NONCOMMUTATIVE DISCRIMINANTS VIA POISSON PRIMES

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Abstract. We present a general method for computing discriminants of noncommutative algebras. It builds a connection with Poisson geometry and expresses the discriminants as products of Poisson primes. The method is applicable to algebras obtained by specialization from families, such as quantum algebras at roots of unity. It is illustrated with the specializations of the algebras of quantum matrices at roots of unity and more generally all quantum Schubert cell algebras.

1. Introduction

1.1. In the commutative setting, the notion of discriminant plays a fundamental role in algebraic number theory, algebraic geometry and combinatorics. We refer the reader to the book of Gelfand, Kapranov, and Zelevinsky [23] for extensive background. The notion also has an analog for noncommutative algebras that has been used in a key way in the study of orders and lattices in central simple algebras, see Reiner's book [35]. Recently, Zhang, Bell, Ceken, Palmieri and Wang found many additional applications of noncommutative discriminants in the study of automorphism groups of PI algebras [6, 7] and related problems, such as the isomorphism problem and the Zariski cancellation problem [1]. Finally, discriminants also have tight relations to the representation theory of the algebra in question.

However, the computation of the discriminant of a noncommutative algebra turns out to be a rather challenging problem. It has been solved for very few families of algebras: quasipolynomial rings and two cases of quantum Weyl algebras [6, 7, 8]. For example, even the discriminants of the algebras of quantum 2×2 matrices at roots of unity are presently unknown. Furthermore, it was observed in [6, 7, 8] that in the known cases the discriminants have an elaborate product form but the meaning of the individual terms remained unclear. The problem of computing noncommutative discriminants has so far been attacked with techniques within noncommutative algebra.

In this paper we present a general method for the computation of the discriminants of algebras obtained as specializations. It establishes a connection between discriminants of noncommutative algebras and Poisson geometry. The method is applicable to broad classes of algebras, such as specializations of quantum algebras at roots of unity. The algebras of quantum matrices of any size at roots of unity are treated as special cases.

The discriminant of an algebra R_ϵ obtained by specialization is an element of a Poisson algebra A sitting inside the center of R_ϵ. We prove that the discriminant is a Poisson normal element of A and that, under conditions satisfied very frequently, it is a product of Poisson prime elements of A. This explains the intrinsic nature of the terms in the

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product formulas for discriminants. The Poisson prime elements are closely related to Poisson hypersurfaces in the Poisson variety Spec$A$.

Thus, the problem of computing noncommutative discriminants becomes one about the interplay between the PI algebra in question, R_ϵ, and the symplectic foliation of the Poisson variety Spec$A$. The precise form of the discriminant is determined by studying its evolution under the hamiltonian flows on Spec$A$, as described in the next subsection.

1.2. In the remaining part of the introduction we formulate the main results of the paper. Let R be an algebra over $\mathbb{K}[q^{\pm 1}]$ for a field $\mathbb{K}$ of characteristic 0. Given $\epsilon \in \mathbb{K}^\times$, we consider the specialization of R at ϵ – the $\mathbb{K}$-algebra $R_\epsilon := R/(q-\epsilon)R$. Denote the projection $\sigma \colon R \to R_\epsilon$. The center $Z(R_\epsilon)$ has a canonical structure of Poisson algebra:

$$\{\sigma(x_1), \sigma(x_2)\} := \sigma\left(\frac{x_1x_2 - x_2x_1}{q-\epsilon}\right), \quad x_i \in \sigma^{-1}(Z(R_\epsilon)).$$

Let C_ϵ be a Poisson subalgebra of $Z(R_\epsilon)$ such that R_ϵ is a free C_ϵ-module of finite rank. A C_ϵ-basis $\mathcal{Y} := \{y_j \mid 1 \leq j \leq N\}$ gives rise to an embedding $R_\epsilon \hookrightarrow M_N(C_\epsilon)$. (Here and below M_n refers to square matrices and $M_{m,n}$ to rectangular matrices of those sizes.) The composition of this embedding with the standard trace map leads to a trace map $\mathrm{tr} \colon R_\epsilon \to C_\epsilon$ which is independent on the choice of basis. The discriminant of R_ϵ over C_ϵ is defined by

$$d(R_\epsilon/C_\epsilon) := \det\left([\mathrm{tr}(y_iy_j)]_{i,j=1}^N\right) \in C_\epsilon,$$

see §2.1 for details on trace maps and discriminants. The notation $[c_{ij}]_{i,j=1}^N$ refers to the square matrix with entries c_{ij}. The element $d(R_\epsilon/C_\epsilon)$ is defined up to associates, because changing the basis $\mathcal{Y}$ multiplies the discriminant by an element of $C_\epsilon^\times$. (As usual, the units of an algebra A will be denoted by $A^\times$; $a, b \in A$ are called associates, denoted by $a =_{A^\times} b$, if $a = ub$ for some $u \in A^\times$.)

An element a of a Poisson algebra A is called Poisson normal if the principal ideal (a) is Poisson (i.e., is closed under the Poisson bracket). An element $p \in A$ is called Poisson prime if (p) is a prime ideal which is also Poisson. The latter happens precisely when the zero locus of p is a union of symplectic leaves of the Poisson variety Spec$A$. Background material on these notions and the notion of noetherian Poisson unique factorization domain is contained in §2.2.

Theorem A. *Let R be a $\mathbb{K}[q^{\pm 1}]$-algebra where $\mathbb{K}$ is a field of characteristic 0 and $\epsilon \in \mathbb{K}^\times$. Assume that $R_\epsilon := R/(q-\epsilon)R$ is a free module of finite rank over a Poisson subalgebra C_ϵ of its center.*

(i) Then $d(R_\epsilon/C_\epsilon)$ is a Poisson normal element of $(C_\epsilon, \{.,.\})$.

(ii) Assume, in addition, that C_ϵ is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain. Then, $d(R_\epsilon/C_\epsilon) = 0$ or

$$d(R_\epsilon/C_\epsilon) =_{C_\epsilon^\times} \prod_{i=1}^m p_i$$

for some (not necessarily distinct) Poisson prime elements $p_1, \ldots, p_m \in C_\epsilon$.

In concrete situations C_ϵ is very close to being the full center of R_ϵ and, as a consequence, the Poisson bracket on C_ϵ is very nontrivial. Because of this, the collection of Poisson primes of C_ϵ is only a very small subset of the set of all primes of the commutative algebra C_ϵ. From this perspective, Theorem A is a rigidity theorem for fully noncommutative extensions (such for which C_ϵ is almost the full center of R_ϵ) in the sense that their discriminants are products of Poisson primes of C_ϵ which altogether form a discrete and often a finite set.

Since Theorem A places a strong restriction on the possible form of the discriminant $d(R_\epsilon/C_\epsilon)$, the latter can be fully determined using a recipe of four methods described in §3.3 (a combination of algebraic methods, such as filtration arguments, with geometric methods, such as evolution under hamiltonian flows). The Poisson primes of C_ϵ are determined from the symplectic foliation of the Poisson variety SpecC_ϵ. The symplectic foliations of Poisson varieties appearing in the theory of quantum groups are well understood in great generality [18, 19, 30, 37].

As a simple example, we first consider the quantum Weyl algebra A_ϵ with generators x_1, x_2 subject to the relation $x_1x_2 = \epsilon x_2x_1 + 1$ where $\epsilon \in \mathbb{K}$ is a primitive l-th root of unity. The induced Poisson structure on its center $Z(A_\epsilon) \cong \mathbb{K}[z_1, z_2]$ (where $z_i := x_i^l$) is

$$\{z_1, z_2\} = l^2\epsilon^{-1}(z_1z_2 - t) \quad \text{where} \quad t := (1-\epsilon)^{-l}.$$

It has only one Poisson prime $z_1z_2 - t$, leading to the discriminant formula

$$d(A_\epsilon, Z(A_\epsilon)) =_{\mathbb{K}^\times} (z_1z_2 - t)^{l(l-1)},$$

see §3.4. This recovers a formula of Chan, Young and Zhang [8] which was obtained in a more complicated fashion.

Let $R_q[M_n]$ be the algebra of square quantum matrices with generators x_{ij} and relations (4.1). The specialization $R_\epsilon[M_n]$ at a primitive l-th root of unity $\epsilon \in \mathbb{K}$ is free over the central subalgebra $C_\epsilon[M_n]$, generated by the images of x_{ij}^l under the projection map $R_q[M_n] \to R_\epsilon[M_n]$. Denote those images by z_{ij}.

For $i \leq j \in \mathbb{Z}$, set $[i, j] := \{i, \dots, j\}$. For two sets $I, J \subseteq [1, n]$, denote by $\Delta_{I;J}$ the minor in z_{ij} corresponding to the rows in the set I and the columns in J.

Theorem B. *For all fields $\mathbb{K}$ of characteristic 0, odd integers $l > 2$, primitive l-th roots of unity $\epsilon \in \mathbb{K}$, and positive integers n, the discriminant of the specialization of the algebra of square quantum matrices $R_\epsilon[M_n]$ is given by*

$$d(R_\epsilon[M_n]/C_\epsilon[M_n]) =_{\mathbb{K}^\times} \prod_{k=1}^{n} \Delta^L_{[n-k+1,n];[1,k]} \prod_{j=1}^{n-1} \Delta^L_{[1,j];[n-j+1,n]}$$

where $L := l^{n^2-1}(l-1)$.

For all simple Lie algebras $\mathfrak{g}$ and Weyl group elements $w \in W$, Lusztig [31] and De Concini, Kac, and Procesi [13] defined a subalgebra $\mathcal{U}^-[w]$ of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ which is a deformation of $\mathcal{U}(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$ for the nilradicals $\mathfrak{n}_\pm$ of a pair of opposite Borel subalgebras of $\mathfrak{g}$. The specialization at a primitive l-th root of unity $\epsilon \in \mathbb{K}$ of the corresponding nonrestricted rational form will be denoted by $\mathcal{U}_\epsilon^-[w]$. The central subalgebra of the latter generated by the images of the l-th powers of Lusztig's root vectors will be denoted by $C_\epsilon^-[w]$. One can canonically identify $C_\epsilon^-[w]$ with the coordinate ring of the Schubert cell $B_+w \cdot B_+ \subset G/B_+$ in the full flag variety of the split, connected, simply connected algebraic $\mathbb{K}$-group G associated to $\mathfrak{g}$. The Schubert cell $B_+w \cdot B_+$ is isomorphic to the unipotent subgroup $U_+ \cap w(U_-)$ of G where $U_\pm$ are the unipotent radicals of $B_\pm$. By restriction, generalized minors $\Delta_{\lambda, w\lambda}$ on G are identified with elements of $C_\epsilon^-[w]$ where λ is a dominant weight.

Theorem C. *Let $\mathfrak{g}$ be a simple Lie algebra, w a Weyl group element, and $l > 2$ an odd* *integer which is $\neq 3$ in the case of G_2. Let $\mathbb{K}$ be a field of characteristic 0 containing a primitive l-th root of unity ϵ. Then*

$$d(\mathcal{U}^-_\epsilon[w]/C^-_\epsilon[w]) =_{\mathbb{K}^\times} \prod_{i \in \mathcal{S}(w)} \Delta^L_{\varpi_i, w\varpi_i}$$

where $L := l^{N-1}(l-1)$, ϖ_i are the fundamental weights of $\mathfrak{g}$ and $\mathcal{S}(w)$ is the support of w.

Theorem A is proved in Section 3. Two companion theorems for n-discriminant ideals without freeness assumptions are obtained in §3.5. These results are extended to the setting of Poisson orders introduced by Brown and Gordon [5] in §3.6. Section 5 contains the proof of Theorem C. It relies on the results of De Concini, Kac and Procesi [11, 12, 13] on specializations of the nonrestricted rational forms of $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}^-[w]$ at root of unity, as well as on results on Poisson algebraic groups and Poisson homogeneous spaces. Theorem B for quantum matrices is proved in Section 4. The proofs are completely elementary and do not use the results in [11, 12, 13]. This section was written so the proofs for the algebras of quantum matrices can be read without any Lie theoretic background. All algebras of quantum matrices $R_q[M_{m,n}]$ arise as special cases of the quantum Schubert cell algebras for $\mathfrak{g} = \mathfrak{sl}_{m+n}$ and particular Weyl group elements. The relation between the corresponding discriminant formulas is described in §5.5.

The results in Theorem C concern quantum Schubert cell algebras for finite dimensional simple Lie algebras as opposed to such for symmetrizable Kac–Moody algebras because of the use of [11, 12] on specializations of $\mathcal{U}_q(\mathfrak{g})$. In the PhD theses of the first two named authors these results will be extended to the Kac–Moody case. This is done by working directly with the algebras $\mathcal{U}_\epsilon[w]$ which are orders in central simple algebras while the specializations of $\mathcal{U}_q(\mathfrak{g})$ are not finite over their centers for general symmetrizable Kac–Moody algebras $\mathfrak{g}$.

In forthcoming papers we apply the techniques in the paper to derive formulas for the discriminants of other important classes of algebras: the specializations at roots of unity of the quantized universal enveloping algebras of simple Lie algebras, the corresponding quantum function algebras, and general quantum Weyl algebras.

The quantum Schubert cell algebras $\mathcal{U}^-[w]$ have quantum cluster algebra structures [21, 24] and $C^-_\epsilon[w]$ are isomorphic to the corresponding classical cluster algebra structures on unipotent cells [21, 25]. In the framework of cluster algebras, Theorems B and C prove that the discriminant $d(\mathcal{U}^-_\epsilon[w]/C^-_\epsilon[w])$ is precisely the product of all frozen variables (raised to the same powers). We expect that the discriminants of the specializations of quantum cluster algebras [2] to primitive l-th roots of unity over the commutative subalgebras of l-th powers of cluster variables will have deep Poisson geometric properties in terms of the Gekhtman–Shapiro–Vainshtein Poisson structure [22] on the corresponding cluster variety.

All algebras in the paper will be assumed to be unital. For a Poisson structure π on an algebraic variety V we will denote by $\{.,.\}_\pi$ the Poisson bracket on the coordinate ring of V.

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2. Discriminants and Poisson primes

This section contains background material on discriminants and Poisson normal and prime elements. The section also contains new results that will be used later in the paper.

2.1. An algebra with trace is an algebra R with a linear map $\mathrm{tr}\colon R \to R$ such that

(1) $\mathrm{tr}(xy) = \mathrm{tr}(yx)$,
(2) $\mathrm{tr}(x)y = y\,\mathrm{tr}(x)$,
(3) $\mathrm{tr}(\mathrm{tr}(x)y) = \mathrm{tr}(x)\,\mathrm{tr}(y)$

for all $x, y \in R$. An algebra with trace is said to be a d-Cayley–Hamilton algebra if it satisfies the d-th Cayley–Hamilton identity and $\mathrm{tr}(1) = d$. A geometric framework for the study of the representations of such algebras was developed by Procesi [33, 34]. De Concini, Kac and Procesi [11, 12, 13] studied the representation theory of quantum algebras at roots of unity in this framework. Orders in central simple algebras give rise to Cayley–Hamilton algebras, and quantized universal enveloping algebras at roots of unity belong to this class of algebras [11, 12, 13]. We refer the reader to [14, Chapter 2] for a detailed and very instructive exposition.

