INTEGRAL QUANTUM CLUSTER STRUCTURES

K. R. GOODEARL AND M. T. YAKIMOV

Abstract. We prove a general theorem for constructing integral quantum cluster algebras over $\mathbb{Z}[q^{\pm 1/2}]$, namely that under mild conditions the integral forms of quantum nilpotent algebras always possess integral quantum cluster algebra structures. These algebras are then shown to be isomorphic to the corresponding upper quantum cluster algebras, again defined over $\mathbb{Z}[q^{\pm 1/2}]$. Previously, this was only known for acyclic quantum cluster algebras. The theorem is applied to prove that for every symmetrizable Kac–Moody algebra $\mathfrak{g}$ and Weyl group element w, the dual canonical form $A_q(\mathfrak{n}_+(w))_{\mathbb{Z}[q^{\pm 1}]}$ of the corresponding quantum unipotent cell has the property that $A_q(\mathfrak{n}_+(w))_{\mathbb{Z}[q^{\pm 1}]} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Z}[q^{\pm 1/2}]$ is isomorphic to a quantum cluster algebra over $\mathbb{Z}[q^{\pm 1/2}]$ and to the corresponding upper quantum cluster algebra over $\mathbb{Z}[q^{\pm 1/2}]$.

1. Introduction

1.1. **Problems for integral quantum custer algebras.** Cluster algebras were introduced by Fomin and Zelevinsky in [6] and have been applied to a number of diverse areas such as representation theory, combinatorics, Poisson and algebraic geometry, mathematical physics and others. Their quantum counterparts, introduced by Berenstein and Zelevinsky [2], are similarly the topic of intensive research from various standpoints. In the uniparameter quantum case it is desirable to work over the minimal ring of definition, namely over

$$\mathcal{A}^{1/2} := \mathbb{Z}[q^{\pm 1/2}], \tag{1.1}$$

where q is the quantum parameter. We will refer to such structures as *integral quantum cluster algebras*. Two fundamental problems that are being investigated are:

(1) Given an algebra R over the rational function field $\mathcal{F}^{1/2} := \mathbb{Q}(q^{1/2})$ and an integral form $R_{\mathcal{A}^{1/2}}$ of R over $\mathcal{A}^{1/2}$ (i.e., $R \cong R_{\mathcal{A}^{1/2}} \otimes_{\mathcal{A}^{1/2}} \mathcal{F}^{1/2}$), when is $R_{\mathcal{A}^{1/2}}$ isomorphic to an integral quantum cluster algebra?
(2) When is the quantum cluster algebra $\mathbf{A}$ in Problem (1) equal to the corresponding upper quantum cluster algebra $\mathbf{U}$ defined over $\mathcal{A}^{1/2}$?

The best known result on Problem (1) is a theorem of Kang, Kashiwara, Kim and Oh [23] that the dual canonical forms (over $\mathcal{A}^{1/2}$) of the quantum unipotent cells for all symmetric Kac–Moody algebras possess integral quantum cluster algebra structures. Berenstein and Zelevinsky [2] proved the equality $\mathbf{A} = \mathbf{U}$ in the acyclic case. Such an equality was proved by Muller [35] for (quantum) cluster algebras that are source-sink decomposable in the case when all frozen variables are inverted. We are not aware of any affirmative solutions of Problem (2) for non-acyclic quantum cluster algebras when frozen variables are not

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inverted. A recent result of Geiß, Leclerc and Schröer [10] establishes an equality of the form

$$\mathbf{A} \otimes_{\mathcal{A}^{1/2}} \mathbb{Q}[q^{\pm 1/2}] = \mathbf{U} \otimes_{\mathcal{A}^{1/2}} \mathbb{Q}[q^{\pm 1/2}]$$

under the assumptions that $\mathbf{A}$ is connected $\mathbb{Z}_{\geq 0}$-graded with homogeneous cluster variables and that such an equality holds on the classical level.

1.2. **Main results.** In this paper we provide affirmative answers to both Problems (1) and (2) in wide generality. As an application, affirmative answers to Problems (1) and (2) are obtained for the dual canonical forms of the quantum unipotent cells for all symmetrizable Kac–Moody algebras.

For an iterated skew polynomial extension

$$R := \mathcal{F}^{1/2}[x_1][x_2; \theta_2, \delta_2] \cdots [x_N; \theta_N, \delta_N]$$

and $1 \leq j \leq k \leq N$, denote by $R_{[j,k]}$ the $\mathcal{F}^{1/2}$-subalgebra generated by $x_j, \ldots, x_k$ and set $R_k := R_{[1,k]}$.

Definition. An iterated skew polynomial extension R is called a *quantum nilpotent algebra* or a *CGL extension* if it is equipped with a rational action of an $\mathcal{F}^{1/2}$-torus $\mathcal{H}$ by $\mathcal{F}^{1/2}$-algebra automorphisms such that:

(i) The elements $x_1, \ldots, x_N$ are $\mathcal{H}$-eigenvectors.
(ii) For every $k \in [2, N]$, δ_k is a locally nilpotent θ_k-derivation of the algebra R_{k-1}.
(iii) For every $k \in [1, N]$, there exists $h_k \in \mathcal{H}$ such that $\theta_k = (h_k \cdot)|_{R_{k-1}}$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

A CGL extension is called *symmetric* if it has the same properties when its generators are adjoined in the opposite order. We will assume throughout Sections 1, 5, 6, 7 that the θ_k-eigenvalues of x_j belong to $q^{\mathbb{Z}/2}$ for $j \leq k$, where we abbreviate $\mathbb{Z}/2 := \mathbb{Z}\frac{1}{2}$. Recall that a nonzero element $p \in R$ is called *prime* if $Rp = pR$ and the ring R/Rp is a domain.

Theorem. [14, Theorem 4.3] *For each CGL extension R and $k \in [1, N]$, the algebra R_k has a unique (up to rescaling) homogeneous prime element y_k which does not belong to R_{k-1}. It either equals x_k or has the property that*

$$y_k - y_{p(k)} x_k \in R_{k-1}$$

for some $p(k) \in [1, k-1]$.

In the following we will work with this choice of sets of homogeneous prime elements (and not with arbitrary $\mathcal{F}^{1/2}$-rescalings of them). For a symmetric CGL extension the theorem can be applied to the interval subalgebras $R_{[p(k),k]}$ to obtain that each of them has a unique (up to rescaling) homogeneous prime element $y_{[p(k),k]}$ which does not belong to the smaller interval subalgebras. An $\mathcal{F}^{1/2}$-rescaling of the generators of a CGL extension R leads to another CGL extension presentation of R. The generators x_k can be always rescaled so that

$$y_{[p(k),k]} = q^m x_{p(k)} x_k - q^{m'} \prod_i p_i^{n_i} \tag{1.2}$$

for some $m, m' \in \mathbb{Z}/2$ and $n_i \in \mathbb{Z}_{\geq 0}$ where the product is over all homogeneous prime elements of $R_{[p(k),k]}$ from the theorem that are different from $y_{[p(k),k]}$ (see §3.3-3.4 and [14]). In the following we will assume that this normalization is made. Denote

$$R_{\mathcal{A}^{1/2}} := \mathcal{A}^{1/2}\langle x_1, \ldots, x_N \rangle \subseteq R.$$

Theorem A. *Let R be a symmetric CGL extension for which $R_{\mathcal{A}^{1/2}}$ is an $\mathcal{A}^{1/2}$-form of R, that is, $R_{\mathcal{A}^{1/2}} \otimes_{\mathcal{A}^{1/2}} \mathcal{F}^{1/2} \cong R$. If the sequence of homogeneous prime elements $y_1, \dots, y_N$ lies in $R_{\mathcal{A}^{1/2}}$, then there exists a quantum cluster algebra $\mathbf{A}$ over $\mathcal{A}^{1/2}$ such that*

$$R \cong \mathbf{A} = \mathbf{U}$$

where $\mathbf{U}$ is the corresponding upper quantum cluster algebra over $\mathcal{A}^{1/2}$. For all $k \in [1, N]$ and $n \in \mathbb{Z}_{>0}$ for which $p^n(k)$ is well defined (as in the previous theorem), $q^m x_k$ and $q^{m'} y_{[p^n(k),k]}$ are cluster variables of $\mathbf{A}$ for some $m, m' \in \mathbb{Z}/2$.

We prove a more general result in Theorem 4.8 which deals with integral forms of multiparameter and arbitrary characteristic CGL extensions and quantum cluster algebras. In §4.3 we illustrate the theorem with various examples which are not connected $\mathbb{Z}_{\geq 0}$-graded, including all quantized Weyl algebras, and with quantum cluster algebras over $\mathbb{F}_p[q^{\pm 1/2}]$.

For each symmetrizable Kac–Moody algebra $\mathfrak{g}$ and a Weyl group element w, De Concini–Kac–Procesi [5] and Lusztig [34] defined a quantum Schubert cell algebra $U^-[w]$ which is a subalgebra of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ defined over $\mathbb{Q}(q)$. The quantum unipotent cells of Geiß–Leclerc–Schröer [9] are $\mathbb{Q}(q)$-algebras $A_q(\mathfrak{n}_+(w))$ which are antiisomorphic to $U^-[w]$. Denote

$$\mathcal{A} := \mathbb{Z}[q^{\pm 1}]. \tag{1.3}$$

The dual canonical forms $A_q(\mathfrak{n}_+(w))_{\mathcal{A}}$ are $\mathcal{A}$-forms of $A_q(\mathfrak{n}_+(w))$ which are obtained by transporting the Kashiwara–Lusztig dual canonical forms $U^-[w]^\vee_{\mathcal{A}}$ of $U^-[w]$.

Theorem B. *Let $\mathfrak{g}$ be an arbitrary symmetrizable Kac–Moody algebra and w a Weyl group element. For the dual canonical form $A_q(\mathfrak{n}_+(w))_{\mathcal{A}}$ of the corresponding quantum unipotent cell, there exists a quantum cluster algebra $\mathbf{A}$ over $\mathcal{A}^{1/2}$ such that*

$$A_q(\mathfrak{n}_+(w))_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}^{1/2} \cong \mathbf{A} = \mathbf{U}$$

where $\mathbf{U}$ is the associated upper quantum cluster algebra defined over $\mathcal{A}^{1/2}$.

Further details about the structure of the quantum cluster algebra $\mathbf{A}$ are given in Theorem 7.3.

The following special cases of parts of the theorem were previously proved: Qin [37] proved that $A_q(\mathfrak{n}_+(w))_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}^{1/2} \cong \mathbf{A}$ for symmetric Kac–Moody algebras $\mathfrak{g}$ and adaptable Weyl group elements w. Kang, Kashiwara, Kim and Oh [23] proved this isomorphism for symmetric Kac–Moody algebras $\mathfrak{g}$ and all Weyl group elements w. Geiß, Leclerc and Schröer [10] proved that

$$\mathbf{A} \otimes_{\mathcal{A}^{1/2}} \mathbb{Q}[q^{\pm 1/2}] = \mathbf{U} \otimes_{\mathcal{A}^{1/2}} \mathbb{Q}[q^{\pm 1/2}]$$

for symmetric Kac–Moody algebras $\mathfrak{g}$ and all Weyl group elements w; however, the fact that $\mathbf{A} = \mathbf{U}$ is new even for simple cases like $\mathfrak{g} = \mathfrak{sl}_n$. For nonsymmetric Kac–Moody algebras $\mathfrak{g}$ the results in the theorem are all new, including the existence of a non-integral quantum cluster structure on $A_q(\mathfrak{n}_+(w))_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}(q^{1/2})$.

The previous approaches to integral quantum cluster structures [4, 19, 23, 26, 36, 37] obtained monoidal categorifications of quantum cluster algebras. At the same time they also relied on extensive knowledge of categorifications which are available for concrete families of algebras. The power of Theorem A for the construction of integral quantum cluster structures lies in its flexibility to adjust to different situations and in the mild assumptions in it: one needs to verify the normalization condition (1.2), that $R_{\mathcal{A}^{1/2}}$ is an $\mathcal{A}^{1/2}$-form of R, and that the sequence of homogeneous prime elements $y_1, \dots, y_N$ belongs to $R_{\mathcal{A}^{1/2}}$.

1.3. **Notation and conventions.** Throughout, $\mathbb{K}$ denotes an infinite field of arbitrary characteristic. For integers $j \le k$, set $[j,k] := \{j, \dots, k\}$. As above, $\mathbb{Z}/2 := \mathbb{Z}\frac{1}{2}$.

An $N \times N$ matrix $\mathbf{t} = (t_{kj})$ over a commutative ring $\mathbb{D}$ is *multiplicatively skew-symmetric* if $t_{jk}t_{kj} = t_{kk} = 1$ for all $j,k \in [1,N]$. Such a matrix gives rise to a skew-symmetric bicharacter $\Omega_{\mathbf{t}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{D}^*$ for which

$$\Omega_{\mathbf{t}}(e_j, e_k) = t_{jk}\,, \quad \forall\, j,k \in [1,N], \tag{1.4}$$

where $e_1, \dots, e_N$ are the standard basis vectors for $\mathbb{Z}^N$. (We denote the group of units of $\mathbb{D}$ by $\mathbb{D}^*$.) When we have need for formulas involving $\mathbb{Z}^N$, we view its elements as column vectors. The transpose of an N-tuple $\mathbf{m} = (m_1, \dots, m_N)$ is denoted $\mathbf{m}^T$.

Given an algebra A over a commutative ring $\mathbb{D}$ and elements $a_1, \dots, a_k \in A$, we write $\mathbb{D}\langle a_1, \dots, a_k\rangle$ to denote the unital $\mathbb{D}$-subalgebra of A generated by $\{a_1, \dots, a_k\}$.

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2. Quantum cluster algebras

We outline notation and conventions for quantum cluster algebras. To connect with the results of [15], we describe a multiparameter setting which extends the uniparameter case originally developed by Berenstein and Zelevinsky [2]. To allow for integral forms, we work over a commutative domain rather than over a field.

Fix a commutative domain $\mathbb{D}$ contained in $\mathbb{K}$ and a positive integer N.

Let $\mathcal{F}$ be a division algebra over $\mathbb{D}$. A *toric frame* (of rank N) for $\mathcal{F}$ (over $\mathbb{D}$) is a mapping

$$M : \mathbb{Z}^N \longrightarrow \mathcal{F}$$

such that

$$M(f)M(g) = \Omega_{\mathbf{r}}(f,g)M(f+g), \quad \forall\, f,g \in \mathbb{Z}^N, \tag{2.1}$$

where

- $\Omega_{\mathbf{r}}$ is a $\mathbb{D}^*$-valued skew-symmetric bicharacter on $\mathbb{Z}^N$ arising from a multiplicatively skew-symmetric matrix $\mathbf{r} \in M_N(\mathbb{D})$ as in (1.4),
- the elements in the image of M are linearly independent over $\mathbb{D}$, and
- $\operatorname{Fract} \mathbb{D}\langle M(\mathbb{Z}^N)\rangle = \mathcal{F}$.

The matrix $\mathbf{r}$ is uniquely reconstructed from the toric frame M, and will be denoted by $\mathbf{r}(M)$. The elements $M(e_1), \dots, M(e_N)$ are called *cluster variables*. Fix a subset $\mathbf{ex} \subset [1,N]$, to be called the set of *exchangeable indices*; the remaining indices, those in $[1,N]\backslash\mathbf{ex}$, will be called *frozen*.

An integral $N \times \mathbf{ex}$ matrix $\widetilde{B}$ will be called an *exchange matrix* if its principal part (the $\mathbf{ex} \times \mathbf{ex}$ submatrix) is skew-symmetrizable. If the principal part of $\widetilde{B}$ is skew-symmetric, then it is represented by a quiver whose vertices are labelled by the integers in $[1,N]$. For $j,k \in [1,N]$, there is a directed edge from the vertex j to the vertex k if and only if $(\widetilde{B})_{jk} > 0$ and the number of such directed edges equals $(\widetilde{B})_{jk}$. In particular, the quiver has no edges between any pair of vertices in $[1,N]\backslash\mathbf{ex}$.

A *quantum seed* for $\mathcal{F}$ (over $\mathbb{D}$) is a pair $(M, \widetilde{B})$ consisting of a toric frame M for $\mathcal{F}$ and an exchange matrix $\widetilde{B}$ compatible with $\mathbf{r}(M)$ in the sense that

$$\Omega_{\mathbf{r}(M)}(b^k, e_j) = 1, \;\; \forall\, k \in \mathbf{ex},\; j \in [1,N],\; k \neq j \quad \text{and}$$
$$\Omega_{\mathbf{r}(M)}(b^k, e_k) \;\text{ are not roots of unity}, \;\; \forall\, k \in \mathbf{ex},$$

where b^k denotes the k-th column of $\widetilde{B}$.

The *mutation in direction* $k \in \mathbf{ex}$ of a quantum seed $(M, \widetilde{B})$ is the quantum seed $(\mu_k(M), \mu_k(\widetilde{B}))$ where $\mu_k(M)$ is described below and $\mu_k(\widetilde{B})$ is the $N \times \mathbf{ex}$ matrix (b'_{ij}) with entries

$$b'_{ij} := \begin{cases} -b_{ij}\,, & \text{if } \ i = k \ \text{ or } \ j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}\,, & \text{otherwise,} \end{cases}$$

[6]. If the principal part of $\widetilde{B}$ is skew-symmetric, then $\mu_k(\widetilde{B})$ has the same property and the pair of quivers corresponding to $\widetilde{B}$ and $\mu_k(\widetilde{B})$ are obtained from each other by quiver mutation at the vertex k, see [7, §§2.1 and 2.7] for details. Corresponding to the column b^k of $\widetilde{B}$ are $\mathbb{D}$-algebra automorphisms $\rho_{b^k,\pm}$ of $\mathcal{F}$ such that

$$\rho_{b^k,\epsilon}(ME_\epsilon(e_j)) = \begin{cases} ME_\epsilon(e_k) + ME_\epsilon(e_k + \epsilon b^k), & \text{if } j = k \\ ME_\epsilon(e_j), & \text{if } j \neq k, \end{cases}$$

[2, Proposition 4.2] and [15, Lemma 2.8], where $E_\epsilon = E_\epsilon^{\widetilde{B}}$ is the $N \times N$ matrix with entries

$$(E_\epsilon)_{ij} = \begin{cases} \delta_{ij}\,, & \text{if } j \neq k \\ -1, & \text{if } i = j = k \\ \max(0, -\epsilon b_{ik}), & \text{if } i \neq j = k. \end{cases}$$

The toric frame $\mu_k(M)$ is defined as

$$\mu_k(M) := \rho_{b^k,\epsilon} ME_\epsilon : \mathbb{Z}^N \longrightarrow \mathcal{F}.$$

It is independent of the choice of ϵ, and, paired with $\mu_k(\widetilde{B})$, forms a quantum seed over $\mathbb{K}$ [15, Proposition 2.9]. (See also [15, Corollary 2.11], and compare with [2, Proposition 4.9] for the uniparameter case.) By [15, Proposition 2.9 and Eq. (2.22)], the entries of $\mathbf{r}(\mu_k(M)) = \mu_k(\mathbf{r}(M))$ are products of powers of the entries of $\mathbf{r}(M)$, so $\mathbf{r}(\mu_k(M)) \in M_N(\mathbb{D})$. It follows that $\mu_k(M)$ is a toric frame for $\mathcal{F}$ over $\mathbb{D}$, so that $(\mu_k(M), \mu_k(\widetilde{B}))$ is a quantum seed over $\mathbb{D}$.

Fix a subset $\mathbf{inv}$ of the set $[1, N]\backslash\mathbf{ex}$ of frozen indices – the corresponding cluster variables will be inverted. The *quantum cluster algebra* $\mathsf{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{D}}$ is the unital $\mathbb{D}$-subalgebra of $\mathcal{F}$ generated by the cluster variables of all seeds obtained from $(M, \widetilde{B})$ by iterated mutations and by $\{M(e_k)^{-1} \mid k \in \mathbf{inv}\}$. To each quantum seed $(M, \widetilde{B})$ and choice of $\mathbf{inv}$, one associates the mixed quantum torus/quantum affine space algebra

$$\mathbb{D}\mathcal{T}_{(M,\widetilde{B},\mathbf{inv})} := \mathbb{D}\langle M(e_k)^{\pm 1},\ M(e_j) \mid k \in \mathbf{ex} \cup \mathbf{inv},\ j \in [1, N]\backslash(\mathbf{ex} \cup \mathbf{inv})\rangle \subset \mathcal{F}. \tag{2.2}$$

The intersection of all such subalgebras of $\mathcal{F}$ associated to all seeds that are obtained by iterated mutation from the seed $(M, \widetilde{B})$ is called the *upper quantum cluster algebra* of $(M, \widetilde{B})$ and is denoted by $\mathsf{U}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{D}}$. The corresponding *Laurent Phenomenon* [15, Theorem 2.15] says that

$$\mathsf{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{D}} \subseteq \mathsf{U}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{D}}\,. \tag{2.3}$$

If $\mathbb{K}$ is the quotient field of $\mathbb{D}$, then $\mathcal{F}$ is also a division algebra over $\mathbb{K}$, and the above constructions may be performed over $\mathbb{K}$. The corresponding quantum cluster algebras over $\mathbb{K}$ are just the $\mathbb{K}$-subalgebras of $\mathcal{F}$ generated by the quantum cluster algebras over $\mathbb{D}$:

$$\mathsf{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}} = \mathbb{K} \cdot \mathsf{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{D}}\,.$$

The uniparameter quantum cluster algebras of Berenstein and Zelevinsky [2] come from the above axiomatics when the following two conditions are imposed:

(1) The base ring is taken to be
$$\mathbb{D} = \mathcal{A}^{1/2} = \mathbb{Z}[q^{\pm 1/2}].$$
So, $\mathbb{D}^* = (\mathcal{A}^{1/2})^* = \pm q^{\mathbb{Z}/2}$.

