(u-w)\varpi_i\|^2/2}\zeta_{w^{-1},u^{-1}}\big(\overline{M}(e_{N+M})\big) - \big(\Delta_{\varpi_i,\varpi_i}\Delta^{-1}_{w^{-1}\varpi_i,\varpi_i}\big)\big(\Delta_{w^{-1}\varpi_i,u\varpi_i}\Delta^{-1}_{w^{-1}(u\varpi_i),u\varpi_i}\big) \in \mathcal{T}(\{M_{mBZ}(e_k) \mid k \in [1, r{+}N{+}M{-}1]\}).$$

Here we also use that $\zeta_{w^{-1},u^{-1}}(\Delta_{\varpi_i,\varpi_i}\Delta^{-1}_{w\varpi_i,\varpi_i}) = \Delta_{\varpi_i,\varpi_i}\Delta^{-1}_{w^{-1}\varpi_i,\varpi_i}$, obtained from (8.10) for $k = N$, and $\zeta_{w^{-1},u^{-1}}(\Delta^{-1}_{-u\varpi_i,-\varpi_i}\Delta_{-\varpi_i,-\varpi_i}) = \Delta_{w^{-1}\varpi_i,u\varpi_i}\Delta^{-1}_{w^{-1}(u\varpi_i),u\varpi_i}$ which is easily deduced from the definition of κ and Proposition 7.1 (i). Hence, (8.16) and (8.17) give

$$t\Delta_{\varpi_i,u\varpi_i} - \Delta_{\varpi_i,\varpi_i}\Delta^{-1}_{w^{-1}\varpi_i,\varpi_i}\Delta_{w^{-1}\varpi_i,u\varpi_i} \in \mathcal{T}(\{M_{mBZ}(e_k) \mid k \in [1, r+N+M-1]\}).$$

From here one gets that $t = 1$ by using that

$$S(S_u^-) := \{c_{\xi_\mu,v}\Delta^{-1}_{\mu,u\mu} \mid \mu \in \mathcal{P}^+, v \in V(\mu)\} \subset R_q[G^{w^{-1},u^{-1}}]$$

(obtained from Proposition 7.2) and applying the antiisomorphism $\varphi_u^- S^{-1} : S(S_u^-) \to \mathcal{U}^+[u]$, cf. Theorem 3.2.

8.7. **Proof of Theorem 8.4 (i).** It follows from (3.19) and (8.11) that

$$\text{(8.18)} \qquad \zeta_{w^{-1},u^{-1}}(\Delta_{w\mu,\mu}) = \Delta_{u^{-1}(u\mu),-u\mu}, \quad \mu \in \mathcal{P}^+.$$

Thus, $\zeta_{w^{-1},u^{-1}}$ extends to an antihomomorphism $R_q[G^{w,u}] \to R_q[G^{w^{-1},u^{-1}}]$. Analogously to §8.6 one shows that

$$\text{(8.19)} \qquad \zeta_{w,u}\big(\Delta_{w^{-1}_{<k}\varpi_{i_k},\varpi_{i_k}}\big) = \sqrt{q}^{\,-\|(w_{\le k}-1)\varpi_{i_k}\|^2/2}\,\overline{M}(e_k)\Delta_{-u(u^{-1}\varpi_{i_k}),-u^{-1}\varpi_{i_k}},$$

$$\text{(8.20)} \qquad \zeta_{w,u}\big(\Delta_{\varpi_{i'_j},u_{\le j}\varpi_{i'_j}}\big) = \sqrt{q}^{\,-\|(u_{\le j}-w)\varpi_{i'_j}\|^2/2}\,\overline{M}(e_{N+j})\Delta_{-u(u^{-1}_{<j}\varpi_{i'_j}),-u^{-1}_{<j}\varpi_{i'_j}}$$

for all $k \in [1, N]$, $j \in [1, M]$. Below we prove the first identity, leaving the second to the reader. Proposition 7.1 (i) and Lemma 8.2 imply that

$$\varphi_w^+\big(\zeta_{w,u}(\Delta_{w^{-1}_{<k}\varpi_{i_k},\varpi_{i_k}}\Delta^{-1}_{w^{-1}\varpi_{i_k},\varpi_k})\big) = (\Delta_{w^{-1}_{<k}\varpi_{i_k},\varpi_{i_k}} \otimes \mathrm{id})\big((T_w^{-1}\otimes 1)\cdot{}^{\omega S\tau\times 1}\mathcal{R}^w\big)$$
$$= (\Delta_{\varpi_{i_k},\varpi_{i_k}} \otimes \mathrm{id})\big((T^{-1}_{w_{\le k}}\otimes 1)\cdot{}^{\omega S\tau\times 1}\mathcal{R}^w\big) = (\Delta_{\varpi_{i_k},\varpi_{i_k}} \otimes \mathrm{id})\big({}^{\tau\times 1}\mathcal{R}^w\cdot(T^{-1}_{(w_{\le k})^{-1}}\otimes 1)\big).$$

This proves (8.19) because $X^+_{\beta_l}v_{w_{\le k}\varpi_{i_k}} = 0$ for $l > k$, which truncates ${}^{\tau\times 1}\mathcal{R}^w$ to ${}^{\tau\times 1}\mathcal{R}^{w_{\le k}}$.