The image of the trace map is a subalgebra of the center $Z(R)$. Slightly more generally, one can consider trace maps $\mathrm{tr}\colon R \to F$ where F is a commutative algebra which is an extension of a subalgebra $C \subset Z(R)$. Such traces are required to be C-linear and cyclic. They arise from embeddings of R into F-algebras.

Definition 2.1. [35] (i) The *discriminant of the set* $\mathcal{Y} := \{y_1, \dots, y_N\} \subset R$ is defined to be

$$d_N(\mathcal{Y} : \mathrm{tr}) := \det\left([\mathrm{tr}(y_i y_j)]_{i,j=1}^N\right) \in F.$$

(ii) The *N-discriminant ideal* $D_N(R/C)$ is the C-submodule of F generated by

$$d_N(\mathcal{Y} : \mathrm{tr}) \quad \text{for the } N\text{-element subsets } \mathcal{Y} \subset R.$$

(iii) If R is a free, rank N module over a subalgebra C of the center $Z(R)$, then the embedding $R \hookrightarrow M_N(C)$ gives rise to a trace map $\mathrm{tr} : R \to C$ which is independent on the choice of basis. The *discriminant* of R over C is defined by

$$d(R/C) :=_{C^\times} d_N(\mathcal{Y} : \mathrm{tr})$$

where $\mathcal{Y} = \{y_1, \dots, y_N\}$ is a C-basis of R.

The discriminant in part (iii) is well defined since for any other C-basis $\mathcal{X} := \{x_1, \dots, x_N\}$ of R,

$$d_N(\mathcal{X} : \mathrm{tr}) = \det(b)^2 d_N(\mathcal{Y} : \mathrm{tr}) \tag{2.1}$$

where $b := (b_{ij}) \in M_N(C)$ is the change of bases matrix given by $x_i = \sum_j b_{ij} y_j$, see [35, Exercise 4.13]. Our notation in Definition 2.1 (iii) differs from the one in [6].

Proposition 2.2. *Assume that R is an algebra with trace $\mathrm{tr}\colon R \to C \subset Z(R)$ which is a free module over $C \subset Z(R)$ of rank N.*

(i) If ∂ is a derivation of R such that $\mathrm{tr}(\partial x) = \partial\,\mathrm{tr}(x)$ for all $x \in R$, then

$$\partial d_N(\mathcal{Y} : \mathrm{tr}) = 2\,\mathrm{tr}(b) d_N(\mathcal{Y} : \mathrm{tr})$$

for any C-basis $\mathcal{Y} := \{y_1, \dots, y_N\}$ of R, where $b = (b_{ij}) \in M_N(C)$ is the matrix with entries given by $\partial y_i = \sum_j b_{ij} y_j$.

(ii) Let $\mathrm{tr}\colon R \to C$ *be the canonical trace map from the embedding* $R \subset M_N(C)$ *associated to any* C*-basis of* R*. Then every derivation* ∂ *of* R*, satisfying* $\partial(C) \subseteq C$*, has the property* $\mathrm{tr}(\partial x) = \partial\,\mathrm{tr}(x)$, $\forall x \in R$.

Proof. (i) The discriminant $d_N(\mathcal{Y} : \mathrm{tr})$ is the same when it is computed for the pair of algebras (R, C) and $(R[t]/(t^2), C[t]/(t^2))$. From the second interpretation and (2.1) we get

$$(1 + t\partial)d_N(\mathcal{Y} : \mathrm{tr}) = d_N((1 + t\partial)\mathcal{Y} : \mathrm{tr}) = \det(I_N + tb)^2 d_N(\mathcal{Y} : \mathrm{tr})$$

where $(1 + t\partial)\mathcal{Y} := \{(1 + t\partial)y_1, \ldots, (1 + t\partial)y_N\}$. The first part now follows from the fact that $d_N(\mathcal{Y} : \mathrm{tr}) \in C$ by comparing the coefficients of t.

(ii) Given a basis of R over C, from the embedding $R[t]/(t^2) \hookrightarrow M_N(C[t]/(t^2))$ we obtain $(1 + t\partial)\,\mathrm{tr}(x) = \mathrm{tr}((1 + t\partial)x)$ for all $x \in R$ where the traces are computed in $R[t]/(t^2)$. This implies the statement since $\mathrm{tr}(x), \mathrm{tr}(\partial x) \in C$. $\square$

The second part is valid in much greater generality for orders in central simple algebras [35, Ch. 9-10], but we will not need this here.

2.2. Let $(A, \{.,.\})$ be a Poisson algebra over a base field of characteristic 0. It is called noetherian if it is noetherian considered as a commutative algebra.

Definition 2.3. (i) An element $a \in A$ is called *Poisson normal* if for every $x \in A$,

$$\{a, x\} = ay \quad \text{for some } y \in A.$$

If A is an integral domain as a (commutative) algebra, this is equivalent to saying that

$$\{a, x\} = a\partial(x) \quad \text{for some Poisson derivation } \partial \text{ of } A.$$

(ii) Assume that A is an integral domain as an algebra. An element $p \in A$ is called *Poisson prime* if it is a prime element of the algebra which is normal in the Poisson sense.

Remark 2.4. (i) An element $a \in A$ is normal if and only if the principal ideal (a) is Poisson.

(ii) An element $p \in A$ is Poisson prime if and only if the ideal (p) is nonzero, prime and Poisson.

(iii) Assume that the base field is $\mathbb{C}$ and Spec$A$ is smooth. View the elements of A as regular functions on the Poisson variety Spec$A$. A prime element $p \in A$ is Poisson prime if and only if its zero locus $\mathcal{V}(p)$ is a union of symplectic leaves of Spec$A$.

This can be proved as follows. If $\mathcal{V}(p)$ is a union of symplectic leaves of Spec$A$, then for all $g \in A$, $\{p, g\}$ vanishes on the smooth locus of $\mathcal{V}(p)$. Thus $\{p, g\}$ vanishes on $\mathcal{V}(p)$ and belongs to (p). In the opposite direction, assume that (p) is Poisson. If $\mathcal{L}$ is a symplectic leaf of Spec$A$ such that $\mathcal{L} \cap \mathcal{V}(p) \neq \varnothing$ and $\mathcal{L} \not\subseteq \mathcal{V}(p)$, then for every smooth point $m \in \mathcal{L} \cap \mathcal{V}(p) \subsetneq \mathcal{L}$ there will exist $g \in A$ such that $\{p, g\}(m) \neq 0$. This would contradict the assumption that (p) is a Poisson ideal.

Definition 2.5. A noetherian Poisson algebra A is called a Poisson unique factorization domain if it is an integral domain as an algebra and every non-zero Poisson prime ideal of A contains a Poisson prime element.

Proposition 2.6. *Let* A *be a Poisson algebra over a field of characteristic 0, satisfying one of the following 2 conditions:*

- A *is a unique factorization domain as a commutative algebra or*
- A *is a noetherian Poisson unique factorization domain.*

Then every non-zero, non-unit Poisson normal element $a \in A$ has a unique factorization of the form

$$a = \prod_{i=1}^{m} p_i$$

for some set of (not necessarily distinct) Poisson prime element $p_1, \ldots, p_m \in A$. The uniqueness is up to taking associates and permutations.

The case of noetherian Poisson UFDs is analogous to the unique factorization property of normal elements in (noncommutative) noetherian UFDs proved by Chatters [10, Proposition 2.1], see also [25, Proposition 2.1]. The case when A is a UFD as a commutative algebra follows from the following lemma.

Lemma 2.7. *Assume that A is a Poisson algebra over a field of characteristic 0 which is a unique factorization domain as a commutative algebra. If $a \in A$ is a Poisson normal element and $p \in A$ is a prime element such that $p \mid a$, then p is a Poisson prime element.*

Proof. Let $a = p^k b$ for some $b \in A$ such that $p \nmid b$. For every $x \in A$, there exists $y \in A$ such that $\{a, x\} = ay$. Then

$$k\{p, x\}p^{k-1}b + p^k\{b, x\} = p^k by.$$

Since the base field has characteristic 0, we have $p^k \mid \{p, x\}p^{k-1}b$ for every $x \in A$, and so $p \mid \{p, x\}$. $\square$

3. General theorems on noncommutative discriminants

In this section we prove two general theorems on discriminants and discriminant ideals of algebras obtained as specializations. We also give a recipe about computing discriminants from the first theorem. In the last subsection we obtain extensions of these results to the setting of Poisson orders introduced by Brown and Gordon [5]. In order to keep the exposition more transparent we first prove the results in the more common setting of specializations, and then extend them to the setting of Poisson orders.

3.1. Let R be an algebra over $\mathbb{K}[q^{\pm 1}]$. For $\epsilon \in \mathbb{K}^\times$, one defines the specialization of R at ϵ which is the $\mathbb{K}$-algebra $R_\epsilon := R/(q-\epsilon)R$. Denote the canonical projection $\sigma \colon R \to R_\epsilon$. The center $Z(R_\epsilon)$ has a canonical structure of Poisson algebra defined as follows. For $z_1, z_2 \in Z(R_\epsilon)$, choose $x_i \in \sigma^{-1}(z_i)$ and set

$$\{z_1, z_2\} := \sigma\left(\frac{x_1x_2 - x_2x_1}{q - \epsilon}\right). \tag{3.1}$$

This definition does not depend on the choice of x_1 and x_2. Indeed, if $\sigma(x_1) = \sigma(x_1')$, then $x_1 - x_1' = (q - \epsilon)x$ for some $x \in R$ and

$$\sigma\left(\frac{[x_1, x_2]}{q - \epsilon}\right) - \sigma\left(\frac{[x_1', x_2]}{q - \epsilon}\right) = \sigma([x, x_2]) = 0$$

because $\sigma(x_2) \in Z(R_\epsilon)$. Similarly one checks that $\{z_1, z_2\} \in Z(R_\epsilon)$ for $z_i \in Z(R_\epsilon)$.

Proposition 3.1. [11, 26] *For every $z \in Z(R_\epsilon)$, the hamiltonian derivation $y \mapsto \{z, y\}$ of the Poisson algebra $(Z(R_\epsilon), \{.,.\})$ has a lift to an algebra derivation of R_ϵ given by*

$$\partial_x(\sigma(\widetilde{y})) := \sigma\left(\frac{x\widetilde{y} - \widetilde{y}x}{q - \epsilon}\right), \quad x \in \sigma^{-1}(z), \widetilde{y} \in R.$$

Note that $\sigma(x) \in Z(R_\epsilon)$ implies that $\sigma(x\widetilde{y} - \widetilde{y}x) = 0$, so $x\widetilde{y} - \widetilde{y}x \in (q - \epsilon)R$. The lifts coming from different elements $x, x' \in \sigma^{-1}(z)$ differ by the inner derivation of R_ϵ corresponding to $\sigma((x - x')/(q - \epsilon))$.

Theorem 3.2. *Let R be a $\mathbb{K}[q^{\pm 1}]$-algebra for a field $\mathbb{K}$ of characteristic 0 and $\epsilon \in \mathbb{K}^\times$. Assume that $R_\epsilon := R/(q - \epsilon)R$ is a free module of finite rank over a Poisson subalgebra C_ϵ of its center.*

(i) Then $d(R_\epsilon/C_\epsilon)$ is a Poisson normal element of $(C_\epsilon, \{.,.\})$.

(ii) Assume, in addition, that C_ϵ is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain. Then, $d(R_\epsilon/C_\epsilon) = 0$ or

$$d(R_\epsilon/C_\epsilon) =_{C_\epsilon^\times} \prod_{i=1}^m p_i$$

for some (not necessarily distinct) Poisson prime elements $p_1, \ldots, p_m \in C_\epsilon$.

As usual, a product of 0 primes is considered to be 1. Let $(A, \{.,.\})$ be a Poisson algebra and $u \in A^\times$. Then $a \in A$ is Poisson normal if and only if ua is Poisson normal. The discriminant $d(R_\epsilon/C_\epsilon)$ is defined up to a unit of C_ϵ, but because of this property it does not matter which representative is considered in part (i) of the theorem.

If R_ϵ is an order in a central simple algebra, then the discriminant $d(R_\epsilon/C_\epsilon)$ is nonzero. Specializations of iterated skewpolynomial extensions fall in this class by [13, Theorem 1.5]. In particular, this is true for the families of algebras considered in the next two sections. The nonvanishing of the discriminants of those algebras also follows from the fact that these algebras have filtrations whose associated graded algebras are quasipolynomial algebras, see (5.20); by [7, Proposition 4.10] the leading terms of the discriminants are nonzero. Generally, nonvanishing of discriminants for Cayley–Hamilton algebras follows from the description of the kernel of the trace form in [15, Proposition 3.4 (2)].

By [28, Example 5.12], there are examples of Poisson structures on polynomial algebras that are not Poisson UFDs. In the opposite direction, it is easy to construct Poisson UFDs that are not UFDs as commutative algebras. In other words the two classes of algebras in Theorem 3.2 (ii) are not properly contained in each other.

3.2. The next result is an explicit version of the statement in Theorem 3.2 (i).

Proposition 3.3. *In the setting of Theorem 3.2 (i), let $\mathcal{Y} := \{y_1, \ldots, y_N\}$ be a C_ϵ-basis of R_ϵ. For all $z \in C_\epsilon$ and $x \in \sigma^{-1}(z)$, we have*

$$\{z, d_N(\mathcal{Y} : \mathrm{tr})\} = 2\,\mathrm{tr}(b(x)) d_N(\mathcal{Y} : \mathrm{tr})$$

where $b(x) := (b_{ij}) \in M_N(C_\epsilon)$ is the matrix with entries given by

$$\partial_x(y_i) := \sum_j b_{ij} y_j.$$

Proof. Set $\delta := d_N(\mathcal{Y} : \mathrm{tr})$. The proposition follows by combining Propositions 2.2 and 3.1:

$$\{z, \delta\} = \partial_x \delta = 2\,\mathrm{tr}(b(x))\delta.$$

□

Part (i) of Theorem 3.2 follows from Proposition 3.3. The second part follows from the first and Proposition 2.6.

3.3. In the situations in which the problem for computing the discriminant $d(R_\epsilon/C_\epsilon)$ was posed, C_ϵ differs only slightly from the full center $Z(R_\epsilon)$. The restriction of the Poisson structure $\{.,.\}$ to C_ϵ is very nontrivial because of the nature of the definition in (3.1). This causes the collection of Poisson primes of C_ϵ to be a small subset of the set of all prime elements of C_ϵ. Theorem 3.2 places a strong restriction on the possible form of the discriminant $d(R_\epsilon/C_\epsilon)$. One can fully determine it using the following 4 methods and sets of existing results from Poisson geometry and algebra:

(1) If the algebra R_ϵ is $\mathbb{Z}^n$-graded and C_ϵ is a homogeneous subalgebra, then one can choose a homogeneous C_ϵ-basis $\mathcal{Y}$ of R_ϵ. Since, in this case, the trace map $\mathrm{tr}\colon R_\epsilon \to C_\epsilon$ will be homogeneous, $d_N(\mathcal{Y} : \mathrm{tr})$ will be graded and

$$\deg d_N(\mathcal{Y} : \mathrm{tr}) = 2 \sum_{y \in \mathcal{Y}} \deg y.$$

Furthermore, the grading assumption implies $C_\epsilon^\times = (C_\epsilon)_0^\times$, thus the class of associates for $d(R_\epsilon/C_\epsilon)$ will consist of homogeneous elements of the same degree. The primes in Theorem 3.2 (ii) will need to be homogeneous and their degrees will satisfy

$$\sum_{i=1}^m \deg p_i = \deg d(R_\epsilon/C_\epsilon) = 2 \sum_{y \in \mathcal{Y}} \deg y.$$

(2) (A) The symplectic foliations of the Poisson manifolds coming up in the theory of quantum groups are well understood: the Belavin–Drinfeld Poisson structures [37], the varieties of Lagrangian subalgebras [18, 19], the Poisson homogeneous spaces of non-standard Poisson structures on simple Lie groups [30]. In light of Remark 2.4 (iii), these facts can be translated into results for the Poisson primes of the corresponding coordinate rings. The results will be for the case when the base field is $\mathbb{C}$, but the algebras in the theory of quantum groups are defined over $\mathbb{Q}[q^{\pm 1}]$ and by base change one can convert the results to any base field of characteristic 0.

