A PROOF OF THE GOODEARL–LENAGAN POLYNORMALITY CONJECTURE

MILEN YAKIMOV

Abstract. The quantum nilpotent algebras $\mathcal{U}^w_-(\mathfrak{g})$, defined by De Concini–Kac–Procesi and Lusztig, are large classes of iterated skew polynomial rings with rich ring theoretic structure. In this paper, we prove in an explicit way that all torus invariant prime ideals of the algebras $\mathcal{U}^w_-(\mathfrak{g})$ are polynormal. In the special case of the algebras of quantum matrices, this construction yields explicit polynormal generating sets consisting of quantum minors for all of their torus invariant prime ideals. This gives a constructive proof of the Goodearl–Lenagan polynormality conjecture [14]. Furthermore we prove that $\mathrm{Spec}\mathcal{U}^w_-(\mathfrak{g})$ is normally separated for all simple Lie algebras $\mathfrak{g}$ and Weyl group elements w, and deduce from it that all algebras $\mathcal{U}^w_-(\mathfrak{g})$ are catenary.

1. Introduction

The algebras of quantum matrices $R_q[M_{m,n}]$ are algebras over a field $\mathbb{K}$ generated by a rectangular array of generators $\{x_{ij} \mid i = 1, \dots, m, j = 1, \dots, n\}$, such that rows and columns generate quantum affine space algebras. Generators along antidiagonals commute, while generators along diagonals satisfy the commutation relation

$$x_{ij}x_{lk} - x_{lk}x_{ij} = (q - q^{-1})x_{ik}x_{lj}, \quad i < l, j < k,$$

where $q \in \mathbb{K}^* = \mathbb{K}\backslash\{0\}$. There is a canonical action of the $\mathbb{K}$-torus $\mathbb{T}^{m+n} = (\mathbb{K}^*)^{\times(m+n)}$ on $R_q[M_{m,n}]$ by algebra automorphisms. The classical minors have natural quantum analogs, which are elements of the algebras $R_q[M_{m,n}]$, see Section 4.

The ring theoretic properties of the algebras of quantum matrices $R_q[M_{m,n}]$ have been heavily investigated since the mid 90's. In this paper we address a conjecture of Goodearl and Lenagan [14] that all $\mathbb{T}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ have polynormal generating sets consisting of quantum minors. This conjecture has been established [14] only for $\min(m, n) \leq 2$ and $m = n = 3$. Shortly after [14], Launois proved in [25] that the $\mathbb{T}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ are generated by the quantum minors contained in those ideals when the base field has characteristic 0 and q is transcendental over $\mathbb{Q}$. In [32] the author constructed explicit generating sets for these ideals under the same conditions on $\mathbb{K}$ and q. These generating sets consist of quantum minors but do not contain all quantum minors in a given ideal. In [7, 15] Casteels, Goodearl,

2010 *Mathematics Subject Classification.* Primary 16T20; Secondary 17B37, 20G42.

Key words and phrases. prime ideals, polynormal generating sets, normal separation, catenarity, iterated skew polynomial rings.

Launois, and Lenagan determined what quantum minors belong to each $\mathbb{T}^{m+n}$-invariant prime ideal of $R_q[M_{m,n}]$, and completed the line of argument of [25] for ideal generators (which results in larger generating sets than those in [32]). In [32] we also constructed explicit generating sets for torus invariant prime ideals of a much larger family of quantum nilpotent algebras. Meanwhile, there was no progress on the main part of the conjecture in [14] on polynormal generating sequences.

In this paper we give a constructive proof of the Goodearl–Lenagan polynormality conjecture [14] – we construct explicit polynormal generating sets consisting of quantum minors for all $\mathbb{T}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$. These sets are smaller than the sets of all quantum minors in a given $\mathbb{T}^{m+n}$-prime ideal. This is done for base fields of characteristic 0 and when q is transcendental over $\mathbb{Q}$. In another theorem we construct polynormal generating sets for all $\mathbb{T}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ under the weaker assumptions that $\mathbb{K}$ is arbitrary and $q \in \mathbb{K}^*$ is not a root of unity. In the second result the generating sets are bigger than those in the first one. The two results are proved by the same type of argument.

The algebras of quantum matrices are a subfamily of a much larger family of iterated skew polynomial rings defined by De Concini, Kac, and Procesi [9], and Lusztig [28]. These algebras $\mathcal{U}_-^w$ are subalgebras of the negative part of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of an arbitrary simple Lie algebra $\mathfrak{g}$ and are parametrized by elements w of the Weyl group W of $\mathfrak{g}$. They are deformations of the universal enveloping algebras $\mathcal{U}(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$, where $\mathfrak{n}_\pm$ denote the nilradicals of a pair of opposite Borel subalgebras of $\mathfrak{g}$. These algebras can be considered as quantizations of the coordinate rings of Schubert cells. The geometry of the corresponding Poisson structures on all (partial) flag varieties was investigated by Brown, Goodearl, and the author in [5, 17]. There is a canonical action of the $\mathbb{K}$-torus $\mathbb{T}^r = (\mathbb{K}^*)^{\times r}$ ($r = \operatorname{rank} \mathfrak{g}$) on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms, which preserves all subalgebras $\mathcal{U}_-^w$. The $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}_-^w$ were classified by Mèriaux and Cauchon [31], and the author [32]. Moreover, [32] described the poset structure of the $\mathbb{T}^r$-spectrum of the algebras $\mathcal{U}_-^w$ and gave an explicit description of all of their torus invariant prime ideals. By the general spectrum stratification theorem [16] of Goodearl and Letzter, $\operatorname{Spec}\mathcal{U}_-^w$ stratifies into a disjoint union of tori, which are parametrized by the $\mathbb{T}^r$-primes of $\mathcal{U}_-^w$. Bell, Casteels, and Launois [2] and the author [34, 36] computed the dimensions of these strata.

In this respect, it is interesting to understand what is the analog of the Goodearl–Lenagan polynormality conjecture for the $\mathbb{T}^r$-spectra of all algebras $\mathcal{U}_-^w$. This is the next problem addressed in the paper. The simple Lie algebras, which are not of A type, have fundamental representations that are not minuscule. Thus one cannot expect to find generating sets for $\mathbb{T}^r$-primes of $\mathcal{U}_-^w$, which consist of elements derived from the braid group orbits of highest weight vectors of fundamental $\mathcal{U}_q(\mathfrak{g})$-modules, since these sets will be too small. We prove that for an arbitrary base field $\mathbb{K}$ of characteristic 0 and $q \in \mathbb{K}^*$ which is transcendental over $\mathbb{Q}$, all $\mathbb{T}^r$-prime ideals of $\mathcal{U}_-^w$ have explicit polynormal generating sets parametrized by certain weight vectors of the Demazure modules corresponding to fundamental weights and the given Weyl group element w. In the special case when w equals the longest element w_0 of the Weyl group W and $\mathbb{K} = \mathbb{C}(q)$ implicit polynormal generating sets were obtained by Caldero [6]. In the A case our generating sets consist of elements which are generalizations of quantum minors (even for arbitrary Weyl group elements w). Furthermore, in the case of an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ which is not a root of unity, for all $\mathbb{T}^r$-prime ideals of $\mathcal{U}^w_-$ we construct polynormal generating sets, which are parametrized by certain weight vectors in possibly higher Demazure modules. Therefore all $\mathbb{T}^r$-prime ideals of the algebras $\mathcal{U}^w_-$ are polynormal for an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ not a root of unity. In fact we prove stronger equivariant polynormality with respect to an action of the weight lattice P of $\mathfrak{g}$, naturally embedded in $\mathbb{T}^r$. We refer to §3.1 for the definitions of equivariant normality and polynormality, and to Theorems 3.4 and 3.6 for the precise statements. Our proofs of these results and the results for quantum matrices rely on theorems of Joseph [21, 23] and Gorelik [18]. These facts appear in Sections 3 and 4. Brown and Goodearl constructed in [3] polynormal generating sets for all torus invariant prime ideals of the quantum algebras of functions on simple groups. The difference with our situation is that the definition of the algebras $\mathcal{U}^w_-$ is not in terms of quantum function algebras and one cannot use R-matrix type commutation relations. As a result, no relation between polynormality for the algebras of quantum matrices and the quantum function algebras on simple groups was previously observed. Only after one realizes the algebras $\mathcal{U}^w_-$ as quantum coordinate rings, this approach to polynormality becomes possible. This realization of the algebras $\mathcal{U}^w_-$ coincides up to an antiisomorphism with Joseph's algebras S^w_+ [22, §10.3.1] (see §2.4), which played an important role in his study [21, 22] of the spectra of quantum groups.

In Section 5 we prove that the spectra of all algebras $\mathcal{U}^w_-$ are normally separated for an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ not a root of unity. The special case of the algebras of quantum matrices is due to Cauchon [8]. The case when $w = w_0$ and $\mathbb{K} = \mathbb{C}(q)$ was obtained by Caldero [6]. We give two proofs of this result. The first applies directly the polynormality and the second relies on a result of Gorelik [18]. For both we use a theorem of Goodearl [11] to pass from graded normal separation of the $\mathbb{T}^r$-spectrum of $\mathcal{U}^w_-$ to normal separation of $\mathrm{Spec}\mathcal{U}^w_-$.

A celebrated theorem of Gabber establishes that the universal enveloping algebras of all solvable Lie algebras are catenary. Since the algebras $\mathcal{U}^w_-$ are quantized universal enveloping algebras of nilpotent Lie algebras, one can conjecture that all of them should be catenary. We prove this here. The special case of catenarity of the algebras of quantum matrices was previously proved by Cauchon [8]. The case when $w = w_0$ and $\mathrm{char}\,\mathbb{K} = 0$ is due to Malliavin [29], Goodearl and Lenagan [13]. We use a general result of Goodearl and Lenagan [13], motivated by Gabber's work, that establishes that catenarity follows from normal separation and certain homological conditions, and a result of Levasseur and Stafford [27] that proves the latter for skew polynomial extensions. Finally, we derive explicit formulas for the heights of all $\mathbb{T}^r$-invariant prime ideals of the algebras $\mathcal{U}^w_-$. This is done in Section 5.

The algebras $\mathcal{U}^w_-$ belong to the large class of so called (torsion free) Cauchon–Goodearl–Letzter (CGL) extensions [26, Definition 3.1]. The latter are iterated skew polynomial rings with a compatible torus action for which the Goodearl–Letzter stratification result produces a finite stratification and the Cauchon method of deleted derivations applies. It is very interesting, yet very difficult to prove or disprove whether all (torsion free) CGL extensions have normal separation and thus are catenary. One should note that general skew polynomial rings do not have this property as shown by Bell and Sigurdsson [1, Example 2.10].

We finish with a notational convention. All algebras A which we consider in this paper are noetherian over an infinite field $\mathbb{K}$. When such algebras are equipped with a rational $\mathbb{T}^r$-action by algebra automorphisms, a result of Brown and Goodearl [4, Proposition II.2.9] applies to give that all of their $\mathbb{T}^r$-primes are $\mathbb{T}^r$-invariant primes. (We refer to [4, Sect. II.2] for a discussion of rational torus actions over arbitrary fields.) Because of this fact, we will use the two terms interchangeably. The corresponding $\mathbb{T}^r$-spectrum will be denoted by $\mathbb{T}^r - \mathrm{Spec} A$.

Acknowledgements. I am grateful to Ken Goodearl and Stéphane Launois for their helpful comments and for sharing their knowledge of the existing literature. I would also like to thank Ken Goodearl for communicating his proof [12] of Proposition 5.9 to me and the referee whose valuable comments and suggestions helped me to improve the exposition.

The research of the author was supported by National Science Foundation grants DMS-0701107 and DMS-1001632.

2. Quantized nilpotent algebras

2.1. In this section we set up our notation for quantized universal enveloping algebras and quantum function algebras, and recall past result on their spectra which will be needed in the paper.

Fix an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ which is not a root of unity. Let $\mathfrak{g}$ be a simple Lie algebra of rank r with Cartan matrix (c_{ij}). Denote by $\mathcal{U}_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over the base field $\mathbb{K}$ with deformation parameter q. It is a Hopf algebra over $\mathbb{K}$ with generators

$$X_i^{\pm}, K_i^{\pm 1}, \ i = 1, \ldots, r$$

and relations

$$K_i^{-1} K_i = K_i K_i^{-1} = 1, \ K_i K_j = K_j K_i,$$
$$K_i X_j^{\pm} K_i^{-1} = q_i^{\pm c_{ij}} X_j^{\pm},$$
$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1 - c_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0, \ i \neq j.$$

Here $q_i = q^{d_i}$ and $\{d_1, \ldots, d_r\}$ is the vector of positive relatively prime integers for which the matrix $(d_i c_{ij})$ is symmetric. The comultiplication, antipode, and counit of $\mathcal{U}_q(\mathfrak{g})$ are given by:

$$\Delta(K_i) = K_i \otimes K_i, \ \Delta(X_i^+) = X_i^+ \otimes 1 + K_i \otimes X_i^+, \\ \Delta(X_i^-) = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-$$

and

$$S(K_i) = K_i^{-1}, \ S(X_i^+) = -K_i^{-1} X_i^+, \ S(X_i^-) = -X_i^- K_i, \ \epsilon(K_i) = 1, \ \epsilon(X_i^{\pm}) = 0.$$

We refer to [20, Ch. 4] for details on the form of $\mathcal{U}_q(\mathfrak{g})$ with this comultiplication.