(2) The toric frame of one seed (and thus of any seed) has a multiplicatively skew-symmetric matrix $\mathbf{r} \in M_N(\mathbb{D})$ of the form
$$\mathbf{r} = (q^{m_{ij}/2})_{i,j=1}^N \quad \text{for some} \quad m_{ij} \in \mathbb{Z}.$$

3. Quantum nilpotent algebras

Quantum nilpotent algebras are iterated skew polynomial algebras over a base field, which we take to be $\mathbb{K}$ in this section. We use the standard notation $S[x;\theta,\delta]$ for a *skew polynomial ring*, or *Ore extension*; it denotes a ring generated by a subring S and an element x satisfying $xs = \theta(s)x + \delta(s)$ for all $s \in S$, where θ is a ring endomorphism of S and δ is a (left) θ-derivation of S. The ring $S[x;\theta,\delta]$ is a free left S-module, with the nonnegative powers of x forming a basis. For all skew polynomial rings $S[x;\theta,\delta]$ considered in this paper, we assume that θ is an *automorphism* of S. Moreover, we work in the context of algebras over a commutative ring $\mathbb{D}$, so our coefficient rings S will be $\mathbb{D}$-algebras, the maps θ will be $\mathbb{D}$-algebra automorphisms, and the maps δ will be $\mathbb{D}$-linear θ-derivations. Under these assumptions, $S[x;\theta,\delta]$ is naturally a $\mathbb{D}$-algebra. Throughout the present section, $\mathbb{D} = \mathbb{K}$.

3.1. CGL extensions.

We focus on iterated skew polynomial extensions
$$R := \mathbb{K}[x_1][x_2;\theta_2,\delta_2]\cdots[x_N;\theta_N,\delta_N], \tag{3.1}$$
where $\mathbb{K}[x_1] = \mathbb{K}[x_1;\mathrm{id}_{\mathbb{K}},0]$. Set
$$R_k := \mathbb{K}\langle x_1,\dots,x_k\rangle = \mathbb{K}[x_1][x_2;\theta_2,\delta_2]\cdots[x_k;\theta_k,\delta_k] \qquad \text{for } k \in [0,N];$$
in particular, $R_0 = \mathbb{K}$.

Definition 3.1. An iterated skew polynomial extension (3.1) is called a *quantum nilpotent algebra* or a *CGL extension* [31, Definition 3.1] if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms such that:

(i) The elements $x_1,\dots,x_N$ are $\mathcal{H}$-eigenvectors.
(ii) For every $k \in [2,N]$, δ_k is a locally nilpotent θ_k-derivation of the algebra R_{k-1}.
(iii) For every $k \in [1,N]$, there exists $h_k \in \mathcal{H}$ such that $\theta_k = (h_k\cdot)|_{R_{k-1}}$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

Conditions (i) and (iii) imply that
$$\theta_k(x_j) = \lambda_{kj}x_j \ \text{ for some } \lambda_{kj} \in \mathbb{K}^*, \ \ \forall\, 1 \le j < k \le N.$$
We then set $\lambda_{kk} := 1$ and $\lambda_{jk} := \lambda_{kj}^{-1}$ for $j < k$. This gives rise to a multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} := (\lambda_{kj}) \in M_N(\mathbb{K}^*)$ and the corresponding skew-symmetric bicharacter $\Omega_{\boldsymbol{\lambda}}$ from (1.4). The elements $h_k \in \mathcal{H}$ interact with the skew derivations δ_k as follows:
$$(h_k\cdot)\circ\delta_k = \lambda_k\delta_k\circ(h_k\cdot), \quad \forall\, k \in [1,N], \tag{3.2}$$
see [15, Eq. (3.15)].

The *length* of R is N and its *rank* is given by by
$$\mathrm{rk}(R) := \{k \in [1,N] \mid \delta_k = 0\} \in \mathbb{Z}_{>0} \tag{3.3}$$
(cf. [14, Eq. (4.3)]). Denote the character group of the torus $\mathcal{H}$ by $X(\mathcal{H})$. The action of $\mathcal{H}$ on R gives rise to an $X(\mathcal{H})$-grading of R, and the $\mathcal{H}$-eigenvectors are precisely the

nonzero homogeneous elements with respect to this grading. The $\mathcal{H}$-eigenvalue of a nonzero homogeneous element $u \in R$ will be denoted by χ_u; this equals its $X(\mathcal{H})$-degree relative to the $X(\mathcal{H})$-grading.

By [31, Proposition 3.2, Theorem 3.7], every CGL extension R is an $\mathcal{H}$-UFD, meaning that each nonzero $\mathcal{H}$-prime ideal of R contains a homogeneous prime element. (A *prime element* of a domain R is a nonzero element $p \in R$ such that $Rp = pR$ – i.e., p is a *normal element* of R – and the ring R/Rp is a domain.) A recursive description of the sets of homogeneous prime elements of the intermediate algebras R_k of a CGL extension R was obtained in [14]. To state the result, we require the standard predecessor and successor functions, $p = p_\eta$ and $s = s_\eta$, of a function $\eta : [1, N] \to \mathbb{Z}$, defined as follows:

$$\begin{aligned} p(k) &:= \max\{j < k \mid \eta(j) = \eta(k)\}, \\ s(k) &:= \min\{j > k \mid \eta(j) = \eta(k)\}, \end{aligned} \tag{3.4}$$

where $\max \varnothing = -\infty$ and $\min \varnothing = +\infty$. Corresponding order functions $O_\pm : [1, N] \to \mathbb{Z}_{\geq 0}$ are defined by

$$\begin{aligned} O_-(k) &:= \max\{m \in \mathbb{Z}_{\geq 0} \mid p^m(k) \neq -\infty\}, \\ O_+(k) &:= \max\{m \in \mathbb{Z}_{\geq 0} \mid s^m(k) \neq +\infty\}. \end{aligned} \tag{3.5}$$

Theorem 3.2. [14, Theorem 4.3] *Let R be a CGL extension of length N and rank* $\mathrm{rk}(R)$ *as in* (3.1). *There exist a function $\eta : [1, N] \to \mathbb{Z}$ whose range has cardinality* $\mathrm{rk}(R)$ *and elements*

$$c_k \in R_{k-1} \ \textit{for all} \ k \in [2, N] \ \textit{with} \ p(k) \neq -\infty$$

such that the elements $y_1, \ldots, y_N \in R$, recursively defined by

$$y_k := \begin{cases} y_{p(k)} x_k - c_k\,, & \textit{if } p(k) \neq -\infty \\ x_k\,, & \textit{if } p(k) = -\infty, \end{cases} \tag{3.6}$$

are homogeneous and have the property that for every $k \in [1, N]$,

$$\{y_j \mid j \in [1, k],\ s(j) > k\} \tag{3.7}$$

is a list of the homogeneous prime elements of R_k up to scalar multiples.

The elements $y_1, \ldots, y_N \in R$ with these properties are unique. The function η satisfying the above conditions is not unique, but the partition of $[1, N]$ into a disjoint union of the level sets of η is uniquely determined by the presentation (3.1) *of R, as are the predecessor and successor functions p and s. The function p has the property that $p(k) = -\infty$ if and only if $\delta_k = 0$.*

The statement of Theorem 3.2 is upgraded as in [15, Theorem 3.6 and following comments]. In the setting of the theorem, the rank of R is also given by

$$\mathrm{rk}(R) = |\{j \in [1, N] \mid s(j) > N\}| \tag{3.8}$$

[14, Eq. (4.3)].

Definition 3.3. Denote by $\prec$ the reverse lexicographic order on $\mathbb{Z}^N_{\geq 0}$:

$$(m'_1, \ldots, m'_N) \prec (m_1, \ldots, m_N) \quad \text{iff there exists} \quad n \in [1, N] \text{ with } m'_n < m_n\,,\ m'_{n+1} = m_{n+1}\,,\ \ldots,\ m'_N = m_N\,. \tag{3.9}$$

A CGL extension R as in (3.1) has the $\mathbb{K}$-basis

$$\{x^f := x_1^{m_1} \cdots x_N^{m_N} \mid f = (m_1, \ldots, m_N)^T \in \mathbb{Z}^N_{\geq 0}\}.$$

We say that a nonzero element $b \in R$ has *leading term* tx^f and *leading coefficient* t where $t \in \mathbb{K}^*$ and $f \in \mathbb{Z}^N_{\geq 0}$ if

$$b = tx^f + \sum_{g \in \mathbb{Z}^N_{\geq 0},\, g \prec f} t_g x^g$$

for some $t_g \in \mathbb{K}$, and we set $\mathrm{lc}(b) := t$ and $\mathrm{lt}(b) := tx^f$.

The leading terms of the prime elements y_k in Theorem 3.2 are given by

$$\mathrm{lt}(y_k) = x_{p^{O_-(k)}(k)} \cdots x_{p(k)} x_k, \ \ \forall\, k \in [1, N]. \tag{3.10}$$

The leading terms of reverse-order monomials $x_N^{m_N} \cdots x_1^{m_1}$ involve symmetrization scalars in $\mathbb{K}^*$ defined by

$$\mathcal{S}_{\boldsymbol{\lambda}}(f) := \prod_{1 \leq j < k \leq N} \lambda_{jk}^{-m_j m_k}, \ \ \forall\, f = (m_1, \ldots, m_N)^T \in \mathbb{Z}^N. \tag{3.11}$$

Namely,

$$\mathrm{lt}(x_N^{m_N} \cdots x_1^{m_1}) = \mathcal{S}_{\boldsymbol{\lambda}}((m_1, \ldots, m_N)^T) x_1^{m_1} \cdots x_N^{m_N}, \ \ \forall\, (m_1, \ldots, m_N)^T \in \mathbb{Z}^N. \tag{3.12}$$

3.2. **Symmetric CGL extensions.** Given an iterated skew polynomial extension R as in (3.1), denote its interval subalgebras

$$R_{[j,k]} := \mathbb{K}\langle x_i \mid j \leq i \leq k \rangle, \ \ \forall\, j, k \in [1, N];$$

in particular, $R_{[j,k]} = \mathbb{K}$ if $j \not\leq k$.

Definition 3.4. A CGL extension R as in Definition 3.1 is called *symmetric* provided

(i) For all $1 \leq j < k \leq N$,

$$\delta_k(x_j) \in R_{[j+1,k-1]}.$$

(ii) For all $j \in [1, N]$, there exists $h_j^* \in \mathcal{H}$ such that

$$h_j^* \cdot x_k = \lambda_{kj}^{-1} x_k = \lambda_{jk} x_k, \ \ \forall\, k \in [j+1, N]$$

and $h_j^* \cdot x_j = \lambda_j^* x_j$ for some $\lambda_j^* \in \mathbb{K}^*$ which is not a root of unity.

Under these conditions, R has a CGL extension presentation with the variables x_k in descending order:

$$R = \mathbb{K}[x_N][x_{N-1}; \theta_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \theta_1^*, \delta_1^*], \tag{3.13}$$

see [14, Corollary 6.4].

Proposition 3.5. [15, Proposition 5.8] *Let R be a symmetric CGL extension of length N. If $l \in [1, N]$ with $O_+(l) = m > 0$, then*

$$\lambda_l^* = \lambda_{s(l)}^* = \cdots = \lambda_{s^{m-1}(l)}^* = \lambda_{s(l)}^{-1} = \lambda_{s^2(l)}^{-1} = \cdots = \lambda_{s^m(l)}^{-1}. \tag{3.14}$$

Definition 3.6. Define the following subset of the symmetric group S_N:

$$\begin{aligned} \Xi_N := \{\sigma \in S_N \mid \sigma(k) = \max \sigma([1, k-1]) + 1 \ \text{ or} \\ \sigma(k) = \min \sigma([1, k-1]) - 1, \ \ \forall\, k \in [2, N]\}. \end{aligned} \tag{3.15}$$

In other words, Ξ_N consists of those $\sigma \in S_N$ such that $\sigma([1, k])$ is an interval for all $k \in [2, N]$. The following subset of Ξ_N will also be needed:

$$\begin{aligned} &\Gamma_N := \{\sigma_{i,j} \mid 1 \leq i \leq j \leq N\}, \ \text{ where} \\ &\sigma_{i,j} := [i+1, \ldots, j, i, j+1, \ldots, N, i-1, i-2, \ldots, 1]. \end{aligned} \tag{3.16}$$

If R is a symmetric CGL extension of length N, then for each $\sigma \in \Xi_N$ there is a CGL extension presentation

$$R = \mathbb{K}[x_{\sigma(1)}][x_{\sigma(2)}; \theta''_{\sigma(2)}, \delta''_{\sigma(2)}] \cdots [x_{\sigma(N)}; \theta''_{\sigma(N)}, \delta''_{\sigma(N)}], \tag{3.17}$$

see [14, Remark 6.5], [15, Proposition 3.9]. Moreover, if $1 \leq i \leq k \leq N$, then the subalgebra $R_{[i,k]}$ of R is a symmetric CGL extension, to which Theorem 3.2 applies. In the case $k = s^m(i)$ we have

Proposition 3.7. [15, Theorem 5.1] *Assume that R is a symmetric CGL extension of length N, and $i \in [1, N]$ and $m \in \mathbb{Z}_{\geq 0}$ are such that $s^m(i) \in [1, N]$. Then there is a unique homogeneous prime element $y_{[i,s^m(i)]} \in R_{[i,s^m(i)]}$ such that*
(i) $y_{[i,s^m(i)]} \notin R_{[i,s^m(i)-1]}$ *and* $y_{[i,s^m(i)]} \notin R_{[i+1,s^m(i)]}$.
(ii) $\mathrm{lt}(y_{[i,s^m(i)]}) = x_i x_{s(i)} \cdots x_{s^m(i)}$.

The elements $y_{[i,s^m(i)]} \in R$ will be called *interval prime elements*. Certain combinations of the homogeneous prime elements from Proposition 3.7 play an important role in the mutation formulas for quantum cluster variables of symmetric CGL extensions. They are given in the following theorem, where we denote

$$\begin{aligned} e_{[j,s^l(j)]} &:= e_j + e_{s(j)} + \cdots + e_{s^l(j)} \in \mathbb{Z}^N, \\ &\qquad \forall\, j \in [1, N],\ l \in \mathbb{Z}_{\geq 0} \text{ such that } s^l(j) \in [1, N], \end{aligned} \tag{3.18}$$

and where we set $y_{[s(i),i]} := 1$.

Theorem 3.8. [15, Corollary 5.11] *Assume that R is a symmetric CGL extension of length N, and $i \in [1, N]$ and $m \in \mathbb{Z}_{>0}$ are such that $s^m(i) \in [1, N]$. Then*

$$u_{[i,s^m(i)]} := y_{[i,s^{m-1}(i)]} y_{[s(i),s^m(i)]} - \Omega_{\boldsymbol{\lambda}}(e_i, e_{[s(i),s^{m-1}(i)]}) y_{[s(i),s^{m-1}(i)]} y_{[i,s^m(i)]} \tag{3.19}$$

is a nonzero homogeneous normal element of $R_{[i+1,s^m(i)-1]}$ which is not a multiple of $y_{[s(i),s^{m-1}(i)]}$ if $m \geq 2$.

The form and properties of the elements $u_{[i,s^m(i)]}$ mainly enter into the proofs of the mutation formulas for symmetric CGL extensions. However, an explicit normalization condition for the leading coefficients of these elements is required; see (3.28) and Proposition 3.10.

3.3. **Rescaling of generators.** Assume R is a CGL extension of length N as in (3.1). Given scalars $t_1, \ldots, t_N \in \mathbb{K}^*$, one can rescale the generators x_j of R in the fashion

$$x_j \longmapsto t_j x_j, \quad \forall\, j \in [1, N], \tag{3.20}$$

meaning that R is also an iterated Ore extension with generators $t_j x_j$; in fact,

$$R = \mathbb{K}[t_1 x_1][t_2 x_2; \theta_2, t_2 \delta_2] \cdots [t_N x_N; \theta_N, t_N \delta_N]. \tag{3.21}$$

This is also a CGL extension presentation of R, and if (3.1) is a symmetric CGL extension, then so is (3.21).

Rescaling as in (3.20), (3.21) does not affect the $\mathcal{H}$-action or the matrix $\boldsymbol{\lambda}$, but various elements computed in terms of the new generators are correspondingly rescaled, such as the homogeneous prime elements from Theorem 3.2 and Proposition 3.7. These transform as follows:

$$y_k \longmapsto \left(\prod_{l=0}^{O_-(k)} t_{p^l(k)} \right) y_k \quad \text{and} \quad y_{[i,s^m(i)]} \longmapsto \left(\prod_{l=0}^{m} t_{s^l(i)} \right) y_{[i,s^m(i)]}\,. \tag{3.22}$$

Consequently, the homogeneous normal elements (3.19) transform via

$$u_{[i,s^m(i)]} \longmapsto \big(t_i t_{s(i)}^2 \cdots t_{s^{m-1}(i)}^2 t_{s^m(i)}\big) u_{[i,s^m(i)]}\,. \tag{3.23}$$

3.4. **Normalization conditions.** In order for the homogeneous prime elements y_k from Theorem 3.2 to function as quantum cluster variables, some normalizations are required. Throughout this subsection, assume that R is a symmetric CGL extension of length N as in Definitions 3.1 and 3.4. Assume also that the following mild conditions on scalars are satisfied:

Condition (A). The base field $\mathbb{K}$ contains square roots $\nu_{kl} = \sqrt{\lambda_{kl}}$ of the scalars λ_{kl} for $1 \le l < k \le N$ such that the subgroup of $\mathbb{K}^*$ generated by the ν_{kl} contains no elements of order 2. Then set $\nu_{kk} := 1$ and $\nu_{kl} := \nu_{lk}^{-1}$ for $k < l$, so that $\boldsymbol{\nu} := (\nu_{kl})$ is a multiplicatively skew-symmetric matrix.

Condition (B). There exist positive integers d_n, for $n \in \eta([1,N])$, such that

$$\lambda_k^{d_{\eta(l)}} = \lambda_l^{d_{\eta(k)}}, \quad \forall\, k,l \in [1,N] \text{ with } p(k), p(l) \neq -\infty.$$

In view of Proposition 3.5, this is equivalent to the condition

$$(\lambda_k^*)^{d_{\eta(l)}} = (\lambda_l^*)^{d_{\eta(k)}}, \quad \forall\, k,l \in [1,N] \text{ with } s(k), s(l) \neq +\infty.$$

Remark 3.9. Note that Condition (B) is always satisfied if all $\lambda_k = q^{m_k}$ for some $m_k \in \mathbb{Z}$ and $q \in \mathbb{K}$ (which has to be a non-root of unity due to assumption (iii) in Definition 3.1). This is the setting of Theorem A in the introduction.

In parallel with (3.11), define

$$\mathcal{S}_{\boldsymbol{\nu}}(f) := \prod_{1 \le j < k \le N} \nu_{jk}^{-m_j m_k}, \;\; \forall\, f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N. \tag{3.24}$$

Then set

$$\overline{e}_j := e_j + e_{p(j)} + \cdots + e_{p^{\mathcal{O}_-(j)}(j)} \quad \text{and} \quad \overline{y}_j := \mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_j) y_j\,, \;\; \forall\, j \in [1,N]. \tag{3.25}$$

We analogously normalize the homogeneous prime elements described in Proposition 3.7:

$$\begin{aligned} &\overline{y}_{[i,s^m(i)]} := \mathcal{S}_{\boldsymbol{\nu}}(e_{[i,s^m(i)]}) y_{[i,s^m(i)]}\,, \\ &\qquad\qquad \forall\, i \in [1,N],\; m \in \mathbb{Z}_{\ge 0} \text{ such that } s^m(i) \in [1,N]. \end{aligned} \tag{3.26}$$

A final normalization, for the leading coefficients of the elements $u_{[i,s^m(i)]}$, is needed in order to establish mutation formulas for the quantum cluster variables $\overline{y}_k$. For $i \in [1,N]$ and $m \in \mathbb{Z}_{\ge 0}$ such that $s^m(i) \in [1,N]$, write

$$\begin{aligned} &\mathrm{lt}(u_{[i,s^m(i)]}) = \pi_{[i,s^m(i)]} x^{f_{[i,s^m(i)]}}\,, \\ &\qquad \pi_{[i,s^m(i)]} \in \mathbb{K}^*, \;\; f_{[i,s^m(i)]} \in \sum_{j=1+1}^{s^m(i)-1} \mathbb{Z}_{\ge 0}\, e_j \subset \mathbb{Z}_{\ge 0}^N\,. \end{aligned} \tag{3.27}$$

We will require the condition

$$\pi_{[i,s(i)]} = \mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s(i)]}), \;\; \forall\, i \in [1,N] \text{ such that } s(i) \neq +\infty. \tag{3.28}$$

This can always be achieved after a suitable rescaling of the x_j, as follows.