(B) The Poisson primes of all algebras in the very large class of so called Poisson–CGL extensions are described in [25].

Combining (A) and (B), gives a description of the Poisson primes needed for Theorem 3.2 (ii) for broad classes of algebras.

(3) If C_ϵ is a domain, Theorem 3.2 (i) implies that $d_N(\mathcal{Y} : \mathrm{tr})$ gives rise to a derivation ∂_{discr} of C_ϵ such that

$$\{d_N(\mathcal{Y} : \mathrm{tr}), z\} = d_N(\mathcal{Y} : \mathrm{tr}) \partial_{discr}(z), \quad \forall z \in C_\epsilon.$$

This derivation is explicitly given by Proposition 3.3. Every Poisson prime $p \in C_\epsilon$ also gives rise to a derivation ∂_p of C_ϵ such that

$$\{p, z\} = p \partial_p(z), \quad \forall z \in C_\epsilon.$$

The primes in Theorem 3.2 need to satisfy

$$\sum_{i=1}^m \partial_{p_i} = \partial_{discr}.$$

In fact, when phrased differently, the procedure (3) can be applied to the general situation when the conditions in Theorem 3.2 (ii) are not satisfied. Proposition 3.3 determines the Poisson brackets of $d_N(\mathcal{Y} : \mathrm{tr})$ with all hamiltonians on $\mathrm{Spec} C_\epsilon$ from which one can determine the evolution of $d_N(\mathcal{Y} : \mathrm{tr})$ under all hamiltonian flows on $\mathrm{Spec} C_\epsilon$.

(4) Filtrations of the algebra R can be used to obtain leading term results for $d(R_\epsilon/C_\epsilon)$, see [7, Proposition 4.10]. They put further restrictions on what Poisson primes can appear in the expansion in Theorem 3.2 (ii) by comparing the leading terms of the two sides. In concrete situations these filtrations are different from the gradings in (1).

In the next 2 sections we show how one can use these methods to compute explicitly the discriminants of the specializations at roots of unity of the algebras of quantum matrices and more generally those of all quantum Schubert cell algebras.

3.4. Next, we illustrate how Theorem 3.2 and the recipe (1-4) give easy derivations for previous results for discriminants that were obtained by more involved methods.

The 2 dimensional quantum Weyl algebra is the $\mathbb{K}[q^{\pm 1}]$-algebra A_q with generators x_1, x_2 and relation

$$x_1 x_2 = q x_2 x_1 + 1.$$

Let $l \in \mathbb{Z}$, $l > 1$ and ϵ be a primitive l-th root of unity in $\mathbb{K}$. Consider the specialization $A_\epsilon := A_q/(q-\epsilon)A_q$ and the canonical projection $\sigma \colon A_q \to A_\epsilon$. The center $Z(A_\epsilon)$ is isomorphic to the polynomial algebra in $z_1 := \sigma(x_1)^l$ and $z_2 := \sigma(x_2)^l$ over $\mathbb{K}$. A_ϵ is a free $Z(A_\epsilon)$-module with basis

$$\mathcal{Y} := \{\sigma(x_1)^{k_1}\sigma(x_2)^{k_2} \mid k_1, k_2 \in [0, l-1]\}.$$

Theorem 3.4. [8] *For any base field $\mathbb{K}$ of characteristic 0, $l \in \mathbb{Z}$, $l > 1$ and a primitive l-th root of unity $\epsilon \in \mathbb{K}$, the discriminant of the specialization of the quantum Weyl algebra A_ϵ over its center is given by*

$$d(A_\epsilon/Z(A_\epsilon)) =_{\mathbb{K}^\times} (z_1 z_2 - t)^{l(l-1)} \quad \textit{where} \quad t := (1-\epsilon)^{-l}.$$

Proof. The induced Poisson bracket (3.1) on $Z(A_\epsilon)$ is

$$\{z_1, z_2\} = l^2 \epsilon^{-1}(z_1 z_2 - t). \tag{3.2}$$

This follows from the expansion

$$x_1^l x_2^l - q^{l^2} x_2^l x_1^l = \sum_{i=0}^{l-1} t_i x_2^i x_1^i \quad \text{with} \quad t_0 = \prod_{m=1}^{l} (1 + q + \cdots + q^{m-1}), t_i \in \mathbb{K}[q^{\pm 1}]$$

and the fact that $\sigma(t_i x_2^i x_1^i/(q-\epsilon)) = 0$ for $0 < i < l$ since $\{z_1, z_2\} \in Z(A_\epsilon) = \mathbb{K}[z_1, z_2]$.

It is sufficient to prove the statement of the theorem for $\mathbb{K} = \mathbb{C}$ since the structure constants for the products of the elements of $\mathcal{Y}$ and z_i belong to $\mathbb{Q}(\epsilon)$. When $\mathbb{K} = \mathbb{C}$, the symplectic leaves of the Poisson structure (3.2) are

- The complement to the hyperplane $z_1 z_2 = t$ (2 dimensional leaf),
- The points on the hyperplane $z_1 z_2 = t$ (0 dimensional leaves).

By Remark 2.4 (iii), the only Poisson prime of C_ϵ is $z_1 z_2 - t$. Indeed, the zero locus of any other irreducible polynomial $f \in \mathbb{C}[z_1, z_2]$ will intersect the 2-dimensional symplectic leaf nontrivially. Hence, such an f cannot be Poisson prime. (Since the Poisson bracket is defined over $\mathbb{Q}$, it follows from [25, Theorem 3.4] that $z_1 z_2 - t$ is the only Poisson prime element of C_ϵ for any base field $\mathbb{K}$ of characteristic 0 such that $\epsilon \in \mathbb{K}$.) Since $Z(A_\epsilon)$ is a polynomial algebra and thus a UFD, Theorem 3.2 implies

$$d(A_\epsilon/Z(A_\epsilon)) =_{\mathbb{K}^\times} (z_1 z_2 - t)^m$$

for some $m \in \mathbb{N}$, We are left with computing m which we do by using the method (3) in §3.3. Since $\{z_1, (z_1 z_2 - t)\} = l^2 \epsilon^{-1} z_1 (z_1 z_2 - t)$,

$$\{z_1, d_{l^2}(\mathcal{Y} : \mathrm{tr})\} = m l^2 \epsilon^{-1} z_1 d_{l^2}(\mathcal{Y} : \mathrm{tr}). \tag{3.3}$$

For $r \in A_\epsilon = \oplus_{y \in \mathcal{Y}} \mathbb{K}[z_1, z_2] y$ and $y \in \mathcal{Y}$, denote by $\mathrm{coeff}_{z_1, y}(r)$ the coefficient of $z_1 y$ in r. One easily checks that for $y = \sigma(x_1)^{k_1} \sigma(x_2)^{k_2}$,

$$\mathrm{coeff}_{z_1,y}\big(\partial_{x_1^l}(y)\big) = k_2 l \epsilon^{-1}.$$

Propositions 3.1 and 3.3, and eq. (3.3) imply

$$m l^2 \epsilon^{-1} = \frac{\{z_1, d_{l^2}(\mathcal{Y} : \mathrm{tr})\}}{z_1 d_{l^2}(\mathcal{Y} : \mathrm{tr})} = 2 \sum_{y \in \mathcal{Y}} \mathrm{coeff}_{z_1,y}\big(\partial_{x_1^l}(y)\big) = 2 l^2 \epsilon^{-1} \sum_{k_2=0}^{l-1} k_2 = l^3 (l-1) \epsilon^{-1}.$$

So, $m = l(l-1)$. □

Remark 3.5. In [6, 7, 8] the more general problem of computing discriminants of algebras over integral domains A was considered. One can obtain extensions of Theorems 3.2 (ii), 3.4 and the results below for specializations of algebras R over $A[q^{\pm 1}]$ for an integral domain A as follows. First, apply the theorems to the algebras $R \otimes_A Q(A)$ over $Q(A)[q^{\pm 1}]$ where $Q(A)$ is the field of fractions of A; this would compute the discriminants $d_N(R_\epsilon \otimes_A Q(A), C_\epsilon \otimes_A Q(A))$. Then compute the leading term of $d(R_\epsilon / C_\epsilon)$ over A using [7, Proposition 4.10], i.e., step (4) in §3.3, and convert the formula for $d_N(R_\epsilon \otimes_A Q(A), C_\epsilon \otimes_A Q(A))$ to one for $d(R_\epsilon / C_\epsilon)$ by clearing the denominators and introducing the necessary extra factor from A in $d(R_\epsilon / C_\epsilon)$.

3.5. Next we prove two general results for the n-discriminant ideals of specializations of algebras, see Definition 2.1 (ii) and [35, p. 126] for background on this notion. These results do not assume any freeness conditions like the one in Theorem 3.2.

Theorem 3.6. *Let R be a $\mathbb{K}[q^{\pm 1}]$-algebra for an infinite field $\mathbb{K}$ and $\epsilon \in \mathbb{K}^\times$. Assume that C_ϵ is a Poisson subalgebra of the center of $R_\epsilon := R/(q-\epsilon)R$ and that R_ϵ is equipped with a trace function $\mathrm{tr} : R_\epsilon \to C_\epsilon$ which commutes with all derivations ∂ of R_ϵ such that $\partial(C_\epsilon) \subseteq C_\epsilon$.*

Then, for all positive integers n, the discriminant ideal $D_n(R_\epsilon / C_\epsilon)$ is a Poisson ideal of C_ϵ. Furthermore, it has the property that $\partial(D_n(R_\epsilon / C_\epsilon)) \subseteq D_n(R_\epsilon / C_\epsilon)$ for all derivations ∂ of R_ϵ such that $\partial(C_\epsilon) \subseteq C_\epsilon$.

The first statement in the theorem follows from the second in view of Proposition 3.1. The second statement of the theorem follows from the next proposition.

Proposition 3.7. *Assume that $\mathrm{tr}\colon S \to C \subset Z(S)$ is a trace for an algebra S over an infinite field $\mathbb{K}$ which commutes with all derivations ∂ of S such that $\partial(C) \subseteq C$. Then $\partial(D_n(S/C)) \subseteq D_n(S/C)$ for all derivations ∂ of S such that $\partial(C) \subseteq C$.*

Given a positive integer n, define

$$\langle .,. \rangle \colon S^n \times S^n \to C \quad \text{by} \ \langle \mathcal{X}, \mathcal{Y} \rangle := \det\big([\mathrm{tr}(x_i y_j)]_{i,j=1}^n\big)$$

for $\mathcal{X} := (x_1, \dots, x_n)$, $\mathcal{Y} := (y_1, \dots, y_n) \in S^n$. This is obviously a symmetric form on S^n which is C-polylinear in the sense that $\langle (x_1, \dots, c x_k, \dots, x_n), \mathcal{Y} \rangle = c \langle \mathcal{X}, \mathcal{Y} \rangle$ for all $c \in C$ and $k \in [1, n]$. For a derivation ∂ of S, define

$$\overline{\partial}(\mathcal{X}) := (\partial(x_1), \dots, \partial(x_n))$$

and

$$\partial(\mathcal{X}) := \sum_{k=1}^{n} (x_1, \ldots, \partial(x_k), \ldots, x_n).$$

Proof of Proposition 3.7. For $p(t) \in S[t]$, denote by $\mathrm{coeff}_{t^i} p(t) \in S$ the coefficient of t^i in $p(t)$. Using several times the differentiation property of ∂ and the assumption that ∂ commutes with tr, gives

$$\partial(d_n(\mathcal{X} : \mathrm{tr})) = 2\langle \mathcal{X}, \partial(\mathcal{X})\rangle = \mathrm{coeff}_t(d_n(\mathcal{X} + t\overline{\partial}(\mathcal{X}) : \mathrm{tr}))$$

for all $\mathcal{X} \in S^n$. The proposition follows from the fact that $d_n(\mathcal{X}+t\overline{\partial}(\mathcal{X}) : \mathrm{tr}) \in D_n(S/C)$, $\forall t \in \mathbb{K}$ and the assumption that $\mathbb{K}$ is infinite. □

Theorem 3.6 and Proposition 3.7 have natural bilinear analogs. Let S be an algebra with trace $\mathrm{tr} \colon S \to C$ where C is a subalgebra of $Z(S)$. Following [7, Definition 1.2 (2)], define the n-th *modified discriminant ideal* $MD_n(S/C)$ to be the ideal of C, generated by

$$\langle \mathcal{X}, \mathcal{Y}\rangle \quad \text{for all } \; \mathcal{X}, \mathcal{Y} \in S^n.$$

Thus, $D_n(S/C) \subseteq MD_n(S/C)$. If S is a free rank N module over C with a basis $\mathcal{X} \in S^N$, then

$$MD_N(S/C) = D_N(S/C) = (d_N(\mathcal{X} : \mathrm{tr}))$$

by an argument similar to the identity (2.1). We refer the reader to [7, Sect. 1] for other properties of modified discriminant ideals.

Theorem 3.8. *Assume that $\mathbb{K}$ is an infinite field and n is a positive integer.*

(i) Let $\mathrm{tr} \colon S \to C \subset Z(S)$ be a trace for a $\mathbb{K}$-algebra S which commutes with all derivations ∂ of S such that $\partial(C) \subseteq C$. Then $\partial(MD_n(S/C)) \subseteq MD_n(S/C)$ for all derivations ∂ of S such that $\partial(C) \subseteq C$.

(ii) In the setting of Theorem 3.6, for all $\epsilon \in \mathbb{K}^\times$, the modified discriminant ideal $MD_n(R_\epsilon/C_\epsilon)$ is a Poisson ideal of C_ϵ with respect to the induced Poisson structure on C_ϵ. Moreover, $\partial(MD_n(R_\epsilon/C_\epsilon)) \subseteq MD_n(R_\epsilon/C_\epsilon)$ for all derivations ∂ of R_ϵ such that $\partial(C_\epsilon) \subseteq C_\epsilon$.

Theorem 3.8 (i) is proved analogously to Proposition 3.7 using the identity

$$\partial\langle \mathcal{X}, \mathcal{Y}\rangle = \langle \mathcal{X}, \partial(\mathcal{Y})\rangle + \langle \partial(\mathcal{X}), \mathcal{Y}\rangle = \mathrm{coeff}_t\langle \mathcal{X} + t\overline{\partial}(\mathcal{X}), \mathcal{Y} + t\overline{\partial}(\mathcal{Y})\rangle$$

for all $\mathcal{X}, \mathcal{Y} \in S^n$, obtained by applying the differentiation property of ∂ and the assumption that ∂ commutes with tr. The second part of the theorem follows from the first.