Denote by P and P^+ the sets of integral and dominant integral weights of $\mathfrak{g}$. The sets of simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$ will be denoted by $\{\alpha_i\}_{i=1}^r$, $\{\alpha_i^\vee\}_{i=1}^r$, and $\{\omega_i\}_{i=1}^r$, respectively. Set $Q = \sum_{i=1}^r \mathbb{Z}\alpha_i$ and $Q^+ = \sum_{i=1}^r \mathbb{N}\alpha_i$. Let $\langle .,. \rangle$ be the symmetric bilinear form on $\mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \dots, \alpha_r\}$ such that $\langle \alpha_i, \alpha_j \rangle = d_i c_{ij}$. Recall the standard partial order on P:

$$\text{for } \ \mu_1, \mu_2 \in P, \ \mu_1 < \mu_2, \ \text{ if and only if } \ \mu_2 - \mu_1 \in Q^+ \backslash \{0\}. \tag{2.1}$$

Let H be the group generated by $\{K_i^{\pm 1}\}_{i=1}^r$, which consists of all group like elements of $\mathcal{U}_q(\mathfrak{g})$. The q-weight spaces of an H-module V are defined by

$$V_\lambda = \{ v \in V \mid K_i v = q^{\langle \lambda, \alpha_i \rangle} v, \ \forall i = 1, \dots, r \}, \ \lambda \in P.$$

A $\mathcal{U}_q(\mathfrak{g})$-module is called a type one module if it is the sum of its q-weight spaces. Each finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-module is completely reducible [20, Theorem 5.17], see the remark on p. 85 of [20] for the validity of this for general base fields $\mathbb{K}$ and $q \in \mathbb{K}^*$ not a root of unity. The category of (left) finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules is closed under taking tensor products and duals (where the latter are defined as left modules using the antipode of $\mathcal{U}_q(\mathfrak{g})$). The irreducible modules in this category are parametrized by P^+, see [20, Theorem 5.10]. We will denote by $V(\lambda)$ the irreducible finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-module of highest weight $\lambda \in P^+$.

Let W and $\mathcal{B}_{\mathfrak{g}}$ denote the Weyl and braid groups corresponding to $\mathfrak{g}$. Let $s_1, \dots, s_r$ be the simple reflections of W corresponding to the roots $\alpha_1, \dots, \alpha_r$, and $T_1, \dots, T_r$ be the related standard generators of $\mathcal{B}_{\mathfrak{g}}$. Recall that the canonical projection $\mathcal{B}_{\mathfrak{g}} \to W_{\mathfrak{g}}$ has a section $W_{\mathfrak{g}} \to \mathcal{B}_{\mathfrak{g}}$, $w \mapsto T_w$ such that for each reduced expression $w = s_{i_1} \dots s_{i_l}$, $T_w = T_{i_1} \dots T_{i_l}$. The Bruhat order on W will be denoted by $\leq$. For $w \in W$, we set $W^{\leq w} = \{ y \in W \mid y \leq w \}$. We will use the $\mathcal{B}_{\mathfrak{g}}$-action on $\mathcal{U}_q(\mathfrak{g})$ given by

$$T_i(X_i^+) = -X_i^- K_i, \ T_i(X_i^-) = -K_i^{-1} X_i^+, \ T_i(K_j) = K_j K_i^{-c_{ij}},$$

$$T_i(X_j^+) = \sum_{k=0}^{-c_{ij}} (-q_i)^{-k} (X_i^+)^{(-c_{ij}-k)} X_j^+ (X_i^+)^{(k)}, \ j \neq i,$$

$$T_i(X_j^-) = \sum_{k=0}^{-c_{ij}} (-q_i)^{k} (X_i^-)^{(k)} X_j^- (X_i^-)^{(-c_{ij}-k)}, \ j \neq i,$$

where $(X_i^{\pm})^{(n)} = X_i^{\pm}/[n]_{q_i}!$, see [20, §8.14] and [28, §37.1]. The braid group $\mathcal{B}_{\mathfrak{g}}$ acts on all finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules V by

$$T_i(v) = \sum_{l,m,n} (-1)^m q_i^{m-ln} (X_i^+)^{(l)} (X_i^-)^{(m)} (X_i^+)^{(n)} v, \quad v \in V_\mu, \mu \in P,$$

where the sum is over all $l, m, n \in \mathbb{N}$ such that $-l+m-n = \langle \mu, \alpha_i^\vee \rangle$, see [20, §8.6] and [28, §5.2]. These actions satisfy

$$T_w(x.v) = (T_w x).(T_w v) \;\text{ and }\; T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu} \tag{2.2}$$

for all $w \in W$, $x \in \mathcal{U}_q(\mathfrak{g})$, $v \in V(\lambda)$, $\lambda \in P^+$, $\mu \in P$, see [20, eq. 8.14(1)].

2.2. Let $\mathcal{U}_\pm$ be the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_i^\pm\}_{i=1}^r$. For a reduced decomposition

$$w = s_{i_1} \ldots s_{i_l} \tag{2.3}$$

of an element $w \in W$, define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_l = s_{i_1} \ldots s_{i_{l-1}}(\alpha_{i_l}) \tag{2.4}$$

and Lusztig's root vectors

$$X_{\beta_1}^\pm = X_{i_1}^\pm, X_{\beta_2}^\pm = T_{s_{i_1}}(X_{i_2}^\pm), \ldots, X_{\beta_l}^\pm = T_{s_{i_1}} \ldots T_{s_{i_{l-1}}}(X_{i_l}^\pm), \tag{2.5}$$

see [28, §39.3] and [20, §8.24]. In [9, Proposition 2.2] De Concini, Kac and Procesi proved that the subalgebra $\mathcal{U}_\pm^w$ of $\mathcal{U}_\pm$ generated by $X_{\beta_j}^\pm$, $j = 1, \ldots, k$ does not depend on the choice of a reduced decomposition of w and that it has the PBW type basis

$$(X_{\beta_l}^\pm)^{n_k} \ldots (X_{\beta_1}^\pm)^{n_1}, \;\; n_1, \ldots, n_k \in \mathbb{N}. \tag{2.6}$$

The fact that the space spanned by the monomials (2.6) does not depend on the choice of a reduced decomposition of w was independently proved by Lusztig [28, Proposition 40.2.1]. The elements $X_{\beta_l}^\pm$ satisfy the Levendorskii–Soibelman straightening rule, see (5.3) below. A consequence of this is that all algebras $\mathcal{U}_\pm^w$ are iterated skew polynomial rings and therefore are domains. There is a unique involutive algebra automorphism ω of $\mathcal{U}_q(\mathfrak{g})$ defined by

$$\omega(X_i^\pm) = X_i^\mp, \; \omega(K_i) = K_i^{-1}, \;\; i = 1, \ldots, r.$$

It satisfies

$$\omega(T_i(x)) = (-q_i)^{\langle \alpha_i^\vee, \gamma \rangle} T_i(\omega(x)), \;\; \forall x \in (\mathcal{U}_q(\mathfrak{g}))_\gamma, \gamma \in Q, i = 1, \ldots, r, \tag{2.7}$$

see [20, eq. 8.14(9)]. This implies that for all $w \in W$, $x \in (\mathcal{U}_q(\mathfrak{g}))_\gamma$, $\gamma \in Q$ we have $\omega(T_w(x)) = t' T_w(\omega(x))$ for some $t' \in \mathbb{K}^*$. Therefore for all $w \in W$, ω induces an algebra isomorphism between $\mathcal{U}_+^w$ and $\mathcal{U}_-^w$.

Denote the $\mathbb{K}$-torus $\mathbb{T}^r = (\mathbb{K}^*)^{\times r}$. We have the group embeddings:

$$H \hookrightarrow P \hookrightarrow \mathbb{T}^r, \tag{2.8}$$

where the first one is given by $K_i \mapsto \alpha_i$, $i = 1, \ldots, r$. The second one is given by $\omega_i \mapsto (1, \ldots, 1, q, 1, \ldots, 1)$, where q is in position i. For $\gamma \in Q$ denote the character of $\mathbb{T}^r$:

$$t \mapsto t^\gamma = \prod_{i=1}^r t_i^{\langle \gamma, \omega_i \rangle}, \quad t = (t_1, \ldots, t_r) \in \mathbb{T}^r.$$

The algebra $\mathcal{U}_q(\mathfrak{g})$ is Q-graded by

$$\deg X_i^{\pm} = \pm\alpha_i, \ \deg K_i = 0. \tag{2.9}$$

The homogeneous components of $\mathcal{U}_q(\mathfrak{g})$ with respect to this grading will be denoted by $(\mathcal{U}_q(\mathfrak{g}))_\gamma$, $\gamma \in Q$. There is a rational $\mathbb{T}^r$-action on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms given by

$$t.x = t^\gamma x, \qquad \text{for } \ x \in (\mathcal{U}_q(\mathfrak{g}))_\gamma, \gamma \in Q. \tag{2.10}$$

The conjugation action of H on $\mathcal{U}_q(\mathfrak{g})$ coincides with the pull back of the action (2.10) under the embedding (2.8). The action (2.10) restricts to the following P-action on $\mathcal{U}_q(\mathfrak{g})$:

$$\mu \cdot x = q^{\langle \mu, \gamma \rangle} x, \quad \text{for } \ x \in (\mathcal{U}_q(\mathfrak{g}))_\gamma, \gamma \in Q. \tag{2.11}$$

The actions of H, P and $\mathbb{T}^r$ on $\mathcal{U}_q(\mathfrak{g})$ preserve the subalgebras $\mathcal{U}_\pm^w$.

2.3. The quantum function algebra $R_q[G]$ is the Hopf subalgebra of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by all matrix coefficients of the finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules. We think of G as of the connected, simply connected algebraic group with Lie algebra $\mathfrak{g}$, but G is only used as a symbol since the base field $\mathbb{K}$ is arbitrary (except that it cannot be finite, because it is assumed that $q \in \mathbb{K}^*$ is not a root of unity). For $\lambda \in P^+$ the matrix coefficient of $\xi \in V(\lambda)^*$ and $v \in V(\lambda)$ will be denoted by $c^\lambda_{\xi,v}$; that is $c^\lambda_{\xi,v}(u) = \xi(u.v)$, $\forall u \in \mathcal{U}_q(\mathfrak{g})$. Let R^+ be the subalgebra of $R_q[G]$ spanned by the matrix coefficients $c^\lambda_{\xi,v}$, where $\lambda \in P^+$, $\xi \in V(\lambda)^*$ and $v \in V(\lambda)_\lambda$. There are two canonical H-actions on $R_q[G]$ by algebra automorphisms:

$$K_i.c^\lambda_{\xi,v} = q^{\langle \nu, \alpha_i \rangle} c^\lambda_{\xi,v}, \quad K_i.c^\lambda_{\xi,v} = q^{\langle \mu, \alpha_i \rangle} c^\lambda_{\xi,v}, \quad \text{for } \ \xi \in (V(\lambda)^*)_\nu, v \in V(\lambda)_\mu. \tag{2.12}$$

One has a related $P \times P$-grading of $R_q[G]$

$$c^\lambda_{\xi,v} \in (R_q[G])_{\nu,\mu}, \quad \text{for } \ \xi \in (V(\lambda)^*)_\nu, v \in V(\lambda)_\mu. \tag{2.13}$$

Fix highest weight vectors $v_\lambda \in V(\lambda)_\lambda$, $\lambda \in P^+$. For simplicity of the notation denote

$$c^\lambda_\xi = c^\lambda_{\xi, v_\lambda} \quad \text{for } \ \lambda \in P^+, \xi \in V(\lambda)^*. \tag{2.14}$$

All weight spaces $V(\lambda)_{w\lambda} = T_w(V(\lambda)_\lambda)$ are one dimensional. For $\lambda \in P^+$ and $w \in W$ define $\xi_{w,\lambda} \in (V(\lambda)^*)_{-w\lambda}$ such that $\langle \xi_{w,\lambda}, T_w v_\lambda \rangle = 1$. Let

$$e^\lambda_w = c^\lambda_{\xi_{w,\lambda}} = c^\lambda_{\xi_{w,\lambda}, v_\lambda}. \tag{2.15}$$

Then

$$e^{\lambda_1}_w e^{\lambda_2}_w = e^{\lambda_1+\lambda_2}_w, \ \ \forall \lambda_1, \lambda_2 \in P^+, \tag{2.16}$$

see [35, eq. (2.18)] for our particular normalization and [22, §9.1.10] in general. Joseph proved [22, Lemma 9.1.10] that

$$E^+_w = \{ e^\lambda_w \mid \lambda \in P^+ \}$$

is an Ore subset of R^+. The two actions (2.12) of H on $R_q[G]$ and the $P \times P$-grading (2.13) descend to the localization

$$R^w = R^+[(E^+_w)^{-1}].$$

The invariant subalgebra of R^w with respect to the H-action induced from the second action in (2.12) will be denoted by R^w_0. It was introduced by Joseph [22, §10.4.8] and called the quantum translated Bruhat cell. Its spectrum was studied by Gorelik in [18]. Given $\mu \in P$, we decompose it as $\mu = \lambda_+ - \lambda_-$ for some $\lambda_\pm \in P^+$ and define $e^\mu_w = e^{\lambda_+}_w (e^{\lambda_-}_w)^{-1}$. This does not depend on the choice of $\lambda_\pm$ because of (2.16). We have that

$$R^w_0 = \{ c^\lambda_\xi e^{-\lambda}_w \mid \lambda \in P^+, \xi \in V(\lambda)^* \} \tag{2.17}$$

(in particular we do not need to take span in the right hand side), since

$$\forall \lambda_1, \lambda_2 \in P^+, \; \xi \in V(\lambda_1)^*, \quad c^{\lambda_1}_\xi e^{-\lambda_1}_w = c^{\lambda_1+\lambda_2}_{\xi'} e^{-\lambda_1-\lambda_2}_w, \tag{2.18}$$
$$\text{where } \; \xi' = (\xi \otimes \xi_{w,\lambda_2})|_{\mathcal{U}_q(\mathfrak{g})(v_{\lambda_2} \otimes v_{\lambda_1})} \in V(\lambda_1 + \lambda_2)^*.$$

Note that

$$R^w_0 = \bigoplus_{\gamma \in Q} (R^w_0)_{\gamma, 0} \tag{2.19}$$

in terms of the induced grading from (2.13) and

$$c^\lambda_\xi e^{-\lambda}_w \in (R^w_0)_{\nu + w(\lambda), 0}, \quad \forall \xi \in (V(\lambda)^*)_\nu, \lambda \in P^+, \nu \in P. \tag{2.20}$$

For $y \in W$ define the ideals

$$Q(y)^\pm = \mathrm{Span}\{ c^\lambda_\xi \mid \lambda \in P^+, \, \xi \in V(\lambda)^*, \, \xi \perp \mathcal{U}_\pm T_y v_\lambda \} \tag{2.21}$$

of R^+ and the ideals

$$Q(y)^\pm_w = \{ c^\lambda_\xi e^{-\lambda}_w \mid \lambda \in P^+, \xi \in V(\lambda)^*, \, \xi \perp \mathcal{U}_\pm T_y v_\lambda \} \tag{2.22}$$

of R^w_0, see [22, 18] for details. In the setting of (2.18), one easily verifies that $\xi \perp \mathcal{U}_\pm T_y v_{\lambda_1}$ implies $\xi' \perp \mathcal{U}_\pm T_y (v_{\lambda_2} \otimes v_{\lambda_2})$. Because of this, there is no need to take span in the right hand side of (2.22).