It follows from (8.10)–(8.11) and (8.19)–(8.20) that $\zeta_{w,u}$ and $\zeta_{w^{-1},u^{-1}}$ are inverse homomorphisms because the set $\{\overline{M}(e_k) \mid k \in [1, N+M]\}$ generates $\mathrm{Fract}(S_w^+ \bowtie S_u^-)$ and the set $\{M_{mBZ}(e_k) \mid k \in [1, N+M+r]\}$ generates $\mathrm{Fract}(R_q[G^{w^{-1},u^{-1}}])$ (by the definition of toric frame). This completes the proof of Theorem 8.4 (i).

9. Quantum cluster algebra structures on $R_q[G^{w,u}]$, the Berenstein–Zelevinsky conjecture

In this section we relate the quantum seeds for the reduced double Bruhat cells from Theorem 6.2 to the BZ quantum seeds via the composition of a reduction map, the quantum twist map (Section 8) and the map ωS (Section 7), and complete the proof of the Berenstein–Zelevinsky conjecture.

9.1. **Reduction of graded quantum cluster algebras.** We will need an easy general fact on such reductions. We will work in the setting of §2.1. Let $\mathcal{P}$ be a free abelian group.

Assume that R is a $\mathcal{P}$-graded Ore domain. Let $(M, \widetilde{B})$ be a $\mathcal{P}$-graded seed in $\mathrm{Fract}(R)$ by which we mean that every cluster variable in a mutation equivalent seed is homogeneous. This is equivalent to saying that the $M(e_k)$ are homogeneous for $k \in [1, N]$ and $\deg M(b^k) = 0$ for all columns b^k of $\widetilde{B}$ (i.e., $k \in \mathbf{ex}$). We will make the following assumption:

(i) The set $\mathbf{inv}$ of inverted frozen indices contains $[1, n]$ for some $n \in [1, N]$ and the function $\phi : \mathbb{Z}^n \to \mathcal{P}$, $\phi(f) := \deg M(f)$ is an isomorphism.

For each quantum seed $(M', \widetilde{B}')$ mutation equivalent to $(M, \widetilde{B})$, we define a quantum seed $(M', \widetilde{B}')_{\mathrm{red}} := (M'_{\mathrm{red}}, \widetilde{B}'_{\mathrm{red}})$ of $\mathrm{Fract}(R_0)$. The cluster variables in the latter will be indexed by $[n+1, N]$, the exchangeable ones by $\mathbf{ex}$ and the inverted frozen ones by $\mathbf{inv}\backslash[1, n]$. Denote by $\mathbb{Z}^{[n+1,N]}$ the set of integer vectors whose components are indexed by $[n+1, N]$. The matrix $\widetilde{B}'_{\mathrm{red}}$ is obtained from $\widetilde{B}'$ by removing the first n rows. The toric frame M'_{red} is defined by

$$M'_{\mathrm{red}}(f) := M'(f - \phi^{-1}(\nu)) \ \text{ where } \ \nu := \deg M'(f), \quad \forall\, f \in \mathbb{Z}^{[n+1,N]}.$$

The matrix $\mathbf{r}'_{\mathrm{red}}$ of M'_{red} is given by

$$(\mathbf{r}'_{\mathrm{red}})_{kj} = \Omega_{\mathbf{r}}(e_k - \phi^{-1}(\nu_k),\, e_j - \phi^{-1}(\nu_j)), \quad k, j \in [n+1, N]$$

(recall (2.1)), where $\nu_k := \deg M'(e_k)$ and $\mathbf{r}'$ is the matrix of M'. The graded condition on $(M, \widetilde{B})$ yields at once that $(M'_{\mathrm{red}}, \widetilde{B}'_{\mathrm{red}})$ is a quantum seed of $\mathrm{Fract}(R_0)$. The definition of reduced toric frames and (2.2) directly imply that,

$$\text{if } (M'', \widetilde{B}'') = \mu_k(M', \widetilde{B}'),\ k \in \mathbf{ex}, \text{ then } (M''_{\mathrm{red}}, \widetilde{B}''_{\mathrm{red}}) := \mu_k(M'_{\mathrm{red}}, \widetilde{B}'_{\mathrm{red}}). \tag{9.1}$$

The following theorem is inverse to [21, Theorem 4.6].