Proposition 3.10. [15, Propositions 6.3, 6.1] *Let R be a symmetric CGL extension of length N, satisfying condition* (A).

(i) *There exist N-tuples $(t_1, \dots, t_N) \in (\mathbb{K}^*)^N$ such that after the rescaling* (3.20)*, condition* (3.28) *holds.*

(ii) *If* (3.28) *holds, then*

$$\pi_{[i,s^m(i)]} = \mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})^{-2}\mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s^m(i)]})$$

for all $i \in [1,N]$, $m \in \mathbb{Z}_{\geq 0}$ *with* $s^m(i) \in [1,N]$.

3.5. **Quantum cluster algebra structures for symmetric CGL extensions.** We present in this subsection the main theorem from [15].

Recall the notation on quantum cluster algebras from Section 2. There is a right action of S_N on the set of toric frames for a division algebra $\mathcal{F}$, given by re-enumeration,

$$(M \cdot \tau)(e_k) := M(e_{\tau(k)}), \;\; \mathbf{r}(M \cdot \tau)_{jk} = \mathbf{r}(M)_{\tau(j),\tau(k)}\,, \quad \tau \in S_N\,, \;\; j,k \in [1,N]. \tag{3.29}$$

Fix a symmetric CGL extension R of length N such that Conditions (A) and (B) hold. Define the multiplicatively skew-symmetric matrix $\boldsymbol{\nu}$ as in Condition (A), with associated bicharacter $\Omega_{\boldsymbol{\nu}}$ as in (1.4), and define a second multiplicatively skew-symmetric matrix $\mathbf{r} = (r_{kj})$ by

$$r_{kj} := \Omega_{\boldsymbol{\nu}}(\overline{e}_k, \overline{e}_j), \;\; \forall\, k,j \in [1,N]. \tag{3.30}$$

Let $\overline{y}_1, \ldots, \overline{y}_N$ be the sequence of normalized homogeneous prime elements given in (3.25). (We recall that each of these is a prime element in some of the subalgebras R_l, but not necessarily in the full algebra $R = R_N$.) There is a unique toric frame $M : \mathbb{Z}^N \to \mathrm{Fract}(R)$ whose matrix is $\mathbf{r}(M) := \mathbf{r}$ and such that $M(e_k) := \overline{y}_k$ for all $k \in [1,N]$ [15, Proposition 4.6].

Next, consider an arbitrary element $\sigma \in \Xi_N \subset S_N$, recall (3.15). For any $k \in [1,N]$, we see that

$$\eta^{-1}\eta\sigma(k) \cap \sigma([1,k]) = \begin{cases} \{p^n(\sigma(k)), \ldots, p(\sigma(k)), \sigma(k)\}, & \text{if } \sigma(1) \leq \sigma(k) \\ \{\sigma(k), s(\sigma(k)), \ldots, s^n(\sigma(k))\}, & \text{if } \sigma(1) \geq \sigma(k) \end{cases} \tag{3.31}$$

for some $n \in \mathbb{Z}_{\geq 0}$. Corresponding to σ, we have the CGL extension presentation (3.17), whose $\boldsymbol{\lambda}$-matrix is the matrix $\boldsymbol{\lambda}_\sigma$ with entries $(\boldsymbol{\lambda}_\sigma)_{ij} := \boldsymbol{\lambda}_{\sigma(i)\sigma(j)}$. Analogously we define the matrix $\boldsymbol{\nu}_\sigma$, and denote by $\mathbf{r}_\sigma$ the corresponding multiplicatively skew-symmetric matrix derived from $\boldsymbol{\nu}_\sigma$ by applying (3.30) to the presentation (3.17). Explicitly,

$$(\mathbf{r}_\sigma)_{kj} = \prod\{\nu_{il} \mid i \in \sigma([1,k]),\; \eta(i) = \eta\sigma(k),\; l \in \sigma([1,j]),\; \eta(l) = \eta\sigma(j)\}, \tag{3.32}$$

cf. (3.31). Let $\overline{y}_{\sigma,1}, \ldots, \overline{y}_{\sigma,N}$ be the sequence of normalized prime elements given by applying (3.25) to the presentation (3.17). By [15, Proposition 4.6], there is a unique toric frame $M_\sigma : \mathbb{Z}^N \to \mathrm{Fract}(R)$ whose matrix is $\mathbf{r}(M_\sigma) := \mathbf{r}_\sigma$ and such that for all $k \in [1,N]$,

$$M_\sigma(e_k) := \overline{y}_{\sigma,k} = \begin{cases} \overline{y}_{[p^n(\sigma(k)),\sigma(k)]}, & \text{if } \sigma(1) \leq \sigma(k) \\ \overline{y}_{[\sigma(k),s^n(\sigma(k))]}, & \text{if } \sigma(1) \geq \sigma(k) \end{cases} \tag{3.33}$$

in the two cases of (3.31), respectively. The last equality is proved in [15, Theorem 5.2].

Recall that the set $P(N) := \{k \in [1,N] \mid s(k) = +\infty\}$ parametrizes the set of homogeneous prime elements of R, i.e.,

$$\{y_k \mid k \in P(N)\} \text{ is a list of the homogeneous prime elements of } R$$

up to scalar multiples (Theorem 3.2). Define

$$\mathbf{ex} := [1,N] \setminus P(N) = \{l \in [1,N] \mid s(l) \neq +\infty\}.$$

Since $|P(N)| = \mathrm{rk}(R)$, the cardinality of the set $\mathbf{ex}$ is $N - \mathrm{rk}(R)$. For $\sigma \in \Xi_N$, define the set

$$\mathbf{ex}_\sigma = \{l \in [1,N] \mid \exists\, k > l \text{ with } \eta\sigma(k) = \eta\sigma(l)\}$$

of the same cardinality. Finally, recall the notation χ_u from Definition 3.1.

In [15, Theorem 8.2] we re-indexed all toric frames M_σ in such a way that the right action in Theorem 3.11 (c) was trivialized and the exchangeable variables in all such seeds were parametrized just by $\mathbf{ex}$, rather than by $\mathbf{ex}_\sigma$. We omit the re-indexing here, to simply the exposition. This affects the upper cluster algebra in the following way: When considering the quantum seed $(M_\sigma, \widetilde{B}_\sigma)$, the set $\mathbf{ex}$ must be replaced by $\mathbf{ex}_\sigma$ in relations such as (2.2).

Theorem 3.11. [15, Theorem 8.2] *Let R be a symmetric CGL extension of length N and rank* $\mathrm{rk}(R)$ *as in Definitions* 3.1 *and* 3.4. *Assume that Conditions* (A), (B) *hold, and that the sequence of generators $x_1, \dots, x_N$ of R is normalized* (*rescaled*) *so that condition* (3.28) *is satisfied. Then the following hold:*

(a) *For all $\sigma \in \Xi_N$* (*see* (3.15)) *and $l \in \mathbf{ex}_\sigma$, there exists a unique vector $b^l_\sigma \in \mathbb{Z}^N$ such that $\chi_{M_\sigma(b^l_\sigma)} = 1$ and*

$$\Omega_{\mathbf{r}_\sigma}(b^l_\sigma, e_j) = 1, \ \ \forall\, j \in [1, N],\ j \neq l \quad \text{and} \quad \Omega_{\mathbf{r}_\sigma}(b^l_\sigma, e_l)^2 = \lambda^*_{\min \eta^{-1}\eta(\sigma(l))}\,. \tag{3.34}$$

Denote by $\widetilde{B}_\sigma \in M_{N\times|\mathbf{ex}|}(\mathbb{Z})$ the matrix with columns b^l_σ, $l \in \mathbf{ex}_\sigma$. Let $\widetilde{B} := \widetilde{B}_{\mathrm{id}}$.

(b) *For all $\sigma \in \Xi_N$, the pair $(M_\sigma, \widetilde{B}_\sigma)$ is a quantum seed for* $\mathrm{Fract}(R)$. *The principal part of $\widetilde{B}_\sigma$ is skew-symmetrizable via the integers $d_{\eta(k)}$, $k \in \mathbf{ex}_\sigma$ from Condition* (B).

(c) *All such quantum seeds are mutation-equivalent to each other up to the S_N action. They are linked by the following one-step mutations. Let $\sigma, \sigma' \in \Xi_N$ be such that*

$$\sigma' = (\sigma(k), \sigma(k+1)) \circ \sigma = \sigma \circ (k, k+1)$$

for some $k \in [1, N-1]$. If $\eta(\sigma(k)) \neq \eta(\sigma(k+1))$, then $M_{\sigma'} = M_\sigma \cdot (k, k+1)$ in terms of the action (3.29). *If $\eta(\sigma(k)) = \eta(\sigma(k+1))$, then $M_{\sigma'} = \mu_k(M_\sigma) \cdot (k, k+1)$.*

(d) *The CGL extension R equals the quantum cluster and upper cluster algebras associated to M, $\widetilde{B}$, $\varnothing$:*

$$R = \mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} = \mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}\,.$$

In particular, $\mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$ and $\mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$ are affine and noetherian, and more precisely $\mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$ is generated by the cluster variables in the seeds parametrized by the finite subset Γ_N of Ξ_N, recall (3.16).

(e) *Let* $\mathbf{inv}$ *be any subset of the set $P(N)$ of frozen variables. Then*

$$R[y_k^{-1} \mid k \in \mathbf{inv}] = \mathbf{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}} = \mathbf{U}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}}\,.$$

4. Integral quantum cluster structures on quantum nilpotent algebras

We introduce integral forms of CGL extensions and show that the quantum cluster algebra structure on a symmetric CGL extension R satisfying the hypotheses of Theorem 3.11 passes to appropriate integral forms of R.

Throughout the section, let R be a CGL extension of length N as in Definition 3.1, with associated torus $\mathcal{H}$, scalars λ_{kj} and λ_k, and other notation as in Section 3. Let $\mathbb{D} \subseteq \mathbb{K}$ be a unital subring of $\mathbb{K}$, and write $\mathbb{D}^*$ for the group of units of $\mathbb{D}$.

4.1. Integral forms of CGL extensions.

Definition 4.1. We say that the $\mathbb{D}$-subalgebra $\mathbb{D}\langle x_1, \dots, x_N\rangle$ of R is a $\mathbb{D}$-*form* of the CGL presentation (3.1) – and therefore that (3.1) *has a $\mathbb{D}$-form* – provided this subalgebra is an iterated skew polynomial extension of the form

$$\mathbb{D}\langle x_1, \dots, x_N\rangle = \mathbb{D}[x_1][x_2; \theta_2, \delta_2] \cdots [x_N; \theta_N, \delta_N], \tag{4.1}$$

where we let θ_k (resp., δ_k) also denote the restriction of the original θ_k (resp., δ_k) to a $\mathbb{D}$-algebra automorphism (resp., θ_k-derivation) of $\mathbb{D}\langle x_1, \dots, x_{k-1}\rangle$.

Remark 4.2. (a) The CGL presentation (3.1) has a $\mathbb{D}$-form if and only if

- $\lambda_{kj} \in \mathbb{D}^*$ for $1 \le j < k \le N$;
- δ_k maps $\mathbb{D}\langle x_1, \dots, x_{k-1}\rangle$ into itself for each $k \in [2, N]$.

(b) Whether (3.1) has a $\mathbb{D}$-form depends on the choice of $\mathbb{D}$ as well as the choice of CGL presentation (3.1). For instance, if $N = 2$ and $\delta_2(x_1) \in \mathbb{K} \setminus \mathbb{D}$, then (3.1) does not have a $\mathbb{D}$-form. However, if $\gamma = \delta_2(x_1)$, then R has the CGL presentation $\mathbb{K}[x_1][\gamma^{-1}x_2; \sigma_2, \gamma^{-1}\delta_2]$, which does have a $\mathbb{D}$-form.

(c) Even if (3.1) has a $\mathbb{D}$-form, the homogeneous prime elements $y_1, \dots, y_N$ from Theorem 3.2 need not belong to $\mathbb{D}\langle x_1, \dots, x_N\rangle$. For instance, if R is the quantized Weyl algebra

$$A_1^q(\mathbb{K}) = \mathbb{K}\langle x_1, x_2 \mid x_1x_2 = qx_2x_1 + 1\rangle$$

with $q \in \mathbb{K}^*$ transcendental over the prime field of $\mathbb{K}$ and $\mathbb{D} = (\mathbb{Z}\cdot 1_{\mathbb{K}})[q^{\pm 1}]$, then the above CGL presentation has a $\mathbb{D}$-form, but $\mathbb{D}\langle x_1, x_2\rangle$ does not contain the element $y_2 = x_1x_2 + (q-1)^{-1}$.

The problems indicated in Remark 4.2 can typically be corrected by rescaling the generators x_k as in §3.3, as we now show.

When working with a $\mathbb{D}$-subalgebra $R' = \mathbb{D}\langle x_1, \dots, x_N\rangle$ of R, we adapt previous notation and write

$$R'_k := \mathbb{D}\langle x_1, \dots, x_k\rangle \quad \text{and} \quad R'_{[j,k]} := \mathbb{D}\langle x_j, \dots, x_k\rangle, \quad \forall\, j \le k \in [1, N].$$

Proposition 4.3. *Assume that $\mathbb{K} = \operatorname{Fract}\mathbb{D}$ and that $\lambda_{kj} \in \mathbb{D}^*$ for $1 \le j < k \le N$. Then there exist $t_1, \dots, t_N \in \mathbb{D} \setminus \{0\}$ such that*

(a) *$R_{\mathbb{D}} := \mathbb{D}\langle t_1x_1, \dots, t_Nx_N\rangle$ is a $\mathbb{D}$-form of the CGL presentation* (3.21).

(b) *The elements $y_1, \dots, y_N$ from Theorem* 3.2 *for the presentation* (3.21) *all lie in $R_{\mathbb{D}}$.*

Proof. Set $R' := R_{\mathbb{D}}$ for the proof. We induct on N. The case $N = 1$ holds trivially by taking $t_1 = 1$.

Now assume that $N > 1$ and that there exist $t_1, \dots, t_{N-1} \in \mathbb{D} \setminus \{0\}$ such that the algebra $R'_{N-1} := \mathbb{D}\langle t_1x_1, \dots, t_{N-1}x_{N-1}\rangle$ satisfies condtions (a), (b). In particular, R'_{N-1} is a $\mathbb{D}$-form of the CGL presentation

$$R_{N-1} = \mathbb{K}[t_1x_1][t_2x_2; \theta_2, t_2\delta_2] \cdots [t_{N-1}x_{N-1}; \theta_{N-1}, t_{N-1}\delta_{N-1}]. \tag{4.2}$$

Since $\lambda_{Nj}^{\pm 1} \in \mathbb{D}$ for all $j \in [1, N-1]$, the automorphism θ_N restricts to an automorphism of R'_{N-1}.

Write $\delta_N(x_1), \dots, \delta_N(x_{N-1})$ as $\mathbb{K}$-linear combinations of monomials

$$(t_1x_1)^{m_1} \cdots (t_{N-1}x_{N-1})^{m_{N-1}}$$

in the standard PBW basis for the presentation (4.2), and let κ_i for $i \in I$ be a list of the nonzero coefficients that appear. Choose a nonzero element $b \in \mathbb{D}$ such that $b\kappa_i \in \mathbb{D}$ for all i. Set

$$t_N := \begin{cases} b & (\text{if } p(N) = -\infty) \\ (\lambda_N - 1)b & (\text{if } p(N) \ne -\infty). \end{cases}$$

Since $b\kappa_i \in \mathbb{D}$ for all i, we have $b\delta_N(x_j) \in R'_{N-1}$ for all $j \in [1, N-1]$, and so $b\delta_N$ maps R'_{N-1} into itself. Then also $t_N\delta_N$ maps R'_{N-1} into itself. Therefore $R' = R'_{N-1}\langle t_Nx_N\rangle$ is an Ore extension $R'_{N-1}[t_Nx_N; \theta_N, t_N\delta_N]$ and (a) holds.

It remains to show that the element y_N for the CGL presentation (3.21) lies in R'. If $p(N) = -\infty$, then $y_N = t_N x_N$ and we are done. Now assume that $p(N) \neq -\infty$. Then $y_N = y_{p(N)} x_N - c_N$ where $c_N \in R_N$ and $y_{p(N)}$ is the $p(N)$-th y-element for (3.21). By our induction hypotheses, $y_{p(N)} \in R'_{N-1}$. From [14, Proposition 4.7(b)], we have

$$(\lambda_N - 1) b\delta_N(y_{p(N)}) = t_N \delta_N(y_{p(N)}) = \prod_{m=1}^{O_-(N)} \lambda_{N,p^m(N)} (\lambda_N - 1) c_N \,.$$

Since $\lambda_{N,p^m(N)} \in \mathbb{D}^*$ for all $m \in [1, O_-(N)]$ and $b\delta_N(y_{p(N)}) \in R'_{N-1}$, we conclude that $c_N \in R'_{N-1}$. Therefore $y_N \in R'$, as required. □

Lemma 4.4. *If the CGL presentation* (3.1) *has a $\mathbb{D}$-form, then $\lambda_k \in \mathbb{D}^*$ for all $k \in [2, N]$ such that $p(k) \neq -\infty$.*

Proof. If $k \in [2, N]$ and $p(k) \neq -\infty$, then $\delta_k \neq 0$ (recall Theorem 3.2). Choose $i \in [1, k-1]$ such that $\delta_k(x_i) \neq 0$, and choose a monomial x^f, for some $f = (m_1, \dots, m_{k-1})^T \in \mathbb{Z}_{\geq 0}^{k-1}$, which appears with a nonzero coefficient in $\delta_k(x_i)$. In view of (3.2), $h_k.\delta_k(x_i) = \lambda_k \lambda_{ki} \delta_k(x_i)$. Since all monomials in $x_1, \dots, x_N$ are h_k-eigenvectors, it follows that $h_k.x^f = \lambda_k \lambda_{ki} x^f$. On the other hand, $h_k.x^f = \theta_k(x^f) = \prod_{j=1}^{k-1} \lambda_{kj}^{m_j} x^f$, and consequently

$$\lambda_k = \lambda_{ki}^{-1} \prod_{j=1}^{k-1} \lambda_{kj}^{m_j} \in \mathbb{D}^* \,,$$

since all $\lambda_{kj} \in \mathbb{D}^*$ (Remark 4.2(a)). □

In case R is symmetric and (3.1) has a $\mathbb{D}$-form, the alternative CGL extension presentations of R given in (3.17) also have $\mathbb{D}$-forms, as we now show.

Lemma 4.5. *Assume that $R_{\mathbb{D}} = \mathbb{D}\langle x_1, \dots, x_N \rangle$ is a $\mathbb{D}$-form for* (3.1)*, and that R is a symmetric CGL extension.*

(a) *For $1 \leq j < k \leq N$, the algebra $(R_{\mathbb{D}})_{[j,k]}$ is a $\mathbb{D}$-form for the CGL presentation*

$$R_{[j,k]} = \mathbb{K}[x_j][x_{j+1}; \theta_{j+1}, \delta_{j+1}] \cdots [x_k; \theta_k, \delta_k]. \tag{4.3}$$

(b) *For each $\sigma \in \Xi_N$, the algebra $R_{\mathbb{D}}$ is a $\mathbb{D}$-form for the CGL presentation* (3.17) *of R.*

Proof. Set $R' := R_{\mathbb{D}}$.

(a) The symmetry assumption on R implies that the $\mathbb{K}$-subalgebra $R_{[j,k]}$ of R is itself a CGL extension of the form (4.3), as noted following Definition 3.6. For $l \in [j+1, k]$, closure of both $R_{[j,l-1]}$ and R'_{l-1} under $\theta_l^{\pm 1}$ and δ_l implies that $R'_{[j,l-1]}$ is closed under $\theta_l^{\pm 1}$ and δ_l. It follows that $R'_{[j,k]}$ is an iterated Ore extension of the form $\mathbb{D}[x_j] \cdots [x_k; \theta_k, \delta_k]$, as required.

(b) We first consider the reverse CGL extension presentation (3.13). As shown in [14, §6.2] (where θ_j^*, δ_j^* are denoted σ'_j, δ'_j), we have

$$\theta_j^*(x_k) = \lambda_{jk} x_k \quad \text{and} \quad \delta_j^*(x_k) = -\lambda_{jk} \delta_k(x_j), \quad \forall\, 1 \leq j < k \leq N.$$

Consequently, $\mathbb{D}\langle x_{j+1}, \dots, x_N \rangle$ is stable under $(\theta_j^*)^{\pm 1}$ and δ_j^* for each $j \in [1, N-1]$. This allows us to write R' as an iterated Ore extension in the form

$$R' = \mathbb{D}[x_N][x_{N-1}; \theta_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \theta_1^*, \delta_1^*], \tag{4.4}$$

which shows that R' is a $\mathbb{D}$-form for (3.13).

