Quantum cluster algebras and quantum nilpotent algebras

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Submitted to Proceedings of the National Academy of Sciences of the United States of America

A major direction in the theory of cluster algebras is to construct (quantum) cluster algebra structures on the (quantized) coordinate rings of various families of varieties arising in Lie theory. We prove that all algebras in a very large axiomatically defined class of noncommutative algebras possess canonical quantum cluster algebra structures. Furthermore, they coincide with the corresponding upper quantum cluster algebras. We also establish analogs of these results for a large class of Poisson nilpotent algebras. Many important families of coordinate rings are subsumed in the class we are covering, which leads to a broad range of application of the general results to the above mentioned types of problems. As a consequence, we prove the Berenstein–Zelevinsky conjecture for the quantized coordinate rings of double Bruhat cells and construct quantum cluster algebra structures on all quantum unipotent groups, extending the theorem of Geiß, Leclerc and Schröer for the case of symmetric Kac–Moody groups. Moreover, we prove that the upper cluster algebras of Berenstein, Fomin and Zelevinsky associated to double Bruhat cells coincide with the corresponding cluster algebras.

quantum cluster algebras | noncommutative unique factorization domains | quantum groups

Significance statement: Cluster Algebras are used to study in a unified fashion phenomena from many areas of mathematics. In this paper we present a new approach to cluster algebras based on noncommutative ring theory. It deals with large, axiomatically defined classes of algebras and does not require initial combinatorial data. Because of this it has a broad range of applications to open problems on constructing cluster algebra structures on coordinate rings and their quantum counterparts.

The theory of cluster algebras provides a unified framework for treating a number of problems in diverse areas of mathematics such as combinatorics, representation theory, topology, mathematical physics, algebraic and Poisson geometry, and dynamical systems [1, 2, 3, 4, 5, 6, 7]. The construction of cluster algebras was invented by Fomin and Zelevinsky [1] who also obtained a number of fundamental results on them. This construction builds algebras in a novel way by producing infinite generating sets via a process of mutation rather than the classical approach using generators and relations.

The main algebraic approach to cluster algebras relies on representations of finite dimensional algebras and derived categories [5, 8]. In this paper we describe a new algebraic approach based on noncommutative ring theory.

An important range of problems in the theory of cluster algebras is to prove that the coordinate rings of certain algebraic varieties coming from Lie theory admit cluster algebra structures. The idea is that, once this is done, one can use cluster algebras to study canonical bases in such coordinate rings. Analogous problems deal with the corresponding quantizations. The approach via representation theory to this type of problem needs combinatorial data for quivers as a starting point. Such might not be available a priori. This approach also differs from one family of varieties to another, which means that in each case one needs to design an appropriate categorification process.

We prove that all algebras in a very general, axiomatically defined class of quantum nilpotent algebras are quantum cluster algebras. The proof is based on constructing quantum clusters by considering sequences of prime elements in chains of subalgebras which are noncommutative unique factorization domains. These clusters are canonical relative to the mentioned chains of subalgebras, which are determined by the presentation of the quantum nilpotent algebra. The construction does not rely on any initial combinatorics of the algebras. On the contrary, the construction itself produces intricate combinatorial data for prime elements in chains of subalgebras. When this is applied to special cases, we recover the Weyl group combinatorics which played a key role in categorification earlier [7, 9, 10]. Because of this, we expect that our construction will be helpful in building a unified categorification of quantum nilpotent algebras. Finally, we also prove similar results for (commutative) cluster algebras using Poisson prime elements.

The results of the paper have many applications since a number of important families of algebras arise as special cases in our axiomatics. Berenstein, Fomin, and Zelevinsky proved [10] that the coordinate rings of all double Bruhat cells in every complex simple Lie group are upper cluster algebras. It was an important problem to decide whether the latter coincide with the corresponding cluster algebras. We resolve this problem positively. On the quantum side, we prove the Berenstein–Zelevinsky conjecture [11] on cluster algebra structures for all quantum double Bruhat cells. Finally, we establish that the quantum Schubert cell algebras for all complex simple Lie groups have quantum cluster algebra structures. Previously this was known for symmetric Kac–Moody groups due to Geiß, Leclerc and Schröer [12].

Prime elements of quantum nilpotent algebras

Definition of quantum nilpotent algebras. Let $\mathbb{K}$ be an arbitrary base field. A skew polynomial extension of a $\mathbb{K}$-algebra A,

$$A \mapsto A[x; \sigma, \delta],$$

is a generalization of the classical polynomial algebra $A[x]$. It is defined using an algebra automorphism σ of A and a skew-

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derivation δ. The algebra $A[x; \sigma, \delta]$ is isomorphic to $A[x]$ as a vector space and the variable x commutes with the elements of A as follows:

$$xa = \sigma(a)x + \delta(a) \quad \text{for all } \ a \in A.$$

For every nilpotent Lie algebra $\mathfrak{n}$ of dimension m there exists a chain of ideals of $\mathfrak{n}$

$$\mathfrak{n} = \mathfrak{n}_m \rhd \mathfrak{n}_{m-1} \rhd \ldots \rhd \mathfrak{n}_1 \rhd \mathfrak{n}_0 = \{0\}$$

such that $\dim(\mathfrak{n}_k/\mathfrak{n}_{k-1}) = 1$ and $[\mathfrak{n}, \mathfrak{n}_k] \subseteq \mathfrak{n}_{k-1}$, for $1 \leq k \leq m$. Choosing an element x_k in the complement of $\mathfrak{n}_{k-1}$ in $\mathfrak{n}_k$ for each $1 \leq k \leq m$ leads to the following presentation of the universal enveloping algebra $\mathcal{U}(\mathfrak{n})$ as an iterated skew polynomial extension:

$$\mathcal{U}(\mathfrak{n}) \cong \mathbb{K}[x_1][x_2; \mathrm{id}, \delta_2] \ldots [x_m; \mathrm{id}, \delta_m]$$

where all the derivations $\delta_2, \ldots, \delta_m$ are locally nilpotent.

Definition 1. *An iterated skew polynomial extension*

$$R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N] \qquad \textbf{[1]}$$

is called a quantum nilpotent algebra if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(a) *The elements $x_1, \ldots, x_N$ are $\mathcal{H}$-eigenvectors.*

(b) *For every $2 \leq k \leq N$, δ_k is a locally nilpotent σ_k-derivation of $\mathbb{K}[x_1] \cdots [x_{k-1}; \sigma_{k-1}, \delta_{k-1}]$.*

(c) *For every $1 \leq k \leq N$, there exists $h_k \in \mathcal{H}$ such that $\sigma_k = (h_k \cdot)$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.*

The universal enveloping algebras of finite dimensional nilpotent Lie algebras satisfy all of the conditions in the definition except for the last part of the third one. More precisely, in that case one can take $\mathcal{H} = \{1\}$, conditions (a)–(b) are satisfied, and in condition (c) we have $\lambda_k = 1$. Thus, condition (c) is the main feature that separates the class of quantum nilpotent algebras from the class of universal enveloping algebras of nilpotent Lie algebras. The torus $\mathcal{H}$ is needed in order to define the eigenvalues λ_k.