3.6. We finish the section with a generalization of the results in §3.1 and §3.5 to the framework of Poisson orders introduced by Brown and Gordon.

Definition 3.9. [5] Assume that S is an affine algebra over a field $\mathbb{K}$ of characteristic 0 which is a finite module over a central subalgebra C. The algebra S is called a Poisson C-order if there is a $\mathbb{K}$-linear map $\Delta \colon C \to \mathrm{Der}_{\mathbb{K}}(S)$ such that

(1) C is stable under Δ_z for all $z \in C$ and
(2) the induced bracket $\{.,.\}$ on C given by $\{z_1, z_2\} := \Delta_{z_1}(z_2)$ turns C into a Poisson algebra.

Proposition 3.1 implies that, in the setting of the proposition, R_ϵ is a Poisson $Z(R_\epsilon)$-order. The map Δ is given as follows. Choose a $\mathbb{K}$-linear map $\omega \colon Z(R_\epsilon) \to R$ such that $\sigma\omega = \mathrm{id}_{Z(R_\epsilon)}$ and set $\Delta_z = \partial_{\omega(z)}$.

Theorem 3.10. *Let S be a $\mathbb{K}$-algebra over a field $\mathbb{K}$ (of characteristic 0) which is a Poisson C-order. Let $\operatorname{tr}\colon S \to C$ be a trace map that commutes with all derivations of S that preserve C.*

(i) If S is a free C-module (of finite rank), then $d(S/C)$ is a Poisson normal element of C. If, in addition, C is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain, then either $d(S/C) = 0$ or

$$d(S/C) =_{C^\times} \prod_{i=1}^{m} p_i$$

for some (not necessarily distinct) Poisson prime elements $p_1, \dots, p_m \in C$.

(ii) For all positive integers n, the discriminant and modified discriminant ideals $D_n(S/C)$ and $MD_n(S/C)$ are Poisson ideals of C. Furthermore, $\partial(D_n(S/C)) \subseteq D_n(S/C)$ and $\partial(MD_n(S/C)) \subseteq MD_n(S/C)$ for all derivations ∂ of S such that $\partial(C) \subseteq C$.

The theorem follows from Propositions 2.2, 2.6 and 3.7 and Theorem 3.8 (i). We leave the details to the reader.

4. Discriminants of quantum matrices

In this section we derive a formula for the discriminants of the specializations at odd roots of unity of the algebras of square quantum matrices. The results do not require any Lie theoretic background and are proved independently from the results of De Concini, Kac and Procesi [11, 12] on the specializations of quantized universal enveloping algebras at roots of unity. The case of rectangular quantum matrices can be obtained along the same lines, but is somewhat more technical. Because of this, it appears in §5.5 and is obtained by applying the Lie theoretic results from the next section.

4.1. Throughout this section, we fix a field $\mathbb{K}$ of characteristic 0 and an odd positive integer $l > 2$ such that $\mathbb{K}$ contains a primitive l-th root of unity ϵ.

Let $n \in \mathbb{Z}$, $n > 1$. The algebra of square quantum matrices $R_q[M_n]$ is the $\mathbb{K}[q^{\pm 1}]$-algebra with generators x_{ij}, $i, j \in [1, n]$ and relations

$$\begin{aligned} x_{ij}x_{kj} &= q x_{kj} x_{ij}, \quad \text{for } i < k, \\ x_{ij}x_{ir} &= q x_{ir} x_{ij}, \quad \text{for } j < r, \\ x_{ij}x_{kr} &= x_{kr} x_{ij}, \quad \text{for } i < k, j > r, \\ x_{ij}x_{kr} - x_{kr}x_{ij} &= (q - q^{-1}) x_{ir} x_{kj}, \quad \text{for } i < k, j < r. \end{aligned} \tag{4.1}$$

Denote the specialization $R_\epsilon[M_n] := R_q[M_n]/(q - \epsilon)R_q[M_n]$ and the natural projection $\sigma\colon R_q[M_n] \to R_\epsilon[M_n]$. One easily verifies that the following elements are central

$$z_{ij} := \sigma(x_{ij})^l \in Z(R_\epsilon[M_n]).$$

Denote by $C_\epsilon[M_n]$ the subalgebra of $Z(R_\epsilon[M_n])$ generated by them. It is isomorphic to a polynomial algebra:

$$C_\epsilon[M_n] \cong \mathbb{K}[z_{ij}, 1 \le i, j \le n]. \tag{4.2}$$

For $k \in [1, n]$, denote the minors

$$\Delta_k := \Delta_{[n-k+1,n];[1,k]}, \quad \overline{\Delta}_j := \Delta_{[1,j];[n-j+1,n]} \in C_\epsilon[M_n].$$

As in the introduction, $\Delta_{I;J}$ denotes the minor in z_{ij} for the rows in the set $I \subseteq [1, n]$ and the columns in $J \subseteq [1, n]$. Note that $\Delta_n = \overline{\Delta}_n$.

The algebra $R_\epsilon[M_n]$ is a free module over $C_\epsilon[M_n]$ with basis

$$\mathcal{Y} := \{\sigma(x_{11})^{k_{11}} \dots \sigma(x_{nn})^{k_{nn}} \mid 0 \le k_{11}, \dots, k_{nn} \le l-1\} \tag{4.3}$$

where the (noncommuting) $\sigma(x_{ij})$-elements are listed down the first column, second, etc., to the n-the column. The facts in (4.2) and (4.3) hold because $\{x_{11}^{k_{11}} \dots x_{nn}^{k_{nn}} \mid k_{11}, \dots, k_{nn} \in \mathbb{N}\}$ is a $\mathbb{K}[q^{\pm 1}]$-basis of $R_q[M_n]$.

Theorem 4.1. *Let $\mathbb{K}$ be a field of characteristic 0, $l > 2$ an odd integer and $\epsilon \in \mathbb{K}$ a primitive l-th root of unity. Then,*

$$d(R_\epsilon[M_n]/C_\epsilon[M_n]) =_{\mathbb{K}^\times} \prod_{k=1}^{n} \Delta_k^L \prod_{j=1}^{n-1} \overline{\Delta}_j^L$$

where $L := l^{n^2-1}(l-1)$.

The theorem is proved in §4.3 and §4.2 contains the needed results for the related Poisson structures.

4.2.

Lemma 4.2. *For an odd integer $l > 2$, $C_\epsilon[M_n]$ is a Poisson subalgebra of $Z(R_\epsilon[M_n])$ with respect to the Poisson structure* (3.1) *and the corresponding Poisson bracket is given by*

$$\{z_{ij}, z_{km}\} := l^2 \epsilon^{-1} (\operatorname{sign}(k-i) + \operatorname{sign}(m-j)) z_{im} z_{kj}. \tag{4.4}$$

The lemma can be derived from [11, Theorem 7.6]. We sketch a direct proof below that shows at what stages the condition of l being odd is needed.

Proof. For all $1 \le i < k \le n$, $1 \le j < m \le n$, we have the algebra embeddings $R_q[M_2] \hookrightarrow R_q[M_n]$ given by

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} x_{ij} & x_{im} \\ x_{kj} & x_{km} \end{bmatrix}. \tag{4.5}$$

Thus, it suffices to check the statement for $n = 2$. In that case the only identity that is not immediate is

$$\{z_{11}, z_{22}\} = 2 l^2 \epsilon^{-1} z_{12} z_{21}.$$

To obtain this, we first compute the derivation

$$\begin{aligned} \partial_{x_{11}^l}(\sigma(x_{22})) &= \sigma\left(\frac{[x_{11}^l, x_{22}]}{q-\epsilon}\right) = \sigma\left(\sum_{i=0}^{l-1} \frac{x_{11}^i [x_{11}, x_{22}] x_{11}^{l-i-1}}{q-\epsilon}\right) \\ &= \sigma\left(\frac{q^{2l}-1}{q(q-\epsilon)} x_{12} x_{21} x_{11}^{l-1}\right) = 2l\epsilon^{-2} \sigma(x_{12})\sigma(x_{21})\sigma(x_{11})^{l-1}. \end{aligned}$$

This is valid for all integers $l > 2$. Applying Proposition 3.1 gives

$$\{z_{11}, z_{22}\} = \partial_{x_{11}^l}(\sigma(x_{22})^l) = 2l\epsilon^{-2} \sum_{i=0}^{l-1} \sigma(x_{22})^i \sigma(x_{12})\sigma(x_{21})\sigma(x_{11})^{l-1}\sigma(x_{22})^{l-i-1}. \tag{4.6}$$

From the commutation relations in $R_q[M_n]$ we obtain that the right hand side can be uniquely represented in the form

$$\{z_{11}, z_{22}\} = \sum_{i=1}^{l} c_i \sigma(x_{12})^i \sigma(x_{21})^i \sigma(x_{22})^{l-i} \sigma(x_{11})^{l-i}$$

for some $c_i \in \mathbb{K}$.

Since $\{z_{11}, z_{22}\} \in Z(R_\epsilon[M_2])$, it commutes with $\sigma(x_{22})$. This gives the recursion relation

$$\epsilon^{2i-1}(\epsilon^{2(l-i+1)}-1)c_{i-1} = (\epsilon^{2i}-1)c_i$$

from which we get $c_{l-1} = c_{l-2} = \ldots = c_1 = 0$. Here we use in an essential way the condition that l is odd which gives that $\epsilon^{2i}-1 \neq 0$ for $i \in [1, l-1]$. The coefficient c_l is computed directly from (4.6):

$$c_l = 2l\epsilon^{-1}\prod_{i=1}^{l-1}(\epsilon^{2i}-1) = 2l^2\epsilon^{-1}.$$

Here again we used that l is odd, so ϵ^2 is a primitive l-th root of unity and thus $\prod_{i=1}^{l-1}(t-\epsilon^{2i}) = \sum_{i=0}^{l-1} t^i$. □

Remark 4.3. In the case of even l, $C_\epsilon[M_n]$ is not a Poisson subalgebra of $Z(R_\epsilon[M_n])$. For example, for $l = 4$, $\epsilon = \pm i$,

$$\{z_{11}, z_{22}\} = -32\epsilon\sigma(x_{22})^2\sigma(x_{12})^2\sigma(x_{21})^2\sigma(x_{11})^2.$$

The algebra $R_q[M_n]$ is $\mathbb{Z}^{2n}$-graded by setting $\deg x_{ij} := e_i + e_{n+j}$ where $\{e_1, \ldots e_{2n}\}$ is the standard basis of $\mathbb{Z}^{2n}$. This induces $\mathbb{Z}^{2n}$-grading on $R_\epsilon[M_n] = R_q[M_n]/(q-\epsilon)R_q[M_n]$. The subalgebra $C_\epsilon[M_n]$ is homogeneous ($\deg z_{ij} := l(e_i + e_{n+j})$) and the trace map $\mathrm{tr}\colon R_\epsilon[M_n] \to C_\epsilon[M_n]$ is graded. Denote the symmetric bilinear form

$$\langle .,. \rangle \colon \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to \mathbb{Z}, \quad \text{given by } \langle e_i, e_j \rangle = \delta_{ij}.$$

For $i \leq j$, denote

$$e_{[i,j]} = e_i + \cdots + e_j.$$

Proposition 4.4. [25] *The homogeneous prime elements of $C_\epsilon[M_n]$ with respect to the Poisson bracket* (4.4) *are*

$$\Delta_1, \ldots, \Delta_n, \overline{\Delta}_1, \ldots, \overline{\Delta}_{n-1}.$$

They satisfy

$$\{\Delta_k, z_{ij}\} = l^2\epsilon^{-1}(\langle e_{[n-k+1,n]}, e_i\rangle - \langle e_{[n+1,n+k]}, e_{n+j}\rangle)\Delta_k z_{ij}$$

and

$$\{\overline{\Delta}_k, z_{ij}\} = -l^2\epsilon^{-1}(\langle e_{[1,k]}, e_i\rangle - \langle e_{[2n-k+1,2n]}, e_{n+j}\rangle)\overline{\Delta}_k z_{ij}$$

for all $i, j \in [1, n]$.

In the case $\mathbb{K} = \mathbb{C}$, Proposition 4.4 has a very illuminating proof using Poisson geometry which we give below. This is sufficient to deduce the case of arbitrary base field in Theorem 4.1 because the structure constants for the products of the elements in $\mathcal{Y}$ and z_{ij} are in $\mathbb{Q}(\epsilon)$; that is, proving Theorem 4.1 over $\mathbb{C}$, will imply the statement over any base field of characteristic 0 such that $\epsilon \in \mathbb{K}$.

Proof. Let $\mathbb{C} = \mathbb{K}$. We identify $C_\epsilon[M_n] \cong \mathbb{C}[z_{ij}, 1 \leq i, j \leq n]$ with the coordinate ring of the affine space M_n of $n \times n$ complex matrices. The action of $H := (\mathbb{C}^\times)^{2n}$ on M_n, given by

$$(t_1, \ldots, t_{2n}) \cdot [z_{ij}]_{i,j=1}^n := [t_i t_{n+j} z_{ij}]_{i,j=1}^n, \tag{4.7}$$

preserves the Poisson structure (4.4). An element $f \in C_\epsilon[M_n]$ which is not a power of another element is a homogeneous Poisson prime if and only if its zero locus is an irreducible subvariety of M_n which is a union of H-orbits of symplectic leaves. By [4, Theorem 0.4 (a)-(b)] the H-orbits of symplectic leaves of M_n are of the following 3 types:

- A Zariski dense H-orbit of leaves $\mathcal{L}$.
- $2n-1$, H-orbits of leaves $\mathcal{L}_1, \dots, \mathcal{L}_{2n-1}$ of codimension 1. They are given by
$$\mathcal{L}_i = \Big\{ g \in M_n \mid \begin{bmatrix} I_n & w_n^\circ g \\ 0 & I_n \end{bmatrix} \in B_- s_i B_+ c^n \Big\}.$$
and are locally closed, irreducible. Here $B_\pm \subset \mathrm{GL}_{2n}(\mathbb{C})$ are the standard Borel subgroups of upper and lower triangular matrices, s_i are the simple reflections of S_{2n} realized as permutation matrices, $w_n^\circ \in S_n \subset S_{2n}$ is the longest element of S_n, $c = (12 \dots (2n))$ is the standard Coxeter element of S_{2n}.
- Finitely many H-orbits of leaves of codimension > 1.

By [38, Theorem 5.3] the zero loci of $\Delta_1, \dots, \Delta_n, \overline{\Delta}_{n-1}, \dots, \overline{\Delta}_1$ are precisely the Zariski closures of $\mathcal{L}_1, \dots, \mathcal{L}_{2n-1}$. Since $\mathcal{L}_i$ are irreducible and are H-orbits of symplectic leaves, the elements $\Delta_i, \overline{\Delta}_j$ are Poisson primes. The Poisson brackets in Proposition 4.4 are well known, see [22, Lemma 3.2].