Now let σ be an arbitrary element of Ξ_N and consider the corresponding CGL extension presentation (3.17) of R. As indicated in [14, Remark 6.5], the automorphisms θ''_j and skew

derivations δ''_j appearing in (3.17) are restrictions of either θ_j, δ_j or θ^*_j, δ^*_j. Combined with the results of the previous paragraph, we conclude that R' is an iterated Ore extension of the form

$$\mathbb{D}[x_{\sigma(1)}][x_{\sigma(2)};\theta''_{\sigma(2)},\delta''_{\sigma(2)}]\cdots[x_{\sigma(N)};\theta''_{\sigma(N)},\delta''_{\sigma(N)}].$$

Therefore R' is a $\mathbb{D}$-form for (3.17). □

Lemma 4.6. *Assume that the CGL presentation* (3.1) *has a $\mathbb{D}$-form $R_{\mathbb{D}} = \mathbb{D}\langle x_1,\dots,x_N\rangle$ which contains the elements $y_1,\dots,y_N$ from Theorem* 3.2.

(a) *For each $k\in[1,N]$, the element y_k is normal in $(R_{\mathbb{D}})_k$.*

(b) *For any subset $I\subseteq[1,N]$, the multiplicative set generated by $\mathbb{D}^*\cup\{y_i \mid i\in I\}$ is a denominator set in $R_{\mathbb{D}}$.*

Proof. Set $R' := R_{\mathbb{D}}$.

(a) By [14, Corollary 4.8], y_k quasi-commutes with those x_j such that $j<s(k)$ according to the rule

$$y_k x_j = \left(\prod_{m=0}^{O_-(k)} \lambda_{j,p^m(k)}\right)^{-1} x_j y_k\,.$$

Since the $\lambda_{j,p^m(k)}$ all lie in $\mathbb{D}^*$, it follows that $y_k R'_k = R'_k y_k$.

(b) It suffices to show that $\mathbb{D}^* y_k^{\mathbb{N}}$, the multiplicative set generated by $\mathbb{D}^*\cup\{y_k\}$, is a denominator set in R' for each $k\in[1,N]$. By part (a), $\mathbb{D}^* y_k^{\mathbb{N}}$ is a denominator set in R'_k.

Since y_k is homogeneous (with respect to the $X(\mathcal{H})$-grading on R), it is an eigenvector for each $h\in\mathcal{H}$ and thus for $\theta_{k+1},\dots,\theta_N$. The leading term of y_k is $x_{p^{O_-(k)}(k)}\cdots x_{p(k)}x_k$, and so

$$\theta_l(y_k) = \left(\prod_{m=0}^{O_-(k)} \lambda_{l,p^m(k)}\right) y_k\,, \quad \text{for } \ 1\le k<l\le N.$$

Consequently, $\theta_l(\mathbb{D}^* y_k^{\mathbb{N}}) = \mathbb{D}^* y_k^{\mathbb{N}}$ for all $l>k$. It therefore follows from [11, Lemma 1.4], by induction on l, that $\mathbb{D}^* y_k^{\mathbb{N}}$ is a denominator set in R'_l for $l=k+1,\dots,N$. □

Proposition 4.7. *Assume that R is a symmetric CGL extension and that the CGL presentation* (3.1) *has a $\mathbb{D}$-form $R_{\mathbb{D}} = \mathbb{D}\langle x_1,\dots,x_N\rangle$ which contains the elements $y_1,\dots,y_N$.*

(a) *The elements $y_{[i,s^m(i)]}$ of Proposition* 3.7 *all belong to $R_{\mathbb{D}}$.*

(b) *The elements $u_{[i,s^m(i)]}$ of Theorem* 3.8 *all belong to $R_{\mathbb{D}}$, and their leading coefficients $\pi_{[i,s^m(i)]}$ belong to $\mathbb{D}$.*

(c) *The elements $y_{\sigma,k}$, for $\sigma\in\Xi_N$ and $k\in[1,N]$, all belong to $R_{\mathbb{D}}$.*

Proof. Set $R' := R_{\mathbb{D}}$.

(a) We first recall that by the case $\tau=\mathrm{id}$ of [15, Theorem 5.3], y_k is a scalar multiple of $y_{[p^{O_-(k)}(k),k]}$ for all $k\in[1,N]$. However, these elements both have leading coefficient 1, so they are equal. Taking $k=s^m(i)$, we obtain

$$y_{[i,s^m(i)]} = y_{s^m(i)}\,, \quad \forall\, i\in[1,N] \ \text{ with } \ p(i)=-\infty. \tag{4.5}$$

This verifies that $y_{[i,s^m(i)]}\in R'$ whenever $p(i)=-\infty$.

We next show, by induction on i, that all $y_{[i,s^m(i)]}\in R'$. The case $i=1$ follows from the previous result, since $p(1)=-\infty$. Now assume that $i>1$ and that $y_{[j,s^m(j)]}\in R'$ for all $j\in[1,i-1]$ and $m\in[0,O_+(j)]$. If $p(i)=-\infty$, we are done by the previous result, so we may assume that $p(i)=j\in[1,i-1]$. Set $k=s^m(i)=s^{m+1}(j)$. By the induction hypothesis, $y_{[j,k]}\in R'$. According to [15, Theorem 5.1(d)],

$$y_{[j,k]} = x_j y_{[i,k]} - c'$$

for some $c' \in R_{[j+1,k]}$. Since $R_{[j,k]}$ (resp., $R'_{[j,k]}$) is a free right module over $R_{[j+1,k]}$ (resp., $R'_{[j+1,k]}$) with basis $\{1, x_j, x_j^2, \dots\}$, the assumption $y_{[j,k]} \in R'$ implies $y_{[i,k]} \in R'$. This concludes the induction step.

(b) Since all values of the bicharacter $\Omega_{\boldsymbol{\lambda}}$ lie in $\mathbb{D}^*$, the formula (3.19) together with part (a) yields $u_{[i,s^m(i)]} \in R'$. Consequently, its leading coefficient, $\pi_{[i,s^m(i)]}$, must lie in $\mathbb{D}$.

(c) Fix $\sigma \in \Xi_N$. We proceed by induction on $k \in [1,N]$ to show that $y_{\sigma,k} \in R'$. The case $k=1$ holds trivially, since $y_{\sigma,1} = x_{\sigma(1)}$. Now let $k>1$ and assume that $y_{\sigma,j} \in R'$ for all $j \in [1,k-1]$.

If $p(\sigma(k)) \notin \sigma([1,k-1])$, then $y_{\sigma,k} = x_{\sigma(k)}$ and we are done. Assume now that $p(\sigma(k)) = \sigma(l)$ for some $l \in [1,k-1]$. Then $y_{\sigma,l} \in R'$ by induction, and

$$y_{\sigma,k} = y_{\sigma,l} x_{\sigma(k)} - c, \quad 0 \neq c \in R_{\sigma([1,k-1])}\,. \tag{4.6}$$

By [15, Theorem 5.3], one of the following cases holds:

(i) $\sigma(k) > \sigma(1)$, $y_{\sigma,k} = \lambda y_{[p^m(\sigma(k)),\sigma(k)]}$, $m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\sigma(k)) \in \sigma([1,k])\}$,

(ii) $\sigma(k) < \sigma(1)$, $y_{\sigma,k} = \lambda y_{[\sigma(k),s^m(\sigma(k))]}$, $m = \max\{n \in \mathbb{Z}_{\geq 0} \mid s^n(\sigma(k)) \in \sigma([1,k])\}$,

for some $\lambda \in \mathbb{K}^*$.

Case (i). By the definition (3.15) of Ξ_N,

$$\sigma(k) = \max \sigma([1,k]) \quad \text{and} \quad \sigma([1,k-1]) \subseteq [1, \sigma(k)-1].$$

As $p(\sigma(k)) = \sigma(l) \in \sigma([1,k])$, we also have $m \geq 1$, and so

$$y_{[p^m(\sigma(k)),\sigma(k)]} = y_{[p^m(\sigma(k)),p(\sigma(k))]} x_{\sigma(k)} - c', \quad 0 \neq c' \in R_{[p^m(\sigma(k)),\sigma(k)-1]}\,.$$

Comparing terms in $R_{[1,\sigma(k)]} = R_{[1,\sigma(k)-1]} x_{\sigma(k)} + R_{[1,\sigma(k)-1]}$, we find that

$$y_{\sigma,l} = \lambda y_{[p^m(\sigma(k)),p(\sigma(k))]}\,.$$

Since $\mathrm{lc}(y_{[p^m(\sigma(k)),p(\sigma(k))]}) = 1$ and $y_{\sigma,l} \in R'$, we find that $\lambda \in \mathbb{D}$. In view of part (a), we conclude that

$$y_{\sigma,k} = \lambda y_{[p^m(\sigma(k)),\sigma(k)]} \in R'\,.$$

Case (ii). Now $\sigma(k) = \min \sigma([1,k])$ and $\sigma([1,k-1]) \subseteq [\sigma(k)+1, N]$. If $m=0$, we would have $y_{\sigma,k} = \lambda y_{\sigma(k),\sigma(k)]} = \lambda x_{\sigma(k)}$, contradicting (4.6). Thus, $m>0$. By [15, Theorem 5.1(d)],

$$y_{[\sigma(k),s^m(\sigma(k))]} = x_{\sigma(k)} y_{[s(\sigma(k)),s^m(\sigma(k))]} - c', \quad 0 \neq c' \in R_{[\sigma(k)+1,s^m(\sigma(k))]}\,.$$

We may rewrite $y_{\sigma,k}$ in the form

$$y_{\sigma,k} = \mu^{-1} x_{\sigma(k)} y_{\sigma,l} - \widetilde{c},$$

where $\mu \in \mathbb{K}^*$ arises from $\theta''_{\sigma(k)}(y_{\sigma,l}) = \mu y_{\sigma,l}$ and

$$\widetilde{c} = \mu^{-1} \delta''_{\sigma(k)}(y_{\sigma,l}) + c \in R_{\sigma([1,k-1])} \subseteq R_{[\sigma(k)+1,N]}\,.$$

Now $y_{\sigma,l}$ is a homogeneous element of R' and R' is a $\mathbb{D}$-form for (3.17). Moreover, $y_{\sigma,l}$ has leading coefficient 1 with respect to the presentation (3.17), so $\theta''_{\sigma(k)}(y_{\sigma,l})$ must be a $\mathbb{D}^*$-multiple of $y_{\sigma,l}$. Hence, $\mu \in \mathbb{D}^*$.

Comparing terms in $R_{[\sigma(k),N]} = x_{\sigma(k)} R_{[\sigma(k)+1,N]} + R_{[\sigma(k)+1,N]}$, we find that

$$\mu^{-1} y_{\sigma,l} = \lambda y_{[s(\sigma(k)),s^m(\sigma(k))]}\,.$$

Since $\mathrm{lc}(y_{[s(\sigma(k)),s^m(\sigma(k))]}) = 1$ while $y_{\sigma,l} \in R'$ and $\mu \in \mathbb{D}^*$, we obtain $\lambda \in \mathbb{D}$, and therefore $y_{\sigma,k} = \lambda y_{[\sigma(k),s^m(\sigma(k))]} \in R'$ in view of part (a). This concludes the second case of the inductive step. □

4.2. **Quantum cluster algebra structures on integral forms.** For integral forms of appropriately normalized symmetric CGL extensions, we have the following exact analog of Theorem 3.11. Fix a symmetric CGL extension R of length N such that Conditions (A) and (B) hold. Set $\mathcal{F} := \mathrm{Fract}(R)$, and let $\mathbb{D}$ be a commutative domain whose field of fractions is $\mathbb{K}$. Define toric frames $M_\sigma : \mathbb{Z}^N \to \mathcal{F}$, multiplicatively skew-symmetric matrices $\mathbf{r}_\sigma \in M_N(\mathbb{K})$, and sets $\mathbf{ex}_\sigma \subseteq [1, N]$ as in Subsection 3.5. (Recall the notation $M = M_{\mathrm{id}}$, $\mathbf{r} = \mathbf{r}_{\mathrm{id}}$, $\mathbf{ex} = \mathbf{ex}_{\mathrm{id}}$.) Provided the matrices $\mathbf{r}_\sigma$ have entries from $\mathbb{D}^*$, the frames M_σ also qualify as toric frames over $\mathbb{D}$, and we shall view them as such.

Theorem 4.8. *Let R be a symmetric CGL extension of length N as in Definitions* 3.1, 3.4*, and assume that Conditions* (A), (B)*, and* (3.28) *hold. Let $\mathbb{D}$ be a* (*commutative*) *domain with quotient field* $\mathrm{Fract}(\mathbb{D}) = \mathbb{K}$*, such that the scalars ν_{kl} in Condition* (A) *all lie in $\mathbb{D}^*$. Assume that the CGL presentation* (3.1) *has a $\mathbb{D}$-form $R_{\mathbb{D}} = \mathbb{D}\langle x_1, \dots, x_N\rangle$ which contains the homogeneous prime elements $y_1, \dots, y_N$ from Theorem* 3.11.

(a) *For each $\sigma \in \Xi_N$, let $\widetilde{B}_\sigma$ be the $N \times |\mathbf{ex}|$ integer matrix determined as in Theorem* 3.11(a). *Then the pair $(M_\sigma, \widetilde{B}_\sigma)$ is a quantum seed for $\mathcal{F} := \mathrm{Fract}(R) = \mathrm{Fract}(R_{\mathbb{D}})$ over $\mathbb{D}$, and the principal part of $\widetilde{B}_\sigma$ is skew-symmetrizable via the integers $d_{\eta(k)}$, $k \in \mathbf{ex}_\sigma$ from Condition* (B).

(b) *All the quantum seeds $(M_\sigma, \widetilde{B}_\sigma)$ from part* (a) *are mutation-equivalent to each other up to the S_N action. They are linked by sequences of one-step mutations of the following kind. Suppose $\sigma, \sigma' \in \Xi_N$ are such that*

$$\sigma' = (\sigma(k), \sigma(k+1)) \circ \sigma = \sigma \circ (k, k+1)$$

for some $k \in [1, N-1]$. If $\eta(\sigma(k)) \neq \eta(\sigma(k+1))$, then $M_{\sigma'} = M_\sigma \cdot (k, k+1)$ in terms of the action (3.29). *If $\eta(\sigma(k)) = \eta(\sigma(k+1))$, then $M_{\sigma'} = \mu_k(M_\sigma) \cdot (k, k+1)$.*

(c) *The algebra $R_{\mathbb{D}}$ equals the quantum cluster and upper cluster algebras over $\mathbb{D}$ associated to M, $\widetilde{B}$, $\varnothing$:*

$$R_{\mathbb{D}} = \mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}} = \mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{D}}\,.$$

In particular, $\mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}}$ is a finitely generated $\mathbb{D}$-algebra, and it is noetherian if $\mathbb{D}$ is noetherian. In fact, $\mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}}$ is generated by the cluster variables in the seeds parametrized by the finite subset Γ_N of Ξ_N, recall (3.16).

(d) *For any subset* **inv** *of the set $P(N)$ of frozen variables, there are equalities*

$$R_{\mathbb{D}}[y_k^{-1} \mid k \in \mathbf{inv}] = \mathbf{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{D}} = \mathbf{U}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{D}}\,.$$

Proof. (a) We already have from Theorem 3.11(a) that $(M_\sigma, \widetilde{B}_\sigma)$ is a quantum seed for $\mathcal{F}$ over $\mathbb{K}$ and that the principal part of $\widetilde{B}_\sigma$ is skew-symmetrizable via the $d_{\eta(k)}$, $k \in \mathbf{ex}_\sigma$. The entries of $\mathbf{r}(M_\sigma) = \mathbf{r}_\sigma$, given in (3.32), lie in $\mathbb{D}^*$ due to the assumption that all $\nu_{kl} \in \mathbb{D}^*$. Since $\mathbb{K} = \mathrm{Fract}(\mathbb{D})$, we have $\mathrm{Fract}\,\mathbb{D}\langle M_\sigma(\mathbb{Z}^N)\rangle = \mathrm{Fract}\,\mathbb{K}\langle M_\sigma(\mathbb{Z}^N)\rangle = \mathcal{F}$, and so $(M_\sigma, \widetilde{B}_\sigma)$ is also a quantum seed for $\mathcal{F}$ over $\mathbb{D}$.

(b) This is immediate from Theorem 3.11(c).

(c) and (d) are proved below. □

4.3. **Examples.**

Example 4.9. Consider a uniparameter quantized Weyl algebra $R = A_n^{q,\boldsymbol{\alpha}}(\mathbb{K})$, for a non-root of unity $q \in \mathbb{K}^*$ and a skewsymmetric matrix $\boldsymbol{\alpha} = (a_{ij}) \in M_n(\mathbb{Z})$. This algebra is

presented by generators $v_1, w_1, \dots, v_n, w_n$ and relations

$$\begin{aligned} w_i w_j &= q^{a_{ij}} w_j w_i\,, && (\text{all } i,j),\\ v_i v_j &= q^{1+a_{ij}} v_j v_i\,, && (i<j),\\ v_i w_j &= q^{-a_{ij}} w_j v_i\,, && (i<j),\\ v_i w_j &= q^{1-a_{ij}} w_j v_i\,, && (i>j),\\ v_j w_j &= 1 + q w_j v_j + (q-1)\sum_{l<j} w_l v_l\,, && (\text{all } j). \end{aligned} \tag{4.7}$$

The torus $\mathcal{H} = (\mathbb{K}^*)^n$ acts rationally on R with

$$(\alpha_1, \dots, \alpha_n).v_i = \alpha_i v_i \qquad \text{and} \qquad (\alpha_1, \dots, \alpha_n).w_i = \alpha_i^{-1} w_i$$

for $(\alpha_1, \dots, \alpha_n) \in \mathcal{H}$ and $i \in [1, n]$. With the variables $v_1, w_1, \dots, v_n, w_n$ in the listed order, R is a CGL extension, but that presentation is not symmetric. There is a symmetric CGL extension presentation with the variables in the order $w_n, \dots, w_1, v_1, \dots, v_n$, and $\mathbb{D}\langle w_n, \dots, w_1, v_1, \dots, v_n\rangle$ is a $\mathbb{D}$-form for this presentation, where $\mathbb{D} = (\mathbb{Z}\cdot 1_{\mathbb{K}})[q^{\pm 1}]$. However, the homogeneous prime elements $y_1, \dots, y_{2n}$ from Theorem 3.2 do not lie in this $\mathbb{D}$-form; see Remark 4.2(c). This can be rectified by rescaling the generators as in Proposition 4.3. One such rescaling leads to the CGL presentation

$$R = \mathbb{K}[(q-1)w_n] \cdots [(q-1)w_1; \theta_n][v_1; \theta_{n+1}, \delta_{n+1}] \cdots [v_n; \theta_{2n}, \delta_{2n}], \tag{4.8}$$

and $\mathbb{D}\langle (q-1)w_n, \dots, (q-1)w_1, v_1, \dots, v_n\rangle$ is a $\mathbb{D}$-form for this presentation which contains all the y_k.

If either R or a $\mathbb{D}$-form of R is $\mathbb{Z}_{\geq 0}$-graded, then in view of the final relations in (4.7) all the generators v_i, w_j must be homogeneous of degree 0. Thus, R and its $\mathbb{D}$-forms have no nontrivial $\mathbb{Z}_{\geq 0}$-gradings. □

Example 4.10. Let $R = A_n^{q,\boldsymbol{\alpha}}(\mathbb{K})$ and $\mathcal{H} = (\mathbb{K}^*)^n$ as in Example 4.9, and take the symmetric CGL presentation (4.8). Set $\mathbb{D} = (\mathbb{Z}\cdot 1_{\mathbb{K}})[q^{\pm 1}]$. Then

$$\mathbb{D}\langle (q-1)w_n, \dots, (q-1)w_1, v_1, \dots, v_n\rangle$$

is a $\mathbb{D}$-form for the presentation (4.8) which contains the homogeneous prime elements $y_1, \dots, y_{2n}$ from Theorem 3.2.