The algebras in Definition 1 are also known as Cauchon–Goodearl–Letzter (CGL) extensions. The axiomatics came from the works [13, 14] which investigated in this generality the stratification of the prime spectrum of an algebra into strata associated to its $\mathcal{H}$-prime ideals.

The dimension of the algebra in Eq. **[1]** in the sense of Gelfand–Kirillov equals N.

Example 1. *For two positive integers m and n, and $q \in \mathbb{K}^*$, define the algebra of quantum matrices $R_q[M_{m\times n}]$ as the algebra with generators t_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, and relations*

$$\begin{aligned} t_{ij}t_{kj} &= qt_{kj}t_{ij}, && \text{for } i < k, \\ t_{ij}t_{il} &= qt_{il}t_{ij}, && \text{for } j < l, \\ t_{ij}t_{kl} &= t_{kl}t_{ij}, && \text{for } i < k,\ j > l, \\ t_{ij}t_{kl} - t_{kl}t_{ij} &= (q - q^{-1})t_{il}t_{kj}, && \text{for } i < k,\ j < l. \end{aligned}$$

It is an iterated skew polynomial extension where

$$R_q[M_{m\times n}] = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \ldots [x_N; \sigma_N, \delta_N],$$

$N = mn$, and $x_{(i-1)n+j} = t_{ij}$. It is easy to write explicit formulas for the automorphisms σ_k and skew derivations δ_k from the above commutation relations, and to check that each δ_k is locally nilpotent. The torus $\mathcal{H} = (\mathbb{K}^*)^{m+n}$ acts on $R_q[M_{m\times n}]$ by algebra automorphisms by the rule

$$(\xi_1, \ldots, \xi_{m+n}) \cdot t_{ij} := \xi_i \xi_{m+j}^{-1} t_{ij}$$

for all $(\xi_1, \ldots, \xi_{m+n}) \in (\mathbb{K}^*)^{m+n}$. Define

$$h_{ij} := (1, \ldots, 1, q^{-1}, 1, \ldots, 1, q, 1, \ldots, 1) \in \mathcal{H}$$

where q^{-1} and q reside in positions i and $m + j$, respectively. Then $\sigma_{(i-1)n+j} = (h_{ij} \cdot)$ and

$$h_{ij} \cdot t_{ij} = q^{-2} t_{ij}.$$

Thus, for all $q \in \mathbb{K}^*$ which are not roots of unity, the algebras $R_q[M_{m\times n}]$ are examples of quantum nilpotent algebras.

Unique factorization domains. The notion of unique factorization domain plays an important role in algebra and number theory. Its noncommutative analog was introduced by Chatters in [15]. A nonzero, non-invertible element p of a domain R (a ring without zero divisors) is called prime if $pR = Rp$ and the factor R/Rp is a domain. A noetherian domain R is called a unique factorization domain (UFD) if every nonzero prime ideal of R contains a prime element. Such rings possess the unique factorization property for all of their nonzero normal elements – the elements $u \in R$ with the property that $Ru = uR$. If the ring R is acted upon by a group G, then one can introduce an equivariant version of this property: Such an R is called a G-UFD if every nonzero G-invariant prime ideal of R contains a prime element which is a G-eigenvector.

It was shown in [16] that every quantum nilpotent algebra R is a noetherian $\mathcal{H}$-UFD. An $\mathcal{H}$-eigenvector of such a ring R will be called a homogeneous element since this corresponds to the homogeneity property with respect to the canonical induced grading of R by the character lattice of $\mathcal{H}$. In particular, we will use the more compact term of homogeneous prime element of R instead of a prime element of R which is an $\mathcal{H}$-eigenvector.

Sequences of prime elements. Next, we classify the set of all homogeneous prime elements of the chain of subalgebras

$$\{0\} \subset R_1 \subset R_2 \subset \ldots \subset R_N = R$$

of a quantum nilpotent algebra R, where R_k is the subalgebra of R generated by the first k variables $x_1, \ldots, x_k$.

We will denote by $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ the sets of all integers and nonnegative integers, respectively. Given two integers $l \leq k$, set $[l, k] := \{l, l + 1, \ldots, k\}$.

For a function $\eta : [1, N] \to \mathbb{Z}$, define the predecessor function $p : [1, N] \to [1, N] \sqcup \{-\infty\}$ and successor function $s : [1, N] \to [1, N] \sqcup \{+\infty\}$ for its level sets by

$$p(k) = \begin{cases} \max\{l < k \mid \eta(l) = \eta(k)\}, & \text{if such } l \text{ exists,} \\ -\infty, & \text{otherwise,} \end{cases}$$

and

$$s(k) = \begin{cases} \min\{l > k \mid \eta(l) = \eta(k)\}, & \text{if such } l \text{ exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem 1. *Let R be a quantum nilpotent algebra of dimension N. There exist a function $\eta : [1, N] \to \mathbb{Z}$ and elements*

$$c_k \in R_{k-1} \ \text{ for all } \ 2 \leq k \leq N \ \text{ with } \ \delta_k \neq 0$$

such that the elements $y_1, \ldots, y_N \in R$, recursively defined by

$$y_k := \begin{cases} y_{p(k)} x_k - c_k, & \text{if } \ \delta_k \neq 0 \\ x_k, & \text{if } \ \delta_k = 0, \end{cases}$$

are homogeneous and have the property that for every $k \in [1, N]$ the homogeneous prime elements of R_k are precisely the nonzero scalar multiples of the elements

$$y_l \ \text{ for } \ l \in [1, k] \ \text{ with } \ s(l) > k.$$

In particular, y_k is a homogeneous prime element of R_k, for all $k \in [1, N]$. The sequence $y_1, \ldots, y_N$ and the level sets of a function η with these properties are both unique.

Example 2. *Given two subsets $I = \{i_1 < \cdots < i_k\} \subset [1, m]$ and $J = \{j_1 < \cdots < j_k\} \subset [1, n]$, define the quantum minor $\Delta_{I,J} \in R_q[M_{m\times n}]$ by*

$$\Delta_{I,J} = \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} t_{i_1 j_{\sigma(1)}} \ldots t_{i_k j_{\sigma(k)}}$$

where S_k denotes the symmetric group and $\ell : S_k \to \mathbb{Z}_{\geq 0}$ the standard length function.