Theorem 9.1. *Assume that $(M, \widetilde{B})$ is a graded quantum seed of the $\mathcal{P}$-graded Ore domain R that satisfies the condition* (i)*. Then the quantum cluster algebra associated to the reduced seed $(M_{\mathrm{red}}, \widetilde{B}_{\mathrm{red}})$ satisfies*

$$\mathcal{A}(M, \widetilde{B}, \mathbf{inv}) = \mathcal{A}(M_{\mathrm{red}}, \widetilde{B}_{\mathrm{red}}, \mathbf{inv}\backslash[1, n])[M(e_k)^{\pm 1},\, 1 \le k \le n],$$
$$\mathcal{U}(M, \widetilde{B}, \mathbf{inv}) = \mathcal{U}(M_{\mathrm{red}}, \widetilde{B}_{\mathrm{red}}, \mathbf{inv}\backslash[1, n])[M(e_k)^{\pm 1},\, 1 \le k \le n].$$

In particular, $\mathcal{A}(M, \widetilde{B}, \mathbf{inv}) = \mathcal{U}(M, \widetilde{B}, \mathbf{inv})$ if and only if $\mathcal{A}(M_{\mathrm{red}}, \widetilde{B}_{\mathrm{red}}, \mathbf{inv}\backslash[1, n]) = \mathcal{U}(M_{\mathrm{red}}, \widetilde{B}_{\mathrm{red}}, \mathbf{inv}\backslash[1, n])$.

The theorem follows from the fact that every graded subalgebra A of R that contains the frozen variables $M(e_k)^{\pm 1}$, $k \in [1,n]$ satisfies $A = A_0[M(e_k)^{\pm 1},\ 1 \leq k \leq n]$ and that

$$\mathcal{T}(M', \widetilde{B}') = \mathcal{T}(M'_{\mathrm{red}}, \widetilde{B}'_{\mathrm{red}})[M(e_k)^{\pm 1},\ 1 \leq k \leq n]$$

for every seed $(M', \widetilde{B}')$ mutation equivalent to $(M, \widetilde{B})$, recall §2.1.

9.2. **Cluster structures on quantum double Bruhat cells.** Let $\zeta : R \to A$ be an antiisomorphism of Ore domains. To a quantum seed $(M : \mathbb{Z}^N \to R, \widetilde{B})$ of R, we associate a quantum seed $\zeta(M, \widetilde{B}) := (\zeta M, -\widetilde{B})$ of A as follows. Set $\zeta M(e_k) := \zeta \circ M(e_k)$, for $k \in [1,N]$ and let the matrix of ζM be (r_{kj}^{-1}) where $\mathbf{r} = (r_{kj})$ is the matrix of M. It is obvious that $(\zeta M, -\widetilde{B})$ is a quantum seed of A. We carry over the set $\mathbf{ex}$ of exchangeable variables and the set $\mathbf{inv}$ of invertible frozen variables from the first to the second.

For the Berenstein–Zelevinsky quantum seed $(M_{BZ}, \widetilde{B}_{BZ})$ of $R_q[G^{u,w}]$ (recall §7.1), we define the set of exchangeable indices (7.6) to be

$$\mathbf{ex}_{BZ} := \{k \in [r+1, r+N+M] \mid s(k) \neq +\infty\}$$

for the successor function s in §7.1 and declare all frozen variables to be invertible,

$$\mathbf{inv}_{BZ} := [1, r+N+M] \backslash \mathbf{ex}_{BZ}.$$

For the modified Berenstein–Zelevinsky quantum seeds $(M_{mBZ}, \widetilde{B}_{BZ})$ of $R_q[G^{w^{-1},u^{-1}}]$ (see §7.4) set $\mathbf{ex}_{mBZ} := \mathbf{ex}_{BZ}$ and $\mathbf{inv}_{mBZ} = \mathbf{inv}_{BZ}$.

We apply the graded reduction from Theorem 9.1 to the seeds $(M_{BZ}, \widetilde{B}_{BZ})$ of $R_q[G^{u,w}]$ and $(M_{mBZ}, \widetilde{B}_{BZ})$ of $R_q[G^{w^{-1},u^{-1}}]$. In the first case we choose the grading $R_q[G^{u,w}]_{\nu,-}$, $\nu \in \mathcal{P}$ and in the second $R_q[G^{w^{-1},u^{-1}}]_{-,\nu}$, $\nu \in \mathcal{P}$. The frozen variables used for the reduction in the two respective cases are

$$M_{BZ}(e_i)^{\pm 1} := \Delta^{\pm 1}_{\varpi_i, w^{-1}\varpi_i}, \quad M_{mBZ}(e_i)^{\pm 1} := \Delta^{\pm 1}_{w^{-1}\varpi_i, \varpi_i}, \quad i \in [1,r].$$

The next facts describe the relations between the quantum seeds of reduced double Bruhat cells and of double Bruhat cells from Sections 6 and 7, respectively.