The CGL presentation (4.8) satisfies Condition (B) with all $d_i = 1$, and to obtain Condition (A) we just need to assume that $\mathbb{K}$ contains a square root of q. Choose one, and label it $q^{1/2}$. The condition (3.28), however, only holds after a further rescaling of the generators. Namely, write R as an iterated Ore extension with variables $x_1, \dots, x_{2n}$ where

$$x_i := \begin{cases} (q-1)w_{n+1-i} & (\text{if } i \in [1,n]),\\ (-1)^{i-n} q^{(i-n-1)/2} v_{i-n} & (\text{if } i \in [n+1, 2n]). \end{cases}$$

In order to express the relations among these x_i in a convenient form, we use the following notation:

$$l' := 2n+1-l \ \ (\text{for } l \in [1,2n]) \qquad \text{and} \qquad c_{ij} := a_{n+1-i,n+1-j} \ \ (\text{for } i,j \in [1,n]).$$

Then R has the presentation with generators $x_1, \dots, x_{2n}$ and defining relations

$$
\begin{aligned}
x_i x_j &= q^{c_{ij}} x_j x_i \,, && (i, j \in [1, n]) \\
x_i x_j &= q^{1+c_{i'j'}} x_j x_i \,, && (n < i < j \le 2n) \\
x_i x_j &= q^{-c_{i'j}} x_j x_i \,, && (j \le n < i < j' \le 2n) \\
x_i x_j &= q^{1-c_{i'j}} x_j x_i \,, && (j \le n < j' < i \le 2n) \\
x_{j'} x_j &= (-1)^{n+1-j} q^{(n-j)/2}(q-1) + q x_j x_{j'} \\
&\quad + (q-1) \sum_{1 \le j < l \le n} (-1)^{l-j} q^{(l-j)/2} x_l x_{l'} \,, && (j \in [1, n]).
\end{aligned}
\tag{4.9}
$$

With the presentation (4.9), R is a symmetric CGL extension satisfying the required hypotheses (A), (B), and (3.28) of Theorem 4.8. It has a $\mathbb{D}$-form

$$
A_n^{q,\boldsymbol{\alpha}}(\mathbb{D}) := \mathbb{D}\langle x_1, \dots, x_{2n} \rangle \tag{4.10}
$$

where we now take $\mathbb{D} = (\mathbb{Z} \cdot 1_{\mathbb{K}})[q^{\pm 1/2}]$. There are two possibilities for $\mathbb{D}$:

$$
\begin{aligned}
&\mathbb{D} \cong \mathcal{A}^{1/2} = \mathbb{Z}[q^{1/2}], && \text{if} \quad \operatorname{char} \mathbb{K} = 0, \\
&\mathbb{D} \cong \mathbb{F}_p[q^{1/2}], && \text{if} \quad \operatorname{char} \mathbb{K} = p
\end{aligned}
$$

(recall (1.1)). The scalars ν_{kl} from Condition (A) all lie in $\mathbb{D}^*$, as do the nonzero coefficients of the homogeneous prime elements $y_1, \dots, y_{2n}$ from Theorem 3.11. Therefore

$$
A_n^{q,\boldsymbol{\alpha}}(\mathbb{D}) = \mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}} = \mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{D}}
$$

by Theorem 4.8. The matrix $\mathbf{r} = \mathbf{r}(M)$ of the initial toric frame has the form

$$
\mathbf{r} = \begin{bmatrix}
1 & s^{c_{12}} & \cdots & s^{c_{1,n-1}} & s^{c_{1n}} & 1 & 1 & \cdots & 1 & s^{-1} \\
s^{c_{21}} & 1 & \cdots & s^{c_{2,n-1}} & s^{c_{2n}} & 1 & 1 & \cdots & s^{-1} & s^{-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
s^{c_{n1}} & s^{c_{n2}} & \cdots & s^{c_{n,n-1}} & 1 & s^{-1} & s^{-1} & \cdots & s^{-1} & s^{-1} \\
1 & 1 & \cdots & 1 & s & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & s & s & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
s & s & \cdots & s & s & 1 & 1 & \cdots & 1 & 1
\end{bmatrix},
$$

where $s := q^{1/2}$. The quiver of the initial seed is acyclic, namely it equals

$$
\begin{tikzcd}
1 \arrow[dr] & 2 & \cdots & n-1 \arrow[dr] & n \\
2n \arrow[u] & 2n-1 \arrow[u] & \cdots & n+2 \arrow[u] & n+1 \arrow[u]
\end{tikzcd}
$$

where the top vertices are mutable and the bottom ones are frozen. □

Next, we illustrate Theorem 4.8 with a CGL extension which is not $\mathbb{Z}_{\ge 0}$-graded connected and whose quiver is not acyclic.

Example 4.11. Recall the notation $\mathcal{F}^{1/2} := \mathbb{Q}(q^{1/2})$. Let R be the $\mathcal{F}^{1/2}$-algebra with generators $x_1, \dots, x_6$ and relations

$$\begin{aligned}
&x_2x_1 = qx_1x_2\,, && x_3x_1 = qx_1x_3 + (1-q)x_2^2\,, && x_3x_2 = qx_2x_3\,,\\
&x_4x_1 = qx_1x_4 + (1-q^2)x_2x_3\,, && x_4x_2 = qx_2x_4 + (q-1)x_3^2\,, && x_4x_3 = qx_3x_4\,,\\
&x_5x_1 = q^{-1}x_1x_5\,, && x_5x_2 = q^{-1}x_2x_5\,, && x_5x_3 = q^{-1}x_3x_5\,,\\
& && x_5x_4 = q^{-1}x_4x_5 + (q-1)^3\,, &&\\
& && x_6x_1 = q^{-1}x_1x_6 + (q^{-1}-q)x_2yx_5 + (1-q)x_3^2x_5^2\,, &&\\
& && x_6x_2 = q^{-1}x_2x_6 + (q-q^{-1})x_3yx_5\,, &&\\
&x_6x_3 = q^{-1}x_3x_6 + (q-1)y^2\,, && x_6x_4 = q^{-1}x_4x_6\,, && x_6x_5 = qx_5x_6\,,
\end{aligned}$$

where

$$y := x_4x_5 - q(1-q)^2\,.$$

The algebra R is a symmetric CGL extension for the torus $\mathcal{H} := ((\mathcal{F}^{1/2})^\times)^2$ acting so that for the corresponding grading by $X(\mathcal{H}) \cong \mathbb{Z}^2$, the variables $x_1, \dots, x_6$ have degrees

$$(4,3),\ (3,2),\ (2,1),\ (1,0),\ (-1,0),\ (-2,-1).$$

The h-elements for this CGL extension are

$$h_3 = h_4 = (q, q^{-1}),\ \ h_5 = h_6 = (q^{-1}, q) \in \mathcal{H}.$$

Consequently, $\lambda_k = q$ for $k \in [3,6]$. The (nonunique) elements $h_1, h_2 \in \mathcal{H}$ can be also chosen so that $\lambda_k = q$ for $k = 1,2$. Obviously Conditions (A) and (B) hold.

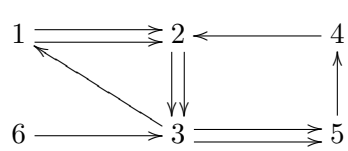
Denote by $R_{\mathcal{A}^{1/2}}$ the $\mathcal{A}^{1/2}$-subalgebra of R generated by $x_1, \dots, x_6$. The homogeneous prime elements $y_1, \dots, y_6$ belong to $R_{\mathcal{A}^{1/2}}$ and are given by

$$\begin{aligned}
&y_1 = x_1\,, \qquad y_2 = x_2\,, \qquad y_3 = x_1x_3 + q^{-1}x_2^2\,,\\
&y_4 = x_2x_4 - q^{-1}x_3^2\,, \qquad y_5 = x_2x_4x_5 - q^{-1}x_3^2x_5 - q(1-q)^2x_2\,,\\
&y_6 = x_1x_3x_6 + q^{-1}x_2^2x_6 - qx_1y^2 - (1+q^{-1})x_2x_3yx_5 + q^{-2}x_3^3x_5^2\,.
\end{aligned}$$

(The element y is precisely the interval prime element $y_{[3,5]}$.) Consequently, the η-function from Theorem 3.2 is given by $\eta(1) = \eta(3) = \eta(6) = 1$ and $\eta(2) = \eta(4) = \eta(5) = 2$. Hence, the predecessor function p maps $6 \mapsto 3 \mapsto 1$ and $5 \mapsto 4 \mapsto 2$. So, $\mathbf{ex} = [1,4]$. One easily verifies that the condition (3.28) is satisfied. The matrix of the initial toric frame for $R_{\mathcal{A}^{1/2}}$ from Theorem 4.8 is given by

$$\mathbf{r} = \begin{bmatrix}
1 & s^{-1} & s^{-1} & s^{-2} & s^{-1} & 1\\
s & 1 & 1 & s^{-1} & 1 & s\\
s & 1 & 1 & s^{-2} & 1 & s^2\\
s^2 & s & s^2 & 1 & s^2 & s^4\\
s & 1 & 1 & s^{-2} & 1 & s\\
1 & s^{-1} & s^{-2} & s^{-4} & s^{-1} & 1
\end{bmatrix},$$

where $s := q^{1/2}$. The quiver of the initial quantum seed of $R_{\mathcal{A}^{1/2}}$ is



where the vertices $5, 6$ are frozen and the rest are mutable. Theorem 4.8 implies that $R_{\mathcal{A}^{1/2}}$ is isomorphic to the corresponding cluster and upper cluster algebras over $\mathcal{A}^{1/2}$ where the two frozen variables are not inverted.

All statements in the example hold if $\mathcal{A}^{1/2}$ and $\mathcal{F}^{1/2}$ are replaced by $\mathbb{F}_p[q^{\pm 1/2}]$ and $\mathbb{F}_p(q^{1/2})$, respectively. □

Remark 4.12. The algebras in Examples 4.9–4.11 do not come from quantum unipotent cells in any symmetrizable Kac–Moody algebra, because the algebras in those examples are $\mathbb{Z}$-graded but they are not $\mathbb{Z}_{\geq 0}$-graded connected algebras while all quantum unipotent cells are $\mathbb{Z}_{\geq 0}$-graded connected algebras. In particular, these examples concern applications of Theorem 4.8 that are not covered by [23] or the results in Sect. 7 of this paper.

Remark 4.13. There are also simple examples of symmetric CGL extensions R which cannot be "untwisted" into a uniparameter form. More precisely, there are such R for which no twist of R relative to a $\mathbb{K}^*$-valued cocycle on a natural grading group turns R into a uniparameter CGL extension. For instance, this is true of the multiparameter quantized Weyl algebra $A_n^{Q,P}(\mathbb{K})$ when the parameters in the vector $Q = (q_1, \dots, q_n)$ generate a non-cyclic subgroup of $\mathbb{K}^*$ (see [16, Example 5.10]). One can show that the quantized Weyl algebras $A_n^{Q,P}(\mathbb{K})$ have integral forms over subrings $\mathbb{Z}[q_1^{\pm 1/2}, \dots, q_n^{\pm 1/2}]$ of $\mathbb{K}$. Theorem 4.8 can be applied to prove that the integral forms are isomorphic to quantum cluster algebras over $\mathbb{Z}[q_1^{\pm 1/2}, \dots, q_n^{\pm 1/2}]$.

4.4. **Proof of parts (c), (d) of Theorem 4.8.** For the first part of this subsection, we assume only that $\mathbb{K} = \mathrm{Fract}(\mathbb{D})$. The normalization assumptions in Theorem 4.8 will be invoked only in the proof of parts (c), (d) of the theorem.

In the following lemma and proposition, divisibility refers to divisibility within the ring $R_{\mathbb{D}}$.

Lemma 4.14. *Assume that* (3.1) *has a* $\mathbb{D}$*-form* $R_{\mathbb{D}} = \mathbb{D}\langle x_1, \dots, x_N\rangle$*. Let* $d \in \mathbb{D}\backslash\{0\}$ *and* $u, v \in R_{\mathbb{D}}\backslash\{0\}$ *such that* $d \mid uv$*. If* $\mathrm{lc}(v) \in \mathbb{D}^*$*, then* $d \mid u$*.*

Proof. Let $\mathrm{lt}(u) = bx^f$ and $\mathrm{lt}(v) = cx^g$ where $b, c \in \mathbb{D}\backslash\{0\}$ and $f, g \in \mathbb{Z}_{\geq 0}^N$. By assumption, $c \in \mathbb{D}^*$ and $uv = dw$ for some $w \in R'\backslash\{0\}$. We proceed by induction on f with respect to $\prec$. If $f = 0$, we have $u = b$ and $bcx^g = \mathrm{lt}(uv) = d\ \mathrm{lt}(w)$. In this case, $d \mid bc$, whence d divides $b = u$ because c is a unit in $\mathbb{D}$.

Now assume that $f \succ 0$. In view of [15, Eq. (3.20)], we have

$$\lambda bc = \mathrm{lc}(uv) = d\ \mathrm{lc}(w)$$

for some $\lambda \in \mathbb{D}$ which is a product of $\lambda_{k,j}$ s. By assumption, λ is a unit in $\mathbb{D}$, whence $b = de$ for some $e \in \mathbb{D}$. Now $u = dex^f + u'$ where either $u' = 0$ or $\mathrm{lt}(u') = b'x^{f'}$ with $b' \in \mathbb{D}$ and $f' \prec f$. In the second case,

$$u'v = uv - dex^f v = d(w - ex^f v).$$

By induction, $d \mid u'$, and thus $d \mid u$. This verifies the induction step. □

Proposition 4.15. *Assume that* (3.1) *has a* $\mathbb{D}$*-form* $R_{\mathbb{D}} = \mathbb{D}\langle x_1, \dots, x_N\rangle$ *which contains* $y_1, \dots, y_N$*. If* Y *is the multiplicative set generated by* $\mathbb{D}^* \cup \{y_1, \dots, y_N\}$*, then*

$$R_{\mathbb{D}}[Y^{-1}] \cap R = R_{\mathbb{D}}\,. \tag{4.11}$$

Recall from Lemma 4.6(b) that Y is a denominator set in $R_{\mathbb{D}}$.

Proof. If $r \in R_{\mathbb{D}}[Y^{-1}] \cap R$, then $r = ay^{-1}$ for some $a \in R_{\mathbb{D}}$ and $y \in Y$. Since $r \in \mathbb{K}R_{\mathbb{D}}$, we also have $r = d^{-1}b$ for some $d \in \mathbb{D}\backslash\{0\}$ and $b \in R_{\mathbb{D}}$. Now $da = by$. Since $\mathrm{lc}(y_j) = 1$ for all $j \in [1, N]$, we see via [15, Eq. (3.20)] that $\mathrm{lc}(y) \in \mathbb{D}^*$. By Lemma 4.14, $b = db'$ for some $b' \in R_{\mathbb{D}}$. Thus $a = b'y$ and therefore $r = ay^{-1} = b' \in R_{\mathbb{D}}$. □

From now on, assume that R is a symmetric CGL extension and that (3.1) has a $\mathbb{D}$-form $R_{\mathbb{D}} = \mathbb{D}\langle x_1, \dots, x_N\rangle$ which contains $y_1, \dots, y_N$.. For each $\sigma \in \Xi_N$, we have the CGL presentation (3.17) for R, and $R_{\mathbb{D}}$ is a $\mathbb{D}$-form of this presentation by Lemma 4.5(b). Let $y_{\sigma,1}, \dots, y_{\sigma,N}$ be the (unnormalized) sequence of homogeneous prime elements from Theorem 3.11 for the presentation (3.17), and let E_σ denote the multiplicative set generated by

$$\mathbb{D}^* \cup \{y_{\sigma,l} \mid l \in [1, N],\ s_\sigma(l) \neq +\infty\} = \mathbb{D}^* \cup \{y_{\sigma,l} \mid l \in \mathbf{ex}_\sigma\},$$

where s_σ is the successor function for the level sets of $\eta\sigma$. (By [15, Corollary 5.6(b)], $\eta\sigma$ can be chosen as the η-function for the presentation (3.17).) By Proposition 4.7(c) and Lemma 4.6(b), E_σ is a denominator set in $R_{\mathbb{D}}$.

Proposition 4.16. *The ring $R_{\mathbb{D}}$ equals the following intersection of localizations:*

$$R_{\mathbb{D}} = \bigcap_{\sigma \in \Gamma_N} R_{\mathbb{D}}[E_\sigma^{-1}]. \tag{4.12}$$

Proof. Let T denote the right hand side of (4.12). Since $\bigcap_{\sigma\in\Gamma_N} R[E_\sigma^{-1}] = R$ by [15, Theorem 8.19(d)], we have $T \subseteq R$. On the other hand, E_{id} is contained in the denominator set Y of Proposition 4.15, and so $T \subseteq R_{\mathbb{D}}[Y^{-1}]$. Proposition 4.15 thus implies $T \subseteq R_{\mathbb{D}}$, yielding (4.12). □

Corollary 4.17. *If* **inv** *is any subset of* $[1, N]\backslash\mathbf{ex}$*, then*

$$R_{\mathbb{D}}[y_k^{-1} \mid k \in \mathbf{inv}] = \bigcap_{\sigma \in \Gamma_N} R_{\mathbb{D}}[E_\sigma^{-1}][y_k^{-1} \mid k \in \mathbf{inv}]. \tag{4.13}$$

Proof. This follows from Proposition 4.16 in the same way that [15, Theorem 8.19(e)] follows from [15, Theorem 8.19(d)]. □

Proof of Theorem 4.8(c)(d). Note that the scalars $\mathcal{S}_{\boldsymbol{\nu}}(f)$ from (3.24), for $f \in \mathbb{Z}^N$, lie in $\mathbb{D}^*$ because of our assumption that all $\nu_{kl} \in \mathbb{D}^*$. Hence, invoking Proposition 4.7(a), the normalized elements $\overline{y}_j$ and $\overline{y}_{[i,s^m(i)]}$ from (3.25) and (3.26) belong to $R_{\mathbb{D}}$. By (3.33), we thus have $\overline{y}_{\sigma,k} \in R_{\mathbb{D}}$ for all $\sigma \in \Xi_N$ and $k \in [1, N]$.

We next show that

$$R_{\mathbb{D}} = \mathbb{D}\langle \overline{y}_{\sigma,k} \mid \sigma \in \Gamma_N\,,\ k \in [1, N]\rangle. \tag{4.14}$$

The proof is parallel to that for the corresponding statement in [15, Theorem 8.2(b)]. For each $j \in [1, N]$, there is an element $\sigma \in \Gamma_N$ with $\sigma(1) = j$. By (3.33), $\overline{y}_{\sigma,1}$ is a $\mathbb{D}^*$-multiple of $y_{[j,j]} = x_j$, and so $x_j \in \mathbb{D}^*\overline{y}_{\sigma,1}$. Therefore all x_j lie in the right hand side of (4.14), and the equation is established. Since all the $\overline{y}_{\sigma,k} = M_\sigma(e_k)$ are cluster variables, it follows that $R_{\mathbb{D}} \subseteq \mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}}$.

We have $\mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}} \subseteq \mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{D}}$ by the Laurent Phenomenon (2.3), and

$$\mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{D}} \subseteq \bigcap_{\sigma \in \Xi_N} \mathbb{D}\mathcal{T}_{(M_\sigma, \widetilde{B}_\sigma, \varnothing)} = \bigcap_{\sigma \in \Xi_N} \mathbb{D}\langle \overline{y}_{\sigma,k}^{\pm 1},\ \overline{y}_{\sigma,j} \mid k \in \mathbf{ex}_\sigma,\ j \in [1, N]\backslash\mathbf{ex}_\sigma\rangle,$$

where $\mathbf{ex}_\sigma$ appears instead of $\mathbf{ex}$ for the indexing reasons explained ahead of Theorem 3.11. Since $\mathbb{D}\langle \overline{y}_{\sigma,k}^{\pm 1}, \overline{y}_{\sigma,j} \mid k \in \mathbf{ex}_\sigma,\, j \in [1,N]\backslash \mathbf{ex}_\sigma\rangle \subseteq R_{\mathbb{D}}[E_\sigma^{-1}]$ for each $\sigma \in \Xi_N$, we obtain

$$\mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{D}} \subseteq \bigcap_{\sigma \in \Xi_N} R_{\mathbb{D}}[E_\sigma^{-1}].$$

In view of Proposition 4.16, we have the following sequence of inclusions:

$$R_{\mathbb{D}} \subseteq \mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}} \subseteq \mathbf{U}(M, \widetilde{B}, \varnothing)_{\mathbb{D}} \subseteq \bigcap_{\sigma \in \Gamma_N} R_{\mathbb{D}}[E_\sigma^{-1}] = R_{\mathbb{D}}\,. \tag{4.15}$$

All the inclusions in (4.15) must be equalities, which establishes the first part of Theorem 4.8(c). The finite generation statements concerning $\mathbf{A}(M, \widetilde{B}, \varnothing)_{\mathbb{D}}$ now follow from (4.14). If $\mathbb{D}$ is noetherian, the iterated Ore extension $R_{\mathbb{D}}$ is noetherian by standard skew polynomial ring results. This concludes the proof of part (c).