Theorem 2.1. *(Gorelik)* [18, Lemmas 6.6 and 6.10, Corollary 7.1.2] *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity and a Weyl group element w we have:*

(a) The H-invariant prime ideals of the quantum translated Bruhat cell algebra R^w_0 (with respect to the first action (2.12)*) which contain the ideal $Q(w)^+_w$ are the ideals*

$$Q(y)^-_w + Q(w)^+_w$$

for $y \in W^{\leq w}$. All such ideals are completely prime.

(b) The poset of such ideals of R^w_0 ordered under inclusion, is isomorphic to $W^{\leq w}$ equipped with the Bruhat order, i.e.

$$Q(y_1)^-_w + Q(w)^+_w \subseteq Q(y_2)^-_w + Q(w)^+_w,$$

if and only if $y_1 \leq y_2$. (In particular, all such ideals are distinct.)

Although we will not need this here, we note that Gorelik also described in [18] all H-invariant prime ideals of R^w_0 (with respect to the first action (2.12)) in terms of the ideals $Q(y)^\pm_w$. Gorelik stated the above results under the assumption that $\mathbb{K}$ has characteristic 0 and q is transcendental over $\mathbb{Q}$. However, her proofs work in the more general case when $q \in \mathbb{K}^*$ is not a root of unity, without any restrictions on $\mathbb{K}$, see [32, §3.2-3.4].

2.4. The quantum R-matrix associated to $w \in W$ is defined by

$$\mathcal{R}^w = \prod_{j=k,\dots,1} \exp_{q_{i_j}} \Big((q_{i_j}^{-1} - q_{i_j}) X^+_{\beta_j} \otimes X^-_{\beta_j} \Big) \tag{2.23}$$

in terms of Lusztig's root vectors (2.5). In (2.23) the noncommuting factors are multiplied in the order $j = k, \dots, 1$, see e.g. [20, eqs. 8.30(1) and 8.30(2)]. The q-exponential function is given by

$$\exp_{q_i}(y) = \sum_{n=0}^{\infty} q_i^{-n(n-1)/2} \frac{y^n}{[n]_{q_i}!}.$$

The R-matrix $\mathcal{R}^w$ belongs to a completion of $\mathcal{U}^w_+ \otimes \mathcal{U}^w_-$ and does not depend on the choice of a reduced decomposition of w, see [28, §4.1.1].

There is a unique graded algebra antiautomorphism τ of $\mathcal{U}_q(\mathfrak{g})$ defined by

$$\tau(X_i^\pm) = X_i^\pm, \; \tau(K_i) = K_i^{-1}, \;\; i = 1, \dots, r, \tag{2.24}$$

cf. [20, Lemma 4.6(b)]. It satisfies

$$\tau(T_w x) = T^{-1}_{w^{-1}}(\tau(x)), \;\; \forall w \in W, x \in \mathcal{U}_q(\mathfrak{g}), \tag{2.25}$$

see [20, eq. 8.18(6)].

We will need the following result from [32, 35].

Theorem 2.2. [35, Theorem 2.6] *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, simple Lie algebras $\mathfrak{g}$ and $w \in W$, the map*

$$\phi_w \colon R_0^w \to \mathcal{U}^w_-, \;\; \phi_w(c^\lambda_\xi e^{-\lambda}_w) = (c^\lambda_{\xi, T_w v_\lambda} \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathcal{R}^w), \; \lambda \in P^+, \xi \in V(\lambda)^*$$

is a (well defined) surjective algebra antihomomorphism. It is H-equivariant with respect to the first action (2.12) *of H on R_0^w and the conjugation action of H on $\mathcal{U}^w_-$. The kernel of ϕ_w is $Q(w)^+_w$.*

In the definition of ϕ_w the elements of $R_q[G]$ are viewed as functionals on $\mathcal{U}_q(\mathfrak{g})$. The H-equivariance property is equivalent to saying that ϕ_w is graded, namely that $\phi_w((R_0^w)_{\gamma,0}) = (\mathcal{U}^w_-)_\gamma$, $\forall \gamma \in Q$, cf. (2.9), (2.13) and (2.19). A version of this theorem for $\mathcal{U}_q(\mathfrak{g})$ equipped with the opposite comultiplication and for different braid group action and choice of Lusztig's root vectors was established in [32, Theorem 3.7]. To prove that such a map ϕ_w is well defined and is an algebra antihomomorphism, in [32, 35] we first defined it in terms of module algebras for Hopf algebras and then proved that it takes the above form.

We will use an interpretation of the algebras $\mathcal{U}^w_-$ as quantized algebras of functions on Schubert cells using matrix coefficients of Demazure modules from [32]. For $\lambda \in P^+$, $w \in W$ consider the Demazure modules $V_w(\lambda) = \mathcal{U}_+ T_w v_\lambda = \mathcal{U}^w_+ T_w v_\lambda$, see [22, §4.4 and §6.3] for details. For $\eta \in V_w(\lambda)^*$ define

$$d^{w,\lambda}_\eta \in (\mathcal{U}_+)^*, \;\; d^{w,\lambda}_\eta(x) = \langle \eta, x T_w v_\lambda \rangle, \; x \in \mathcal{U}_+.$$

Set $U^w_+ = U_+ \cap w U_- w^{-1}$ where $U_\pm \subset G$ are the unipotent radicals of a pair of opposite Borel subgroups of the connected, simply connected algebraic group G with Lie algebra $\mathfrak{g}$. (We need those just as symbols for reference purposes to the needed quantized coordinate rings, defined over an arbitrary base field.) Denote by $R_q[U^w_+]$ the subset of $(\mathcal{U}_+)^*$ consisting of

$$d^{w,\lambda}_\eta, \quad \lambda \in P^+, \eta \in V_w(\lambda)^*.$$

It is a $\mathbb{K}$-vector space because of (2.18). In [32, §3.8] we proved that

$$d^{w,\lambda_1}_{\eta_1} d^{w,\lambda_2}_{\eta_2} = q^{\langle \lambda_1, \lambda_1 + w^{-1}(\nu_1)\rangle} d^{w,\lambda_1+\lambda_2}_\eta, \tag{2.26}$$

where

$$\eta := \eta_1 \otimes \eta_2|_{\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2})} \text{ and } \eta_i \in (V_w(\lambda_i)^*)_{\nu_i}$$

defines an algebra structure on $R_q[U^w_+]$. In particular, (2.26) is a well defined multiplication in $R_q[U^w_+]$. (The result in [32, §3.8] concerned the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ equipped with the opposite comultiplication and because of this there is a small difference in the power of q. The proofs are the same in both cases.) We have:

Theorem 2.3. [32, §3.8] *For an arbitrary base field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, a simple Lie algebra $\mathfrak{g}$, and a Weyl group element $w \in W$, we have:*

(a) The map

$$\varphi_w \colon R^w_0 \to R_q[U^w_+], \quad \varphi_w(c^\lambda_\xi e^{-\lambda}_w) = d^{w,\lambda}_{\xi|_{V_w(\lambda)}}, \ \lambda \in P^+, \xi \in V(\lambda)^* \tag{2.27}$$

is a (well defined) surjective algebra homomorphism with kernel $Q(w)^+_w$.

(b) The algebras $R_q[U^w_+]$ and $\mathcal{U}^w_-$ are antiisomorphic with an antiisomorphism given by

$$\psi_w \colon R_q[U^w_+] \to \mathcal{U}^w_-, \quad \psi_w(d^{w,\lambda}_\eta) = (d^{w,\lambda}_\eta \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathcal{R}^w), \ \lambda \in P^+, \eta \in V_w(\lambda)^*.$$

We note that $\phi_w = \psi_w \varphi_w$. In [32, §3.8] we established the analog of Theorem 2.3 for the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ equipped with the opposite comultiplication to the one considered here. Theorem 2.3 follows from Theorem 2.2 along the lines of the same argument as in [32, §3.8].

Since $Q(w)^+_w = \ker \varphi_w$ is a graded ideal of R^w_0 with respect to the Q-grading (2.19) of R^w_0, one can push forward under φ_w this grading to a Q-grading on $R_q[U^w_+]$. Comparing (2.20) and (2.27) gives that

$$d^{w,\lambda}_\eta \in (R_q[U^w_+])_{\nu + w(\lambda)} \quad \text{for } \eta \in (V_w(\lambda)^*)_\nu, \tag{2.28}$$

where the weight spaces are computed with respect to the the action of H on the dual of the Demazure module via the antipode of $\mathcal{U}_q(\mathfrak{g})$. Since $\phi_w = \psi_w \varphi_w$ and ϕ_w is an antihomomorphism of Q-graded algebras, we have that ψ_w is a graded antiisomorphism with respect to the Q-gradings (2.28) and (2.9) of $R_q[U^w_+]$ and $\mathcal{U}^w_-$, respectively.

It follows from Theorem 2.3 (a) that the algebras $\mathcal{U}^w_-$ are antiisomorphic to Joseph's algebras S^+_w, which are defined as the invariant subalgebras of the algebras $(R^+/Q(w)^+)[(E^+_w)^{-1}]$ with respect to the H-action induced from the second action in (2.12) (i.e. $S^+_w \cong R^w_0/Q(w)^+_w$). We refer the reader to [22, §10.3.1] for details. These algebras played a key role in Joseph's work [21, 22] on the spectrum of $R_q[G]$.

As it is customary in the area, here and below we denote by the same symbols the images of elements of $R_q[G]$ and R^+ in their various quotients.

3. Polynormal generating sets of the $\mathbb{T}^r$-primes of $\mathcal{U}_-^w$

3.1. In this section we construct explicit P-polynormal generating sequences for all $\mathbb{T}^r$-prime ideals of the algebras $\mathcal{U}_-^w$. In the case when the base field has characteristic 0 and q is transcendental over $\mathbb{Q}$, the polynormal generating sets are very small, see Theorem 3.4. They correspond to certain subsets of the weight vectors of the duals of the Demazure modules of $\mathcal{U}_q(\mathfrak{g})$ corresponding to fundamental weights. In the general case of an arbitrary base field $\mathbb{K}$ and $q \in \mathbb{K}^*$ not a root of unity, in Theorem 3.6 we construct P-polynormal generating sets for all $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}_-^w$. The generating sets correspond to (possibly) bigger sets derived from Demazure modules for other highest weights. The proofs of Theorems 3.4 and 3.6 are analogous.

For the convenience of the reader we recall several definitions regarding polynormality. Assume that I is an ideal of a ring R. A sequence $u_1, \ldots, u_n \in R$ is called a polynormal generating sequence if the set $\{u_1, \ldots, u_n\}$ generates I and for all $i = 1, \ldots, n$, the element u_i is normal in R modulo the ideal generated by $u_1, \ldots, u_{i-1}$. In particular u_1 should be a normal element of R. If $u_1, \ldots, u_n$ is a polynormal generating sequence of I, then I is generated both as a left and right ideal of R by the set $\{u_1, \ldots, u_n\}$.

If a group Γ acts on the ring R by algebra automorphisms, we say that an element $u \in R$ is Γ-normal if it is a Γ-eigenvector and if there exists $g \in \Gamma$ such that $ur = (g.r)u$ for all $r \in R$. We note that sometimes Γ-normality is defined requiring only the second condition, see [11]. In all cases we will be able to construct elements satisfying both conditions. We also note that requiring only the second condition will not be sufficient to extend this definition to Γ-polynormality, as we do next. We say that an element $u \in R$ is Γ-normal modulo a Γ-stable ideal I, if its image in R/I is Γ-normal.

We say that a sequence $u_1, \ldots, u_n \in R$ is a Γ-polynormal generating sequence of a Γ-stable ideal I if $\{u_1, \ldots, u_n\}$ generates I and for all $i = 1, \ldots, n$, the element u_i is a Γ-normal element of R modulo the ideal generated by $u_1, \ldots, u_{i-1}$. We note that the conditions posed on the elements $u_1, \ldots, u_{i-1}$ imply that the ideal of R generated by them is Γ-stable.