For the algebra of quantum matrices $R_q[M_{m\times n}]$, the sequence of prime elements from Theorem 1 consists of solid quantum minors; more precisely,

$$y_{(i-1)n+j} = \Delta_{[i-\min(i,j)+1,i],[j-\min(i,j)+1,j]}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Furthermore, the function $\eta : [1, mn] \to \mathbb{Z}$ can be chosen as

$$\eta((i-1)n + j) := j - i.$$

Definition 2. *The cardinality of the range of the function η from Theorem 1 is called the rank of the quantum nilpotent algebra R and is denoted by* $\mathrm{rk}(R)$.

For example, the algebra of quantum matrices $R_q[M_{m\times n}]$ has rank $m + n - 1$.

Embedded quantum tori. An $N \times N$ matrix $\mathbf{q} := (q_{kl})$ with entries in $\mathbb{K}$ is called multiplicatively skewsymmetric if

$$q_{kl} q_{lk} = q_{kk} = 1 \quad \text{for } 1 \leq l, k \leq N.$$

Such a matrix gives rise to the quantum torus $\mathcal{T}_{\mathbf{q}}$ which is the $\mathbb{K}$-algebra with generators $Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}$ and relations

$$Y_k Y_l = q_{kl} Y_l Y_k \quad \text{for } 1 \leq l, k \leq N.$$

Let $\{e_1, \ldots, e_N\}$ be the standard basis of the lattice $\mathbb{Z}^N$. For a quantum nilpotent algebra R of dimension N, define the eigenvalues $\lambda_{kl} \in \mathbb{K}$:

$$h_k \cdot x_l = \lambda_{kl} x_l \quad \text{for } 1 \leq l < k \leq N. \qquad [\mathbf{2}]$$

There exists a unique group bicharacter $\Omega : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^*$ such that

$$\Omega(e_k, e_l) = \begin{cases} 1, & \text{if } k = l, \\ \lambda_{kl}, & \text{if } k > l, \\ \lambda_{lk}^{-1}, & \text{if } k < l. \end{cases}$$

Set $e_{-\infty} := 0$. Define the vectors

$$\overline{e}_k := e_k + e_{p(k)} + \cdots \in \mathbb{Z}^N, \qquad [\mathbf{3}]$$

noting that only finitely many terms in the sum are nonzero. Then $\{\overline{e}_1, \ldots, \overline{e}_N\}$ is another basis of $\mathbb{Z}^N$.

Theorem 2. *For each quantum nilpotent algebra R, the sequence of prime elements from Theorem 1 defines an embedding of the quantum torus $\mathcal{T}_{\mathbf{q}}$ associated to the $N \times N$ multiplicatively skewsymmetric matrix with entries*

$$q_{kl} := \Omega(\overline{e}_k, \overline{e}_l), \quad 1 \leq k, l \leq N$$

into the division ring of fractions $\mathrm{Fract}(R)$ *of R such that $Y_k^{\pm 1} \mapsto y_k^{\pm 1}$, for all $1 \leq k \leq N$.*

Cluster structures on quantum nilpotent algebras

Symmetric quantum nilpotent algebras.

Definition 3. *A quantum nilpotent algebra R as in Definition 1 will be called symmetric if it can be presented as an iterated skew polynomial extension for the reverse order of its generators,*

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \sigma_1^*, \delta_1^*],$$

in such a way that conditions (a)–(c) *in Definition* 1 *are satisfied for some choice of $h_N^*, \ldots, h_1^* \in \mathcal{H}$.*

A quantum nilpotent algebra R is symmetric if and only if it satisfies the Levendorskii–Soibelman type straightening law

$$x_k x_l - \lambda_{kl} x_l x_k = \sum_{n_{l+1}, \ldots, n_{k-1} \in \mathbb{Z}_{\geq 0}} \xi_{n_{l+1}, \ldots, n_{k-1}} x_{l+1}^{n_{l+1}} \ldots x_{k-1}^{n_{k-1}}$$

for all $l < k$ (where the $\xi_\bullet$ are scalars) and there exist $h_k^* \in \mathcal{H}$ such that $h_k^* \cdot x_l = \lambda_{lk}^{-1} x_l$ for all $l > k$. The defining commutation relations for the algebras of quantum matrices $R_q[M_{m\times n}]$ imply that they are symmetric quantum nilpotent algebras.

Denote by Ξ_N the subset of the symmetric group S_N consisting of all permutations that have the property that

$$\tau(k) = \max \tau([1, k-1]) + 1 \quad \text{or} \qquad [\mathbf{4}]$$

$$\tau(k) = \min \tau([1, k-1]) - 1 \qquad [\mathbf{5}]$$

for all $2 \leq k \leq N$. In other words, Ξ_N consists of those $\tau \in S_N$ such that $\tau([1, k])$ is an interval for all $2 \leq k \leq N$. For each $\tau \in \Xi_N$, a symmetric quantum nilpotent algebra R has the presentation

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}] \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}] \qquad [\mathbf{6}]$$

where $\sigma''_{\tau(k)} := \sigma_{\tau(k)}$ and $\delta''_{\tau(k)} := \delta_{\tau(k)}$ if Eq. [**4**] is satisfied, while $\sigma''_{\tau(k)} := \sigma^*_{\tau(k)}$ and $\delta''_{\tau(k)} := \delta^*_{\tau(k)}$ if Eq. [**5**] holds. This presentation satisfies the conditions (a)–(c) in Definition 1 for the elements $h''_{\tau(k)} \in \mathcal{H}$, given by $h''_{\tau(k)} := h_{\tau(k)}$ in case of Eq. [**4**] and $h''_{\tau(k)} := h^*_{\tau(k)}$ in case of Eq. [**5**]. The use of the term symmetric in Definition 3 is motivated by the presentations in Eq. [**6**] parametrized by the elements of the subset Ξ_N of the symmetric group S_N.

Proposition 3. *For every symmetric quantum nilpotent algebra R, the h_k^*-eigenvalues of x_k, to be denoted by λ_k^*, satisfy*

$$\lambda_k^* = \lambda_l^*$$

for all $1 \leq k, l \leq N$ such that $\eta(k) = \eta(l)$ and $s(k) \neq +\infty$, $s(l) \neq +\infty$. They are related to the eigenvalues λ_l by

$$\lambda_k^* = \lambda_l^{-1}$$

for all $1 \leq k, l \leq N$ such that $\eta(k) = \eta(l)$ and $s(k) \neq +\infty$, $p(l) \neq -\infty$.

Construction of exchange matrices. Our construction of a quantum cluster algebra structure on a symmetric quantum nilpotent algebra R of dimension N will have as the set of exchangeable indices

$$\mathbf{ex} := \{k \in [1, N] \mid s(k) \neq +\infty\}. \qquad [\mathbf{7}]$$

We will impose the following two mild conditions:

(A) The field $\mathbb{K}$ contains square roots $\sqrt{\lambda_{kl}}$ of the scalars λ_{kl} for $1 \leq l < k \leq N$ such that the subgroup of $\mathbb{K}^*$ generated by all of them contains no elements of order 2.