Assume that $f \in C_\epsilon[M_n]$ is any other homogeneous Poisson prime. Its zero locus $\mathcal{V}(f)$ cannot intersect $\mathcal{L}$, since $\mathcal{L}$ is a single H-orbit of leaves and this would imply that $\mathcal{V}(f) \supset \overline{\mathcal{L}} = M_n$. (Here and in §5.2–5.3, $\overline{X}$ denotes the Zariski closure of a locally closed subset X.) Because the third type of H-orbits of symplectic leaves of M_n are of codimension > 1, there exists $i \in [1, 2n-1]$ such that $\mathcal{V}(f) \cap \mathcal{L}_i \neq \varnothing$. Using one more time that $\mathcal{L}_i$ is a single H-orbit of leaves implies $\mathcal{V}(f) \supseteq \overline{\mathcal{L}_i}$. Since f is prime, this is only possible if $\mathcal{V}(f) = \overline{\mathcal{L}_i}$. Thus, f is one of the Poisson primes $\Delta_i, \overline{\Delta}_j$. $\square$

4.3. The idea of the proof of Theorem 4.1 is to apply Theorem 3.2 and use the methods (1-3) in §3.3: We use the degree of $d(R_\epsilon[M_n]/C_\epsilon[M_n])$ and its Poisson brackets with the elements of $C_\epsilon[M_n]$ which are obtained from Proposition 3.3. Denote for brevity
$$\delta := d_{l^{n^2}}(\mathcal{Y} : \mathrm{tr})$$
for the $C_\epsilon[M_n]$-basis $\mathcal{Y}$ of $R_\epsilon[M_n]$ given by (4.3). $C_\epsilon[M_n]$ is a polynomial ring and thus a UFD. Theorem 3.2 and Proposition 4.4 imply
$$\delta = t \prod_{i=1}^{n} \Delta_i^{m_i} \prod_{j=1}^{n-1} \overline{\Delta}_j^{\overline{m}_j} \tag{4.8}$$
for some $t \in C_\epsilon[M_n]^\times = \mathbb{K}^\times$ and $m_i, \overline{m}_j \in \mathbb{N}$. Since $\mathrm{tr}\colon R_\epsilon[M_n] \to C_\epsilon[M_n]$ is $\mathbb{Z}^{2n}$-graded,
$$\deg \delta = 2 \sum_{k_{11}, \dots, k_{nn}=0}^{l-1} \deg x_{11}^{k_{11}} \dots x_{nn}^{k_{nn}} = n l^{n^2}(l-1) e_{[1,2n]}.$$
Taking into account that
$$\deg \Delta_k = l(e_{[n-k+1,n]} + e_{[n+1,n+k]}), \quad \deg \overline{\Delta}_k = l(e_{[1,k]} + e_{[2n-k+1,2n]})$$
and comparing the degrees of the two sides of (4.8), gives
$$m_1 + \dots + m_n = n l^{n^2-1}(l-1), \quad \overline{m}_i = m_{n-i}, \;\; i \in [1, n-1]. \tag{4.9}$$

Since $C_\epsilon[M_n] \cong \mathbb{K}[z_{ij}, 1 \le i, j \le n]$ and $R_\epsilon[M_n] = \oplus_{y \in \mathcal{Y}} C_\epsilon[M_n] y$, every element $r \in R_\epsilon[M_n]$ can be uniquely represented in the form
$$r = \sum_{y \in \mathcal{Y}} \sum_{\mu} t_{\mu,y} \mu y$$

for some $t_{\mu,y} \in \mathbb{K}$ where the second sum runs over all monomials μ of $\mathbb{K}[z_{ij}, 1 \leq i, j \leq n]$. For a monomial μ in z_{ij} and $y \in \mathcal{Y}$, denote

$$\mathrm{coeff}_{\mu,y}(r) := t_{\mu,y}.$$

Lemma 4.5. *For all $y = \sigma(x_{11})^{k_{11}} \dots \sigma(x_{nn})^{k_{nn}} \in \mathcal{Y}$,*

$$\mathrm{coeff}_{z_{ij},y}\big(\partial_{x_{ij}^l}(y)\big) = \Big(\sum_{a=1}^{n} \mathrm{sign}(a-j)k_{ia} + \sum_{a=1}^{n} \mathrm{sign}(a-i)k_{aj}\Big) l\epsilon^{-1}. \tag{4.10}$$

In particular,

$$\sum_{y \in \mathcal{Y}} \mathrm{coeff}_{z_{ij},y}\big(\partial_{x_{ij}^l}(y)\big) = (n-i-j+1)(l-1)l^{n^2+1}\epsilon^{-1}.$$

Proof. For $\mathbf{k} = (k_{11}, \dots, k_{nn}) \in \mathbb{N}^{n^2}$, denote

$$x^{\mathbf{k}} := x_{11}^{k_{11}} \dots x_{nn}^{k_{nn}} \tag{4.11}$$

where the product is ordered down columns 1, 2, etc., of the entries of an $n \times n$ matrix. These elements form a $\mathbb{K}[q^{\pm 1}]$-basis of $R_q[M_n]$. Eq. (4.10) in the lemma follows from the definition of the derivations ∂_x in Proposition 3.1 and the following commutation relations which are obtained from the defining relations for $R_q[M_n]$:

$$x_{ij}^l x^{\mathbf{k}} = q^{-\sum_{a=1}^{j-1} k_{ia} l - \sum_{a=1}^{i-1} k_{aj} l} x^{\mathbf{k}'} + \text{ lower order terms},$$
$$x^{\mathbf{k}} x_{ij}^l = q^{-\sum_{a=j+1}^{n} k_{ia} l - \sum_{a=i+1}^{n} k_{aj} l} x^{\mathbf{k}'} + \text{ lower order terms}$$

where $\mathbf{k}' = (k'_{11}, \dots, k'_{nn})$ with $k'_{ab} := k_{ab} + l\delta_{ai}\delta_{bj}$. The elements (4.11) are ordered using the (linear) lexicographic order on $\mathbb{Z}^{n^2}$ from right to left, meaning that $(k_1, \dots, k_{n^2}) \prec (m_1, \dots, m_{n^2})$ if $k_{n^2} = m_{n^2}, \dots, k_{i+1} = m_{i+1}$ and $k_i < m_i$. The second equation in the lemma follows directly from the first by summation. $\square$

To determine the integers $m_i, \overline{m}_j$, we use the Poisson brackets of δ with the generators of $C_\epsilon[M_n]$, keeping in mind Theorem 3.2 (i). Proposition 4.4 and and eq. (4.8) imply that $\{z_{ij}, \delta\}/(z_{ij}\delta) \in \mathbb{K}$. Combining this with Proposition 2.2, gives

$$2\sum_{y \in \mathcal{Y}} \mathrm{coeff}_{z_{ij},y}\big(\partial_{x_{ij}^l}(y)\big) z_{ij}\delta = \{z_{ij}, \delta\} = M_{ij} z_{ij}\delta$$

where

$$\frac{M_{ij}}{l^2\epsilon^{-1}} = -\sum_{k=n-i+1}^{n} m_k + \sum_{k=i}^{n-1} \overline{m}_k + \sum_{k=j}^{n} m_k - \sum_{k=n-j+1}^{n-1} \overline{m}_k.$$

Since $\overline{m}_i = m_{n-i}$, M_{ij} simplifies to

$$\frac{M_{ij}}{l^2\epsilon^{-1}} = \sum_{k=1}^{n-i} m_k - \sum_{k=n-i+1}^{n} m_k - \sum_{k=1}^{j-1} m_k + \sum_{k=j}^{n} m_k.$$

Thus, for $j \in [2, n]$,

$$\begin{aligned} 2m_j l^2 \epsilon^{-1} = M_{i,j-1} - M_{ij} &= 2\sum_{y \in \mathcal{Y}} \mathrm{coeff}_{z_{i,j-1},y}\big(\partial_{x_{i,j-1}^l} y\big) - 2\sum_{y \in \mathcal{Y}} \mathrm{coeff}_{z_{ij},y}\big(\partial_{x_{ij}^l} y\big) \\ &= 2(l-1)l^{n^2+1}\epsilon^{-1} \end{aligned}$$

where at the end we used the second part of Lemma 4.5. Finally, $m_1 = (l-1)l^{n^2-1}$ from (4.9). This proves Theorem 4.1.

5. Discriminants of quantum Schubert cell algebras

In this section we prove an explicit formula for the discriminants of the specializations at roots of unity of the quantum Schubert cell algebras for all simple Lie algebras $\mathfrak{g}$ and Weyl group elements $w \in W$.

For two subgroups B_1 and B_2 of a group G, we will denote by $g \cdot B_2$ the elements of G/B_2, by $B_1 g \cdot B_2$ the B_1-orbit of $g \cdot B_2 \in G/B_2$, and by $B_1 g B_2$ the corresponding double coset in G.

5.1. Let $[c_{ij}] \in M_r(\mathbb{Z})$ be a Cartan matrix of finite type and $\mathcal{U}_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra defined over $\mathbb{K}(q)$ where $\mathbb{K}$ is a field of characteristic 0. We will follow the notation of [27] except for denoting the Chevalley generators of $\mathcal{U}_q(\mathfrak{g})$ by $E_i, F_i, K_i^{\pm 1}$, $i \in [1, r]$ (in [27] they were indexed by the simple roots of $\mathfrak{g}$).

Let W be the Weyl group of $[c_{ij}]$, $\{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots, and $\{s_1, \ldots, s_r\} \subset W$ the corresponding set of simple reflections. Denote by $\langle ., . \rangle$ the W-invariant bilinear form on $\bigoplus_j \mathbb{Q}\alpha_i$ normalized by $\|\alpha_i\|^2 = 2$ for short roots α_i. Set $q_i := q^{\|\alpha_i\|/2}$.

Given a Weyl group element w and a reduced expression

$$w = s_{i_1} \ldots s_{i_N},$$

consider the roots $\beta_k = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$, $k \in [1, N]$. They are precisely the roots of the nilpotent Lie algebra $\mathfrak{n}_+ \cap w(\mathfrak{n}_-)$ where $\mathfrak{n}_\pm$ are the nilradicals of a pair of opposite Borel subalgebras of $\mathfrak{g}$. The quantum Schubert cell algebras $\mathcal{U}^-[w]$ are the $\mathbb{K}[q^{\pm 1}]$-subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by the quantum root vectors

$$F_{\beta_j} := T_{i_1} \ldots T_{i_{j-1}}(F_{i_j}), \quad j \in [1, N] \tag{5.1}$$

where T_i refers to the action [27, 31] of the braid group of W on $\mathcal{U}_q(\mathfrak{g})$. The algebras $\mathcal{U}^-[w]$ do not depend on the choice of a reduced expression of w, [31, 13]. Their generators satisfy the Levendorskii–Soibelman straightening relations: for $1 \le j < m \le N$,

$$F_{\beta_m} F_{\beta_j} - q^{-\langle \beta_m, \beta_j \rangle} F_{\beta_j} F_{\beta_m} = \sum_{k_{j+1}, \ldots, k_{m-1} \in \mathbb{N}} t_{k_{j+1}, \ldots, k_{m-1}} F_{\beta_{j+1}}^{k_{j+1}} \ldots F_{\beta_{m-1}}^{k_{m-1}} \tag{5.2}$$

for some $t_{k_{j+1}, \ldots, k_{m-1}} \in \mathbb{Q}[q^{\pm 1}]$. As a consequence, $\mathcal{U}^-[w]$ has the PBW basis

$$\{ F_{\beta_1}^{k_1} \ldots F_{\beta_N}^{k_N} \mid k_1, \ldots, k_N \in \mathbb{N} \}. \tag{5.3}$$

Remark 5.1. The algebras $\mathcal{U}^-[w]$ can be defined as the algebras with generators $F_{\beta_1}, \ldots, F_{\beta_N}$ and relations (5.2). In particular, they are defined over $\mathbb{Q}[q^{\pm 1}]$ and their specializations at a root of unity ϵ are defined over $\mathbb{Q}(\epsilon)$. All formulas for discriminants proved for one field of characteristic 0 are valid for any other field of characteristic 0 by a direct base change.

Let $\epsilon \in \mathbb{K}$ be a primitive l-th root of unity. Denote the specialization $\mathcal{U}_\epsilon^-[w] := \mathcal{U}^-[w]/(q - \epsilon)\mathcal{U}^-[w]$ and the canonical projection $\sigma \colon \mathcal{U}^-[w] \to \mathcal{U}_\epsilon^-[w]$. Set

$$z_{\beta_j} := (\epsilon^{\|\alpha_{i_j}\|/2} - \epsilon^{-\|\alpha_{i_j}\|/2})^l \sigma(F_{\beta_j})^l \in \mathcal{U}_\epsilon^-[w], \quad j \in [1, N]. \tag{5.4}$$

Denote by $C_\epsilon^-[w]$ the $\mathbb{K}$-subalgebra of $\mathcal{U}_\epsilon^-[w]$ generated by z_{β_j}, $j \in [1, N]$.

Theorem 5.2. [11] *For all integers $l > 1$, $C_\epsilon^-[w]$ is a subalgebra of $Z(\mathcal{U}_\epsilon^-[w])$. It is isomorphic to the polynomial algebra in the generators z_{β_j}, $j \in [1, N]$ and is independent of the choice of reduced expression of w.*

The last part was stated in [11, Proposition 3.3] for the longest element of W; the proof works for all $w \in W$. The algebra $\mathcal{U}_\epsilon^-[w]$ is a free $C_\epsilon^-[w]$-module with basis

$$\mathcal{Y} := \{\sigma(F_{\beta_1})^{k_1} \dots \sigma(F_{\beta_N})^{k_N} \mid k_1, \dots, k_N \in [0, l-1]\}. \tag{5.5}$$

This and the second part of Theorem 5.2 follow from the PBW basis (5.3).

Denote by G the split, connected, simply connected algebraic $\mathbb{K}$-group with Lie algebra $\mathfrak{g}$. Let $B_\pm$ be a pair of opposite Borel subgroups of G and $U_\pm$ be their unipotent radicals. Let $\{e_i, f_i\}$ be a set of Chevalley generators of $\mathfrak{g}$ that generate $\mathrm{Lie}\,(U_\pm)$. Denote by $\dot{s}_i$ the representatives of s_i in the normalizer of the maximal torus $H := B_+ \cap B_-$ of G given by

$$\dot{s}_i := \exp(f_i) \exp(-e_i) \exp(f_i).$$

They are extended (in a unique way) to Tits' representatives of the elements $u \in W$ in $N_G(H)$ by setting $\dot{v} := \dot{u}\dot{s}_i$ if $v = us_i$ and $\ell(v) = \ell(u) + 1$ where $\ell \colon W \to \mathbb{N}$ is the length function. For a positive root β of $\mathfrak{g}$, denote the root vectors

$$e_\beta = \mathrm{Ad}_{\dot{u}}(e_{\alpha_i}) \quad \text{and} \quad f_\beta := \mathrm{Ad}_{\dot{u}}(f_{\alpha_i}) \tag{5.6}$$

where $u \in W$ is any element satisfying $\beta = u(\alpha_i)$ (it is well known that this does not depend on the choice of $u \in W$ and α_i).