For $y, w \in W$, $y \leq w$, define

$$\begin{aligned} (3.1) \quad I_w(y) &= \phi_w(Q(y)^-_w + Q(w)^+_w) = \phi_w(Q(y)^-_w) \\ &= \{(d^{w,\lambda}_\eta \otimes \mathrm{id})(\mathcal{R}^w) \mid \lambda \in P^+, \eta \in (V_w(\lambda) \cap \mathcal{U}_- T_y v_\lambda)^\perp\} \subset \mathcal{U}_-^w. \end{aligned}$$

Theorem 2.1 of Gorelik [18] and Theorem 2.2 imply the first two parts of the following theorem. The third part of the theorem is [32, Theorem 1.1].

Theorem 3.1. *For an arbitrary base field $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, a simple Lie algebra $\mathfrak{g}$, and a Weyl group element $w \in W$, we have:*

(a) If $y \in W^{\leq w}$, then $I_w(y)$ is a $\mathbb{T}^r$-invariant completely prime ideal of $\mathcal{U}_-^w$ with respect to the action (2.10)*. All $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}_-^w$ are of this form.*

(b) The correspondence $y \in W^{\leq w} \mapsto I_w(y)$ is an isomorphism from the poset $W^{\leq w}$ equipped with the Bruhat order to the poset of $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}_-^w$ ordered under inclusion; that is $I_w(y) \subseteq I_w(y')$ for $y, y' \in W^{\leq w}$ if and only if $y \leq y'$.

(c) Assume that $\mathbb{K}$ *has characteristic* 0 *and* q *is transcendental over* $\mathbb{Q}$*. Then* $I_w(y)$ *is generated as a right ideal by*

$$\psi_w(d^{w,\omega_i}_\eta) = (d^{w,\omega_i}_\eta \otimes \mathrm{id})(\mathcal{R}^w) \quad \textit{for} \quad \eta \in (V_w(\omega_i) \cap \mathcal{U}_- T_y v_{\omega_i})^\perp, i = 1, \dots, r,$$

where $\omega_1, \dots, \omega_r$ *are the fundamental weights of* $\mathfrak{g}$*.*

For part (c) one needs the stronger assumptions on q and $\mathbb{K}$ (that $\mathbb{K}$ has characteristic 0 and q is transcendental over $\mathbb{Q}$), because it relies on Joseph's result [23, Théorème 3] which uses a specialization argument.

Note that the sets of invariant subspaces of $\mathcal{U}^w_-$ with respect to the conjugation action of H, the P-action (2.11), and the $\mathbb{T}^r$-action (2.10) coincide. Thus

$$H - \mathrm{Spec}\mathcal{U}^w_- = P - \mathrm{Spec}\mathcal{U}^w_- = \mathbb{T}^r - \mathrm{Spec}\mathcal{U}^w_-. \tag{3.2}$$

We use $\mathbb{T}^r$-invariance in Theorem 3.1 to align our treatment to the Goodearl–Letzter framework [16]. In §2.3-2.4 we used H-invariance instead, because it was more convenient to state the results within the framework of the adjoint action of the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ on itself, although an appropriate torus invariance could have been used as well.

3.2. Denote by w_0 the longest element of W and set $\mathcal{R} = \mathcal{R}^{w_0}$. For $\gamma \in Q^+$, $\gamma \neq 0$ denote $m(\gamma) = \dim(\mathcal{U}_+)_\gamma = \dim(\mathcal{U}_-)_{-\gamma}$, and fix a pair of dual bases $\{u_{\gamma,k}\}_{k=1}^{m(\gamma)}$ and $\{u_{-\gamma,k}\}_{k=1}^{m(\gamma)}$ of $(\mathcal{U}_+)_\gamma$ and $(\mathcal{U}_-)_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [20, Ch. 6]. Then

$$\mathcal{R} = 1 \otimes 1 + \sum_{\gamma \in Q^+, \gamma \neq 0} \sum_{k=1}^{m(\gamma)} u_{\gamma,k} \otimes u_{-\gamma,k}. \tag{3.3}$$

Recall the standard R-matrix commutation relations:

Lemma 3.2. *For all* $\lambda_i \in P^+$*,* $\nu_i \in P$*,* $\xi_i \in V(\lambda_i)^*_{\nu_i}$*,* $i = 1, 2$*:*

$$c^{\lambda_1}_{\xi_1} c^{\lambda_2}_{\xi_2} = q^{\langle \lambda_1, \lambda_2 \rangle - \langle \nu_1, \nu_2 \rangle} c^{\lambda_2}_{\xi_2} c^{\lambda_1}_{\xi_1} + \sum_{\gamma \in Q^+, \gamma \neq 0} \sum_{k=1}^{m(\gamma)} q^{\langle \lambda_1, \lambda_2 \rangle - \langle \nu_1 - \gamma, \nu_2 + \gamma \rangle} c^{\lambda_2}_{S^{-1}(u_{\gamma,k})\xi_2} c^{\lambda_1}_{S^{-1}(u_{-\gamma,k})\xi_1}.$$

For details we refer to [4, Theorem I.8.15]. Lemma 3.2 implies that for all $\lambda_1, \lambda_2 \in P^+$, $\nu_2 \in P$, $\xi_2 \in V(\lambda_2)^*_{\nu_2}$

$$c^{\lambda_1}_{\xi_1} e^{\lambda_2}_w = q^{\langle \lambda_1, \lambda_2 \rangle + \langle \nu_1, w(\lambda_2) \rangle} e^{\lambda_2}_w c^{\lambda_1}_{\xi_1} \mod Q(w)^+, \tag{3.4}$$

recall (2.21). Combining Lemma 3.2 and (3.4) leads to the following result.

Lemma 3.3. *For all* $\lambda_i \in P^+$*,* $\nu_i \in P$*,* $\xi_i \in V(\lambda_i)^*_{\nu_i}$*,* $i = 1, 2$

$$(c^{\lambda_1}_{\xi_1} e^{-\lambda_1}_w)(c^{\lambda_2}_{\xi_2} e^{-\lambda_2}_w) - q^{\langle w(\lambda_1) - \nu_1, w(\lambda_2) + \nu_2 \rangle} (c^{\lambda_2}_{\xi_2} e^{-\lambda_2}_w)(c^{\lambda_1}_{\xi_1} e^{-\lambda_1}_w) - \sum_{\gamma \in Q^+, \gamma \neq 0} \sum_{k=1}^{m(\gamma)} q^{\langle w(\lambda_1) - \nu_1 + \gamma, w(\lambda_2) + \nu_2 + \gamma \rangle - \langle w(\lambda_1), \gamma \rangle} (c^{\lambda_2}_{S^{-1}(u_{\gamma,k})\xi_2} e^{-\lambda_2}_w)(c^{\lambda_1}_{S^{-1}(u_{-\gamma,k})\xi_1} e^{-\lambda_1}_w)$$

belongs to $Q(w)^+_w$*.*

3.3. Fix $y \in W^{\leq w}$. For each $i = 1, \dots, r$ choose a basis Ω_i of the orthogonal complement $(V_w(\omega_i) \cap \mathcal{U}_- T_y v_{\omega_i})^\perp$ inside $V_w(\omega_i)^*$, consisting of weight vectors (with respect to the H-action). Let $\Omega_w(y) = \Omega_1 \sqcup \dots \sqcup \Omega_r$. For $\eta \in (V_w(\omega_i))^*_\nu$ denote $\iota(\eta) = i$ and $\mathrm{wt}(\eta) = \nu$. In particular, we have maps $\iota \colon \Omega_w(y) \to \{1, \dots, r\}$ and $\mathrm{wt} \colon \Omega_w(y) \to P$.

Theorem 3.4. *Assume that* $\mathbb{K}$ *has characteristic* 0 *and* q *is transcendental over* $\mathbb{Q}$. *Choose any linear ordering on* $\Omega_w(y)$ *with the property that if* $\eta_1, \eta_2 \in \Omega_w(y)$, $\iota(\eta_1) = \iota(\eta_2)$ *and* $\mathrm{wt}(\eta_1) < \mathrm{wt}(\eta_2)$, *then* $\eta_1 < \eta_2$ *(recall* (2.1)*). Denote* $\Omega_w(y) = \{\eta_1 < \eta_2 < \dots < \eta_{|\Omega_w(y)|}\}$. *Then*

$$\psi_w\Big(d^{w,\omega_{\iota(\eta_j)}}_{\eta_j}\Big) = \Big(d^{w,\omega_{\iota(\eta_j)}}_{\eta_j} \otimes \mathrm{id}\Big)(\mathcal{R}^w), \quad j = 1, \dots, |\Omega_w(y)| \tag{3.5}$$

is a P*-polynormal generating sequence for the* $\mathbb{T}^r$*-prime ideal* $I_w(y)$ *of* $\mathcal{U}^w_-$, *with respect to the action* (2.11).

In [6] Caldero obtained a nonconstructive proof of the polynormality in the case $w = w_0$ (the longest element of the Weyl group W) and $\mathbb{K} = \mathbb{C}(q)$.

Proof of Theorem 3.4. Theorem 3.1 (c) implies that the set (3.5) generates the ideal $I_w(y)$. Its elements are homogeneous with respect to the grading (2.9) and are thus P-eigenvectors with respect to the action (2.11). More precisely

$$\psi_w\Big(d^{w,\omega_{\iota(\eta_j)}}_{\eta_j}\Big) \in (\mathcal{U}^w_-)_{\mathrm{wt}(\eta_j) + w(\omega_{\iota(\eta_j)})}, \quad j = 1, \dots, |\Omega_w(y)|,$$

cf. (2.28). Denote $i_j = \iota(\eta_j)$ and $\nu_j = \mathrm{wt}(\eta_j)$. Since

$$\eta_j \in (V_w(\omega_{i_j}))^*_{\nu_j} \;\text{ and }\; \eta_j \perp (V_w(\omega_{i_j}) \cap \mathcal{U}_- T_y v_{\omega_{i_j}}),$$

there exist preimages $\xi_j \in V(\omega_{i_j})^*$ with the following properties

$$\xi_j \in (V(\omega_{i_j}))^*_{\nu_j},\; \xi_j \perp \mathcal{U}_- T_y v_{\omega_{i_j}} \;\text{ and }\; \xi_j|_{V_w(\omega_{i_j})} = \eta_j.$$

We fix a family of such preimages. Applying Theorem 2.3 and Lemma 3.3, we obtain that for all $\xi \in (V(\lambda))^*_\nu$, $\lambda \in P^+$, $\nu \in P$:

$$\begin{aligned} &\phi_w(c^\lambda_\xi e^{-\lambda}_w)\psi_w\Big(d^{w,\omega_{i_j}}_{\eta_j}\Big) - \psi_w\Big(d^{w,\omega_{i_j}}_{\eta_j}\Big)\Big((w(\omega_{i_j}) - \nu_j) \cdot \phi_w(c^\lambda_\xi e^{-\lambda}_w)\Big) \\ &= \sum_{\gamma \in Q^+, \gamma \neq 0} \sum_{k=1}^{m(\gamma)} q^{\langle w(\omega_{i_j}) - \nu_j + \gamma, w(\lambda) + \nu + \gamma\rangle - \langle w(\omega_{i_j}), \gamma\rangle} \phi_w\Big(c^{\omega_{i_j}}_{S^{-1}(u_{-\gamma,k})\xi_j} e^{-\omega_{i_j}}_w\Big) \\ &\qquad \times \phi_w(c^\lambda_{S^{-1}(u_{\gamma,k})\xi} e^{-\lambda}_w) \end{aligned}$$

in terms of the P-action (2.11) and the dual bases $\{u_{\pm\gamma,k}\}_{k=1}^{m(\gamma)}$ from §3.2. Recall from Theorem 2.2 that $\phi_w \colon R^w_0 \to \mathcal{U}^w_-$ is a graded antihomomorphism. Thus $\phi_w(c^\lambda_\xi e^{-\lambda}_w) \in (\mathcal{U}^w_-)_{\nu + w(\lambda)}$ and

$$(w(\omega_{i_j}) - \nu_j) \cdot \phi_w(c^\lambda_\xi e^{-\lambda}_w) = q^{\langle w(\omega_{i_j}) - \nu_j, w(\lambda) + \nu\rangle} \phi_w(c^\lambda_\xi e^{-\lambda}_w).$$

Since $\xi_j \in (\mathcal{U}_- T_y v_{\omega_{i_j}})^\perp$ and $(\mathcal{U}_- T_y v_{\omega_{i_j}})^\perp$ is a $\mathcal{U}_-$-submodule of $V(\omega_{i_j})^*$, we have that

$$S^{-1}(u_{-\gamma,k})\xi_j \in (\mathcal{U}_- T_y v_{\omega_{i_j}})^\perp,$$

for all $\gamma \in Q^+\backslash\{0\}$ and $k = 1, \dots, m(w)$. The order relation on $\Omega_w(y)$ implies that for all $\gamma' \in Q^+$, $\gamma' \neq 0$

$$((\mathcal{U}_- T_y v_{\omega_{i_j}})^\perp)_{\nu_j - \gamma'} \subseteq \mathrm{Span}\{\xi_n \mid n = 1, \dots, j-1, \ \iota(n) = i_j\} + V_w(\omega_{i_j})^\perp.$$

Therefore

$$S^{-1}(u_{-\gamma,k})\xi_j \in \mathrm{Span}\{\xi_n \mid n = 1, \dots, j-1, \ \iota(n) = i_j\} + V_w(\omega_{i_j})^\perp$$

and

$$\phi_w\Big(c^{\omega_{i_j}}_{S^{-1}(u_{-\gamma,k})\xi_j} e_w^{-\omega_{i_j}}\Big) \in \Big\langle \phi_w\Big(c^{\omega_{i_1}}_{\xi_1} e_w^{-\omega_{i_1}}\Big), \dots, \phi_w\Big(c^{\omega_{i_{j-1}}}_{\xi_{j-1}} e_w^{-\omega_{i_{j-1}}}\Big)\Big\rangle$$
$$= \Big\langle \psi_w\Big(d^{w,\omega_{i_1}}_{\eta_1}\Big), \dots, \psi_w\Big(d^{w,\omega_{i_{j-1}}}_{\eta_{j-1}}\Big)\Big\rangle$$

for all $\gamma \in Q^+$, $\gamma \neq 0$ and $k = 1, \dots, m(\gamma)$. Therefore

$$\psi_w\Big(d^{w,\omega_{i_j}}_{\eta_j}\Big)u - ((\nu_j - w(\omega_{i_j})) \cdot u)\psi_w\Big(d^{w,\omega_{i_j}}_{\eta_j}\Big) \in \Big\langle \psi_w\Big(d^{w,\omega_{i_1}}_{\eta_1}\Big), \dots, \psi_w\Big(d^{w,\omega_{i_{j-1}}}_{\eta_{j-1}}\Big)\Big\rangle$$

for all $u \in \mathcal{U}_-^w$ which proves the statement of the theorem. $\square$

Remark 3.5. Because of the embedding (2.8), Theorem 3.4 (and Theorem 3.6 below) are stronger results than constructing $\mathbb{T}^r$-polynormal generating sequences for the $\mathbb{T}^r$-prime ideals of $\mathcal{U}_-^w$. Recalling the definition of P-polynormality and $\mathbb{T}^r$-polynormality §3.1, one should note that the P-eigenvectors and $\mathbb{T}^r$-eigenvectors in $\mathcal{U}_-^w$ are the same, as they are simply the homogeneous elements of $\mathcal{U}_-^w$ with respect to the grading (2.9).