(B) There exist positive integers d_n, $n \in \mathrm{range}(\eta)$, for the function η from Theorem 1 such that

$$(\lambda_k^*)^{d_{\eta(l)}} = (\lambda_l^*)^{d_{\eta(k)}}$$

for all $k, l \in \mathbf{ex}$.

Remark 1. *With the exception of some two-cocycle twists, for all of the quantum nilpotent algebras R coming from the theory of quantum groups, the subgroup of $\mathbb{K}^*$ generated by $\{\lambda_{kl} \mid 1 \le l < k \le N\}$ is torsionfree. For all such algebras, condition* (A) *only requires that* $\mathbb{K}$ *contains square roots of the scalars* λ_{kl}.

All symmetric quantum nilpotent algebras that we are aware of satisfy

$$\lambda_k^* = q^{m_k} \quad \text{for } 1 \le k \le N$$

for some non-root of unity $q \in \mathbb{K}^*$ and positive integers $m_1, \ldots, m_N$. Proposition 3 implies that all of them satisfy the condition (B).

The set $\mathbf{ex}$ has cardinality $N - \mathrm{rk}(R)$ where $\mathrm{rk}(R)$ is the rank of the quantum nilpotent algebra R. By an $N \times \mathbf{ex}$ matrix we will mean a matrix of size $N \times (N - \mathrm{rk}(R))$ whose rows and columns are indexed by the sets $[1, N]$ and $\mathbf{ex}$, respectively. The set of such matrices with integer entries will be denoted by $M_{N\times\mathbf{ex}}(\mathbb{Z})$.

Theorem 4. *For every symmetric quantum nilpotent algebra R of dimension N satisfying conditions* (A) *and* (B), *there exists a unique matrix $\widetilde{B} = (b_{lk}) \in M_{N\times\mathbf{ex}}(\mathbb{Z})$ whose columns satisfy the following two conditions:*

- $\Omega\Big(\sum_{l=1}^N b_{lk}\overline{e_l}, \overline{e}_n\Big) = \begin{cases} \lambda_n^*, & \text{if } k = n \\ 1, & \text{if } k \neq n, \end{cases}$ *for all $k \in \mathbf{ex}$ and $n \in [1, N]$* (*a system of linear equations written in a multiplicative form*), *and*
- *the products $y_1^{b_{1k}} \ldots y_N^{b_{Nk}}$ are fixed under $\mathcal{H}$ for all $k \in \mathbf{ex}$* (*a homogeneity condition*).

The second condition can be written in an explicit form using the fact that y_k is an $\mathcal{H}$-eigenvector and its eigenvalue equals the product of the $\mathcal{H}$-eigenvalues of $x_k, \ldots, x_{p^{n_k}(k)}$ where n_k is the maximal nonnegative integer n such that $p^n(k) \neq -\infty$. This property is derived from Theorem 1.

Example 3. *It follows from Example 2 that in the case of the algebras of quantum matrices $R_q[M_{m\times n}]$,*

$$\mathbf{ex} = \{(i-1)n + j \mid 1 \le i < m,\ 1 \le j < n\}.$$

The bicharacter $\Omega : \mathbb{Z}^{mn} \times \mathbb{Z}^{mn} \to \mathbb{K}^*$ is given by

$$\Omega(e_{(i-1)n+j}, e_{(k-1)n+l}) = q^{\delta_{jl}\mathrm{sign}(k-i)+\delta_{ik}\mathrm{sign}(l-j)}$$

for all $1 \le i, k \le m$ and $1 \le j, l \le n$. Furthermore,

$$\lambda_s^* = q^2 \quad \text{for } 1 \le s \le mn.$$

After an easy computation one finds that the unique solution of the system of equations in Theorem 4 is given by the matrix $\widetilde{B} = (b_{(i-1)n+j,(k-1)n+j}) \in M_{mn\times\mathbf{ex}}(\mathbb{Z})$ with entries

$$b_{(i-1)n+j,(k-1)n+l} = \begin{cases} \pm 1, & \text{if } i = k,\ l = j \pm 1 \\ & \text{or } j = l,\ k = i \pm 1 \\ & \text{or } i = k \pm 1,\ j = l \pm 1, \\ 0, & \text{otherwise} \end{cases}$$

for all $i, k \in [1, m]$ and $j, l \in [1, n]$.

Cluster algebra structures. Let us consider a symmetric quantum nilpotent algebra R of dimension N. When Theorem 1 is applied to the presentation of R from Eq. [**6**] associated to the element $\tau \in \Xi_N$, we obtain a sequence of prime elements

$$y_{\tau,1}, \ldots, y_{\tau,N} \in R.$$

Similarly, applying Theorem 4 to the presentation from Eq. [**6**] we obtain the integer matrix

$$\widetilde{B}_\tau \in M_{N\times\mathbf{ex}}(\mathbb{Z}).$$

For $\tau = \mathrm{id}$ we recover the original sequence $y_1, \ldots, y_N$ and matrix $\widetilde{B}$.

Theorem 5. *Every symmetric quantum nilpotent algebra R of dimension N satisfying the conditions* (A) *and* (B) *possesses a canonical structure of quantum cluster algebra for which no frozen cluster variables are inverted. Its initial seed has:*

- *Cluster variables $\zeta_1 y_1, \ldots, \zeta_N y_N$ for some $\zeta_1, \ldots, \zeta_N \in \mathbb{K}^*$, among which the variables indexed by the set $\mathbf{ex}$ from Eq. [**7**] are exchangeable and the rest are frozen.*
- *Exchange matrix $\widetilde{B}$ given by Theorem 4.*

Furthermore, this quantum cluster algebra aways coincides with the corresponding upper quantum cluster algebra.

*After an appropriate rescaling, each of the generators x_k of such an algebra R given by Eq. [**1**] is a cluster variable. Moreover, for each element τ of the subset Ξ_N of the symmetric group S_N, R has a seed with cluster variables obtained by reindexing and rescaling the sequence of prime elements $y_{\tau,1}, \ldots, y_{\tau,N}$. The exchange matrix of this seed is the matrix $\widetilde{B}_\tau$.*

The base fields of the algebras covered by this theorem can have arbitrary characteristic. We refer the reader to Theorem 8.2 in [17] for a complete statement of the theorem, which includes additional results, and gives explicit formulas for the scalars $\zeta_1, \ldots, \zeta_N$ and the necessary reindexing and rescaling of the sequences $y_{\tau,1}, \ldots, y_{\tau,N}$.