Part (d) is proved analogously, using Corollary 4.17 in place of Proposition 4.16. □

5. Quantum Schubert cell algebras, canonical bases and quantum function algebras

5.1. **Quantized universal enveloping algebras.** Fix a (finite) index set $I = [1, r]$ and consider a *Cartan datum* $(A, P, \Pi, P^\vee, \Pi^\vee)$ consisting of the following:

(i) A *generalized Cartan matrix* $A = (a_{ij})_{i,j \in I}$ such that $a_{ii} = 2$ for $i \in I$, $-a_{ij} \in \mathbb{Z}_{\geq 0}$ for $i \neq j \in I$, and there exists a diagonal matrix $D = \mathrm{diag}(d_i)_{i \in I}$ with relatively prime entries $d_i \in \mathbb{Z}_{>0}$ for which DA is symmetric.
(ii) A free abelian group P (*weight lattice*).
(iii) A subset $\Pi = \{\alpha_i \mid i \in I\} \subset P$ (*set of simple roots*).
(iv) The dual group $P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ (*coweight lattice*).
(v) Two linearly independent subsets $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ (*set of simple coroots*) such that $\langle h_i, \alpha_j\rangle = a_{ij}$ for $i, j \in I$, and $\{\varpi_i \in P \mid i \in I\}$ (*set of fundamental weights*) such that $\langle h_i, \varpi_j\rangle = \delta_{ij}$.

Let $\mathfrak{g}$ be the *symmetrizable Kac–Moody algebra* over $\mathbb{Q}$ corresponding to this Cartan datum. Denote

$$Q := \oplus_{i \in I} \mathbb{Z}\alpha_i \subset P, \;\; Q_+ := \oplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$$

and

$$P_+ := \{\gamma \in P \mid \langle h_i, \gamma\rangle \in \mathbb{Z}_{\geq 0}, \forall i \in I\}, \;\; P_{++} := \{\gamma \in P \mid \langle h_i, \gamma\rangle \in \mathbb{Z}_{>0}, \forall i \in I\}.$$

Set $\mathfrak{h} := \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$. There exists a $\mathbb{Q}$-valued nondegenerate symmetric bilinear form $(.,.)$ on $\mathfrak{h}^* = \mathbb{Q} \otimes_{\mathbb{Z}} P$ such that

$$\langle h_i, \mu\rangle = \frac{2(\alpha_i, \mu)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad (\alpha_i, \alpha_i) = 2d_i \quad \text{for } i \in I,\; \mu \in \mathfrak{h}^*. \tag{5.1}$$

Set $\|\gamma\|^2 := (\gamma, \gamma)$ for $\gamma \in \mathfrak{h}^*$. Denote by W the Weyl group of $\mathfrak{g}$ acting by isometries on $(\mathfrak{h}^*, (.,.))$. Denote by s_i its generators, by $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function on W, and by $\geq$ the Bruhat order on W. We will also denote by $(.,.)$ the transfer of this bilinear form to $\mathfrak{h}$, satisfying $(h_i, h_j) = (\alpha_i, \alpha_j)/d_i d_j$ for all $i, j \in I$.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ over the rational function field $\mathbb{Q}(q)$. It has generators q^h, e_i, f_i for $i \in I$, $h \in P^\vee$ and the following relations for

$h, h' \in P^\vee$, $i, j \in I$:

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'},$$
$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0, \quad i \neq j,$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0, \quad i \neq j,$$

where

$$q_i := q^{d_i}, \quad [n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! := [1]_i \cdots [n]_i \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_i := \frac{[n]_i}{[k]_i [n-k]_i}$$

for $k \leq n$ in $\mathbb{Z}_{\geq 0}$ and $i \in I$. The algebra $U_q(\mathfrak{g})$ is a Hopf algebra with coproduct, antipode and counit such that

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + q^{d_i h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-d_i h_i} + 1 \otimes f_i,$$
$$S(q^h) = q^{-h}, \quad S(e_i) = -q^{-d_i h_i} e_i, \quad S(f_i) = -f_i q^{d_i h_i},$$
$$\epsilon(q^h) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0$$

for $h \in P^\vee$, $i \in I$. The Hopf algebra $U_q(\mathfrak{g})$ is Q-graded with

$$\deg e_i = \alpha_i, \quad \deg f_i = -\alpha_i, \quad \deg q^h = 0. \tag{5.2}$$

For a Q-graded subalgebra R of $U_q(\mathfrak{g})$, its graded components will be denoted by R_γ, where $\gamma \in Q$. For a homogeneous $x \in U_q(\mathfrak{g})_\gamma$, set $\operatorname{wt} x := \gamma$. Define the torus

$$\mathcal{H} := (\mathbb{Q}(q)^\times)^I.$$

For $\gamma = \sum n_i \alpha_i \in Q$, let $t \mapsto t^\gamma$ denote the character of $\mathcal{H}$ given by $(r_i)_{i \in I} \mapsto \prod_i r_i^{n_i}$. This identifies the rational character lattice of $\mathcal{H}$ with Q. The torus $\mathcal{H}$ acts on $U_q(\mathfrak{g})$ by

$$t \cdot x = t^\gamma x \quad \text{for} \quad x \in U_q(\mathfrak{g})_\gamma, \ \gamma \in Q. \tag{5.3}$$

Let $\Delta_+ \subset Q_+$ be the set of positive roots of $\mathfrak{g}$. For $w \in W$, denote the following Lie subalgebras of the Kac–Moody algebra $\mathfrak{g}$,

$$\mathfrak{n}_\pm := \oplus_{\alpha \in \Delta_+} \mathfrak{g}^{\pm\alpha}, \quad \mathfrak{n}_\pm(w) := \oplus_{\alpha \in \Delta_+ \cap w^{-1}(-\Delta_+)} \mathfrak{g}^{\pm\alpha}, \tag{5.4}$$

where for $\alpha \in \Delta_+$, $\mathfrak{g}^{\pm\alpha}$ are the corresponding root spaces in $\mathfrak{g}$. Let $\mathfrak{b}_\pm$ be the corresponding Borel subalgebras of $\mathfrak{g}$. Denote by $U_q(\mathfrak{n}_\pm)$ and $U_q(\mathfrak{h})$ the unital subalgebras of $U_q(\mathfrak{g})$ respectively generated by $\{e_i \mid i \in I\}$, $\{f_i \mid i \in I\}$ and $\{q^h \mid h \in P^\vee\}$. Denote the Hopf subalgebras $U_q(\mathfrak{b}_\pm) := U_q(\mathfrak{n}_\pm) U_q(\mathfrak{h})$ of $U_q(\mathfrak{g})$.

Consider the $\mathbb{Q}(q)$-linear anti-automorphisms $*$ and φ of $U_q(\mathfrak{g})$ defined by

$$e_i^* := e_i, \quad f_i^* := f_i, \quad (q^h)^* := q^{-h}, \quad \text{and}$$
$$\varphi(e_i) := f_i, \quad \varphi(f_i) := e_i, \quad \varphi(q^h) := q^h$$

for $i \in I$, $h \in P^\vee$. Their composition $\varphi^* := \varphi \circ * = * \circ \varphi$ is the $\mathbb{Q}(q)$-linear automorphism of $U_q(\mathfrak{g})$ satisfying

$$\varphi^*(e_i) = f_i, \quad \varphi^*(f_i) = e_i \quad \text{and} \quad \varphi^*(q^h) = q^{-h}.$$

Denote by $c \mapsto \overline{c}$ the automorphism of the field $\mathbb{Q}(q)$ given by $\overline{q} = q^{-1}$. The bar involution $x \mapsto \overline{x}$ of $U_q(\mathfrak{g})$ is its $\mathbb{Q}(q)$-skewlinear automorphism such that $\overline{cx} = \overline{c}\,\overline{x}$ for $c \in \mathbb{Q}(q)$, $x \in U_q(\mathfrak{g})$ and $\overline{f}_i = f_i$, $\overline{e}_i = e_i$, $\overline{q^h} = q^{-h}$ for $i \in I$, $h \in P^\vee$. Denote the $\mathbb{Q}(q)$-skewlinear antiautomorphism $\overline{\varphi}$ of $U_q(\mathfrak{g})$,

$$\overline{\varphi}(x) := \varphi(\overline{x}) = \overline{\varphi(x)}, \quad \forall x \in U_q(\mathfrak{g}).$$

A $U_q(\mathfrak{g})$-module V is called *integrable* if e_i and f_i act locally nilpotently on V and

$$V = \oplus_{\mu \in P} V_\mu \ \text{ with } \ \dim V_\mu < \infty, \quad \text{where} \quad V_\mu = \{v \in M \mid q^h \cdot v = q^{\langle h,\mu\rangle} v, \ \forall h \in P^\vee\}.$$

The category $\mathcal{O}_{\mathrm{int}}(\mathfrak{g})$ consists of the integrable $U_q(\mathfrak{g})$ modules whose nontrivial graded subspaces have weights in $\cup_j(\mu_j + Q)$ for finitely many $\mu_1, \dots, \mu_n \in P$ (depending on the module). It is a semisimple monoidal category with respect to the tensor product of $U_q(\mathfrak{g})$-modules and with simple objects given by the irreducible highest weight modules $V(\mu)$ with highest weights $\mu \in P_+$.

For $V \in \mathcal{O}_{\mathrm{int}}(\mathfrak{g})$ its restricted dual module with respect to the antiautomorphism φ is a module in $\mathcal{O}_{\mathrm{int}}(\mathfrak{g})$ defined by

$$D_\varphi V := \oplus_{\mu \in P} V_\mu^*, \quad \text{where } V_\mu^* \text{ is the dual } \mathbb{Q}(q)\text{-vector space of } V_\mu.$$

The $U_q(\mathfrak{g})$-action on $D_\varphi V$ is given by $\langle x \cdot \xi, v\rangle = \langle \xi, \varphi(x) \cdot v\rangle$ for $v \in V$, $\xi \in D_\varphi V$.

Denote by $\{T_i \mid i \in I\}$ the generators of the braid group of W. For $w \in W$, let $T_w := T_{i_1} \cdots T_{i_N}$ for a reduced expression $s_{i_1} \cdots s_{i_N}$ of w. We will denote by the same notation Lustig's braid group action [34] on $U_q(\mathfrak{g})$ and on the modules in $\mathcal{O}_{\mathrm{int}}(\mathfrak{g})$. We will follow the conventions of [20].

5.2. **Two bilinear forms.** Consider the $\mathbb{Q}(q)$-linear skew-derivations e_i'' of $U_q(\mathfrak{n}_-)$,

$$e_i''(f_j) = \delta_{ij} \quad \text{and} \quad e_i''(xy) = e_i''(x)y + q_i^{-\langle h_i,\gamma\rangle} x e_i''(y)$$

for all $i, j \in I$, $x \in U_q(\mathfrak{n}_-)_\gamma$, $y \in U_q(\mathfrak{n}_-)$. The Kashiwara–Lusztig nondegenerate, symmetric bilinear form $(-,-)_{KL} : U_q(\mathfrak{n}_-) \times U_q(\mathfrak{n}_-) \to \mathbb{Q}(q)$ is the unique bilinear form such that

$$(1,1)_{KL} = 1 \quad \text{and} \quad (f_i x, y)_{KL} = (q_i^{-1} - q_i)^{-1}(x, e_i''(y))_{KL}, \quad \forall i \in I, \ x, y \in U_q(\mathfrak{n}_-).$$

Remark 5.1. The Lusztig form uses the scalars $(1 - q_i^{-2})^{-1}$ instead of $(q_i^{-1} - q_i)^{-1}$, see [33, Eq. (1.2.13)(a)]. For the Kashiwara form $(q_i^{-1} - q_i)^{-1}$ is replaced by 1, and e_i'' are replaced by the skew-derivations e_i' of $U_q(\mathfrak{n}_-)$ satisfying $e_i'(xy) = e_i'(x)y + q^{(\alpha_i,\gamma)} x e_i'(y)$, see [24, Eq. (3.4.4) and Proposition 3.4.4].

The use of the above form leads to minimal rescaling of dual PBW generators, quantum minors, and cluster variables.

Let $d \in \mathbb{Z}_{>0}$ be such that $(P^\vee, P^\vee) \subseteq \mathbb{Z}/d$. The Rosso–Tanisaki form [20, §6.12]

$$(-,-)_{RT} : U_q(\mathfrak{b}_-) \times U_q(\mathfrak{b}_+) \to \mathbb{Q}(q^{1/d})$$

is the Hopf algebra pairing satisfying

$$(x, yy')_{RT} = (\Delta(x), y' \otimes y)_{RT}, \quad (xx', y)_{RT} = (x \otimes x', \Delta(y))_{RT} \tag{5.5}$$

for $x, x' \in U_q(\mathfrak{b}_-)$, $y, y' \in U_q(\mathfrak{b}_+)$, and normalized by

$$(f_i, e_j)_{RT} = \delta_{ij}(q_i^{-1} - q_i)^{-1}, \quad (q^h, q^{h'})_{RT} = q^{-(h,h')}, \quad (f_i, q^h)_{RT} = (q^h, e_i)_{RT} = 0$$

for all $i, j \in I$, $h \in P^\vee$. Its restrictions to $U_q(\mathfrak{n}_-) \times U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-) \times U_q(\mathfrak{n}_+)$ take values in $\mathbb{Q}(q)$. The above two forms are related by

$$(x, x')_{KL} = (x, \varphi^*(x'))_{RT}, \quad \forall x, x' \in U_q(\mathfrak{n}_-), \tag{5.6}$$

see e.g. [29, Lemma 3.8] or [39, Proposition 8.3].

5.3. **Integral forms and canonical bases.** Recall the notation (1.3). The (divided power) *integral forms* $U_q(\mathfrak{n}_\pm)_{\mathcal{A}}$ of $U_q(\mathfrak{n}_\pm)$ are the $\mathcal{A}$-subalgebras generated by $e_i^{(k)} := e_i^k/[k]_i!$ (resp. $f_i^{(k)} := f_i^k/[k]_i!$) for $i \in I$, $k \in \mathbb{Z}_{>0}$. We have $\varphi^*(U_q(\mathfrak{n}_-)_{\mathcal{A}}) = U_q(\mathfrak{n}_+)_{\mathcal{A}}$. The *dual integral form* $U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}}$ of $U_q(\mathfrak{n}_-)$ is the $\mathcal{A}$-subalgebra

$$\begin{aligned} U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}} &= \{x \in U_q(\mathfrak{n}_-) \mid (x, U_q(\mathfrak{n}_-)_{\mathcal{A}})_{KL} \subset \mathcal{A}\} \\ &= \{x \in U_q(\mathfrak{n}_-) \mid (x, U_q(\mathfrak{n}_+)_{\mathcal{A}})_{RT} \subset \mathcal{A}\}. \end{aligned} \tag{5.7}$$

Kashiwara [24] defined a *lower global basis* $\mathbf{B}^{\mathrm{low}}$ of $U_q(\mathfrak{n}_-)_{\mathcal{A}}$ and an *upper global basis* $\mathbf{B}^{\mathrm{up}}$ of $U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}}$. The basis $\mathbf{B}^{\mathrm{up}}$ is defined from $\mathbf{B}^{\mathrm{low}}$ as the dual basis with respect to the form $(-,-)_{KL}$. Lusztig [33] defined related *canonical* and *dual canonical* bases of $U_q(\mathfrak{n}_+)_{\mathcal{A}}$ and a dual integral form of $U_q(\mathfrak{n}_+)$.

5.4. **Quantum Schubert cell algebras, dual integral forms and CGL extensions.** To each $w \in W$, De Concini–Kac–Procesi [5] and Lusztig [34, §40.2] associated *quantum Schubert cell subalgebras* of $U_q(\mathfrak{n}_\pm)$. Given a reduced expression

$$w = s_{i_1} \dots s_{i_N}, \tag{5.8}$$

define

$$w_{\le k} := s_{i_1} \dots s_{i_k}, \;\; w_{[j,k]} := s_{i_j} \dots s_{i_k}, \;\; w_{\le k}^{-1} := (w_{\le k})^{-1}, \;\; w_{[j,k]}^{-1} := (w_{[j,k]})^{-1} \in W$$

for $0 \le j \le k \le N$. Denote the roots and root vectors

$$\beta_k := w_{\le k-1}(\alpha_{i_k}), \; f_{\beta_k} := T^{-1}_{w^{-1}_{\le k-1}}(f_{i_k}) \in U_q(\mathfrak{n}_-)_{\mathcal{A}}, \; e_{\beta_k} := T^{-1}_{w^{-1}_{\le k-1}}(e_{i_k}) \in U_q(\mathfrak{n}_+)_{\mathcal{A}} \tag{5.9}$$

for $k \in [1, N]$. The algebras $U_q(\mathfrak{n}_\pm(w))$ are the unital $\mathbb{Q}(q)$-subalgebras of $U_q(\mathfrak{n}_\pm)$ generated by $e_{\beta_1}, \dots, e_{\beta_N}$ and $f_{\beta_1}, \dots, f_{\beta_N}$, respectively. These definitions are independent of the choice of reduced expression of w. Furthermore,

$$\begin{aligned} U_q(\mathfrak{n}_\pm(w)) &= U_q(\mathfrak{n}_\pm) \cap T^{-1}_{w^{-1}}(U_q(\mathfrak{n}_\mp)), \\ U_q(\mathfrak{n}_\pm) &= \big(U_q(\mathfrak{n}_\pm) \cap T^{-1}_{w^{-1}}(U_q(\mathfrak{n}_\pm))\big) U_q(\mathfrak{n}_\pm(w)). \end{aligned} \tag{5.10}$$

This was conjectured in [1, Conjecture 5.3] and proved in [28, 38].

Note that the algebras considered in [5] (see also [20]) are

$$U_q^\pm[w] = *\big(U_q(\mathfrak{n}_\pm(w))\big).$$

We use $U_q(\mathfrak{n}_\pm(w))$ instead, to avoid making all algebras here antiisomorphic to the ones in [9]. The $\mathcal{A}$-algebra

$$U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}} := U_q(\mathfrak{n}_-(w)) \cap U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}}$$

is called the *dual integral form* of $U_q(\mathfrak{n}_-(w))$. Define the *dual PBW generators* of $U_q(\mathfrak{n}_-(w))$

$$f^*_{\beta_k} := \frac{1}{(f_{\beta_k}, e_{\beta_k})_{RT}} f_{\beta_k} = \frac{1}{(\varphi^*(e_{\beta_k}), \varphi^*(e_{\beta_k}))_{KL}} \varphi^*(e_{\beta_k}) = (q_{i_k}^{-1} - q_{i_k}) f_{\beta_k} \tag{5.11}$$

for $k \in [1, N]$. Note that $\varphi^*(e_{\beta_k})$ differs from f_{β_k} by a unit of $\mathcal{A}$, namely $\varphi^*(e_{\beta_k}) = (-q_{i_k}) \prod_i (-q_i)^{n_i} f_{\beta_k}$ where $n_i \in \mathbb{Z}_{\ge 0}$ are such that $\beta_k = \sum n_i \alpha_i$ (see e.g [20, Eq. 8.14(9)]). The inner products between the dual PBW monomials and the divided-power PBW monomials are given by

$$\Big((f^*_{\beta_1})^{m_1} \cdots (f^*_{\beta_N})^{m_N}, e_{\beta_1}^{(l_1)} \cdots e_{\beta_N}^{(l_N)} \Big)_{RT} = \prod_{k=1}^N \delta_{m_k l_k} q_{i_k}^{m_k(m_k-1)/2}, \quad \forall m_k, l_k \in \mathbb{Z}_{\ge 0} \tag{5.12}$$

(see e.g. [20, §8.29-8.30]), where $e_{\beta_k}^{(l_k)} := e_{\beta_k}^{l_k}/[l_k]_{i_k}$.

Theorem 5.2. (Kimura) [27, Prop. 4.26, Thms. 4.25 and 4.27] *The algebras $U_q(\mathfrak{n}_-(w))_{\mathcal{A}}^{\vee}$ have the following decompositions as free $\mathcal{A}$-modules:*

$$U_q(\mathfrak{n}_-(w))_{\mathcal{A}}^{\vee} = \bigoplus_{m_1,\dots,m_N\in\mathbb{Z}_{\geq 0}} \mathcal{A}\cdot(f_{\beta_1}^*)^{m_1}\cdots(f_{\beta_N}^*)^{m_N} = \bigoplus_{d\in\mathbf{B}^{\mathrm{up}}\cap U_q(\mathfrak{n}_-(w))} \mathcal{A}\cdot d. \tag{5.13}$$

The Levendorskii–Soibelman straightening law *takes on the form*

$$f_{\beta_k}^* f_{\beta_j}^* - q^{(\beta_k,\beta_j)} f_{\beta_j}^* f_{\beta_k}^* = \sum_{\mathbf{m}=(m_{j+1},\dots,m_{k-1})\in\mathbb{Z}_{\geq 0}^{k-j-1}} b_{\mathbf{m}}(f_{\beta_{j+1}}^*)^{m_{j+1}}\dots(f_{\beta_{k-1}}^*)^{m_{k-1}}, \quad b_{\mathbf{m}}\in\mathcal{A} \tag{5.14}$$

for all $1\leq j<k\leq N$.