3.4. Next we prove that all ideals $I_w(y)$ of $\mathcal{U}_-^w$ are P-polynormal under the weaker assumption that $q \in \mathbb{K}^*$ is not a root of unity without any restrictions on the characteristic of the field $\mathbb{K}$.

Recall (3.1). Since $\mathcal{U}_-^w$ is noetherian, for each $y \in W^{\leq w}$ there exists a finite set $\Sigma_w(y) \subset P^+$ such that

$$\{\psi_w(d^{w,\lambda}_\eta) \mid \lambda \in \Sigma_w(y), \eta \in (V_w(\lambda) \cap \mathcal{U}_- T_y v_\lambda)^\perp \subset V_w(\lambda)^*\}$$

generates $I_w(y)$. For $\lambda \in \Sigma_w(y)$ let Γ_λ be a basis of the orthogonal complement $(V_w(\lambda) \cap \mathcal{U}_- T_y v_\lambda)^\perp$ inside $V_w(\lambda)^*$, which consists of weight vectors (with respect to the action of H). Denote $\Gamma_w(y) = \sqcup\{\Gamma_\lambda \mid \lambda \in \Sigma_w(y)\}$. Define the maps

$$\mathrm{hw}\colon \Gamma_w(y) \to \Sigma_w(y) \ \text{ and } \ \mathrm{wt}\colon \Gamma_w(y) \to P$$

by

$$\mathrm{hw}(\eta) = \lambda, \ \ \mathrm{wt}(\eta) = \nu, \ \text{ if } \ \eta \in (V_w(\lambda))^*_\nu,$$

where hw(.) stands for highest weight and wt(.) stands for weight.

Theorem 3.6. *Let $\mathbb{K}$ be an arbitrary base field, $q \in \mathbb{K}^*$ not a root of unity, and $y \in W$, $y \leq w$. Choose a set $\Sigma_w(y)$ as above. Consider any linear ordering on $\Gamma_w(y)$ with the property that if $\eta_1, \eta_2 \in \Gamma_w(y)$, $\mathrm{hw}(\eta_1) = \mathrm{hw}(\eta_2)$ and $\mathrm{wt}(\eta_1) < \mathrm{wt}(\eta_2)$, then $\eta_1 < \eta_2$. Let $\Gamma_w(y) = \{\eta_1 < \eta_2 < \ldots < \eta_{|\Gamma_w(y)|}\}$. Then*

$$\psi_w\Big(d^{w,\mathrm{hw}(\eta_j)}_{\eta_j}\Big) = \Big(d^{w,\mathrm{hw}(\eta_j)}_{\eta_j} \otimes \mathrm{id}\Big)(\mathcal{R}^w), \quad j = 1, \dots, |\Gamma_w(y)| \tag{3.6}$$

is a P-polynormal generating sequence for the $\mathbb{T}^r$-prime ideal $I_w(y)$ of $\mathcal{U}_-^w$. Moreover,

$$\psi_w\Big(d_{\eta_j}^{w,\mathrm{hw}(\eta_j)}\Big)u = [(\mathrm{wt}(\eta_j) - w(\mathrm{hw}(\eta_j)))\cdot u]\psi_w\Big(d_{\eta_j}^{w,\mathrm{hw}(\eta_j)}\Big) \mod \Big\langle \psi_w\Big(d_{\eta_1}^{w,\mathrm{hw}(\eta_1)}\Big), \ldots, \psi_w\Big(d_{\eta_{j-1}}^{w,\mathrm{hw}(\eta_j)}\Big)\Big\rangle \tag{3.7}$$

for all $j = 1, \ldots, |\Gamma_w(y)|$, $u \in \mathcal{U}_-^w$.

Proof. We argue analogously to the proof of Theorem 3.4. The choice of the set $\Gamma_w(y)$ implies that the elements (3.6) generate the ideal $I_w(y)$. They are homogeneous with respect to the grading (2.9)

$$\psi_w\Big(d_{\eta_j}^{w,\mathrm{hw}(\eta_j)}\Big) \in (\mathcal{U}_-^w)_{\mathrm{wt}(\eta_j)+w(\mathrm{hw}(\eta_j))}, \quad j = 1, \ldots, |\Gamma_w(y)|,$$

see (2.28), and are thus P-eigenvectors with respect to the action (2.11).

From the definition of the elements η_j it follows that there exist preimages $\xi_j \in V(\mathrm{hw}(\eta_j))^*$ such that

$$\xi_j \in (V(\mathrm{hw}(\eta_j)))^*_{\mathrm{wt}(\eta_j)},\ \xi_j \perp \mathcal{U}_- T_y v_{\mathrm{hw}(\eta_j)} \ \text{ and } \ \xi_j|_{V_w(\mathrm{hw}(\eta_j))} = \eta_j.$$

We fix a family of such preimages. Theorem 2.3 and Lemma 3.3 imply that for all for all $\xi \in (V(\lambda))^*_\nu$, $\lambda \in P^+$, $\nu \in P$:

$$\begin{aligned}
&\phi_w(c_\xi^\lambda e_w^{-\lambda})\psi_w\Big(d_{\eta_j}^{w,\mathrm{hw}(\eta_j)}\Big) - \psi_w\Big(d_{\eta_j}^{w,\mathrm{hw}(\eta_j)}\Big)\Big((w(\mathrm{hw}(\eta_j)) - \mathrm{wt}(\eta_j))\cdot \phi_w(c_\xi^\lambda e_w^{-\lambda})\Big)\\
&= \sum_{\gamma\in Q^+,\gamma\neq 0}\sum_{k=1}^{m(\gamma)} q^{\langle w(\mathrm{hw}(\eta_j))-\mathrm{wt}(\eta_j)+\gamma, w(\lambda)+\nu+\gamma\rangle - \langle w(\mathrm{hw}(\eta_j)),\gamma\rangle}\phi_w\Big(c^{\mathrm{hw}(\eta_j)}_{S^{-1}(u_{-\gamma,k})\xi_j} e_w^{-\mathrm{hw}(\eta_j)}\Big)\\
&\qquad\times \phi_w(c^\lambda_{S^{-1}(u_{\gamma,k})\xi} e_w^{-\lambda}),
\end{aligned}$$

where $\{u_{\pm\gamma,k}\}_{k=1}^{m(\gamma)}$ are dual bases of $(\mathcal{U}_\pm)_{\pm\gamma}$ as in §3.2. As in the proof of Theorem 3.4, the properties of the linear ordering of $\Gamma_w(y)$ imply that

$$S^{-1}(u_{-\gamma,k})\xi_j \in \mathrm{Span}\{\xi_n \mid n = 1, \ldots, j-1,\ \mathrm{hw}(\eta_n) = \mathrm{hw}(\eta_j)\} + V_w(\mathrm{hw}(\eta_j))^\perp$$

and

$$\phi_w\Big(c^{\mathrm{hw}(\eta_j)}_{S^{-1}(u_{-\gamma,k})\xi_j} e_w^{-\mathrm{hw}(\eta_j)}\Big) \in \Big\langle \psi_w\Big(d_{\eta_1}^{w,\mathrm{hw}(\eta_j)}\Big), \ldots, \psi_w\Big(d_{\eta_{j-1}}^{w,\mathrm{hw}(\eta_j)}\Big)\Big\rangle$$

for all $\gamma \in Q^+$, $\gamma \neq 0$ and $k = 1, \ldots, m(\gamma)$. This implies (3.7). □

4. The Goodearl–Lenagan conjecture

4.1. In this section we specialize the results from the previous one to obtain a constructive proof of the Goodearl–Lenagan conjecture [14] that all torus invariant prime ideals of the algebras of quantum matrices have polynormal generating sets consisting of quantum minors. Since all fundamental representations of $\mathfrak{sl}_{r+1}$ are minuscule, all ideal generators in Theorem 3.4 become quantum minors.

Fix two positive integers m and n. Recall that the algebra of quantum matrices $R_q[M_{m,n}]$ is the $\mathbb{K}$-algebra with generators x_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and relations

$$\begin{aligned} x_{ij}x_{lj} &= qx_{lj}x_{ij}, \quad \text{for } i<l, \\ x_{ij}x_{ik} &= qx_{ik}x_{ij}, \quad \text{for } j<k, \\ x_{ij}x_{lk} &= x_{lk}x_{ij}, \quad \text{for } i<l, j>k, \\ x_{ij}x_{lk} - x_{lk}x_{ij} &= (q-q^{-1})x_{ik}x_{lj}, \quad \text{for } i<l, j<k. \end{aligned}$$

The torus $\mathbb{T}^{m+n} = (\mathbb{K}^*)^{\times(m+n)}$ acts on $R_q[M_{m,n}]$ by algebra automorphisms by

$$(t_1, \dots, t_{m+n}) \cdot x_{ij} = t_i t_{m+j}^{-1} x_{ij}, \quad (t_1, \dots, t_{m+n}) \in \mathbb{T}^{m+n}. \tag{4.1}$$

For two integers $i \leq l$ denote $[i,l] = \{i, i+1, \dots, l\}$. Given two subsets $J = \{j_1 < \dots < j_k\} \subset [1,m]$ and $J' = \{j'_1 < \dots < j'_k\} \subset [1,n]$ define the quantum minor $\Delta^q_{J,J'} \in R_q[M_{m,n}]$ by

$$\Delta^q_{J,J'} = \sum_{w \in S_k} (-q)^{l(w)} x_{j_1 j'_{w(1)}} \dots x_{j_k j'_{w(k)}} = \sum_{w \in S_k} (-q)^{-l(w)} x_{j_{w(k)} j'_k} \dots x_{j_{w(1)} j'_1}.$$

Throughout this subsection we set $\mathfrak{g} = \mathfrak{sl}_{m+n}$, $W = S_{m+n}$, and $w = c^m$, where c is the Coxeter element $(12\dots m+n)$. Mériaux and Cauchon constructed an isomorphism between $\mathcal{U}_+^{c^m}$ and $R_q[M_{m,n}]$ in [31, Proposition 2.1.1]. Recall from §2.2 that the automorphism ω of $\mathcal{U}_q(\mathfrak{sl}_{m+n})$ restricts to an isomorphism $\omega \colon \mathcal{U}_+^{c^m} \to \mathcal{U}_-^{c^m}$. Furthermore we have the antiisomorphism $\psi_{c^m} \colon R_q[U_+^{c^m}] \to \mathcal{U}_-^{c^m}$ from Theorem 2.3 (b). The composition of the above maps provides an antiisomorphism $R_q[U_+^{c^m}] \to R_q[M_{m,n}]$, which was used in [32] to study $\mathbb{T}^{m+n} - \mathrm{Spec} R_q[M_{m,n}]$. We briefly go over the modifications of this construction, which are needed because of the different comultiplication of $\mathcal{U}_q(\mathfrak{g})$, braid group action and Lusztig's root vectors of $\mathcal{U}_q(\mathfrak{g})$ used in this paper.

Consider the reduced expression

$$c^m = (s_m \dots s_1)(s_{m+1} \dots s_2) \dots (s_{m+n-1} \dots s_n).$$

Denote the Lusztig root vectors of $\mathcal{U}_+^{c^m}$ and $\mathcal{U}_-^{c^m}$ constructed in (2.5) by

$$X_{1,m+1}, \dots, X_{m,m+1}; X_{1,m+2}, \dots, X_{m,m+2}; \dots; X_{1,m+n}, \dots, X_{m,m+n}.$$

and

$$X_{m+1,1}, \dots, X_{m+1,m}; X_{m+2,1}, \dots, X_{m+2,m}; \dots; X_{m+n,1}, \dots, X_{m+n,m},$$

respectively. For $i \in [1,m]$ and $j \in [1,n]$ set

$$\zeta_{m,n}(X_{m+j,i}) = (-q)^{i+j-2} x_{ij}. \tag{4.2}$$

Lemma 4.1. [32, Lemma 5.4] *The map* (4.2) *extends (uniquely) to an algebra isomorphism* $\zeta_{m,n} \colon \mathcal{U}_-^{c^m} \to R_q[M_{m,n}]$.