Define the following automorphism of the lattice $\mathbb{Z}^N$:

$$g = l_1 e_1 + \cdots + l_N e_N \ \mapsto\ \overline{g} := l_1\overline{e}_1 + \cdots + l_N\overline{e}_N$$

for all $l_1, \ldots, l_N \in \mathbb{Z}$ in terms of the vectors $\overline{e}_1, \ldots, \overline{e}_N$ from Eq. [**3**]. The construction of seeds for quantum cluster algebras in [11] requires assigning quantum frames to all of them. The quantum frame $M : \mathbb{Z}^N \to \mathrm{Fract}(R)$ associated to the initial seed for the quantum cluster algebra structure in Theorem 5 is uniquely reconstructed from the rules

$$M(e_k) = \zeta_k y_k \quad \text{for } 1 \le k \le N \text{ and}$$
$$M(f+g) = \Omega(\overline{f}, \overline{g}) M(f) M(g) \quad \text{for } f, g \in \mathbb{Z}^N.$$

Analogous formulas describe the quantum frames associated to the elements τ of the set Ξ_N.

Example 4. *The cluster variables in the initial seed from Theorem 5 for $R_q[M_{m\times n}]$ are*

$$\Delta_{[i-\min(i,j)+1,i],[j-\min(i,j)+1,j]}$$

where $1 \le i \le m$ and $1 \le j \le n$. The ones with $i = m$ or $j = n$ are frozen. This example and Example 3 recover the quantum cluster algebra structure of [12] *on $R_q[M_{m\times n}]$.*

We finish the section by raising two questions concerning the line of Theorem 5:

1. If a symmetric quantum nilpotent algebra has two iterated skew polynomial extension presentations that satisfy the assumptions in Definitions 1, 3 and these two presentations are not obtained from each other by a permutation in Ξ_N, how are the corresponding quantum cluster algebra structures on R related?

2. What is the role of the quantum seeds of a symmetric quantum nilpotent algebra R indexed by Ξ_N among the set of all quantum seeds? Is there a generalization of Theorem 5 that constructs a larger family of quantum seeds using sequences of prime elements in chains of subalgebras?

For the first question we expect that the two quantum cluster algebra structures on R are the same (i.e., the corresponding quantum seeds are mutation equivalent) if the maximal tori for the two presentations are the same and act in the same way on R. However, proving such a fact appears to be difficult due to the generality of the setting. The condition on the tori is natural in light of Theorem 5.5 in [18] which proves the existence of a canonical maximal torus for a quantum nilpotent algebra. Without imposing such a condition, the cluster algebra structures can be completely unrelated. For example, every polynomial algebra

$$R = \mathbb{K}[x_1, \dots, x_N]$$

over an infinite field $\mathbb{K}$ is a symmetric quantum nilpotent algebra with respect to the natural action of $(\mathbb{K}^*)^N$. The quantum cluster algebra structure on R constructed in Theorem 5 has no exchangeable indices and its frozen variables are $x_1, \dots, x_N$. Each polynomial algebra has many different presentations associated to the elements of its automorphism group and the corresponding cluster algebra structures are not related in general.

Applications to quantum groups

Quantized universal enveloping algebra. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra of rank r with Cartan matrix (c_{ij}). For an arbitrary field $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$, one defines [19] the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ with generators

$$K_i^{\pm 1}, E_i, F_i, \ 1 \le i \le r$$

and relations

$$K_i^{-1} K_i = K_i K_i^{-1} = 1, \ \ K_i K_j = K_j K_i,$$
$$K_i E_j K_i^{-1} = q_i^{c_{ij}} E_j, \ \ K_i F_j K_i^{-1} = q_i^{-c_{ij}} F_j,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{n=0}^{1-c_{ij}} (-1)^n \begin{bmatrix} 1-c_{ij} \\ n \end{bmatrix}_{q_i} (E_i)^n E_j (E_i)^{1-c_{ij}-n} = 0, \ \ i \ne j,$$

together with the analogous relation for the generators F_i. Here $\{d_1, \dots, d_r\}$ is the collection of relatively prime positive integers such that the matrix $(d_i c_{ij})$ is symmetric, and $q_i := q^{d_i}$. The algebra $\mathcal{U}_q(\mathfrak{g})$ is a Hopf algebra with coproduct

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$
$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i.$$

The quantum Schubert cell algebras $\mathcal{U}^{\pm}[w]$, parametrized by the elements w of the Weyl group W of $\mathfrak{g}$, were introduced by De Concini–Kac–Procesi [20] and Lusztig [21]. In [12] the term quantum unipotent groups was used. These algebras are quantum analogs of the universal enveloping algebras $\mathcal{U}(\mathfrak{n}^{\pm} \cap w(\mathfrak{n}^{\mp}))$ where $\mathfrak{n}^{\pm}$ are the nilradicals of a pair of opposite Borel subalgebras of $\mathfrak{g}$. The torus $\mathcal{H} := (\mathbb{K}^*)^r$ acts on $\mathcal{U}_q(\mathfrak{g})$ by

$$h \cdot K_i^{\pm 1} = K_i^{\pm 1}, \ \ h \cdot E_i = \xi_i E_i, \ \ h \cdot F_i = \xi_i^{-1} F_i, \qquad [8]$$

for all $h = (\xi_1, \dots, \xi_r) \in \mathcal{H}$ and $1 \le i \le r$. The subalgebras $\mathcal{U}^{\pm}[w]$ are preserved by this action.

Denote by $\alpha_1, \dots, \alpha_r$ the set of simple roots of $\mathfrak{g}$ and by $s_1, \dots, s_r \in W$ the corresponding set of simple reflections. All algebras $\mathcal{U}^{\pm}[w]$ are symmetric quantum nilpotent algebras for all base fields $\mathbb{K}$ and non-roots of unity $q \in \mathbb{K}^*$. In fact, to each reduced expression

$$w = s_{i_1} \dots s_{i_N},$$

one associates a presentation of $\mathcal{U}^{\pm}[w]$ that satisfies Definitions 1 and 3 as follows. Consider the Weyl group elements

$$w_{\le k} := s_{i_1} \dots s_{i_k}, \ 1 \le k \le N, \quad \text{and} \quad w_{\le 0} := 1.$$

In terms of them the roots of the Lie algebra $\mathfrak{n}^+ \cap w(\mathfrak{n}^-)$ are

$$\beta_1 = \alpha_{i_1}, \ \beta_2 = w_{\le 1}(\alpha_{i_2}), \ \dots, \ \beta_N := w_{\le N-1}(\alpha_{i_N}).$$

One associates [19, 21] to those roots the Lusztig root vectors $E_{\beta_1}, F_{\beta_1}, \dots, E_{\beta_N}, F_{\beta_N} \in \mathcal{U}_q(\mathfrak{g})$. The quantum Schubert cell algebra $\mathcal{U}^-[w]$, defined as the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $F_{\beta_1}, \dots, F_{\beta_N}$, has an iterated skew polynomial extension presentation of the form

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \dots [F_{\beta_N}; \sigma_N, \delta_N] \qquad [9]$$

for which conditions (a)–(c) of Definition 1 are satisfied with respect to the action Eq. [8]. Moreover, this presentation satisfies the condition for a symmetric quantum nilpotent algebra in Definition 3 because of the Levendorskii–Soibelman straightening law [19] in $\mathcal{U}_q(\mathfrak{g})$. The opposite algebra $\mathcal{U}^+[w]$, generated by $E_{\beta_1}, \dots, E_{\beta_N}$, has analogous properties and is actually isomorphic [19] to $\mathcal{U}^-[w]$.