Remark 5.3. Recall (5.3) and denote

$$t := (q_i^{-1}-q_i)_{i\in I}\in\mathcal{H}. \tag{5.15}$$

The objects associated to $U_q(\mathfrak{n}_+)$ used by Geiß, Leclerc and Schröer in [9] are precisely the images under the isomorphism

$$(t\cdot)\circ\varphi^* : U_q(\mathfrak{n}_-) \xrightarrow{\cong} U_q(\mathfrak{n}_+)$$

of the objects associated to $U_q(\mathfrak{n}_-)$ which we consider. Firstly, [9] uses the canonical basis $\overline{\varphi}(\mathbf{B}^{\mathrm{low}}) = \varphi^*(\mathbf{B}^{\mathrm{low}})$ of $U_q(\mathfrak{n}_+)$ and the PBW generators $e_{\beta_k} = \overline{\varphi}(f_{\beta_k})$. They use the bilinear form $(-,-)$ on $U_q(\mathfrak{n}_+)$ defined by

$$(y,y') := (\varphi^*(y), t^{-1}\varphi^*(y'))_{KL}, \quad \forall y,y'\in U_q(\mathfrak{n}_+) \tag{5.16}$$

leading to the following:

(1) The dual canonical basis of $U_q(\mathfrak{n}_+)$ constructed from the canonical basis $\overline{\varphi}(\mathbf{B}^{\mathrm{low}}) = \varphi^*(\mathbf{B}^{\mathrm{low}})$ and the bilinear form (5.16), thus giving the basis $(t\cdot)\circ\varphi^*(\mathbf{B}^{\mathrm{up}})$;
(2) The dual PBW generators $e_{\beta_k}^* := e_{\beta_k}/(e_{\beta_k},e_{\beta_k}) = (t\cdot)\circ\varphi^*(f_{\beta_k}^*)$ of $U_q(\mathfrak{n}_+(w))$;
(3) The dual integral forms $(t\cdot)\circ\varphi^*(U_q(\mathfrak{n}_-)_{\mathcal{A}}^{\vee}) = \{y\in U_q(\mathfrak{n}_+) \mid (x, U_q(\mathfrak{n}_+)_{\mathcal{A}})\subset\mathcal{A}\}$ of $U_q(\mathfrak{n}_+)$ and $(t\cdot)\circ\varphi^*(U_q(\mathfrak{n}_-(w))_{\mathcal{A}}^{\vee})$ of $U_q(\mathfrak{n}_+(w))$. □

For a reduced expression (5.8) of $w\in W$ and $k\in[1,N]$, fix elements $t_k, t_k^*\in\mathcal{H}$ such that

$$t_k^{\beta_j} = q^{(\beta_k,\beta_j)} \text{ for } j\in[1,k] \quad\text{and}\quad (t_k^*)^{\beta_l} = q^{-(\beta_k,\beta_l)} \text{ for } l\in[k,N], \tag{5.17}$$

cf. (5.3); such t_k, t_k^* exist but are not unique since the restriction of the form $(.,.)$ to Q is degenerate when $\mathfrak{g}$ is not finite dimensional. Note that the algebras $U_q(\mathfrak{n}_-(w))_{\mathcal{A}}^{\vee}$ are preserved by the automorphisms $(t_k\cdot), (t_k^*\cdot)$.

Lemma 5.4. *Let $w\in W$, (5.8) a reduced expression of w, and $t_k\in\mathcal{H}$ satisfying (5.17).*

(a) *For $k\in[1,N]$, the algebra $U_q(\mathfrak{n}_-(w_{\leq k}))_{\mathcal{A}}^{\vee}$ is an Ore extension*

$$U_q(\mathfrak{n}_-(w_{\leq k}))_{\mathcal{A}}^{\vee} \cong U_q(\mathfrak{n}_-(w_{\leq k-1}))_{\mathcal{A}}^{\vee}[f_{\beta_k}^*; (t_k\cdot), \delta_k],$$

where δ_k is the locally nilpotent $(t_k\cdot)$-derivation of $U_q(\mathfrak{n}_-(w_{\leq k-1}))_{\mathcal{A}}^{\vee}$ given by

$$\delta_k(x) := f_{\beta_k}^* x - q^{(\beta_k,\mathrm{wt}\,x)} x f_{\beta_k}^* \quad \textit{for homogeneous} \quad x\in U_q(\mathfrak{n}_+(w_{\leq k-1}))_{\mathcal{A}}^{\vee}.$$

The t_k-eigenvalue of $f_{\beta_k}^*$ equals $q_{i_k}^2$, which is not a root of unity.

(b) *The algebra*

$$U_q(\mathfrak{n}_-(w)) \cong \mathbb{Q}(q)[f^*_{\beta_1}][f^*_{\beta_2};(t_2\cdot),\delta_2]\cdots[f^*_{\beta_N};(t_N\cdot),\delta_N] \tag{5.18}$$

is a symmetric CGL extension. The algebra

$$U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}} \cong \mathcal{A}[f^*_{\beta_1}][f^*_{\beta_2};(t_2\cdot),\delta_2]\cdots[f^*_{\beta_N};(t_N\cdot),\delta_N] \tag{5.19}$$

with the generators $f^*_{\beta_1},\dots,f^*_{\beta_N}$ *is an* $\mathcal{A}$*-form of the CGL extension* (5.18).

(c) *The interval subalgebras of* $U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}}$ *are*

$$(U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}})_{[j,k]} = T^{-1}_{w^{-1}_{\le j-1}}\big(U_q(\mathfrak{n}_-(w_{[j,k]}))^\vee_{\mathcal{A}}\big) \quad \textit{for} \quad 1\le j\le k\le N. \tag{5.20}$$

Proof. Part (a) follows from (5.13) and (5.14).

(b) The facts that $U_q(\mathfrak{n}_-(w))$ is a CGL extension and that $U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}}$ with the generators $f^*_{\beta_1},\dots,f^*_{\beta_N}$ is an $\mathcal{A}$-form of it follow by iterating (a). Its symmetricity is proved analogously to (a).

(c) Applying twice (5.13) and using (5.11), we obtain

$$\begin{aligned}
&(U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}})_{[j,k]} = \oplus_{m_j,\dots,m_k\in\mathbb{Z}_{\ge 0}}\mathcal{A}\cdot(f^*_{\beta_j})^{m_j}\cdots(f^*_{\beta_k})^{m_k}\\
&= T^{-1}_{w^{-1}_{\le j-1}}\big(\oplus_{m_j,\dots,m_k\in\mathbb{Z}_{\ge 0}}\mathcal{A}\cdot((q^{-1}_{i_j}-q_{i_j})f_{i_j})^{m_j}\cdots((q^{-1}_{i_k}-q_{i_k})T^{-1}_{w^{-1}_{[j,k-1]}}f_{i_k})^{m_k}\big)\\
&= T^{-1}_{w^{-1}_{\le j-1}}\big(U_q(\mathfrak{n}_-(w_{[j,k]}))^\vee_{\mathcal{A}}\big),
\end{aligned}$$

which proves (5.20). □

An important feature of the normalization of $(-,-)_{KL}$ is that there are no additional scalars in Lemma 5.4(c) due to the braid group action.

5.5. **The quantum function algebra of $\mathfrak{g}$.** Consider the full dual $\mathbb{Q}(q)$-vector space $U_q(\mathfrak{g})^*$ which is canonically a unital algebra using the coproduct and counit of $U_q(\mathfrak{g})$. It is a $U_q(\mathfrak{g})$-bimodule by

$$\langle x\cdot c\cdot y, z\rangle := \langle c, yzx\rangle \quad \text{for} \quad c\in U_q(\mathfrak{g})^*,\ x,y,z\in U_q(\mathfrak{g}). \tag{5.21}$$

For a right $U_q(\mathfrak{g})$-module V, let V^φ be the left $U_q(\mathfrak{g})$-module structure on the vector space V such that

$$x\cdot v = v\cdot\varphi(x) \quad \text{for} \quad v\in V,\ x\in U_q(\mathfrak{g}).$$

For each $\mu\in P_+$, there exists a unique irreducible right $U_q(\mathfrak{g})$ module $V^{\mathrm{r}}(\mu)$ such that $V^{\mathrm{r}}(\mu)^\varphi\cong V(\mu)$. Analogously to $\mathcal{O}_{\mathrm{int}}(\mathfrak{g})$, one defines an $\mathcal{O}$-type category of integrable right $U_q(\mathfrak{g})$-modules; it is denoted by $\mathcal{O}_{\mathrm{int}}(\mathfrak{g}^{\mathrm{op}})$.

Kashiwara defined [25, Sect. 7] the quantized coordinate ring $A_q(\mathfrak{g})$ of the Kac–Moody group of $\mathfrak{g}$ as the unital subalgebra of $U_q(\mathfrak{g})^*$ consisting of those $f\in U_q(\mathfrak{g})^*$ such that

$$U_q(\mathfrak{g})\cdot f\in\mathcal{O}_{\mathrm{int}}(\mathfrak{g}) \quad \text{and} \quad f\cdot U_q(\mathfrak{g})\in\mathcal{O}_{\mathrm{int}}(\mathfrak{g}^{\mathrm{op}}).$$

Kashiwara also proved [25, Proposition 7.2.2] a quantum version of the Peter-Weyl theorem that there is an isomorphism of $U_q(\mathfrak{g})$-bimodules

$$A_q(\mathfrak{g}) \cong \bigoplus_{\mu\in P_+} V^{\mathrm{r}}(\mu)\otimes V(\mu). \tag{5.22}$$

For $M\in\mathcal{O}_{\mathrm{int}}(\mathfrak{g})$ and $v\in M$, $\xi\in D_\varphi M$ define the matrix coefficient

$$c_{\xi v}\in U_q(\mathfrak{g})^* \quad \text{given by} \quad \langle c_{\xi v}, x\rangle := \langle \xi, x\cdot v\rangle \ \ \forall x\in U_q(\mathfrak{g}). \tag{5.23}$$

It follows from (5.22) that

$$A_q(\mathfrak{g}) = \{c_{\xi v}\mid M\in\mathcal{O}_{\mathrm{int}}(\mathfrak{g}),\ v\in M,\ \xi\in D_\varphi M\} = \oplus_{\mu\in P_+}\{c_{\xi v}\mid v\in V(\mu),\ \xi\in D_\varphi V(\mu)\}.$$

This is the form in which quantum function algebras were defined in the finite dimensional case [32]. The algebra $A_q(\mathfrak{g})$ is $P \times P$-graded by

$$A_q(\mathfrak{g})_{\mu,\nu} = \{c_{\xi v} \mid \xi \in (V_\mu)^* \subset D_\varphi V,\, v \in V_\nu,\, V \in \mathcal{O}_{\mathrm{int}}(\mathfrak{g})\}, \quad \forall \mu, \nu \in P. \tag{5.24}$$

6. Homogeneous prime ideals of $A_q(\mathfrak{n}_+(w))$

6.1. **The algebras $A_q(\mathfrak{n}_+)$ and $A_q(\mathfrak{n}_+(w))$.** It follows from the first identity in (5.5) and the nondegeneracy of the form that the map

$$\iota : U_q(\mathfrak{n}_-) \to U_q(\mathfrak{b}_+)^* \quad \text{given by} \quad \langle \iota(x), y\rangle = (x, y)_{RT}, \ \ \forall x \in U_q(\mathfrak{n}_-),\, y \in U_q(\mathfrak{b}_+) \tag{6.1}$$

is an injective algebra homomorphism. Here $U_q(\mathfrak{b}_+)^*$ denotes the unital algebra which is the full dual of the Hopf algebra $U_q(\mathfrak{b}_+)$ over $\mathbb{Q}(q)$.

Following Geiß–Leclerc–Schröer [9, §4.2], denote the subalgebra $A_q(\mathfrak{n}_+) \subset U_q(\mathfrak{b}_+)^*$ consisting of those $f \in U_q(\mathfrak{b}_+)^*$ such that

(i) $f(xq^h) = f(x)$ for all $x \in U_q(\mathfrak{n}_+)$, $h \in P^\vee$ and
(ii) $f(x) = 0$ for all $x \in U_q(\mathfrak{n}_+)_\gamma$ and $\gamma \in Q_+ \backslash S$ for a finite subset S of Q_+.

The properties

$$(xq^h, yq^{h'})_{RT} = (x, y)_{RT} q^{-(h,h')}, \quad (U_q(\mathfrak{n}_-)_{-\gamma}, U_q(\mathfrak{n}_+)_\delta)_{RT} = 0$$

for $x \in U_q(\mathfrak{n}_-)$, $y \in U_q(\mathfrak{n}_+)$, $h, h' \in P^\vee$, $\gamma \neq \delta$ in Q_+ (see [20, Eq 6.13(1)]) and the nondegeneracy of $(-,-)_{RT}$ imply that $A_q(\mathfrak{n}_+) := \iota(U_q(\mathfrak{n}_-))$. Thus

$$\iota : U_q(\mathfrak{n}_-) \xrightarrow{\cong} A_q(\mathfrak{n}_+) \tag{6.2}$$

is an algebra isomorphism. Following [9, §7.2], define $A_q(\mathfrak{n}_+(w)) := \iota(U_q(\mathfrak{n}_-(w)))$. Hence, ι restricts to the algebra isomorphsim

$$\iota : U_q(\mathfrak{n}_-(w)) \xrightarrow{\cong} A_q(\mathfrak{n}_+(w)). \tag{6.3}$$

Using the isomorphism ι, transport the isomorphisms $T_w : U_q(\mathfrak{n}_-) \cap T_w^{-1}(U_q(\mathfrak{n}_-)) \to T_w(U_q(\mathfrak{n}_-)) \cap U_q(\mathfrak{n}_-)$ to such maps on $A_q(\mathfrak{n}_+)$. Denote the integral forms over $\mathcal{A}$

$$A_q(\mathfrak{n}_+)_{\mathcal{A}} := \iota(U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}}) \quad \text{and} \quad A_q(\mathfrak{n}_+(w))_{\mathcal{A}} := \iota(U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}})$$

of $A_q(\mathfrak{n}_+)$ and $A_q(\mathfrak{n}_+(w))$. The algebra $A_q(\mathfrak{n}_+(w))$ is Q_+-graded by

$$A_q(\mathfrak{n}_+(w))_\gamma := \iota(U_q(\mathfrak{n}_-(w))_{-\gamma}), \quad \forall \gamma \in Q_+.$$

In other words, the isomorphism (6.3) is not $\mathcal{H}$-equivariant, but satisfies $\iota(t \cdot u) = t^{-1} \cdot \iota(u)$ for $t \in \mathcal{H}$, $u \in U_q(\mathfrak{n}_-(w))$.

Remark 6.1. Using the bilinear form (5.16), in [9] the algebra $A_q(\mathfrak{n}_+)$ is identified with $U_q(\mathfrak{n}_+)$ via the isomorphism

$$\Psi \colon U_q(\mathfrak{n}_+) \xrightarrow{\cong} A_q(\mathfrak{n}_+), \quad \langle \Psi(y), y' q^h\rangle := (y, y')_{KL}, \quad \forall y, y' \in U_q(\mathfrak{n}_+),\ h \in P^\vee.$$

Ψ fits in the commutative diagram

$$\begin{tikzcd} & A_q(\mathfrak{n}_+) & \\ U_q(\mathfrak{n}_-) \arrow[ur, "\iota"] \arrow[rr, "(t\cdot)\circ\varphi^*"] & & U_q(\mathfrak{n}_+) \arrow[ul, "\Psi"'] \end{tikzcd}$$

in terms of $t \in \mathcal{H}$ given by (5.15). This and Remark 5.3 imply that $A_q(\mathfrak{n}_+)_{\mathcal{A}}$ and $\iota(\mathbf{B}^{\mathrm{up}})$ are precisely the integral form of $A_q(\mathfrak{n}_+)$ and the dual canonical basis of $A_q(\mathfrak{n}_+)$ considered in [9]. However the braid group action of [9] on $A_q(\mathfrak{n}_+)$ is a conjugate of ours by an element of the torus $\mathcal{H}$, and involves extra scalars compared to our formulas.

6.2. **An algebra isomorphism.** For $\mu \in P_+$, fix a highest weight vector v_μ of $V(\mu)$. For $w \in W$, define the extremal weight vector

$$v_{w\mu} := T_{w^{-1}}^{-1} v_\mu \in V(\mu)_{w\mu}.$$

Denote the associated *Demazure modules*

$$V_w^{\pm}(\mu) := U_q(\mathfrak{b}_\pm) v_{w\mu} \subseteq V(\mu).$$

Let

$$\xi_{w\mu} \in V(\mu)^*_{w\mu} \subset D_\varphi(V(\mu)) \quad \text{be such that} \quad \langle \xi_{w\mu}, v_{w\mu} \rangle = 1.$$

For $u, w \in W$ and $\mu \in P_+$, using the notation (5.23), define the quantum minors

$$\Delta_{u\mu,w\mu} := c_{\xi_{u\mu}, v_{w\mu}} \in A_q(\mathfrak{g}),$$

which are equivalently given by [2, Eq. (9.10)], [9, Eq. (3.5)]. It is well known that

$$T_{w^{-1}}^{-1}(v_\mu \otimes v_\nu) = T_{w^{-1}}^{-1} v_\mu \otimes T_{w^{-1}}^{-1} v_\nu \tag{6.4}$$

for all $\mu, \nu \in P_+$. This implies that

$$\Delta_{u\mu,w\mu} \Delta_{u\nu,w\nu} = \Delta_{u(\mu+\nu),w(\mu+\nu)}, \quad \forall \mu, \nu \in P_+. \tag{6.5}$$

Following Joseph [21, §9.1.6], denote the subalgebra

$$A_q^+(\mathfrak{g}) := \oplus_{\mu \in P_+} \{ c_{\xi v_\mu} \mid \xi \in D_\varphi(V(\mu)) \}$$

of $A_q(\mathfrak{g})$. By [8, Lemma 2.1(i)], the multiplicative set

$$E_w := \{ \Delta_{w\mu,\mu} \mid \mu \in P_+ \}$$

is a denominator set in $A_q^+(\mathfrak{g})$. Denote the subsets

$$J_w^{\pm} := \oplus_{\mu \in P_+} \{ c_{\xi v_\mu} \mid \xi \in D_\varphi(V(\mu)),\ \xi \perp V_w^{\pm}(\mu) \} \subset A_q^+(\mathfrak{g}).$$

By the proofs of Theorems 6.2 and 6.4 below, they are completely prime ideals of $A_q^+(\mathfrak{g})$.

The $P \times P$-grading (5.24) of $A_q(\mathfrak{g})$ extends to a $P \times P$-grading of the localization $A_q^+(\mathfrak{g})[E_w^{-1}]$. For a graded subalgebra $R \subseteq A_q^+(\mathfrak{g})[E_w^{-1}]$, denote the subalgebra

$$R_0 := \oplus_{\nu \in P} R_{\nu,0},$$

noting that R_0 is naturally P-graded. It is easy to show that every element of $(A_q^+(\mathfrak{g})[E_w^{-1}])_0$ has the form $c_{\xi,v_\mu} \Delta_{w\mu,\mu}^{-1}$ for some $\mu \in P_+$, $\xi \in D_\varphi(V(\mu))$; in particular, this algebra is Q-graded. The following theorem was proved in the finite dimensional case in [40].

Theorem 6.2. *For all symmetrizable Kac–Moody algebras $\mathfrak{g}$ and $w \in W$, there exists a Q-graded surjective homomorphism $\psi_w : (A_q^+(\mathfrak{g})[E_w^{-1}])_0 \to A_q(\mathfrak{n}_+(w))$ such that*

$$\langle \psi_w(c_{\xi,v_\mu} \Delta_{w\mu,\mu}^{-1}), y q^h \rangle = \langle \xi, y v_{w\mu} \rangle \tag{6.6}$$

for $\mu \in P_+$, $\xi \in D_\varphi(V(\mu))$, $y \in U_q(\mathfrak{b}_+)$, $h \in P^\vee$. Its kernel equals $(J_w^+[E_w^{-1}])_0$.

We will need the following lemma.

Lemma 6.3. [40, Lemma 3.2] *Let H be a Hopf algebra over $\mathbb{K}$ and A be an H-module algebra equipped with a right H-action. For every algebra homomorphism $\theta : A \to \mathbb{K}$, the map $\psi : A \to H^*$, given by*

$$\psi(a)(h) = \theta(a \cdot h),$$

is an algebra homomorphism.