Proposition 9.2. *For all simple Lie algebras $\mathfrak{g}$ and $w, u \in W$, the following hold:*

(i) *The quantum seed $(\overline{M}, \overline{B})$ of $R_q[G^{w,u}/H]$ in Theorem* 6.2 (d) *is a reduction of the modified Berenstein–Zelevinsky seed of $R_q[G^{w^{-1},u^{-1}}]$ in the sense that*

$$(\overline{M}, \overline{B}) = \zeta_{w,u}\big((M_{mBZ}, \widetilde{B}_{BZ})_{\mathrm{red}}\big)$$

where $\zeta_{w,u} : R_q[G^{w^{-1},u^{-1}}] \to R_q[G^{w,u}]$ is the quantum twist map (*antiisomorphism*) *from Section* 8.

(ii) *The quantum seed $(\overline{M}, \overline{B})$ of $R_q[G^{w,u}/H]$ is a reduction of the Berenstein–Zelevinsky seed of $R_q[G^{u,w}]$,*

$$(\overline{M}, \overline{B}) = \zeta_{w,u} \circ (h_* \cdot) \circ (\omega S)\big((M_{BZ}, \widetilde{B}_{BZ})_{\mathrm{red}}\big),$$

recall the isomorphism $\omega S : R_q[G^{u,w}] \to R_q[G^{w^{-1},u^{-1}}]$ and the definition (7.9) *of $h_* \in \mathcal{H}$.*

Moreover, the sets $\mathbf{ex}$ *and* $\mathbf{inv}$ *in Theorem* 6.2 (d) *match those for $R_q[G^{w^{-1},u^{-1}}]$ by converting the index sets from $[1, N+M]$ to $[r+1, r+N+M]$.*

The key part of (i) is that $M(e_k) = \zeta_{w,u}(M_{mBZ})_{\mathrm{red}}(e_{r+k})$ for $k \in [1, M+N]$. This is exactly what is proved in Theorem 8.4 (ii) (though written in an explicit form there). The facts that the matrices $\mathbf{r}$ of the toric frames and the exchange matrices match are verified directly. Part (ii) follows from Proposition 7.3 and the first part.

Theorem 9.3. [Berenstein–Zelevinsky Conjecture] *For all simple Lie algebras $\mathfrak{g}$, Weyl group elements $w, u \in W$, arbitrary base fields $\mathbb{K}$, and non-roots of unity $q \in \mathbb{K}^*$ such that $\sqrt{q} \in \mathbb{K}$, the quantum coordinate ring $R_q[G^{u,w}]$ equals the quantum cluster algebra $\mathcal{A}(M_{BZ}, \widetilde{B}_{BZ}, \mathbf{inv}_{BZ})$ and the upper quantum cluster algebra $\mathcal{U}(M_{BZ}, \widetilde{B}_{BZ}, \mathbf{inv}_{BZ})$.*

The theorem follows from Theorem 6.2 (d) via the reduction procedure of Theorem 9.1 and the connections in Proposition 9.2. Theorems 6.2 (a)–(b) and 9.1 also give large sets of explicit quantum seeds for the quantum cluster algebra structure on $R_q[G^{u,w}]$. These seeds are parametrized by $\sigma \in \Xi_{N+M}$ (recall (2.15)) and the one in the theorem corresponds to $\sigma = w_N^\circ$. The cluster variables in these seeds have the form

$$q^m \big(\zeta_{w,u}(h_* \cdot) \omega S(\widetilde{M}_\sigma(e_k))\big) \Delta_{\mu, w^{-1}\mu}, \quad k \in [1, N+M],$$

see (6.7). Here $m \in \mathbb{Z}$ and $\mu \in \mathcal{P}$ depend on σ and k. The exact values of m and μ can be determined by applying the sequence of mutations in Theorem 6.2 (b) to get from the seed $(\overline{M}, \overline{B})$ to the seed $(M_\sigma, \widetilde{B}_\sigma)$ and then using (9.1) at each step. The result is more technical and we leave the details to the reader. The principal parts of the exchange matrices of the seeds are the principal parts of the matrices $\widetilde{B}_\sigma$.

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