The algebra $R_q[M_{m \times n}]$ is isomorphic to one of the algebras $\mathcal{U}^-[w]$ for $\mathfrak{g} = \mathfrak{sl}_{m+n}$ and a certain choice of $w \in S_{m+n}$.

Quantized function algebras. The irreducible finite dimensional modules of $\mathcal{U}_q(\mathfrak{g})$ on which the elements K_i act diagonally via powers of the scalars q_i are parametrized by the set P_+ of dominant integral weights of $\mathfrak{g}$. The module corresponding to such a weight λ will be denoted by $V(\lambda)$.

Let G be the connected, simply connected algebraic group with Lie algebra $\mathfrak{g}$. The Hopf subalgebra of $\mathcal{U}_q(\mathfrak{g})^*$ spanned by the matrix coefficients $c^{\lambda}_{f,y}$ of all modules $V(\lambda)$ (where $f \in V(\lambda)^*$ and $y \in V(\lambda)$) is denoted by $R_q[G]$ and called the quantum group corresponding to G. The weight spaces $V(\lambda)_{w\lambda}$ are one dimensional for all Weyl group elements w. Considering the fundamental representations $V(\varpi_1), \dots, V(\varpi_r)$, and a normalized covector and vector in each of those weight spaces, one defines [11] the quantum minors

$$\Delta^i_{w,v} = \Delta_{w\varpi_i, v\varpi_i} \in R_q[G], \quad 1 \le i \le r, \ w, v \in W.$$

The subalgebras of $R_q[G]$ spanned by the elements of the form $c^{\lambda}_{f,y}$ where y is a highest or a lowest weight vector of $V(\lambda)$ and $f \in V(\lambda)^*$ are denoted by $R^{\pm}$. They are quantum analogs of the base affine space of G. With the help of the Demazure modules $V^+_w(\lambda) = \mathcal{U}^+_q(\mathfrak{g}) V(\lambda)_{w\lambda}$ (where $\mathcal{U}^+_q(\mathfrak{g})$ is the unital subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $E_1, \dots, E_r$) one defines the ideal

$$I^+_w = \mathrm{Span}\{c^{\lambda}_{f,y} \mid \lambda \in P_+, \ f|_{V^+_w(\lambda)} = 0, \ y \in V(\lambda)_{\lambda}\}$$

of R^+. Analogously one defines an ideal I^-_w of R^-. For all pairs of Weyl group elements (w, v), the quantized coordinate ring of the double Bruhat cell [9]

$$G^{w,v} := B^+ w B^+ \cap B^- v B^-,$$

where $B^{\pm}$ are opposite Borel subgroups of G, is defined by

$$R_q[G^{w,v}] := (I^+_w R^- + R^+ I^-_v)[(\Delta^i_{w,1})^{-1}, (\Delta^i_{vw_0,w_0})^{-1}]$$

where the localization is taken over all $1 \le i \le r$ and w_0 denotes the longest element of the Weyl group W.

To connect cluster algebra structures on the two kinds of algebras (quantum Schubert cells and quantum double Bruhat cells), we use Joseph's subalgebras S_w^+ of $(R^+/I_w^+)[(\Delta^i_{w,1})^{-1}, 1 \leq i \leq r]$ defined in [22]. They are the subalgebras generated by the elements

$$(\Delta^i_{w,1})^{-1}(c^{\varpi_i}_{f,y} + I^+_w),$$

for $1 \leq i \leq r$, $f \in V(\varpi_i)^*$, and y a highest weight vector of $V(\varpi_i)$. These algebras played a major role in the study of the spectra of quantum groups [22, 23] and the quantum Schubert cell algebras [24]. In [23], the second author constructed an algebra antiisomorphism

$$\varphi_w : S^+_w \to \mathcal{U}^-[w].$$

An earlier variant of it for $\mathcal{U}_q(\mathfrak{g})$ equipped with a different coproduct appeared in [24].

Cluster structures on quantum Schubert cell algebras. Denote by $\langle ., . \rangle$ the Weyl group invariant bilinear form on the vector space $\mathbb{R}\alpha_1 \oplus \cdots \oplus \mathbb{R}\alpha_r$ normalized by $\langle \alpha_i, \alpha_i \rangle = 2$ for short roots α_i. Let $\|\gamma\|^2 := \langle \gamma, \gamma \rangle$.

Fix a Weyl group element w and consider the quantum Schubert cell algebra $\mathcal{U}^-[w]$. A reduced expression $w = s_{i_1} \ldots s_{i_N}$ gives rise to the presentation in Eq. **[9]** of $\mathcal{U}^-[w]$ as a symmetric quantum nilpotent algebra. The result of the application of Theorem 1 to it is as follows. The function η can be chosen as

$$\eta(k) = i_k \quad \text{for all} \;\; 1 \leq k \leq N.$$

The predecessor function p is the function $k \mapsto k^-$ which plays a key role in the works of Fomin and Zelevinsky [1, 9],

$$k^- := \begin{cases} \max\{l < k \mid i_l = i_k\}, & \text{if such } l \text{ exists,} \\ -\infty, & \text{otherwise.} \end{cases}$$

The successor function s is the function $k \mapsto k^+$ in [1, 9]. The sequence of prime elements $y_1, \ldots, y_N$ consists of scalar multiples of the elements

$$\varphi_w\left(\Delta^{i_k}_{w_{\leq p^{n_k}(k)-1},1}(\Delta^{i_k}_{w_{\leq k},1})^{-1}\right), \quad 1 \leq k \leq N$$

where n_k denotes the maximal nonnegative integer n such that $p^n(k) \neq -\infty$. The presentation in Eq. **[9]** of $\mathcal{U}^-[w]$ as a symmetric quantum nilpotent algebra satisfies the conditions (A) and (B) if $\sqrt{q} \in \mathbb{K}$, in which case Theorem 5 produces a canonical cluster algebra structure on $\mathcal{U}^-[w]$. Among all the clusters in Theorem 5, indexed by the elements of the subset Ξ_N of the symmetric group S_N, the one corresponding to the longest element of S_N is closest to the combinatorial setting of [10, 11]. It goes with the reverse presentation of $\mathcal{U}^-[w]$,