Consider the Schubert cell $B_+ w \cdot B_+$ in the full flag variety G/B_+ and the isomorphisms

$$\mathrm{Spec} C_\epsilon^-[w] \cong \mathbb{K}[U_+ \cap w(U_-)] \cong \mathbb{K}[B_+ w \cdot B_+]. \tag{5.7}$$

The first one is given by

$$f \in \mathbb{K}[U_+ \cap w(U_-)] \quad \mapsto \quad f\big(\exp(z_{\beta_1} e_{\beta_1}) \dots \exp(z_{\beta_N} e_{\beta_N})\big) \in C_\epsilon^-[w]$$

and the second is the pull-back map for the algebraic isomorphism $B_+ w \cdot B_+ \cong U_+ \cap w(U_-)$, $g \in U_+ \cap w(U_-) \mapsto gw \cdot B_+$. (The first isomorphism is the presentation of $U_+ \cap w(U_-)$ as the product of the one-parameter unipotent subgroups of G corresponding to the roots $\beta_1, \dots, \beta_N$ whose coordinate rings are identified with $\mathbb{K}[z_{\beta_j}]$.)

Denote by $\mathcal{P}^+$ the set of dominant integral weights of $\mathfrak{g}$ and by $\{\varpi_1, \dots, \varpi_r\}$ the set of fundamental weights. Let

$$\rho = \varpi_1 + \dots + \varpi_r.$$

For $\lambda \in \mathcal{P}^+$ and $u, v \in W$, one defines the generalized minors

$$\Delta_{u\lambda, v\lambda} \in \mathbb{K}[G]$$

as follows. Consider the irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ with highest weight λ. Let b_λ be a highest weight vector of $L(\lambda)$ and ξ_λ be a vector in the dual weight space, normalized by $\langle \xi_\lambda, b_\lambda \rangle = 1$. Set

$$\Delta_{u\lambda, v\lambda}(g) := \langle \xi_\lambda, \dot{u}^{-1} g \dot{v} b_\lambda \rangle, \quad g \in G.$$

Finally, recall that the support of a Weyl group element w is defined by

$$\mathcal{S}(w) := \{i \in [1, r] \mid s_i \text{ occurs in one and thus in any reduced expression of } w\}.$$

Theorem 5.3. *Let $\mathfrak{g}$ be a simple Lie algebra, w a Weyl group element and $l > 2$ an odd integer which is $\neq 3$ in the case of G_2. Assume that $\mathbb{K}$ is a field of characteristic 0 which contains a primitive l-th root of unity ϵ. Then*

$$d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w]) =_{\mathbb{K}^\times} \Delta_{\rho, w\rho}^L =_{\mathbb{K}^\times} \prod_{i \in \mathcal{S}(w)} \Delta_{\varpi_i, w\varpi_i}^L$$

in the first isomorphism in (5.7) *where $L := l^{N-1}(l-1)$.*

More explicitly, under the first isomorphism in (5.7), the minor $\Delta_{\lambda,w\lambda}$ corresponds to

$$\langle \xi_\lambda, \exp(z_{\beta_1} e_{\beta_1}) \dots \exp(z_{\beta_N} e_{\beta_N}) \dot{w} b_\lambda \rangle. \tag{5.8}$$

The equality between the second and third term in Theorem 5.3 follows from the product property

$$\Delta_{u\lambda,v\lambda}\Delta_{u\mu,v\lambda} = \Delta_{u(\lambda+\mu),v(\lambda+\mu)}, \quad u, v \in W, \lambda, \mu \in \mathcal{P}^+$$

and the fact that $\Delta_{\varpi_i,w\varpi_i}|_{U_+\cap w(U_-)} = 1$ for $i \notin \mathcal{S}(w)$.

The algebras $\mathcal{U}_\epsilon^-[w]$ and $C_\epsilon^-[w]$ are defined over $\mathbb{Q}(\epsilon)$ and the structure constants for the $C_\epsilon^-[w]$-action on the basis $\mathcal{Y}$ belong to $\mathbb{Q}(\epsilon)$ because of (5.2). This implies that it is sufficient to prove Theorem 5.3 for any extension $\mathbb{K}$ of $\mathbb{Q}(\epsilon)$. We will do this for $\mathbb{K} = \mathbb{C}$.

From now on we assume that $\mathbb{K} = \mathbb{C}$ to avoid technicalities with Poisson manifolds over general fields of characteristic 0. (All arguments work for general fields of characteristic 0.)

Theorem 5.3 is proved in §5.4. Subsections 5.2 and 5.3 establish results on the Poisson geometry of flag varieties and their relation to the induced Poisson structures on $C_\epsilon^-[w]$. Subsection 5.5 applies Theorem 5.3 to obtain a formula for the discriminants of the specializations at roots of unity of all algebras of rectangular quantum matrices.

5.2. For a G-action on a manifold M, denote by $\chi\colon \mathfrak{g} \to \Gamma(M, TM)$ the corresponding infinitesimal action and its extension to $\wedge^\bullet \mathfrak{g} \to \Gamma(M, \wedge^\bullet TM)$.

Let Δ_+ denote the set of positive roots of $\mathfrak{g}$. Recall the definition of the root vectors (5.6) of $\mathfrak{g}$. The standard r-matrix for $\mathfrak{g}$ is the element

$$r := \sum_{\beta\in\Delta_+} \frac{\|\beta\|^2}{2} e_\beta \wedge f_\beta \in \wedge^2 \mathfrak{g}. \tag{5.9}$$

Define the Poisson bivector field

$$\pi := -\chi(r) \in \Gamma(G/B_+, \wedge^2 T(G/B_+)),$$

called the standard Poisson structure of the flag variety G/B_+. Denote the open Richardson varieties

$$R_{v,w} := B_- v \cdot B_+ \cap B_+ w \cdot B_+ \subset G/B_+, \quad v \leq w \in W,$$

see [3, 16, 36]. We will make repeated use of the following facts:

(A) The H-orbits of symplectic leaves of $(G/B_+, \pi)$ are $R_{v,w}$.

(B) $\overline{R_{v,w}} \cap B_+ w \cdot B_+ = \bigsqcup_{u\in W, u\leq v} R_{u,w}$,

see [19, Theorem 4.14] and [36, Theorem 3.2].

Theorem 5.4. *The composition of the two isomorphisms in* (5.7) *is an isomorphism of Poisson algebras*

$$(C_\epsilon^-[w], \{.,.\}) \to (\mathbb{C}[B_+ w \cdot B_+], l^2 \epsilon^{-1}\{.,.\}_\pi).$$

For the proof of Theorem 5.4 we will need several constructions for Poisson algebraic groups and Poisson homogeneous spaces, see [9, Ch. 1] for background. The standard Poisson structure on G is defined by

$$\pi_{\rm st} := \chi_R(r) - \chi_L(r) \in \Gamma(G, \wedge^2 TG),$$

in terms of (5.9). Here χ_R and χ_L denote the infinitesimal actions for the actions of G on itself on the right and the left. The groups $B_\pm$ are Poisson algebraic subgroups of

$(G, \pi_{\rm st})$. The r-matrix for the Drinfeld double of the Poisson algebraic group $(G, \pi_{\rm st})$ is

$$r_D := \sum_{\beta \in \Delta_+} \frac{\|\beta\|^2}{2} \left((e_\beta, e_\beta) \wedge (f_\beta, 0) - (f_\beta, f_\beta) \wedge (0, e_\beta) \right) \\ + \frac{1}{2} \sum_i (h_i, h_i) \wedge (h_i, -h_i) \in (\mathfrak{g} \oplus \mathfrak{g})^{\otimes 2}$$

where $\{h_i\}$ is an orthonormal basis of $\mathrm{Lie}\,(H)$ with respect to the bilinear form $\langle .,. \rangle$, extending the one in §5.1. The double of $(G, \pi_{\rm st})$ is the group $G \times G$ equipped with the Poisson structure

$$\pi_D := \chi_R(r_D) - \chi_L(r_D).$$

The group

$$G^* := \{ (u_- h^{-1}, u_+ h) \mid u_\pm \in U_\pm, h \in H \} \subset G \times G$$

is a Poisson submanifold of $(G \times G, \pi_D)$; the pair $(G^*, -\pi_D)$ is the dual Poisson algebraic group of $(G, \pi_{\rm st})$. The projection onto the first component $\eta \colon (G \times G, \pi_D) \to (G, \pi_{\rm st})$, $\eta(g_1, g_2) = g_1$ is Poisson. It restricts to the Poisson quotient map

$$\eta \colon (G^*, -\pi_D) \to (B_-, -\pi_{\rm st}). \tag{5.10}$$

Denote by τ and θ the unique antiautomorphism and automorphism of G which on the Lie algebra level are given by

$$\tau(e_i) = e_i, \tau(f_i) = f_i, \tau(\alpha_i^\vee) = -\alpha_i^\vee \quad \text{and} \quad \theta(e_i) = f_i, \theta(f_i) = e_i, \theta(\alpha_i^\vee) = -\alpha_i^\vee$$

for the Chevalley generators of $\mathfrak{g}$. It follows from the definition of $\pi_{\rm st}$ that $\theta\tau \colon (G, \pi_{\rm st}) \to (G, -\pi_{\rm st})$ is a Poisson map. This gives rise to the Poisson isomorphism

$$\theta\tau \colon (B_-, -\pi_{\rm st}) \xrightarrow{\cong} (B_+, \pi_{\rm st}). \tag{5.11}$$

The commutation relations

$$\tau \mathrm{Ad}_{\dot{s}_i} = \mathrm{Ad}_{\dot{s}_i^{-1}} \tau \quad \text{and} \quad \theta \mathrm{Ad}_{\dot{s}_i} = \mathrm{Ad}_{\dot{s}_i^{-1}} \theta$$

and the involutivity of τ and θ imply

$$\theta\tau(f_\beta) = e_\beta \quad \text{for} \quad \beta \in \Delta_+. \tag{5.12}$$

Recall that an action of a Poisson algebraic group (G, π) on a Poisson manifold (M, Π) is Poisson if the map

$$(G, \pi) \times (M, \Pi) \to (M, \Pi)$$

is Poisson. Such a pair (M, Π) is a Poisson homogeneous space of (G, π) if M is a homogeneous G-space. If Π vanishes at a point x, then (M, Π) is a Poisson quotient of (G, π) via the map

$$(G, \pi) \to (M, \Pi), \quad g \mapsto g \cdot x. \tag{5.13}$$

The *nonrestricted rational form* of $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}[q^{\pm 1}]$-subalgebra, generated by E_i, F_i, $K_i^{\pm 1}$ and $(K_i - K_i^{-1})/(q_i - q_i^{-1})$. It will be denoted by $\mathcal{U}_q^{\rm nrf}(\mathfrak{g})$. Consider the specialization $\mathcal{U}_\epsilon^{\rm nrf}(\mathfrak{g}) := \mathcal{U}_q^{\rm nrf}(\mathfrak{g})/(q-\epsilon)\mathcal{U}_q^{\rm nrf}(\mathfrak{g})$ and the canonical projection $\nu \colon \mathcal{U}_q^{\rm nrf}(\mathfrak{g}) \to \mathcal{U}_\epsilon^{\rm nrf}(\mathfrak{g})$. De Concini, Kac and Procesi [11, 12] proved that $\nu(E_i)^l, \nu(F_i)^l, \nu(K_i)^{\pm l} \in Z(\mathcal{U}_\epsilon^{\rm nrf}(\mathfrak{g}))$ and that, for good integers l, the subalgebra $C_\epsilon^{\rm nrf}(\mathfrak{g})$, generated by them, is a Poisson subalgebra of $Z(\mathcal{U}_\epsilon^{\rm nrf}(\mathfrak{g}))$ that contains all elements $\nu(F_{\beta_j})^l$.

Extend the reduced expression $w = s_{i_1} \dots s_{i_N}$ to a reduced expression $w_\circ = s_{i_1} \dots s_{i_M}$ of the longest element of W (here $M := \dim \mathfrak{n}_-$). Extend the set of root vectors $F_{\beta_1}, \dots, F_{\beta_N}$ to a set of root vectors $F_{\beta_1}, \dots, F_{\beta_N}, \dots, F_{\beta_M}$ by (5.1) applied for $j \in [1, M]$.

The algebra $\mathcal{U}^-[w_\circ]$ is the $\mathbb{C}[q^{\pm 1}]$-subalgebra of $\mathcal{U}_q^{\mathrm{nrf}}(\mathfrak{g})$ generated by all negative Chevalley generators $F_1, \ldots, F_r$.

By the definition of the induced Poisson structure in §3.1, the embeddings of $\mathbb{C}[q^{\pm 1}]$-algebras $\mathcal{U}^-[w] \hookrightarrow \mathcal{U}^-[w_\circ] \hookrightarrow \mathcal{U}_q^{\mathrm{nrf}}(\mathfrak{g})$ give rise to the canonical embeddings of Poisson algebras

$$(C_\epsilon^-[w], \{.,.\}) \hookrightarrow (C_\epsilon^-[w_\circ], \{.,.\}) \hookrightarrow (C_\epsilon^{\mathrm{nrf}}(\mathfrak{g}), \{.,.\}) \tag{5.14}$$

where all three Poisson structures are the ones from (3.1). The first embedding is given by sending $z_{\beta_j} \in C_\epsilon^-[w]$ to $z_{\beta_j} \in C_\epsilon^-[w_\circ]$ for $j \in [1, N]$, recall (5.4). The second one is given by $\sigma(F_{\beta_j})^l \mapsto \nu(F_{\beta_j})^l$.