Proof. By [31, Proposition 2.1.1], the map $X_{i,m+j} \mapsto x_{ij}$, $i \in [1,m]$, $j \in [1,n]$ defines an isomorphism $\mathcal{U}_+^{c^m} \cong R_q[M_{m,n}]$. By iterating (2.7), one obtains that the isomorphism $\omega \colon \mathcal{U}_+^{c^m} \to \mathcal{U}_-^{c^m}$ satisfies

$$\omega(X_{i,m+j}) = (-q)^{2-i-j} X_{m+j,i}, \quad \forall i \in [1,m], j \in [1,n].$$

The map $\zeta_{m,n}$ is equal to an appropriate composition of these isomorphisms. $\square$

For all $i \in [1,m]$, $j \in [1,n]$,

$$X_{j+m,i} \in (\mathcal{U}_q(\mathfrak{sl}_{m+n}))_{-\alpha_{m-i+1}-\ldots-\alpha_{m+j-1}}. \tag{4.3}$$

Because of this $\zeta_{m,n}$ defines a bijection between the set of $\mathbb{T}^{m+n-1}$-eigenvectors in $\mathcal{U}_-^{c^m}$ with respect to the action (2.10) and the set of $\mathbb{T}^{m+n}$-eigenvectors in $R_q[M_{m,n}]$ with respect to (4.1). Denote by P the weight lattice of $\mathfrak{sl}_{m+n}$. It follows from (4.3) that the P-action (2.11) on $\mathcal{U}_-^{c^m}$ transfers via $\zeta_{m,n}$ to the following action of P on $R_q[M_{m,n}]$:

$$\omega_k \cdot x_{ij} = q^{-\delta_{k,[m-i+1,m+j-1]}} x_{ij} \tag{4.4}$$

for $k \in [1, m+n-1]$, $i \in [1,m]$, $j \in [1,n]$, where for $a,b,c \in \mathbb{Z}$, $\delta_{a,[b,c]} = 1$ if $a \in [b,c]$ and $\delta_{a,[b,c]} = 0$ otherwise.

For $y \in S_{m+n}^{\leq c^m}$ denote

$$I_{m,n}(y) = \zeta_{m,n}(I_{c^m}(y)). \tag{4.5}$$

Theorem 3.1 implies:

Corollary 4.2. *Let $\mathbb{K}$ be an arbitrary base field and $q \in \mathbb{K}^*$ not a root of unity. For all $y \in S_{m+n}^{\leq c^m}$, $I_{m,n}(y)$ is a $\mathbb{T}^{m+n}$-invariant prime ideal of $R_q[M_{m,n}]$ and all $\mathbb{T}^{m+n}$-primes of $R_q[M_{m,n}]$ are of this form. The map $S_{m+n}^{\leq c^m} \to \mathbb{T}^{m+n} - \mathrm{Spec} R_q[M_{m,n}]$ given by $y \mapsto I_{m,n}(y)$ is an isomorphism of posets with respect to the Bruhat order and the inclusion order on ideals.*

4.2. We will need the following partial order on the set of subsets of $[1, m+n]$ with k elements: for $J = \{j_1 < \ldots < j_k\}$ and $J' = \{j'_1 < \ldots < j'_k\} \subseteq [1, m+n]$ set

$$J \leq J', \quad \text{if } \ j_l \leq j'_l \ \text{ for all } \ l = 1, \ldots, k. \tag{4.6}$$

Set $J < J'$, if $J \leq J'$ and $J \neq J'$.

For $J \subseteq [1, m+n]$ denote

$$p_1(J) = J \cap [1,m] \ \text{ and } \ p_2(J) = J \cap [m+1, m+n].$$

Given $J = \{j_1, \ldots, j_k\} \subseteq [m+1, m+n]$, set $J - m = \{j_1 - m, \ldots, j_k - m\} \subseteq [1,n]$.

For $J \subseteq [1, m+n]$, $|J| = k$ such that $J \leq c^m([1,k])$, define

$$\Delta^q(J) = \Delta^q_{w_m^\circ(p_1(J) \backslash p_1(c^m([1,k]))), (p_2(c^m([1,k])) \backslash p_2(J)) - m}, \tag{4.7}$$

where w_m° denotes the longest element of the copy of S_m inside S_{m+n} acting on the first m indices. First we simplify (4.7) in the cases $k \leq n$ and $k > n$, and verify that the two sets in the definition of the quantum determinant in the right hand side of (4.7) have the same cardinality. Indeed, for $1 \leq k \leq n$ we have $p_1(c^m([1,k])) = \emptyset$ and $p_2(c^m([1,k])) = [m+1, m+k]$, and (4.7) simplifies to

$$\Delta^q(J) = \Delta^q_{w_m^\circ(p_1(J)), ([m+1,m+k] \backslash p_2(J)) - m}. \tag{4.8}$$

Moreover for these values of k, $J \leq c^m([1,k]) = [m+1, m+k]$ implies $p_2(J) \subseteq [m+1, m+k]$ and thus $|w_m^\circ(p_1(J))| = |([m+1, m+k] \backslash p_2(J)) - m|$. Here and below $|S|$ denotes the cardinality of a set S.

For $n+1 \leq k \leq m+n$ we have $p_1(c^m([1,k])) = [1, k-n]$ and $p_2(c^m([1,k])) = [m+1, m+n]$, and (4.7) simplifies to

$$\Delta^q(J) = \Delta^q_{w_m^\circ(p_1(J) \backslash [1,k-n]), ([m+1,m+n] \backslash p_2(J)) - m}. \tag{4.9}$$

For these values of k, $J \leq c^m([1,k]) = [1,k-n] \sqcup [m+1,m+n]$ implies $p_1(J) \supseteq [1,k-n]$, therefore $|w_m^\circ(p_1(J)\backslash[1,k-n])| = |([m+1,m+n]\backslash p_2(J)) - m|$.

Denote by $\Lambda_q(\mathbb{K}^{m+n})$ the quantum exterior algebra in $m+n$ generators. It is a $\mathcal{U}_q(\mathfrak{sl}_{m+n})$-module algebra with generators $v_1, \ldots, v_{m+n}$ and relations $v_i v_j = -q v_j v_i$, $1 \leq j < i \leq m+n$, $v_i^2 = 0$, $i = 1, \ldots, m+n$. The $\mathcal{U}_q(\mathfrak{sl}_{m+n})$-action on it is given by:

$$X_i^+ v_j = \delta_{i+1,j} v_i,\ X_i^- v_j = \delta_{ij} v_{i+1},\ K_i v_j = q^{a_{ij}} v_j,\ \ i \in [1, m+n-1], j \in [1, m+n],$$

where $a_{ij} = 1$ if $j = i$, $a_{ij} = -1$ if $j = i+1$, and $a_{ij} = 0$ otherwise. The algebra $\Lambda_q(\mathbb{K}^{m+n})$ is $\mathbb{Z}$-graded by $\deg v_i = 1$. For $k = 1, \ldots, m+n-1$ its component $\Lambda_q(\mathbb{K}^{m+n})_k$ in degree k is isomorphic to the fundamental representation $V(\omega_k)$ of $\mathcal{U}_q(\mathfrak{sl}_{m+n})$. We will identify $V(\omega_k)$ and $\Lambda_q(\mathbb{K}^{m+n})$. For $J = \{j_1 < \ldots < j_k\} \subseteq [1, m+n]$ define

$$v_J = v_{j_1} \ldots v_{j_k}.$$

When J runs over all subsets of $[1, m+n]$ with k elements we obtain a basis of $V(\omega_k)$. The corresponding dual basis of $V(\omega_k)^*$ will be denoted by $\{\xi_J \mid J \subseteq [1, m+n], |J| = k\}$. The Demazure modules $V_w(\omega_k)$ are given by

$$V_w(\omega_k) = \mathcal{U}_+ T_w v_{[1,k]} = \mathrm{Span}\{v_J \mid J \subset [1, m+n], |J| = k, J \leq w([1,k])\},$$

for all $w \in S_{m+n}$. For $J \subset [1, m+n]$, $|J| = k$, $J \leq w([1,k])$ denote

$$\eta_J = \xi_J|_{V_{c^m}(\omega_k)}.$$

Then

$$\{\eta_J \mid J \subset [1, m+n], |J| = k, J \leq c^m([1,k])\} \ \text{ is a } \mathbb{K}\text{-basis of } \ V_{c^m}(\omega_k)^*. \tag{4.10}$$

For a set J as in (4.10) and $y \in S_{m+n}^{\leq c^m}$ we have:

$$\eta_J \in (V_w(\omega_k) \cap \mathcal{U}_- T_y v_{[1,k]})^\perp \Leftrightarrow J \not\geq y([1,k]). \tag{4.11}$$

Denote

$$\Upsilon(y) = \{J \subseteq [1, m+n] \mid |J| = k, k \in [1, m+n-1], \\ J \leq c^m([1,k]), J \not\geq y([1,k])\}. \tag{4.12}$$

Lemma 4.3. *For all $k = 1, \ldots, m+n$ and $J \subset [1, m+n]$ such that $|J| = k$ and $J \leq c^m([1,k])$ we have*

$$\zeta_{m,n}(\psi_{c^m}(d_{\eta_J}^{c^m, \omega_k})) = t' \Delta^q(J) \tag{4.13}$$

for some $t' \in \mathbb{K}^*$ (depending on J).

Proof. We will prove Lemma 4.3 in the case $k \in [1, n]$. The case $k \in [n+1, m+n-1]$ is analogous and is left to the reader, cf. the proof of [32, eq. (5.18)].

Denote the longest element of S_{m+n} by w_{m+n}° and consider the reduced decomposition

$$w_{m+n}^\circ = s_1(s_2 s_1) \ldots (s_{m+n-1} \ldots s_1).$$

Denote the Lusztig root vectors of $\mathcal{U}_+^{w_{m+n}^\circ} = \mathcal{U}_+$ from eq. (2.5) by

$$Y_{1,2}; Y_{1,3}, Y_{2,3}; \ldots; Y_{1,m+n}, \ldots, Y_{m+n-1,m+n}.$$

By [31, Lemma 2.1.1], $Y_{i,i+1} = X_i^+$ for $i \in [1, m+n-1]$ and for $j > i+1$, Y_{ij} is recursively given by

$$Y_{ij} = Y_{i,j-1}Y_{j-1,j} - q^{-1}Y_{j-1,j}Y_{i,j-1}. \tag{4.14}$$

Using induction on $j-i$, one easily verifies that for all $I \subseteq [1, m+n]$, $|I| = k$ and $i, j \in [1, m+n]$, $i < j$:

$$(\tau Y_{ij})v_I = (-q)^{i-j+|I \cap [i+1,j-1]|+1} v_{(I \backslash \{j\}) \cup \{i\}}, \ \text{ if } \ j \in I \ \text{ and} \tag{4.15}$$

$$(\tau Y_{ij})v_I = 0, \ \text{ if } \ j \notin I, \tag{4.16}$$

recall (2.24). The quantum R-matrix corresponding to $c^m \in S_{m+n}$ is given by

$$\begin{aligned}\mathcal{R}^{c^m} = &\left(\exp_q(q'X_{m,m+n} \otimes X_{m+n,m}) \ldots \exp_q(q'X_{1,m+n} \otimes X_{m+n,1})\right) \ldots \\ &\left(\exp_q(q'X_{m,m+1} \otimes X_{m+1,m}) \ldots \exp_q(q'X_{1,m+1} \otimes X_{m+1,1})\right),\end{aligned}$$

where $q' = q^{-1} - q$. Recall that w_m° denotes the longest element of the copy of S_m inside S_{m+n} acting on the first m indices. Fix $k \in [1, n]$ and $J \subset [1, m+n]$ such that $|J| = k$ and $J \leq c^m([1, k])$. Denote

$$w_m^\circ(p_1(J)) = \{i_1 < \ldots < i_l\}, \ \ [m+1, m+n] \backslash p_2(J) = \{m + j_1 < \ldots < m + j_l\}.$$

By [31, Lemma 2.1.3] $X_{i,m+j} = T^{-1}_{w_m^\circ}(Y_{i,m+j})$, $\forall i \in [1, m]$, $j \in [1, n]$. After some straightforward computations using eqs. (2.2) and (4.15)-(4.16) one deduces that

$$\begin{aligned}&\zeta_{m,n}(\psi_{c^m}(d^{c^m,\omega_k}_{\eta_J})) = t''\zeta_{m,n}((d^{c^m,\omega_k}_{\eta_J}\tau \otimes \mathrm{id})(\mathcal{R}^{c^m})) \\ =&t'' \sum_{w \in S_l} \langle \xi_{\{i_1,\ldots,i_l\}}, (\tau Y_{i_{w(1)},m+j_1}) \ldots (\tau Y_{i_{w(l)},m+j_l}) v_{\{m+j_1,\ldots,m+j_l\}} \rangle x_{i_{w(l)},j_l} \ldots x_{i_{w(1)},j_1} \\ =&t' \sum_{w \in S_l} (-q)^{-l(w)} \langle \xi_{\{i_1,\ldots,i_l\}}, v_{\{i_1,\ldots,i_l\}} \rangle x_{i_{w(l)},j_l} \ldots x_{i_{w(1)},j_1} = t'\Delta^q(J)\end{aligned}$$

for some $t', t'' \in \mathbb{K}^*$, which completes the proof of the lemma. $\square$

Recall from Corollary 4.2 that all $\mathbb{T}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ with respect to the action (4.1) are of the form $I_{m,n}(y)$ for some $y \in S^{\leq c^m}_{m+n}$. The following theorem provides a constructive proof of the Goodearl–Lenagan conjecture [16], showing that each of these ideals posses a polynormal generating sequence. Even more, such a sequence is shown to be P-polynormal with respect to the action (4.4), where P is the weight lattice of $\mathfrak{sl}_{m+n}$.