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_N}][F_{\beta_{N-1}}; \sigma^*_{N-1}, \delta^*_{N-1}] \ldots [F_{\beta_1}; \sigma^*_1, \delta^*_1].$$

The transition from the original presentation in Eq. **[9]** to the above one amounts to interchanging the roles of the predecessor and successor functions. As a result, the set of exchangeable indices for the latter presentation is

$$\mathbf{ex}_w := \{k \in [1, N] \mid k^- \neq -\infty\}. \qquad [10]$$

Theorem 6. *Consider an arbitrary finite dimensional complex simple Lie algebra $\mathfrak{g}$, a Weyl group element $w \in W$, a reduced expression of w, an arbitrary base field $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$ such that $\sqrt{q} \in \mathbb{K}$. The quantum Schubert cell algebra $\mathcal{U}^-[w]$ possesses a canonical quantum cluster algebra structure for which no frozen cluster variables are inverted and the set of exchangeable indices is $\mathbf{ex}_w$. Its initial seed consists of the cluster variables*

$$\sqrt{q}^{\,\|(w - w_{\leq k-1})\varpi_{i_k}\|^2/2}\varphi_w\left(\Delta^{i_k}_{w_{\leq k-1},1}(\Delta^{i_k}_{w,1})^{-1}\right),$$

$1 \leq k \leq N$. The exchange matrix $\widetilde{B}$ of this seed has entries given by

$$b_{kl} = \begin{cases} 1, & \text{if } k = p(l) \\ -1, & \text{if } k = s(l) \\ c_{i_k i_l}, & \text{if } p(k) < p(l) < k < l \\ -c_{i_k, i_l}, & \text{if } p(l) < p(k) < l < k \\ 0, & \text{otherwise} \end{cases}$$

for all $1 \leq k \leq N$ and $l \in \mathbf{ex}_w$. Furthermore, this quantum cluster algebra equals the corresponding upper quantum cluster algebra. For all $k \in [1, N]$, $m \in \mathbb{Z}_{\geq 0}$ such that $s^m(k) \in [1, N]$, the elements

$$\sqrt{q}^{\,\|(w_{\leq s^m(k)} - w_{\leq k-1})\varpi_{i_k}\|^2/2} \times \\ \times \varphi_{w_{\leq s^m(k)}}\left(\Delta^{i_k}_{w_{\leq k-1},1}(\Delta^{i_k}_{w_{\leq s^m(k)},1})^{-1}\right)$$

are cluster variables of $\mathcal{U}^-[w]$.

For symmetric Kac-Moody algebras $\mathfrak{g}$ the theorem is due to Geiß, Leclerc, and Schröer [12]. Our proof also works for all Kac-Moody algebras $\mathfrak{g}$, but here we restrict to the finite dimensional case for simplicity of the exposition. Theorem 6 is proved in Section 10 of [17].

Examples 3 and 4 can be recovered as special cases of Theorem 6 for $\mathfrak{g} = \mathfrak{sl}_{m+n}$ and a particular choice of the Weyl group element $w \in S_{m+n}$. In this case the torus action can be used to kill the power of $\sqrt{q}$.

Remark 2. *It follows from the definition of the antiisomorphism $\varphi_w : S^+_w \to \mathcal{U}^-[w]$ in [23, 24] that the element*

$$\varphi_w\left(\Delta^{i_k}_{w_{\leq k-1},1}(\Delta^{i_k}_{w,1})^{-1}\right) \qquad [11]$$

is obtained by evaluating

$$\left(\Delta^{i_k}_{w_{\leq k-1},w} \otimes \mathrm{id}\right)(\mathcal{R}^w)$$

*where $\mathcal{R}^w$, called the R-matrix for the Weyl group element w, equals the infinite sum $\sum_j u^+_j \otimes u^-_j$ for dual bases $\{u^+_j\}$ and $\{u^-_j\}$ of $\mathcal{U}^+[w]$ and $\mathcal{U}^-[w]$. Because of this, the element in Eq. **[11]** can be identified with $\Delta^{i_k}_{w_{\leq k-1},w}$ and thus can be thought of as a quantum minor. Such a construction of cluster variables of $\mathcal{U}^-[w]$ via quantum minors is due to [12], which used linear maps that are not algebra (anti)isomorphisms.*

More generally,

$$\mathcal{U}^-[w_{\leq j}] \subseteq \mathcal{U}^-[w] \quad \text{for} \;\; 1 \leq j \leq N$$

and the cluster variables in Theorem 6,

$$\varphi_{w_{\leq s^m(k)}}\left(\Delta^{i_k}_{w_{\leq k-1},1}(\Delta^{i_k}_{w_{\leq s^m(k)},1})^{-1}\right) \in \mathcal{U}^-[w_{\leq s^m(k)}],$$

can be identified with the quantum minors $\Delta^{i_k}_{w_{\leq k-1},w_{\leq s^m(k)}}$.

The Berenstein–Zelevinsky conjecture. Consider a pair of Weyl group elements (w, v) with reduced expressions

$$w = s_{i_1} \dots s_{i_N} \quad \text{and} \quad v = s_{i'_1} \dots s_{i'_M}.$$

Let $\eta : [1, r + M + N] \to [1, r]$ be the function given by

$$\eta(k) = \begin{cases} k, & \text{for } 1 \le k \le r, \\ i'_{k-r}, & \text{for } r+1 \le k \le r+M, \\ i_{k-r-M}, & \text{for } r+M+1 \le k \le r+M+N. \end{cases}$$

The following set will be used as the set of exchangeable indices for a quantum cluster algebra structure on $R_q[G^{w,v}]$:

$$\mathbf{ex}_{w,v} := [1, k] \sqcup \{k \in [r+1, r+M+N] \mid s(k) \neq +\infty\}$$

where s is the successor function for the level sets of η. Set $\epsilon(k) := 1$ for $k \le r + M$ and $\epsilon(k) := -1$ for $k > r + M$. Following [11], define the $(r + M + N) \times \mathbf{ex}$ matrix $\widetilde{B}_{w,v}$ with entries