Proof of Theorem 5.4. De Concini, Kac and Procesi [11, Theorem 7.6] constructed an explicit isomorphism of Poisson algebras

$$I_{DKP} \colon (C_\epsilon^{\mathrm{nrf}}(\mathfrak{g}), \{.,.\}) \xrightarrow{\cong} (\mathbb{C}[G^*], -l^2\epsilon^{-1}\{.,.\}_{\pi_D}).$$

It restricts to the Poisson isomorphism

$$I_{DKP} \colon (C_\epsilon^-[w_0], \{.,.\}) \xrightarrow{\cong} (\mathbb{C}[F\backslash G^*], -l^2\epsilon^{-1}\{.,.\}_{\pi_D})$$

where $F := \{(h^{-1}, hu_+) \mid u_+ \in U_+, h \in H\}$ and $\mathbb{C}[F\backslash G^*]$ is viewed as a Poisson subalgebra of $\mathbb{C}[G^*]$. The second isomorphism is explicitly given by $f(z_{\beta_j}) = z_j$, $j \in [1, M]$ where $z_1, \ldots z_M$ are the coordinate functions on $F\backslash G^*$ from the parametrization

$$F\backslash G^* = \{F \cdot \exp(z_M f_{\beta_M}) \ldots \exp(z_1 f_{\beta_1}) \mid z_1, \ldots, z_M \in \mathbb{C}\}.$$

The explicit statement of this result is given in [12, Eq. (4.4.1)]. The factor $-l^2\epsilon^{-1}$ comes from the normalization made in [11, §7.3] and [12, p. 420] for the induced Poisson bracket on $Z(\mathcal{U}_\epsilon^{\mathrm{nrf}}(\mathfrak{g}))$. The extra factor of 2 in [11, 12] comes from the fact that the Poisson structure π_D differs by a factor of 2 from that in [11, 12]. Our choice of π_D and π_{st} matches the Poisson structures in §4.2. Composing I_{DKP} with the Poisson maps η^* and $\tau^*\theta^*$ (see (5.10) and (5.11)) gives the Poisson isomorphism

$$\tau^*\theta^*\eta^* I_{DKP} \colon (C_\epsilon^-[w_0], \{.,.\}) \xrightarrow{\cong} (\mathbb{C}[B_+/H], l^2\epsilon^{-1}\{.,.\}_{\pi_{\mathrm{st}}}) \tag{5.15}$$

where $\mathbb{C}[B_+/H]$ is viewed as a Poisson subalgebra of $(\mathbb{C}[B_+], l^2\epsilon^{-1}\{.,.\}_{\pi_{\mathrm{st}}})$. The definition of I_{DKP} and the property (5.12) of $\tau\theta$ imply that the explicit form of the isomorphism (5.15) is $\tau^*\theta^*\eta^* I_{DKP}(z_{\beta_j}) = \widetilde{z}_j$, $j \in [1, M]$ where $\widetilde{z}_j$ are the coordinate functions on $\mathbb{C}[B_+/H]$ from the parametrization

$$B_+/H = \{\exp(\widetilde{z}_1 e_{\beta_1}) \ldots \exp(\widetilde{z}_M e_{\beta_M}) \cdot H \mid z_1, \ldots, z_M \in \mathbb{C}\}.$$

The flag variety $(G/B_+, \pi)$ is a Poisson homogeneous space for (G, π_{st}). Thus, it is a Poisson (B_+, π_{st})-space. The property (A) in §5.2 implies that the Schubert cell $(B_+ w \cdot B_+, \pi)$ is a Poisson homogeneous space for Poisson algebraic group (B_+, π_{st}). By a direct calculation one checks that π vanishes at the base point $w \cdot B_+$. Thus, the fact (5.13) implies that the quotient map

$$(B_+, \pi_{\mathrm{st}}) \to (B_+ w \cdot B_+, \pi), \quad b_+ \mapsto b_+ w \cdot B_+$$

is Poisson. In the $\widetilde{z}_j$ coordinates the map is given by $\exp(\widetilde{z}_1 e_{\beta_1}) \ldots \exp(\widetilde{z}_M e_{\beta_M}) \mapsto \exp(\widetilde{z}_1 e_{\beta_1}) \ldots \exp(\widetilde{z}_N e_{\beta_N}) w \cdot B_+$. The pull-back map is an embedding of Poisson algebras

$$(\mathbb{C}[B_+ w \cdot B_+], \{.,.\}_\pi) \hookrightarrow (\mathbb{C}[B_+/H], \{.,.\}_{\pi_{\mathrm{st}}}).$$

The theorem follows by combining this embedding, the isomorphism (5.15) and the first embedding in (5.14). $\square$

5.3. Denote by $\mathcal{Q}$ the root lattice of $\mathfrak{g}$. The algebras $\mathfrak{g}$, $\mathcal{U}_q(\mathfrak{g})$, $\mathcal{U}_\epsilon^-[w]$ and $C_\epsilon^-[w]$ are $\mathcal{Q}$-graded and the projection $\sigma\colon \mathcal{U}^-[w] \to \mathcal{U}_\epsilon^-[w]$ is graded. The graded components of these algebras of degree $\gamma \in \mathcal{Q}$ will be denoted by $(.)_\gamma$.

Proposition 5.5. *The homogeneous prime elements of* $(C_\epsilon^-[w], \{.,.\})$ *are* $\Delta_{\varpi_i, w\varpi_i}$ *for* $i \in \mathcal{S}(w)$, *in terms of the first identification in* (5.7). *They satisfy*

$$\{\Delta_{\varpi_i, w\varpi_i}, z\} = -l\epsilon^{-1}\langle (w+1)\varpi_i, \gamma\rangle \Delta_{\varpi_i, w\varpi_i} z, \quad \forall z \in (C_\epsilon^-[w])_\gamma.$$

Proof. For $i \in \mathcal{S}(w)$, the vanishing ideal of $\overline{R_{s_i,w}} \cap B_+ w \cdot B_+$ in $\mathbb{C}[B_+ w \cdot B_+]$ is $(\Delta_{\varpi_i, w\varpi_i})$, [38, Theorem 4.7]. Each of these sets is irreducible and is a union of H-orbits of symplectic leaves. This follows from the properties (A)-(B) in §5.2 and the well known fact that the open Richardson varieties $R_{v,w}$ are irreducible. Remark 2.4 (iii) implies that $\Delta_{\varpi_i, w\varpi_i} \in C_\epsilon^-[w]$ are homogeneous Poisson prime elements.

Assume that $f \in C_\epsilon^-[w]$ is another homogeneous Poisson prime element. By Remark 2.4 (iii), the zero locus $\mathcal{V}(f)$ of f should be a union of H-orbits of symplectic leaves of $(B_+ w \cdot B_+, \pi)$. Since

$$B_+ w \cdot B_+ = \bigsqcup_{v \in W, v \leq w} R_{v,w}$$

and $\dim R_{v,w} = \dim(B_+ w \cdot B_+) - \ell(v)$, either $\mathcal{V}(f) \cap R_{1,w} \neq \varnothing$ or $\mathcal{V}(f) \cap R_{s_i,w} \neq \varnothing$ for some $i \in \mathcal{S}(w)$. The first case is impossible since by (A)-(B), $R_{1,w}$ is a single H-orbit of symplectic leaves and $\overline{R_{1,w}} \supset B_+ w \cdot B_+$. In the second case, $\mathcal{V}(f) \supseteq \overline{R_{s_i,w}} \cap B_+ w \cdot B_+$ because $R_{s_i,w}$ is a single H-orbit of leaves. Since f is prime, $f =_{\mathbb{C}^\times} \Delta_{\varpi_i, w\varpi_i}$.

The formulas for Poisson brackets in the proposition are the specializations at $q = 1$ of eq. (5.1) in [39] for $y_1 = 1$. □

5.4. We proceed with the proof of Theorem 5.3. Recall the $C_\epsilon^-[w]$-basis $\mathcal{Y}$ of $\mathcal{U}_\epsilon^-[w]$ from (5.5). By Theorem 3.2 (ii) and Proposition 5.5,

$$d_{l^N}(\mathcal{Y} : \mathrm{tr}) =_{\mathbb{C}^\times} \Delta_{\lambda, w\lambda} \tag{5.16}$$

for some $\lambda \in \mathcal{P}^+$. ($C_\epsilon^-[w]$ is a polynomial algebra and thus a UFD.) We determine λ by using the methods (1) and (3) in §3.3: We compare the degrees of the two sides of the equality (in the $\mathcal{Q}$-grading) and their Poisson brackets with the elements of $C_\epsilon^-[w]$. (Since $\Delta_{\varpi_i, w\varpi_i}|_{U_+ \cap w(U_-)} = 1$ for $i \notin \mathcal{S}(w)$, λ is only defined up to adding an element of $\oplus_{i \notin \mathcal{S}(w)} \mathbb{Z}\varpi_i$.) Firstly,

$$\deg \Delta_{\lambda, w\lambda} = l(w-1)\lambda.$$

This follows for instance from (5.8) by using that $\deg z_{\beta_j} = -l\beta_j$. For the reduced expression $w = s_{i_1} \ldots s_{i_N}$, denote

$$w_{\leq j} := s_1 \ldots s_{i_j}.$$

In this notation

$$-\beta_j = -w_{\leq j-1}(\alpha_{i_j}) = w_{\leq j}\rho - w_{\leq j-1}\rho. \tag{5.17}$$

Since the map $\mathrm{tr}\colon \mathcal{U}_\epsilon^-[w] \to C_\epsilon^-[w]$ is graded,

$$\begin{aligned}\deg d_{l^N}(\mathcal{Y} : \mathrm{tr}) &= 2\sum_{y \in \mathcal{Y}} \deg y = 2 \sum_{k_1, \ldots, k_N = 0}^{l-1} \deg \sigma(F_{\beta_1}^{k_1} \ldots F_{\beta_N}^{k_N}) \\ &= 2 \sum_{k_1, \ldots, k_N = 0}^{l-1} k_1(w_{\leq 1}\rho - \rho) + \cdots + k_N(w_{\leq N}\rho - w_{\leq N-1}\rho) = (l-1)l^N\rho.\end{aligned}$$

Hence, by comparing degrees in (5.16),

$$(w-1)(\lambda-(l-1)l^{N-1}\rho)=0. \tag{5.18}$$

Proposition 5.5 and the fact that $\deg z_{\beta_j}=-l\beta_j$ imply

$$\{\Delta_{\lambda,w\lambda}, z_{\beta_j}\}=l^2\epsilon^{-1}\langle(w+1)\lambda,\beta_j\rangle\Delta_{\lambda,w\lambda}z_{\beta_j}, \quad j\in[1,N]. \tag{5.19}$$

To evaluate $\{d_{l^N}(\mathcal{Y}:\mathrm{tr}), z_{\beta_j}\}$, we use Proposition 3.3. Since $\mathcal{Y}$ is a $C_\epsilon^-[w]$-basis of $\mathcal{U}_\epsilon^-[w]$ and $C_\epsilon^-[w]\cong\mathbb{C}[z_{\beta_1},\ldots,z_{\beta_N}]$,

$$\mathcal{U}_\epsilon^-[w]=\oplus_{y\in\mathcal{Y}}\mathbb{C}[z_{\beta_1},\ldots,z_{\beta_N}]y.$$

For a monomial μ in $z_{\beta_1},\ldots,z_{\beta_N}$, a basis element $y\in\mathcal{Y}$ and $r\in\mathcal{U}_\epsilon^-[w]$, denote by $\mathrm{coeff}_{\mu,y}(r)$ the coefficient of μy in r. For $\mathbf{k}=(k_1,\ldots,k_N)\in\mathbb{N}^N$, denote the PBW basis element

$$F^{\mathbf{k}}:=F_1^{k_1}\ldots F_N^{k_N}\in\mathcal{U}^-[w].$$

Lemma 5.6. *For all $\mathbf{k}\in[1,l-1]^{\times N}$ and $j\in[1,N]$,*

$$(\epsilon^{\|\alpha_{i_j}\|/2}-\epsilon^{-\|\alpha_{i_j}\|/2})^l\,\mathrm{coeff}_{z_{\beta_j},\sigma(F^{\mathbf{k}})}\big(\partial_{F^l_{\beta_j}}(\sigma(F^{\mathbf{k}}))\big)=\Big(\sum_{m=1}^N \mathrm{sign}(m-j)k_m\langle\beta_m,\beta_j\rangle\Big)l\epsilon^{-1}.$$

Proof. Consider the right-to-left lexicographic order $\prec$ on $\mathbb{N}^N$ given by

$$(k_1,\ldots,k_N)\prec(m_1,\ldots,m_N)\quad\text{if } k_N=m_N,\ldots,k_{j+1}=m_{j+1} \text{ and } k_j<m_j \text{ for some } j.$$

Recursively applying the straightening law (5.2) gives

$$F^{\mathbf{k}}F^{\mathbf{m}}=q^{-\sum_{j>a}k_jm_a\langle\beta_j,\beta_a\rangle}F^{\mathbf{k}+\mathbf{m}}+\sum_{\mathbf{k}'\prec\mathbf{k}+\mathbf{m}}F^{\mathbf{k}'}. \tag{5.20}$$

Thus,

$$F^l_{\beta_j}F^{\mathbf{k}}-F^{\mathbf{k}}F^l_{\beta_j}=\big(q^{-l\sum_{a<j}k_j\langle\beta_j,\beta_a\rangle}-q^{-l\sum_{a>j}k_j\langle\beta_j,\beta_a\rangle}\big)F^{\mathbf{k}+le_j}+\sum_{\mathbf{k}'\prec\mathbf{k}+le_j}F^{\mathbf{k}'}$$

where $\{e_1,\ldots,e_N\}$ denotes the standard basis of $\mathbb{Z}^N$. The lemma follows from this by dividing by $q-\epsilon$ and applying σ. ☐

It follows from (5.16) and (5.19) that

$$\frac{\{d_{l^N}(\mathcal{Y}:\mathrm{tr}),z_{\beta_j}\}}{d_{l^N}(\mathcal{Y}:\mathrm{tr})z_{\beta_j}}\in\mathbb{C}.$$

Now, from Proposition 3.3 we have

$$\begin{aligned}\frac{\{d_{l^N}(\mathcal{Y}:\mathrm{tr}),z_{\beta_j}\}}{d_{l^N}(\mathcal{Y}:\mathrm{tr})z_{\beta_j}}&=-2\sum_{k_1,\ldots,k_N=0}^{l-1}(\epsilon^{\|\alpha_{i_j}\|/2}-\epsilon^{-\|\alpha_{i_j}\|/2})^l\,\mathrm{coeff}_{z_{\beta_j},\sigma(F^{\mathbf{k}})}\big(\partial_{F^l_{\beta_j}}(\sigma(F^{\mathbf{k}}))\big)\\&=-2\sum_{k_1,\ldots,k_N=0}^{l-1}\Big(\sum_{m=1}^N\mathrm{sign}(m-j)k_m\langle\beta_m,\beta_j\rangle\Big)l\epsilon^{-1}\\&=(l-1)l^{N+1}\langle(w+1)\rho,\beta_j\rangle\epsilon^{-1}.\end{aligned}$$

In the last equality we used the identity $-\langle(w+1)\rho,\beta_j\rangle=\sum_{m=1}^N\mathrm{sign}(m-j)\langle\beta_m,\beta_j\rangle$ which follows from (5.17). Comparing this with (5.19), leads to

$$\langle(w+1)(\lambda-(l-1)l^{N-1}\rho),\beta_j\rangle=0\quad\text{for } j\in[1,N].$$

The definition of β_j implies $\beta_j - \alpha_{i_j} \in \oplus_{m<j}\mathbb{Z}\alpha_{i_m}$, and thus,

$$\bigoplus_{j=1}^{N} \mathbb{Z}\beta_j = \bigoplus_{i\in\mathcal{S}(w)} \mathbb{Z}\alpha_i.$$

Therefore,

$$\langle (w+1)(\lambda - (l-1)l^{N-1}\rho), \alpha_i\rangle = 0 \quad \text{for } \ i \notin \mathcal{S}(w).$$

This and the degree formula (5.18) give

$$\langle \lambda - (l-1)l^{N-1}\rho, \alpha_i\rangle = 0 \quad \text{for } \ i \notin \mathcal{S}(w),$$

that is

$$\lambda - (l-1)l^{N-1}\rho \in \bigoplus_{i\notin\mathcal{S}(w)} \mathbb{Z}\varpi_i.$$

Theorem 5.3 now follows from the fact that $\Delta_{\varpi_i, w\varpi_i}|_{U_+\cap w(U_-)} = 1$ for $i \notin \mathcal{S}(w)$.