Theorem 4.4. *Assume that the base field $\mathbb{K}$ has characteristic 0 and $q \in \mathbb{K}$ is transcendental over $\mathbb{Q}$. Fix any linear ordering $\prec$ on the set $\Upsilon(y)$ given by (4.12) with the property that if $J, J' \in \Upsilon(y)$, $|J| = |J'|$ and $J < J'$ (recall (4.6)), then $J \prec J'$. Let $\Upsilon(y) = \{J_1 \prec J_2 \prec \ldots \prec J_{\Upsilon(y)}\}$. Then*

$$\Delta^q(J_1), \ldots, \Delta^q(J_{|\Upsilon(y)|}) \tag{4.17}$$

is a P-polynormal generating sequence for the $\mathbb{T}^{m+n}$-invariant prime ideal $I_{m,n}(y)$ of $R_q[M_{m,n}]$ (with respect to the action (4.4) of the weight lattice P of $\mathfrak{sl}_{m+n}$ on $R_q[M_{m,n}]$).

Proof. Fix $y \in S_{m+n}$, $y \leq c^m$. Theorem 3.1 (c) and Lemma 4.3 imply that the set from (4.17) generates the ideal $I_{m,n}(y)$, see [32, Theorem 5.5] for details. By a direct computation one obtains

$$\mathrm{wt}(\eta_J) = -\omega_k + \sum_{i=1}^{k} (\alpha_i + \ldots + \alpha_{j_i - 1}),$$

for all $J = \{j_1 < \ldots < j_k\} \in \Upsilon(y)$. In the setting of Theorem 3.4 we can choose $\Omega_{c^m}(y) = \{\eta_J \mid J \in \Upsilon(y)\}$. The definition of the linear ordering $\prec$ implies that

$$\eta_{J_1} < \ldots < \eta_{|\Upsilon(y)|}$$

is a linear ordering on the set $\Omega_{c^m}(y)$ satisfying the conditions of Theorem 3.4. It follows from Theorem 3.4 that

$$\psi_{c^m}\left(d^{c^m, |J_1|}_{\eta_{J_1}}\right), \ldots, \psi_{c^m}\left(d^{c^m, |J_{|\Upsilon(y)|}|}_{\eta_{J_{|\Upsilon(y)|}}}\right)$$

is a P-polynormal generating sequence of the ideal $I_{c^m}(y)$ of $\mathcal{U}^y_-$. Recall that $I_{m,n}(y) = \zeta_{m,n}(I_{c^m}(y))$, see (4.5). The statement of the theorem now follows from (4.13) and the fact that $\zeta_{m,n} \colon \mathcal{U}^{c^m}_- \to R_q[M_{m,n}]$ intertwines the P-actions (2.11) and (4.4). $\square$

Remark 4.5. Theorem 3.6 implies that under the weaker assumption that $q \in \mathbb{K}^*$ is not a root of unity and without restrictions on the characteristic of the base field $\mathbb{K}$, all $\mathbb{T}^{m+n}$-invariant prime ideals of $R_q[M_{m,n}]$ are P-polynormal with respect to the action (4.4). (This is sufficient for our applications of polynormality, see Section 5 below.) Currently, there is no proof that under the above weaker assumptions the ideals $I_{m,n}(y)$ are generated by the quantum minors in Theorem 4.4 and more generally that the ideals $I_w(y)$ of $\mathcal{U}^w_-$ are generated by the elements in Theorem 3.4 (for an arbitrary simple Lie algebra $\mathfrak{g}$). We conjecture that this is correct. It is proved for the height one $\mathbb{T}^{m+n}$-primes of quantum matrices [26, Proposition 4.2] and more generally for the height one $\mathbb{T}^r$-prime ideals of all algebras $\mathcal{U}^w_-$, [35, Proposition 6.8].

5. Catenarity of Spec$\mathcal{U}^w_-$

5.1. In this section we prove that Spec$\mathcal{U}^w_-$ is normally separated for all simple Lie algebras $\mathfrak{g}$ and $w \in W$. From this we deduce that all algebras $\mathcal{U}^w_-$ are catenary. Furthermore, we prove a formula for the heights of all $\mathbb{T}^r$-invariant prime ideals $I_w(y)$ of $\mathcal{U}^w_-$, recall Theorem 3.1.

We recall that for a ring R, one says that Spec$R$ is normally separated if for any two prime ideals $I \subsetneq I'$ of R there exists $u \in I'$, which is normal in R modulo I and such that $u \notin I$. If, in addition a group G acts on R by algebra automorphisms, we say that $G -$ Spec$R$ is G-normally separated if for every two G-prime ideals $I \subsetneq I'$ of R there exists $u \in I'$, which is G-normal in R modulo I and such that $u \notin I$, recall §3.1.

Corollary 5.1. *Let $\mathbb{K}$ be an arbitrary base field, $q \in \mathbb{K}^*$ not a root of unity, $\mathfrak{g}$ be a simple Lie algebra and $w \in W$. Then $\mathbb{T}^r -$ Spec$\mathcal{U}^w_-$ is P-normally separated for the action* (2.11)*. In particular, $\mathbb{T}^r -$ Spec$\mathcal{U}^w_-$ is $\mathbb{T}^r$-normally separated with respect to the action* (2.10)*.*

The special case of Corollary 5.1 for the algebras of quantum matrices is due to Cauchon [8]. The case when $w = w_0$ and $\mathbb{K} = \mathbb{C}(q)$ is due to Caldero [6].

Proof of Corollary 5.1. Let $I \subsetneq I'$ be two $\mathbb{T}^r$-prime ideals of $\mathcal{U}_-^w$. By Theorem 3.6, I' possesses a P-polynormal generating sequence $u_1, \ldots, u_n$. Denote the image of u_j in $\mathcal{U}_-^w/I$ by $\overline{u}_j$, $j = 1, \ldots, n$. Let $k = \min\{j \mid \overline{u}_j \neq 0\}$. Then u_k is P-normal modulo the ideal generated by $u_1, \ldots, u_{k-1} \in I'$. Therefore $u_k \in I'$ is a P-normal element of $\mathcal{U}_-^w$ modulo I and $u_k \notin I$. $\square$

Next, we give a second proof to the P-normal separation result for the $\mathbb{T}^r$-invariant prime ideals $\{I_w(y)\}_{y \in W^{\leq w}}$ of $\mathcal{U}_-^w$, using results of Gorelik [18]. This proof also constructs explicit separating elements for all pairs of $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}_-^w$.

Theorem 5.2. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity and $\mathfrak{g}$ is an arbitrary simple Lie algebra. Let $y_1, y_2 \in W^{\leq w}$ and $\lambda' \in P^+$ be such that $y_1 < y_2$ in the Bruhat order and $y_1(\lambda') \neq y_2(\lambda')$. Then $\phi_w(e_{y_1}^{\lambda'} e_w^{-\lambda'}) \in I_w(y_2)$ is a P-normal element of $\mathcal{U}_-^w$ modulo $I_w(y_1)$, which does not belong to $I_w(y_1)$. For all $\lambda \in P^+$, $\nu \in P$, $\xi \in (V(\lambda)^*)_\nu$, we have*

$$
\begin{aligned}
(5.1) \quad \phi_w(e_{y_1}^{\lambda'} e_w^{-\lambda'})\phi_w(c_\xi^\lambda e_w^{-\lambda}) &= q^{\langle -(y_1+w)(\lambda'), \nu + w(\lambda)\rangle} \phi_w(c_\xi^\lambda e_w^{-\lambda}) \phi_w(e_{y_1}^{\lambda'} e_w^{-\lambda'}) \\
&= \Big((-(y_1 + w)(\lambda')).\phi_w(c_\xi^\lambda e_w^{-\lambda}) \Big) \phi_w(e_{y_1}^{\lambda'} e_w^{-\lambda'}) \mod I_w(y_1).
\end{aligned}
$$

Proof. Lemma 3.3 and the fact that $\phi_w \colon R_0^w \to \mathcal{U}_-^w$ is a graded antiisomorphism (see Theorem 2.2) imply (5.1). Recall that

$$e_{y_1}^{\lambda'} = c_{\xi_{y_1,\lambda'}}^{\lambda'}$$

for certain $\xi_{y_1,\lambda'} \in (V(\lambda')^*)_{-y_1\lambda'}$, see (2.15). Since $y_1 < y_2$ and $y_1(\lambda') \neq y_2(\lambda')$, we have that $y_1(\lambda') > y_2(\lambda')$. Therefore $(\mathcal{U}_- T_{y_2} v_{\lambda'})_{y_1(\lambda')} = 0$ and $\xi_{y_1,\lambda'} \perp \mathcal{U}_- T_{y_2} v_{\lambda'}$. Hence $e_{y_1}^{\lambda'} e_w^{-\lambda'} \in Q_w(y_2)^-$ and $\phi_w(e_{y_1}^{\lambda'} e_w^{-\lambda'}) \in I_w(y_2)$.

Proposition 5.3.3 (ii), Lemma 6.6, and Corollary 6.10.1 (i) of Gorelik [18] imply that $\phi_w(e_{y_1}^\lambda e_w^{-\lambda}) \notin I_w(y_1)$ for all $\lambda \in P^+$. These results were formulated in [18] for $\mathbb{K}$ of characteristic 0 and $q \in \mathbb{K}$ transcendetal over $\mathbb{Q}$, but it was shown in [36, Theorem 3.1(b)] that Gorelik's proof works for all fields $\mathbb{K}$ and $q \in \mathbb{K}^*$ which are not roots of unity. This completes the proof of the theorem. $\square$

We proceed with proving that all algebras $\mathcal{U}_-^w$ have normal separation.

Theorem 5.3. *Assume that $\mathbb{K}$ is an arbitrary base field and $q \in \mathbb{K}^*$ is not a root of unity. For all simple Lie algebras $\mathfrak{g}$ and $w \in W$,* Spec$\mathcal{U}_-^w$ *is normally separated.*

Normal separation of Spec$\mathcal{U}_-^w$ was established in two special cases earlier. The case of the algebras of quantum matrices is due to Cauchon [8], who used very different techniques based on his method of deleting derivations. The case when $w = w_0$ (the longest element of the Weyl group W) and $\mathbb{K} = \mathbb{C}(q)$ was obtained by Caldero in [6].

Proof of Theorem 5.3. Goodearl proved [11, Corollary 4.6] that, if R is a right noetherian ring graded by an abelian group and R has graded normal separation, then $\mathrm{Spec} R$ is normally separated. (A graded ring R is said to have graded normal separation if for every two graded prime ideals $I \subsetneq I'$ there exists a homogeneous nonzero element $x \in I'/I$ which is normal in R/I.) The algebras $\mathcal{U}_-^w$ are noetherian, because they are iterated skew polynomial rings. The graded prime ideals of $\mathcal{U}_-^w$ with respect to the Q-grading (2.9) are precisely the $\mathbb{T}^r$-invariant prime ideals with respect to the action (2.10). Corollary 5.1 implies that the the set of Q-graded prime ideals of $\mathcal{U}_-^w$ is P-normally separated. Recall that P-normal elements are P-eigenvectors (see §3.1) and that the P-eigenvectors in $\mathcal{U}_-^w$ are precisely the homogeneous elements of $\mathcal{U}_-^w$ with respect to the grading (2.9). Therefore the algebras $\mathcal{U}_-^w$ have graded normal separation and we can apply Goodearl's result to them, which establishes the theorem. □

5.2. We proceed with proving that all algebras $\mathcal{U}_-^w$ are catenary. Motivated by Gabber's proof of catenarity of the universal enveloping algebras of all solvable Lie algebras, Goodearl and Lenagan proved the following theorem.

Theorem 5.4. *(Goodearl–Lenagan,* [13]*) Assume that A is an affine, noetherian, Auslander–Gorenstein and Cohen–Macaulay algebra over a field, with finite Gelfand–Kirillov dimension. If* $\mathrm{Spec} A$ *is normally separated, then A is catenary. If, in addition, A is a prime ring, then Tauvel's height formula holds.*

We recall that Tauvel's height formula holds for A if for all prime ideals I of A, the height of I is equal to

$$\mathrm{GK}\dim A - \mathrm{GK}\dim(A/I).$$

Here and below $\mathrm{GK}\dim(.)$ denotes the Gelfand–Kirillov dimension of an algebra or a module.

For the convenience of the reader we also recall the notions of Auslander regular, Auslander–Gorenstein, and Cohen–Macaulay rings. A ring R is called Auslander–Gorenstein if the injective dimension of R (as both right and left R-module) is finite, and for all integers $0 \leq i < j$ and finitely generated (right or left) R-modules M, we have $\mathrm{Ext}^i_R(N,R) = 0$ for all R-submodules N of $\mathrm{Ext}^j_R(M,R)$. A ring R is said to be Auslander regular if, in addition, the global dimension of R is finite. The grade of a finitely generated R-module M is defined by

$$j(M) = \inf\{i \geq 0 \mid \mathrm{Ext}^i_R(M,R) \neq 0\}.$$

An algebra R is called Cohen–Macauley if

$$j(M) + \mathrm{GK}\dim M = \mathrm{GK}\dim R$$

for all finitely generated R-modules M.

We apply Theorem 5.4 to the algebras $\mathcal{U}_-^w$. The normal separation of $\mathrm{Spec}\mathcal{U}_-^w$ was established in Theorem 5.3. It is well known that all algebras $\mathcal{U}_-^w$ are CGL extensions (a special kind of iterated skew polynomial rings) in $l(w)$ variables, where $l(w)$ is the length of w. Thus for all base fields $\mathbb{K}^*$, $q \in \mathbb{K}^*$ not a root of unity, and $w \in W$, the algebra $\mathcal{U}_-^w$ are affine, noetherian and

$$\mathrm{GK}\dim \mathcal{U}_-^w = l(w). \tag{5.2}$$

We derive from the following result of Ekström, Levasseur and Stafford [10, 27] that all algebras $\mathcal{U}_-^w$ are Auslander regular and Cohen–Macaulay:

Proposition 5.5. *(Ekström, Levasseur–Stafford,* [10, 27]*) Assume R is a noetherian, Auslander regular ring. Let $S = R[x; \sigma, \delta]$ be a skew polynomial extension of R. Then:*

(a) S is Auslander regular.