$$b_{kl} := \begin{cases} -\epsilon(l), & \text{if } k = p(l), \\ -\epsilon(l) c_{\eta(k),\eta(l)}, & \text{if } k < l < s(k) < s(l),\ \epsilon(l) = \epsilon(s(k)) \\ & \text{or } k < l \le r+M < s(l) < s(k), \\ \epsilon(k) c_{\eta(k),\eta(l)}, & \text{if } l < k < s(l) < s(k),\ \epsilon(k) = \epsilon(s(l)) \\ & \text{or } l < k \le r+M < s(k) < s(l), \\ \epsilon(k), & \text{if } k = s(l), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 7. (Berenstein–Zelevinsky conjecture, [11]) *Let G be an arbitrary complex simple Lie group and (w, v) a pair of elements of the corresponding Weyl group. For any base field $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$ such that $\sqrt{q} \in \mathbb{K}^*$, the quantum double Bruhat cell algebra $R_q[G^{w,v}]$ possesses a canonical structure of quantum cluster algebra for which all frozen cluster variables are inverted and the set of exchangeable indices is $\mathbf{ex}_{w,v}$. The initial seed has exchange matrix $\widetilde{B}_{w,v}$ defined above and cluster variables $y_1, \dots, y_{r+M+N} \in R_q[G^{w,v}]$ given by*

$$y_k = \begin{cases} \Delta^k_{1,v^{-1}}, & \text{for } 1 \le k \le r, \\ \xi_k \Delta^{i'_{k-r}}_{1,v^{-1} v_{\le k-r}}, & \text{for } r+1 \le k \le r+M, \\ \xi_k \Delta^{i_{k-r-M}}_{w_{\le k-r-M},1}, & \text{for } r+M+1 \le k \le r+M+N \end{cases}$$

for some scalars $\xi_k \in \mathbb{K}^*$ of a similar nature to the ones in Theorem 6.

Furthermore, this quantum cluster algebra coincides with the corresponding upper quantum cluster algebra.

We briefly sketch the relationship of the quantum double Bruhat cell algebras $R_q[G^{w,v}]$ to quantum nilpotent algebras and the proof of the theorem. We first show, using results of Joseph [22], that $R_q[G^{w,v}]$ is a localization of

$$(S^+_w \bowtie S^-_v) \,\#\, \mathbb{K}[(\Delta^1_{1,v^{-1}})^{\pm 1}, \dots, (\Delta^r_{1,v^{-1}})^{\pm 1}]$$

where S^-_v is the Joseph subalgebra of R^- defined in a similar way [22] to the subalgebra S^+_w of R^+. The "bicrossed" and smash products are defined [22, 23] from the Drinfeld R-matrix commutation relations of $R_q[G]$. Using the antiisomorphism φ_w and its negative counterpart (which turns out to be an isomorphism [23]), one converts

$$S^+_w \bowtie S^-_v \cong \mathcal{U}^-[w]^{\mathrm{op}} \bowtie \mathcal{U}^+[v],$$

where R^{op} stands for the algebra with opposite product. We then establish that the right hand algebra above is a symmetric quantum nilpotent algebra satisfying the conditions (A) and (B). The proof is completed by applying Theorem 5, and showing that the localization that we started with is, in fact, a localization by all frozen cluster variables.

Poisson nilpotent algebras and cluster algebras

In this subsection, we will assume that the base field $\mathbb{K}$ has characteristic 0. A prime element p of a Poisson algebra R with Poisson bracket $\{.,.\}$ will be called Poisson prime if

$$\{R, p\} = Rp.$$

In other words, this requires that the principal ideal Rp be a Poisson ideal as well as a prime ideal.

For a commutative algebra R equipped with a rational action of a torus $\mathcal{H}$ by algebra automorphisms, we will denote by ∂_h the derivation of R corresponding to an element h of the Lie algebra of $\mathcal{H}$.

Definition 4. *A nilpotent semi-quadratic Poisson algebra is a polynomial algebra $\mathbb{K}[x_1, \dots, x_N]$ with a Poisson structure $\{.,.\}$ and a rational action of a torus $\mathcal{H} = (\mathbb{K}^*)^r$ by Poisson algebra automorphisms for which $x_1, \dots, x_N$ are $\mathcal{H}$-eigenvectors and there exist elements $h_1, \dots, h_N$ in the Lie algebra of $\mathcal{H}$ such that the following two conditions are satisfied for $1 \le k \le N$:*

(a) *For all $b \in R_{k-1} := \mathbb{K}[x_1, \dots, x_{k-1}]$*

$$\{x_k, b\} = \partial_{h_k}(b) x_k + \delta_k(b)$$

for some $\delta_k(b) \in R_{k-1}$ and the map $\delta_k \colon R_{k-1} \to R_{k-1}$ is locally nilpotent.

(b) *The h_k-eigenvalue of x_k is non-zero.*

Such an algebra will be called symmetric if the above condition is satisfied for the reverse order of generators $x_N, \dots, x_1$ (with different choices of elements $h_\bullet$).

The adjective semi-quadratic refers to the leading term in the Poisson bracket $\{x_k, x_l\}$ which is forced to have the form $\lambda_{kl} x_k x_l$ for some $\lambda_{kl} \in \mathbb{K}^*$.

Theorem 8. *Every symmetric nilpotent semi-quadratic Poisson algebra as above satisfying the Poisson analog of condition* (B) *has a canonical structure of cluster algebra for which no frozen variables are inverted and the compatible Poisson bracket in the sense of Gekhtman–Shapiro–Vainshtein* [25] *is $\{.,.\}$. Its initial cluster consists, up to scalar multiples, of those Poisson prime elements of the chain of Poisson subalgebras*

$$\mathbb{K}[x_1] \subset \mathbb{K}[x_1, x_2] \subset \ldots \subset \mathbb{K}[x_1, \dots, x_N]$$

which are $\mathcal{H}$-eigenvectors. Each generator x_k, $1 \le k \le N$, is a cluster variable of this cluster algebra.

Furthermore, this cluster algebra coincides with the corresponding upper cluster algebra.

Applying this theorem in a similar fashion to Theorem 7 we obtain the following result.

Theorem 9. *Let G be an arbitrary complex simple Lie group. For all pairs of elements (w, v) of the Weyl group of G, the Berenstein–Fomin–Zelevinsky upper cluster algebra* [10] *on the coordinate ring of the double Bruhat cell $G^{w,v}$ coincides with the corresponding cluster algebra. In other words, the coordinate rings of all double Bruhat cells $\mathbb{C}[G^{w,v}]$ are cluster algebras with initial seeds constructed in* [10].

ACKNOWLEDGMENTS. We are grateful to S. Fomin, A. Berenstein, Ph. Di Francesco, R. Kedem, B. Keller, A. Knutson, B. Leclerc, N. Reshetikhin, D. Rupel and A. Zelevinsky for helpful discussions and comments. We are also thankful to the referee, whose suggestions helped improve the exposition. Moreover, we would like to thank MSRI for its hospitality during the programs in "Cluster Algebras" and "Noncommutative Algebraic Geometry and Representation Theory" when parts of this project were completed. This work was partially supported by the National Science Foundation grants DMS-0800948 (to K.R.G.), and DMS-1001632 and DMS-1303038 (to M.T.Y.).

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