5.5. Let $1 \le m \le n \in \mathbb{Z}$. The algebra of (rectangular) quantum matrices $R_q[M_{m,n}]$ is the $\mathbb{K}[q^{\pm 1}]$-algebra with generators x_{ij}, $i \in [1,m]$, $j \in [1,n]$ and the four relations in (4.1) for $i,k \in [1,n]$, $j,r \in [1,m]$. The algebra is isomorphic to $\mathcal{U}^-[c^m]$ for $\mathfrak{g} = \mathfrak{sl}_{m+n}$ and the Coxeter element $c = (12\dots(m+n)) \in S_{m+n}$, see [32, Proposition 2.1.1] and [39, Lemma 4.1]. The isomorphism $\zeta\colon \mathcal{U}^-[c^m] \xrightarrow{\cong} R_q[M_{m,n}]$ has the form

$$\zeta(F_{\beta_k}) = (-q)^{i+j-2}x_{ij} \tag{5.21}$$

for appropriate indices $i \in [1,m]$, $j \in [1,n]$ depending on $k \in [1,mn]$. Because of this, the isomorphism in [32, 39], stated over $\mathbb{K}(q)$, is valid over $\mathbb{K}[q^{\pm 1}]$.

Let $l > 2$ be an odd integer and $\epsilon \in \mathbb{K}$ be an l-th primitive root of unity. Denote the specialization $R_\epsilon[M_{m,n}] := R_q[M_{m,n}]/(q-\epsilon)R_q[M_{m,n}]$ and the canonical projection $\sigma\colon R_q[M_{m,n}] \to R_\epsilon[M_{m,n}]$. Analogously to §4.1, $z_{ij} := \sigma(x_{ij})^l \in Z(R_\epsilon[M_{m,n}])$ and the subalgebra $C_\epsilon[M_{m,n}]$ of $Z(R_\epsilon[M_{m,n}])$ generated by them is a polynomial algebra in z_{ij}. The algebra $R_\epsilon[M_{m,n}]$ is a free module over $C_\epsilon[M_{m,n}]$ with basis

$$\{\sigma(x_{11})^{k_{11}} \dots \sigma(x_{mn})^{k_{mn}} \mid 0 \le k_{11}, \dots, k_{mn} \le l-1\}$$

where the elements $\sigma(x_{ij})$ are listed in any (fixed) order. It follows from (5.21) that ζ induces in isomorphism $\overline{\zeta}\colon C_\epsilon^-[c^m] \xrightarrow{\cong} C_\epsilon[M_{m,n}]$ satisfying

$$\overline{\zeta}(z_{\beta_k}) = (\epsilon - \epsilon^{-1})^l(-\epsilon)^{i+j-2}z_{ij}$$

where the relation between i,j and k is the same as in (5.21). Any minor in these elements is a scalar multiple of the corresponding minor in z_{ij}.

Applying Theorem 5.3 for $G = \mathrm{SL}_{m+n}$ and $w = c^m$, and computing explicitly the restriction of the minors $\Delta_{\varpi_i, c^m\varpi_i}$ to $U_+ \cap c^m(U_-)$, identified with the affine $\mathbb{K}$-space $M_{m,n}$, leads to the following:

Theorem 5.7. *Assume that $\mathbb{K}$ is a field of characteristic 0, $l > 2$ is an odd integer, $\epsilon \in \mathbb{K}$ is a primitive l-th root of unity and $m \le n$ are positive integers. The discriminants of the algebras of rectangular quantum matrices are given by*

$$d(R_\epsilon[M_{m,n}]/C_\epsilon[M_{m,n}]) =_{\mathbb{K}^\times} \prod_{j=1}^{m-1} \big(\Delta^L_{[m-j+1,m];[1,j]}\Delta^L_{[1,j];[n-j+1,n]}\big) \prod_{k=1}^{n-m+1} \Delta^L_{[1,m];[k,m+k-1]}$$

where $L := l^{mn-1}(l-1)$ and $\Delta_{I;J}$ denotes the minor in z_{ij} for the rows in the set I and the columns in J as in the square case.

5.6. The key properties of a discriminant of an algebra which are used in [1, 6, 7] are those of being locally dominating or dominating. Let A be a finitely generated algebra. Fix a generating subspace V of A such that $V \cap \mathbb{K} = \{0\}$. Denote by $F_j A := (V \oplus \mathbb{K})^j$ the spaces of the induced $\mathbb{N}$-filtration on A. Assume that $\operatorname{gr} A$ is connected and locally finite. For $a \in A \backslash \{0\}$, denote by $\deg a$ the minimal integer j such that $a \in F_j A$. Fix a $\mathbb{K}$-basis $a_1, \ldots, a_n$ of V (so, $\deg a_i = 1$).

Definition 5.8. An element $f = f(a_1, \ldots, a_n) \in A$ is called locally dominating [6] if for every $h \in \operatorname{Aut}(A)$ the following conditions hold:
(i) $\deg f(h(a_1), \ldots, h(a_n)) \geq \deg f$ and
(ii) if $\deg h(a_i) > 1$ for some i, then $\deg f(h(a_1), \ldots, h(a_n)) > \deg f$.

A stronger property of *dominating element* was defined in [6].

Definition 5.9. We will say that $f \in A$ has a unique leading term if it has the form

$$f = t a_1^{k_1} \ldots a_n^{k_n} + \ (cwlt), \quad t \in \mathbb{K}^\times, k_1, \ldots, k_n \in \mathbb{Z}_+$$

where the tail $(cwlt)$ is a linear combination of monomials $a_1^{m_1} \ldots a_n^{m_n}$ whose n-tuple of powers $(m_1, \ldots, m_n)$ is component wise less than $(k_1, \ldots, k_n)$, i.e., $m_i \leq k_i$ for all i and for at least one i the inequality is strict.

It was proved in [6, Lemma 2.2] that every element that has a unique leading term is dominating and thus locally dominating.

The discriminants computed in [1, 6, 7] posses the stronger property – those discriminants have a unique leading term. The determinants of the matrices $[z_{ij}]_{i,j=1}^n$, considered as elements of $\mathbb{K}[z_{ij}, 1 \leq i, j \leq n]$, do not have a unique leading term unless they are of size 1×1. It follows from Theorem 5.3 that the discriminants of the quantum Schubert cell algebras $d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w]) \in C_\epsilon^-[w]$ do not have a unique leading term unless $\mathcal{U}^-[w]$ is isomorphic to a quantum affine space algebra. One can deduce from [24, Theorem 9.5] that this is precisely the case when w is a subword of a Coxeter element of W, (i.e., when every simple reflection of W appears at most one time in a reduced expression of w). Recently, this subclass of algebras $\mathcal{U}^-[w]$ appeared in a different context (inner faithful Hopf algebra actions) in [17].

The next proposition shows that the algebras $\mathcal{U}_\epsilon^-[w]$ exhibit a new phenomenon: they can have locally dominating discriminants without possessing the stronger property of having a unique leading term.

Proposition 5.10. *For all fields $\mathbb{K}$ of characteristic 0, odd integers integers l, and l-th roots of unity $\epsilon \in \mathbb{K}$, the discriminants $d(R_\epsilon[M_2]/C_\epsilon[M_2])$ and $d(R_\epsilon[M_{2,3}]/C_\epsilon[M_{2,3}])$ are locally dominating.*

Proof. Fix an integer $n \geq 2$. Denote the generators of $R_\epsilon[M_{2,n}]$ by $a_{ij} := \sigma(x_{ij})$. They satisfy the relations (4.1) with q replaced by ϵ. Consider the $\mathbb{N}$-grading of $R_\epsilon[M_{2,n}]$ defined by assigning a_{ij} degree one. The grading is connected and locally finite. The grading components will be denoted by $R_\epsilon[M_{2,n}]_k$ and we set $R_\epsilon[M_{2,n}]_{\geq k} := \oplus_{k \geq 2} R_\epsilon[M_{2,n}]_k$. The central elements z_{ij} in Theorem 5.7 are $z_{ij} = a_{ij}^l$ and

$$(5.22) \quad d(R_\epsilon[M_{2,n}]/C_\epsilon[M_{2,n}]) =_{\mathbb{K}^\times} \big(a_{21}(a_{11}a_{22} - \epsilon a_{12}a_{21}) \ldots (a_{1,n-1}a_{2,n} - \epsilon a_{1,n}a_{2,n-1})a_{1,n}\big)^{l^{2n}(l-1)}.$$

One can show that an identity of this type holds for all $\mathcal{U}^-[w]$; we give an elementary proof for the case of $R_\epsilon[M_{2,n}]$. Denote $\Delta := a_{11}a_{22} - \epsilon a_{12}a_{21} \in Z(R_\epsilon[M_2])$. The q-binomial formula and $(a_{11}^{-1}\Delta)(a_{11}^{-1}a_{12}a_{21}) = \epsilon^{-2}(a_{11}^{-1}a_{12}a_{21})(a_{11}^{-1}\Delta)$ give

$$a_{22}^l = (a_{11}^{-1}\Delta + \epsilon a_{11}^{-1}a_{12}a_{21})^l = (a_{11}^{-1}\Delta)^l + (a_{11}^{-1}a_{12}a_{21})^l = a_{11}^{-l}\big(\Delta^l + a_{12}^l a_{21}^l\big).$$

Thus, $\Delta^l = a_{11}^l a_{22}^l - a_{12}^l a_{21}^l$, and (5.22) follows from Theorem 5.7 and the embeddings (4.5).

Fix $h \in \mathrm{Aut}(R_\epsilon[M_{2,n}])$. By [29, Proposition 3.2], it follows from $a_{i1}a_{ij} = \epsilon a_{ij}a_{i1}$ ($i = 1, 2,\ j \in [2, n]$) that $h(a_{ij}) \in R_\epsilon[M_{2,n}]_{\geq 1}$. Let $h_0(a_{ij})$ be the component of $h(a_{ij})$ in $R_\epsilon[M_{2,n}]_1$. It follows at once that h_0 extends to a graded automorphism of $R_\epsilon[M_{2,n}]$ and

$$h(a_{ij}) - h_0(a_{ij}) \in R_\epsilon[M_{2,n}]_{\geq 2}.$$

So,

$$\begin{aligned} &h(a_{1,j-1}a_{2,j} - \epsilon a_{1,j}a_{2,j-1}) - h_0(a_{1,j-1}a_{2,j} - \epsilon a_{1,j}a_{2,j-1}) \in R_\epsilon[M_{2,n}]_{\geq 3}, \\ &h_0(a_{1,j-1}a_{2,j} - \epsilon a_{1,j}a_{2,j-1}) \in R_\epsilon[M_{2,n}]_2 \backslash \{0\}. \end{aligned} \tag{5.23}$$

for $j \in [1, n-1]$. As in Definition 5.8, consider the filtration $F_k R_\epsilon[M_2] := R_\epsilon[M_2]_{\leq k}$ and, for $a \in R_\epsilon[M_{2,n}]$, denote by $\deg a$ the minimal k such that $a \in R_\epsilon[M_{2,n}]_{\leq k}$. Eq. (5.23) implies that

$$\deg h(a_{1,j-1}a_{2,j} - \epsilon a_{1,j}a_{2,j-1}) \geq 2, \deg h(a_{ij}) \geq 1 \tag{5.24}$$

for $j \in [1, n-1]$ in the first inequality and $i = 1, 2,\ j \in [1, n]$ in the second. This, combined with (5.22), leads to

$$\deg h(d(R_\epsilon[M_{2,n}]/C_\epsilon[M_{2,n}])) \geq 2nl^{2n}(l-1) = \deg d(R_\epsilon[M_{2,n}]/C_\epsilon[M_{2,n}]), \tag{5.25}$$

which verifies condition (i) in Definition 5.8 for $d(R_\epsilon[M_{2,n}]/C_\epsilon[M_{2,n}])$. This part of the proof works in the same way for the discriminants of all algebras $\mathcal{U}^-[w]$ that have the property that *the $\mathcal{Q}$-grading of $\mathcal{U}^-[w]$ can be specialized to an $\mathbb{N}$-grading in such way that $\mathcal{U}^-[w]$ is generated in degree 1.*

Next we verify that $d(R_\epsilon[M_2]/C_\epsilon[M_2])$ satisfies the second condition in Definition 5.8. Assume that $h \in \mathrm{Aut}(R_\epsilon[M_2])$ and $\deg h(a_{ij}) > 1$ for at least one generator a_{ij}. If $\deg h(a_{21}) > 1$ or $\deg h(a_{12}) > 1$, then (5.24) implies that the inequality (5.25) is strict. Assume that $\deg h(a_{12}) = \deg h(a_{21}) = 1$. Then either $\deg h(a_{11}) > 1$ or $\deg h(a_{22}) > 1$, so, $\deg h(a_{11}a_{22}) > 2$ and $\deg h(a_{12}a_{21}) = 2$. Thus,

$$\deg h(a_{11}a_{22} - \epsilon a_{12}a_{21}) > 2,$$

which implies that the inequality (5.25) is strict. This proves that $d(R_\epsilon[M_2]/C_\epsilon[M_2])$ satisfies the condition (ii) in Definition 5.8.

Finally, let $h \in \mathrm{Aut}(R_\epsilon[M_{2,3}])$ and $\deg h(a_{ij}) > 1$ for at least one generator a_{ij}. If $\deg h(a_{13}) > 1$ or $\deg h(a_{21}) > 1$, then, again (5.24) implies that the inequality (5.25) is strict. Consider the case $\deg h(a_{13}) = 1$ and $\deg h(a_{21}) = 1$; so, $h(a_{13}) = h_0(a_{13})$ and $h(a_{21}) = h_0(a_{21})$. We prove by contradiction that $d(R_\epsilon[M_{2,3}]/C_\epsilon[M_{2,3}])$ satisfies the condition (ii) in Definition 5.8. Assume that this is not the case, then $\deg h(d(R_\epsilon[M_{2,3}]/C_\epsilon[M_{2,3}])) = 6l^6(l-1)$ and

$$\begin{aligned} &h(a_{11})h(a_{22}) - \epsilon h(a_{12})h(a_{21}) \in R_\epsilon[M_{2,3}]_2, \\ &h(a_{12})h(a_{23}) - \epsilon h(a_{13})h(a_{22}) = h(a_{23})h(a_{12}) - \epsilon^{-1}h(a_{13})h(a_{22}) \in R_\epsilon[M_{2,3}]_2. \end{aligned} \tag{5.26}$$

Therefore,

$$\deg h(a_{11}) + \deg h(a_{22}) = 1 + \deg h(a_{12}),$$
$$\deg h(a_{12}) + \deg h(a_{23}) = 1 + \deg h(a_{22}),$$

which imply that $\deg h(a_{11}) = \deg h(a_{23}) = 1$ and $\deg h(a_{12}) = \deg h(a_{22}) = k > 1$ (the latter holds because of the assumption that $\deg h(a_{ij}) > 1$ for at least one generator a_{ij}). Denote by b_{12} and b_{22} the (nonzero) components of $h(a_{12})$ and $h(a_{22})$ in $R_\epsilon[M_{2,3}]_k$. From (5.26) we obtain

$$h_0(a_{11})b_{22} - \epsilon h_0(a_{21})b_{12} = 0 = h_0(a_{23})b_{12} - \epsilon^{-1} h_0(a_{13})b_{22}.$$

Thus, $b_{22}b_{12}^{-1} = \epsilon h_0(a_{11})^{-1} h_0(a_{21}) = \epsilon h_0(a_{13})^{-1} h_0(a_{23})$ and $a_{11}^{-1}a_{21} = a_{13}^{-1}a_{23}$ which is a contradiction. Therefore, $d(R_\epsilon[M_{2,3}]/C_\epsilon[M_{2,3}])$ satisfies the condition (ii) in Definition 5.8. $\square$

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