(b) If $R = \oplus_{k \geq 0} R_k$ is a connected graded Cohen–Macauley $\mathbb{K}$-algebra over a field $\mathbb{K}$ such that $\sigma(R_k) \subseteq R_k$ for all $k \geq 0$, then S is Cohen–Macauley.

Proposition 5.6. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, simple Lie algebras $\mathfrak{g}$, and Weyl group elements $w \in W$, the algebras $\mathcal{U}_-^w$ are Auslander regular and Cohen–Macauley.*

Proof. Fix $w \in W$ and a reduced expression $w = s_{i_1} \dots s_{i_l}$ of it. Denote $w' = w s_{i_l}$. In the notation of §2.2, the subalgebra of $\mathcal{U}_-^w$ generated by $X^-_{\beta_1}, \dots, X^-_{\beta_{l-1}}$ coincides with $\mathcal{U}_-^{w'}$. Recall the Levendorskii–Soibelman straightening rule

$$\begin{aligned} X^-_{\beta_i} X^-_{\beta_j} - q^{\langle \beta_i, \beta_j \rangle} X^-_{\beta_j} X^-_{\beta_i} \\ = \sum_{\mathbf{n} = (n_{i+1}, \dots, n_{j-1}) \in \mathbb{N}^{\times (j-i-2)}} p_{\mathbf{n}} (X^-_{\beta_{i+1}})^{n_{i+1}} \dots (X^-_{\beta_{j-1}})^{n_{j-1}}, \;\; p_{\mathbf{n}} \in \mathbb{K}, \end{aligned} \tag{5.3}$$

for all $1 \leq i < j \leq l$. It implies that $\mathcal{U}_-^w$ is a skew polynomial extension of $\mathcal{U}_-^{w'}$:

$$\mathcal{U}_-^w \cong \mathcal{U}_-^{w'}[x; \sigma, \delta],$$

where $\sigma \in \operatorname{Aut}(\mathcal{U}_-^{w'})$ is given by

$$\sigma(x) = q^{-\langle \gamma, \beta_l \rangle} x, \quad \forall x \in (\mathcal{U}_-^{w'})_{-\gamma}, \gamma \in Q^+, \tag{5.4}$$

recall (2.10). By repeated applications of Proposition 5.5 (a) one obtains that the algebra $\mathcal{U}_-^w$ is Auslander regular. Fix $\mu \in \sum_{i=1}^r \mathbb{Z}_+ \omega_i$ and specialize the $-Q^+$-grading of $\mathcal{U}_-^{w'}$ to an $\mathbb{N}$-grading by

$$(\mathcal{U}_-^{w'})_k = \oplus_{\mu \in Q^+} \{ (\mathcal{U}_-^{w'})_{-\gamma} \mid \langle \gamma, \mu \rangle = k \}, \quad k \in \mathbb{N}.$$

Obviously $\mathcal{U}_-^w$ is connected. It follows from (5.4) that $\sigma((\mathcal{U}_-^{w'})_{-\gamma}) \subseteq (\mathcal{U}_-^{w'})_{-\gamma}$ for all $\gamma \in Q^+$ and thus $\sigma((\mathcal{U}_-^{w'})_k) \subseteq (\mathcal{U}_-^{w'})_k$ for all $k \in \mathbb{N}$. Repeatedly applying Proposition 5.5 (b), we obtain that the algebra $\mathcal{U}_-^w$ is Cohen–Macauley. □

Combining Theorems 5.3 and 5.4, Proposition 5.6, and the fact that the algebras $\mathcal{U}_-^w$ are affine, noetherian domains of Gelfand–Kirillov dimension $l(w)$, we obtain:

Theorem 5.7. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, simple Lie algebras $\mathfrak{g}$ and Weyl group elements $w \in W$, the algebras $\mathcal{U}_-^w$ are catenary and Tauvel's height formula holds.*

The special case of Theorem 5.7 for the algebras of quantum matrices is due to Cauchon [8]. The case when $w = w_0$ and $\operatorname{char} \mathbb{K} = 0$ was obtained by Malliavin [29], Goodearl and Lenagan [13].

5.3. In this subsection we establish formulas for the heights of all $\mathbb{T}^r$-invariant prime ideals $I_w(y)$ of $\mathcal{U}^w_-$ (recall Theorem 3.1) and the Gelfand–Kirillov dimensions of the quotients $\mathcal{U}^w_-/I_w(y)$.

Theorem 5.8. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, simple Lie algebras $\mathfrak{g}$ and Weyl group elements $y, w \in W$, $y \leq w$, the height of the $\mathbb{T}^r$-invariant prime ideal $I_w(y)$ equals $l(y)$ and*

$$\mathrm{GK}\dim(\mathcal{U}^w_-/I_w(y)) = l(w) - l(y).$$

We will need the following proposition. Its proof was communicated to us by Ken Goodearl [12].

Proposition 5.9. *Let A be a noetherian algebra over an infinite field $\mathbb{K}$, equipped with a rational action of a $\mathbb{K}$-torus T by algebra automorphisms. If $T-\mathrm{Spec}A$ is T-normally separated, then for each pair of T-invariant prime ideals $I \subsetneq I'$ there exists a saturated chain of prime ideals $I \subsetneq I_1 \subsetneq \ldots \subsetneq I_m \subsetneq I'$ consisting entirely of T-invariant prime ideals.*

Proof. Arguing by induction, it suffices to prove that:

If $I \subsetneq I'$ is a pair of T-invariant prime ideals of A such that there is no T-invariant prime ideal of A lying strictly between I and I', then there is no prime ideal of A lying strictly between I and I'.

By changing A to A/I, we see that it is sufficient to prove the special case when $I = \{0\}$ (and thus A is a prime algebra). The assumption on normal separation implies that there exists a normal element c of A such that $c \in I'$, $c \neq 0$ and c is a T-eigenvector. By [4, Proposition II.2.9] all T-primes of A are prime. Therefore all minimal primes over cA are T-invariant prime ideals. Denote one of them that is contained in I' by J. We have that $\{0\} \subsetneq J \subseteq I'$. The assumption that there is no T-invariant prime ideal of A lying strictly between $\{0\}$ and I' implies that $J = I'$. By the principle ideal theorem [30, Theorem 4.1.11] the height of J equals 0 or 1. The former is impossible since $\{0\} \subsetneq I'$ and $\{0\}$ is a prime ideal. Thus the height of I' is equal to 1, and there are no prime ideals of A lying strictly between $\{0\}$ and I'. $\square$

Proof of Theorem 5.8. By Corollary 5.1, $\mathbb{T}^r - \mathrm{Spec}\mathcal{U}^w_-$ is $\mathbb{T}^r$-normally separated. The base field $\mathbb{K}$ is infinite since $q \in \mathbb{K}^*$ is not a root of unity. Applying Proposition 5.9, we obtain that there exists a saturated chain of prime ideals $\{0\} \subsetneq I_1 \subsetneq \ldots \subset I_m \subsetneq I_y(w)$ consisting entirely of $\mathbb{T}^r$-invariant prime ideals. It follows from Theorem 3.1 (b) that the length of this chain is equal to $l(y)$. Therefore the height of $I_y(w)$ equals $l(y)$. Recall from (5.2) that $\mathrm{GK}\dim\mathcal{U}^w_- = l(w)$. Applying the fact that Tauvel's height formula holds for $\mathcal{U}^w_-$ (Theorem 5.7), we obtain that

$$\mathrm{GK}\dim(\mathcal{U}^w_-/I_w(y)) = \mathrm{GK}\dim\mathcal{U}^w_- - l(y) = l(w) - l(y),$$

which completes the proof of the theorem. $\square$

References

[1] A. D. Bell and G. Sigurdsson, *Catenarity and Gel'fand-Kirillov dimension in Ore extensions*, J. Algebra **127** (1989), 409–425.

[2] J. Bell, K. Casteels, and S. Launois, *Primitive ideals in quantum Schubert cells: dimension of the strata,* preprint, ArXiv:1009.1347.
[3] K. A. Brown and K. R. Goodearl, *Prime spectra of quantum semisimple groups*, Trans. Amer. Math. Soc. **348** (1996), 2465–2502.
[4] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2002.
[5] K. A. Brown, K. R. Goodearl, and M. Yakimov, *Poisson structures of affine spaces and flag varieties. I. Matrix affine Poisson space,* Adv. Math. **206** (2006), 567–629.
[6] P. Caldero, *Étude des q-commutations dans l'algèbre* $U_q(\mathfrak{n}^+)$, J. Algebra **178** (1995), 444–457.
[7] K. Casteels, *A graph theoretic method for determining generating sets of prime ideals in quantum matrices*, J. Algebra **330** (2011), 188–205.
[8] G. Cauchon, *Spectre premier de* $O_q(M_n(k))$*: image canonique et séparation normale*, J. Algebra 260 (2003), 519–569.
[9] C. De Concini, V. Kac, and C. Procesi, *Some quantum analogues of solvable Lie groups,* In: Geometry and analysis (Bombay, 1992), pp. 41–65, Tata Inst. Fund. Res., Bombay, 1995.
[10] E. K. Ekström, *The Auslander condition on graded and filtered Noetherian rings*, In: Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), 220–245, Lecture Notes in Math., 1404, Springer, Berlin, 1989.
[11] K. R. Goodearl, *Prime spectra of quantized coordinate rings*, in: Interactions between ring theory and representations of algebras (Murcia), 205–237, Lect. Notes in Pure and Appl. Math., 210, Dekker, New York, 2000.
[12] K. R. Goodearl, private communication.
[13] K. R. Goodearl and T. H. Lenagan, *Catenarity in quantum algebras,* J. Pure Appl. Algebra **111** (1996), 123–142.
[14] K. R. Goodearl and T. H. Lenagan, *Winding-invariant prime ideals in quantum* 3×3 *matrices*, J. Algebra **260** (2003), 657–687.
[15] K. R. Goodearl, S. Launois, and T. H. Lenagan, $\mathcal{H}$*-primes in quantum matrices, totally nonnegative cells and symplectic leaves*, Math. Z. **269** (2011), 29–45,
[16] K. R. Goodearl and E. S. Letzter, *The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras,* Trans. Amer. Math. Soc. **352** (2000) 1381–1403.
[17] K. R. Goodearl and M. Yakimov, *Poisson structures of affine spaces and flag varieties. II*, Trans. Amer. Math. Soc. **361** (2009), 5753–5780.
[18] M. Gorelik, *The prime and the primitive spectra of a quantum Bruhat cell translate,* J. Algebra **227** (2000), 211–253.
[19] T. J. Hodges and T. Levasseur, *Primitive ideals of* $\mathbb{C}_q[\mathrm{SL}(n)]$, J. Algebra **168** (1994), 455–468.
[20] J. C. Jantzen, *Lectures on quantum groups,* Grad. Studies in Math., 6. Amer. Math. Soc., Providence, RI, 1996.
[21] A. Joseph, *On the prime and primitive spectra of the algebra of functions on a quantum group*, J. Algebra **169** (1994), 441–511.
[22] A. Joseph, *Quantum groups and their primitive ideals,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1995.
[23] A. Joseph, *Sur les idéaux génériques de l'agèbre des fonctions sur un groupe quantique,* C. R. Acad. Sci. Paris 321 (1995), 135–140.
[24] S. Kolb, *The AS-Cohen-Macaulay property for quantum flag manifolds of minuscule weight*, J. Algebra **319** (2008), 3518–3534.
[25] S. Launois, *Combinatorics of* $\mathcal{H}$*-primes in quantum matrices*, J. Algebra **309** (2007), 139–167.
[26] S. Launois, T. H. Lenagan and L. Rigal, *Quantum unique factorisation domains*, J. London Math. Soc. (2) **74** (2006), 321–340.
[27] T. Levasseur and J. T. Stafford, *The quantum coordinate ring of the special linear group,* J. Pure Appl. Algebra **86** (1993), 181–186.

[28] G. Lusztig, *Introduction to quantum groups*, Progr. Math. 110, Birkhäuser, 1993.
[29] M.-P. Malliavin, *La caténarité de la partie positive de l'algèbre enveloppante quantifiée de l'algèbre de Lie simple de type* B_2, In: Festschrift on the occasion of the 65th birthday of Otto Krötenheerdt. Beiträge Algebra Geom. **35** (1994), 73–83.
[30] J.C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*. With the cooperation of L. W. Small. Revised ed. Grad. Stud. in Math., 30. Amer. Math. Soc., Providence, RI, 2001.
[31] A. Mériaux and G. Cauchon, *Admissible diagrams in quantum nilpotent algebras and combinatoric properties of Weyl groups,* Represent. Theory **14** (2010), 645-687.
[32] M. Yakimov, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Proc. London Math. Soc. (3) **101** (2010), 454–476,
[33] M. Yakimov, *A classification of* H*-primes of quantum partial flag varieties*, Proc. Amer. Math. Soc. **138** (2010), 1249–1261.
[34] M. Yakimov, *Strata of prime ideals of De Concini–Kac–Procesi algebras and Poisson geometry*, preprint, ArXiv:1103.3451.
[35] M. Yakimov, *On the spectra of quantum groups*, preprint, ArXiv:1106.3821.
[36] M. Yakimov, *Spectra and catenarity of multiparameter quantum Schubert cells*, preprint, ArXiv:1112.2235.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.
E-mail address: `yakimov@math.lsu.edu`