Proof of Theorem 6.2. Eq. (6.4) implies that

$$\theta_w : A_q^+(\mathfrak{g}) \to \mathbb{Q}(q) \quad \text{given by} \quad \theta_w(c_{\xi v_\mu}) := \langle \xi, v_{w\mu} \rangle, \ \ \forall \mu \in P_+, \ \xi \in D_\varphi(V(\mu))$$

is an algebra homomorphism. We apply the lemma to it and to the right action (5.21) of $U_q(\mathfrak{b}_+)$ on $A_q^+(\mathfrak{g})$. It shows that the map $\psi_w : A_q^+(\mathfrak{g}) \to U_q(\mathfrak{b}_+)^*$, given by

$$\langle \psi_w(c_{\xi v_\mu}), y \rangle := \langle \xi, y v_{w\mu} \rangle, \quad \forall \mu \in P_+, \ \xi \in D_\varphi(V(\mu)), \ y \in U_q(\mathfrak{b}_+),$$

is an algebra homomorphism. The element $\psi_w(\Delta_{w\mu,\mu})$ is a unit of $U_q(\mathfrak{b}_+)^*$ because

$$\langle \psi_w(\Delta_{w\mu,\mu}), y q^h \rangle = \epsilon(y) q^{\langle h, \mu \rangle}, \quad \forall y \in U_q(\mathfrak{n}_+), \ h \in P^\vee.$$

Hence, ψ_w extends to $A_q^+(\mathfrak{g})[E_w^{-1}]$, $\psi_w((A_q^+(\mathfrak{g})[E_w^{-1}])_0) \subset A_q(\mathfrak{n}_+)$, and the restriction of ψ_w to $(A_q^+(\mathfrak{g})[E_w^{-1}])_0$ is given by (6.6). From now on we will denote by ψ_w this restriction. The formula (6.6) implies at once that the kernel of ψ_w equals $(J_w^+[E_w^{-1}])_0$ and

$$\langle \operatorname{Im} \psi_w, U_q(\mathfrak{n}_+(w)) y \rangle = 0, \quad \forall y \in \big(U_q(\mathfrak{n}_+) \cap T_{w^{-1}}^{-1}(U_q(\mathfrak{n}_+))\big)_\gamma, \ \gamma \in Q_+ \backslash \{0\}.$$

For each $\gamma \in Q_+$ such that $U_q(\mathfrak{n}_+(w))_\gamma \neq 0$, there exists $\mu \in P_+$ such that the pairing

$$(V_w(\mu)_{\gamma + w\mu})^* \times U_q(\mathfrak{n}_+(w))_\gamma \quad \text{given by} \quad \xi, y \mapsto \langle \xi, y v_{w\mu} \rangle$$

is nondegenerate. This, the second equality in (5.10) and the fact that

$$(U_q(\mathfrak{n}_-(w)), U_q(\mathfrak{n}_+(w)) y)_{RT} = 0, \quad \forall y \in \big(U_q(\mathfrak{n}_+) \cap T_{w^{-1}}^{-1}(U_q(\mathfrak{n}_+))\big)_\gamma, \ \gamma \in Q_+ \backslash \{0\}$$

imply that $\operatorname{Im} \psi_w = A_q(\mathfrak{n}_+(w))$. □

Theorem 6.4. *In the setting of Theorem* 6.2*, there exists a* (Q*-graded*) *homomorphism* $\psi_w^- : (A_q^+(\mathfrak{g})[E_w^{-1}])_0 \to U_q(\mathfrak{b}_-)^*$ *such that*

$$\langle \psi_w(c_{\xi, v_\mu} \Delta_{w\mu,\mu}^{-1}), y q^h \rangle = \langle \xi, y^* v_{w\mu} \rangle$$

for $\mu \in P_+$, $\xi \in D_\varphi(V(\mu))$, $y \in U_q(\mathfrak{b}_-)$, $h \in P^\vee$. *Its kernel equals* $(J_w^-[E_w^{-1}])_0$. *Its image is contained in the image of the antiembedding* $U_q(\mathfrak{n}_+(w)) \to (U_q(\mathfrak{b}_-))^*$ *coming from the second component of the Rosso-Tanisaki form.*

The proof of the theorem is analogous to that of Theorem 6.2.

6.3. **The prime spectrum of $A_q(\mathfrak{n}_+(w))$.** The fact that $\mathcal{O}_{\mathrm{int}}(\mathfrak{g})$ is a braided monoidal category gives rise to $\mathcal{R}$-matrix commutation relations in $A_q(\mathfrak{g})$, [21, Proposition 9.1.5]. Particular cases of those are the relations

$$\Delta_{w\mu,\mu} x = q^{\pm((w\mu,\nu)-(\mu,\gamma))} x \Delta_{w\mu,\mu} \mod J_w^\pm, \quad \forall x \in A_q^+(\mathfrak{g})_{\nu,\gamma}, \ \mu \in P_+, \ \nu, \gamma \in P. \tag{6.7}$$

For $u \in W$, $\mu \in P_+$, denote the unipotent quantum minors

$$D_{u\mu,w\mu} := \psi_w(\Delta_{u\mu,\mu} \Delta_{w\mu,\mu}^{-1}) \in A_q(\mathfrak{n}_+(w)).$$

They are alternatively defined as the elements of $A_q(\mathfrak{n}_+(w))_{(u-w)\mu} \subset A_q(\mathfrak{n})$ such that

$$\langle D_{u\mu,w\mu}, x q^h \rangle = \langle \xi_{u\mu}, x v_{w\mu} \rangle, \quad \forall x \in U_q^+(\mathfrak{g}), \ h \in P^\vee, \tag{6.8}$$

which implies that they are precisely the elements of $A_q(\mathfrak{n}_+(w))$ defined in [9, Eqs. (5.3)-(5.4)]. Set

$$W^{\leq w} = \{u \in W \mid u \leq w\}.$$

For $u \in W^{\leq w}$, denote the ideals

$$I_w(u) := \psi_w\big((J_u^-[E_w^{-1}])_0\big) \quad \text{of} \quad A_q(\mathfrak{n}_+(w)).$$

It follows from (6.5) and (6.7) that

$$D_{u\mu,w\mu} D_{u\nu,w\nu} = q^{(w\mu,u\nu)-(\mu,\nu)} D_{u(\mu+\nu),w(\mu+\nu)}, \quad \forall \mu, \nu \in P_+$$

and that

$$D_{u\mu,w\mu}x = q^{((w+u)\mu,\mathrm{wt}\,x)}xD_{u\mu,w\mu} \mod I_w(u), \quad \forall\mu\in P_+,\ \text{homogeneous } x\in A_q(\mathfrak{n}_+(w)).$$

We have $I_w(1)=0$, thus

$$D_{\mu,w\mu}x = q^{((w+1)\mu,\mathrm{wt}\,x)}xD_{\mu,w\mu}, \quad \forall\mu\in P_+,\ \text{homogeneous } x\in A_q(\mathfrak{n}_+(w)). \tag{6.9}$$

Denote the multiplicative sets

$$E_w(u) := q^{\mathbb{Z}}\{D_{u\mu,w\mu} \mid \mu\in P_+\} \quad \text{in} \quad A_q(\mathfrak{n}_+(w)).$$

Analogously to (5.3), we use the Q_+-grading of $A_q(\mathfrak{n}_+(w))$ to construct an action of the torus $\mathcal{H}$ on it.

Theorem 6.5. *For all symmetrizable Kac–Moody algebras $\mathfrak{g}$ and $w\in W$, the following hold:*

(a) *The graded prime ideals of $A_q(\mathfrak{n}_+(w))$ are the ideals $I_w(u)$ for $u\in W^{\le w}$. The map $u\mapsto I_w(u)$ is an isomorphism of posets from $W^{\le w}$ with the Bruhat order to the set of graded prime ideals of $A_q(\mathfrak{n}_+(w))$ with the inclusion order.*

(b) *All prime ideals of $A_q(\mathfrak{n}_+(w))$ are completely prime and*

$$\operatorname{Spec} A_q(\mathfrak{n}_+(w)) = \bigsqcup_{u\in W^{\le w}} \operatorname{Spec}_u A_q(\mathfrak{n}_+(w)),$$

where $\operatorname{Spec}_u A_q(\mathfrak{n}_+(w)) := \{\mathcal{J}\in\operatorname{Spec} A_q(\mathfrak{n}_+(w)) \mid \cap_{t\in\mathcal{H}}(t\cdot\mathcal{J}) = I_w(u)\}$.

The following hold for $u\in W^{\le w}$:

(c) *$I_w(u)\cap E_w(u)=\varnothing$ and the localization $R_{u,w} = \big(A_q(\mathfrak{n}_+(w))/I_w(u)\big)[E_w(u)^{-1}]$ is an $\mathcal{H}$-simple domain.*

(d) *For $u\in W^{\le w}$, the center $Z(R_{u,w})$ is a Laurent polynomial ring over $\mathbb{Q}(q)$ and there is a homeomorphism*

$$\eta_u\colon \operatorname{Spec} Z(R_{u,w}) \xrightarrow{\cong} \operatorname{Spec}_u A_q(\mathfrak{n}_+(w))$$

where for $\mathcal{J}\in\operatorname{Spec} Z(R_{u,w})$, $\eta_u(\mathcal{J})$ is the ideal of $A_q(\mathfrak{n}_+(w))$ containing $I_w(u)$ such that $\eta_u(\mathcal{J})/I_w(u) = \mathcal{J}R_{u,w}\cap(A_q(\mathfrak{n}_+(w))/I_w(u))$.

Denote for brevity the algebra

$$A_w^+ := (A_q^+(\mathfrak{g})[E_w^{-1}])_0 \subseteq A_q^+(\mathfrak{g})[E_w^{-1}].$$

It is Q-graded by

$$(A_w^+)_\nu := (A_q^+(\mathfrak{g})[E_w^{-1}])_{\nu,0} \quad \text{for} \quad \nu\in Q$$

in terms of the $P\times P$-grading (5.24) of $A_q^+(\mathfrak{g})[E_w^{-1}]$. Define the commuting (inner) automorphisms $\tau_w^\mu\in\operatorname{Aut}(A_w^+)$ for $\mu\in P_+$ by

$$\tau_w^\mu(c) := \Delta_{w\mu,\mu}^{-1}c\Delta_{w\mu,\mu}.$$

For each $i\in I$, define the automorphism $\kappa_i\in\operatorname{Aut}(A_q^+(\mathfrak{g}))$ by $\kappa_i(c) := c\cdot q^{d_ih_i}$ and the locally nilpotent (right skew) κ_i-derivation ∂_i of $A_q^+(\mathfrak{g})$ by $\partial_i(c) := c\cdot f_i$ in terms of the second action in (5.21). It easy to check that $\kappa_i\partial_i\kappa_i^{-1} = q_i\partial_i$. Following Joseph [21, §A.2.9], for $c\in A_q^+(\mathfrak{g})\backslash\{0\}$ set

$$\deg_i(c) := \max\{n\in\mathbb{Z}_{>0} \mid \partial_i^n(c)\neq 0\}$$

and

$$\partial_w^*(c) := \partial_{i_1}^{n_1}\dots\partial_{i_N}^{n_N}(c)\neq 0$$

where $n_N,\dots,n_1\in\mathbb{Z}_{\ge 0}$ are recursively defined by $n_k := \deg_{i_k}(\partial_{i_{k+1}}^{n_{k+1}}\dots\partial_{i_N}^{n_N}(c))$ in terms of the reduced expression (5.8). Set $\partial_w^*(0) := 0$.

Proof. We carry out the proof in four steps as follows:

Step 1. For all $u \in W^{\leq w}$, the ideals $I_w(u)$ of $A_q(\mathfrak{n}_+(w))$ are completely prime.

The image of ψ_w is an iterated skew polynomial extension, and thus is a domain. Similarly one shows that the image of ψ_w^- is also a domain. Therefore $(J_w^\pm[E_w^{-1}])_0$ are completely prime ideals of A_w^+. By direct extension and contraction arguments one gets that $J_u^\pm$ are completely prime ideals of $A_q^+(\mathfrak{g})$ for $u \in W$, and that the same is true for the ideals $(J_u^\pm[E_w^{-1}])_0$ of A_w^+. The remaining part of the proof of the statement of step 1 uses elements of Gorelik's and Joseph's proofs [18, 21] of related facts in the finite dimensional case. We prove the stronger fact that there exists an embedding

$$A_q(\mathfrak{n}_+(w))/I_w(u) \hookrightarrow A_w^+/(J_w^-[E_w^{-1}])_0$$

which we construct next. For a linear map τ on a $\mathbb{Q}(q)$-vector space V and $t \in \overline{\mathbb{Q}(q)}$, denote by $\mathcal{E}_\tau(t)$ the generalized t-eigenspace of τ. Using the first action (5.21), one shows that for all $i \in I$, $w \in W$ such that $\ell(s_i w) < \ell(w)$ and $\nu \in P$, $\lambda \in P_+$, $t \in \overline{\mathbb{Q}(q)}$:

$$\text{If} \quad c \in \mathcal{E}_{\tau_w^\mu}(t) \cap A_q^+(\mathfrak{g})_{\nu,\lambda}, \quad \text{then} \quad \partial_i^n(c) \in \mathcal{E}_{\tau_{s_i w}^\mu}(tq^{(w\mu,\nu)-(s_i w\mu,\nu+n\alpha_i)})$$

where $n := \deg_i(c)$; the proof of this is analogous to [18, Lemma 6.3.1]. By induction on the length of w, this implies that

$$A_w^+ = \oplus_{\gamma\in Q_+}(A_w^+)[2\gamma] \quad \text{where} \tag{6.10}$$
$$(A_w^+)[2\gamma] := \bigoplus_{\nu\in Q}\{c \in (A_w^+)_\nu \mid c \in \mathcal{E}_{\tau_w^\mu}(q^{-(w^{-1}\nu+2\gamma,\mu)}),\ \forall \mu \in P_+\},$$

and that for $\gamma \in Q_+$, $\lambda \in P_+$,

$$c_{\xi,v_\lambda}\Delta_{w\lambda,\lambda}^{-1} \in (A_w^+)[2\gamma] \quad \Rightarrow \quad (\partial_{w^{-1}}^*(c_{\xi,v_\lambda}))\Delta_{\lambda,\lambda}^{-1} \in (A_1^+)[2\gamma]. \tag{6.11}$$

The base of the induction for $w = 1$ follows from (6.7) applied to $J_1^- = 0$, which gives that $\tau_1^\mu(c) = q^{(\mu,\nu)}c$ for all $c \in (A_q^+(\mathfrak{g})[E_1^{-1}])_{\nu,0}$, $\nu \in -Q_+$; that is

$$(A_1^+)[2\gamma] = \bigoplus_{\lambda\in P_+}\{c_{\xi,v_\lambda}\Delta_{\lambda,\lambda}^{-1} \mid \xi \in (V(\lambda)_{\lambda-\gamma}))^* \subset D_\varphi(V(\lambda))\}, \quad \forall \gamma \in Q_+. \tag{6.12}$$

Furthermore, we have

$$(J_w^+[E_w^{-1}])_0 = \oplus_{\gamma\in Q_+\setminus\{0\}}A_w^+[2\gamma]. \tag{6.13}$$

By (6.7), applied to J_w^+, the right hand side is contained in the left one. Because of (6.10) it remains to show that $(J_w^+[E_w^{-1}])_0 \cap A_w^+[0] = 0$. Assume the opposite, that $(J_w^+[E_w^{-1}])_0 \cap A_w^+[0] \neq 0$. By putting elements over a common denominator, each element of A_w^+ can be represented in the form $c_{\xi,v_\lambda}\Delta_{w\lambda,\lambda}^{-1}$ for some $\lambda \in P_+$, $\xi \in D_\varphi(V(\lambda))$. Choose a nonzero element of this form in $(J_w^+[E_w^{-1}])_0 \cap A_w^+[0]$. By (6.11), $(\partial_{w^{-1}}^*(c_{\xi,v_\lambda}))\Delta_{\lambda,\lambda}^{-1} \in (A_1^+)[0]$. Hence, (6.12) implies that $\partial_{w^{-1}}^*(c_{\xi,v_\lambda}) = rc_{\xi_\lambda,v_\lambda}$ for some $r \in \mathbb{Q}(t)^*$. The definition of $\partial_{w^{-1}}^*$ gives that

$$\langle \xi, f_{i_1}^{n_1}\dots f_{i_N}^{n_N}v_\lambda\rangle \neq 0 \quad \text{for some} \quad n_1,\dots,n_N \in \mathbb{Z}_{\geq 0}$$

in terms of the reduced expression (5.8). However, $f_{i_1}^{n_1}\dots f_{i_N}^{n_N}v_\lambda \in V_w^+(\lambda)$ by the standard presentation of Demazure modules [21, Lemma 4.4.3(v)]. This contradicts with $c_{\xi,v_\lambda}\Delta_{\lambda,\lambda}^{-1} \in (J_w^+[E_w^{-1}])_0$ and proves (6.13).

Since $\tau_w^\mu \in \mathrm{Aut}(A_w^+)$, $A_w^+[0]$ is a subalgebra of A_w^+. Theorem 6.2 and (6.10), (6.13) imply

$$\begin{aligned} A_q(\mathfrak{n}_+(w))/I_w(u) &\cong A_w^+/((J_w^+[E_w^{-1}])_0 + (J_u^-[E_w^{-1}])_0) \\ &\cong A_w^+[0]/(A_w^+[0] \cap (J_u^-[E_w^{-1}])_0) \hookrightarrow A_w^+/(J_u^-[E_w^{-1}])_0. \end{aligned}$$

Step 2. For all $u \in W^{\leq w}$, $I_w(u) \cap E_w(u) = \varnothing$.

Denote by $G^{\min}$ the minimal Kac–Moody group associated to $\mathfrak{g}$, see [30, §7.4] for details. Let H be the Cartan subgroup of $G^{\min}$, and $N_+^{\min}$ and N_- the subgroups of $G^{\min}$ generated by its one-parameter unipotent subgroups for positive and negative roots, respectively. Denote by $\mathcal{B}_+^{\min}$ and $\mathcal{B}_-$ the associated Borel subgroups of $G^{\min}$. Denote by $N_+(w)$ the unipotent subgroup of $N_+^{\min}$ corresponding to $\mathfrak{n}_+(w)$. By [27, Theorem 4.44], we have the specialization isomorphism

$$A_q(\mathfrak{n}_+(w))_{\mathcal{A}} \otimes \mathbb{C} \cong \mathbb{C}[N_+(w)] \tag{6.14}$$

for the map $\mathcal{A} \to \mathbb{C}$ given by $q \mapsto 1$. By [39, Proposition 9.7], $I_w(u) \cap A_q(\mathfrak{n}_+(w))_{\mathcal{A}}$ is an $\mathcal{A}$-form of $I_w(u)$. The definitions of J_u^- and $I_w(u)$ in terms of Demazure modules imply that under the specialization isomorphism (6.14), $I_w(u) \cap A_q(\mathfrak{n}_+(w))_{\mathcal{A}}$ is mapped to functions that vanish on the nonempty set

$$N(w) \cap \mathcal{B}_- u \mathcal{B}_+^{\min} w^{-1}, \tag{6.15}$$

which is isomorphic to the open Richardson variety in the flag scheme of $G^{\min}$ corresponding to the pair $u \leq w \in W$. Let $\mu \in P_+$. Analogously to the quantum situation, using special representatives of $w \in W$ in the normalizer of H in $G^{\min}$, one defines the generalized minor $\overline{\Delta}_{u\mu,w\mu}$ which is a strongly regular function on $G^{\min}$. It is well known that under the specialization isomorphism (6.14), the element $D_{u\mu,w\mu} \in E_w(u)$ corresponds to the restriction of $\overline{\Delta}_{u\mu,w\mu}$ to $N_+^{\min}$. This function is nowhere vanishing on the set (6.15). Therefore, the specializations $I_w(u)$ and $E_w(u)$ are disjoint, so $I_w(u) \cap E_w(u) = \varnothing$.

For the next step, we denote for brevity

$$c_\xi := \psi_w(c_{\xi,v_\lambda} \Delta_{w\lambda,\lambda}^{-1}) \in A_q(\mathfrak{n}_+(w)) \quad \text{for} \quad \xi \in D_\varphi(V(\lambda)),\ \lambda \in P_+.$$

For $\mathcal{J} \in \operatorname{Spec} A_q(\mathfrak{n}_+(w))$ and $\lambda \in P_+$, denote

$$C_{\mathcal{J}}(\lambda) = \{\nu \in P \mid \exists \xi \in (V(\lambda)_\nu)^* \subset D_\varphi(V(\lambda)) \text{ such that } c_\xi \notin \mathcal{J}\}.$$

Since $c_{\xi_{w\lambda}} = 1 \notin \mathcal{J}$, $w\lambda \in C_{\mathcal{J}}(\lambda)$. Denote by $M_{\mathcal{J}}(\lambda)$ the set of maximal elements of $C_{\mathcal{J}}(\lambda)$ with respect to the partial order $\nu \preceq \nu'$ if $\nu' - \nu \in Q_+$.

Step 3. For every $\mathcal{J} \in \operatorname{Spec} A_q(\mathfrak{n}_+(w))$, there exists a unique $u \in W^{\leq w}$ such that $M_{\mathcal{J}}(\lambda) = \{u\lambda\}$ for all $\lambda \in P_+$.

This step is similar to [21, Proposition 9.3.8]. Let $\lambda \in P_+$ and $\nu \in M_{\mathcal{J}}(\lambda)$, so there exists $\xi \in (V(\lambda)^*)_\nu$ such that $c_\xi \notin \mathcal{J}$. The $\mathcal{R}$-matrix commutation relations in $A_q(\mathfrak{g})$ (see e.g. [21, Proposition 9.1.5]) and the homomorphism from Theorem 6.2 imply that

$$c_\xi x \equiv q^{-(\nu+w\lambda,\gamma)} x c_\xi \mod \mathcal{J},\ \ \forall x \in A_q(\mathfrak{n}_+(w))_\gamma,\ \gamma \in Q_+.$$

Take any other pair $\lambda' \in P_{++}$ and $\nu' \in M_{\mathcal{J}}(\lambda')$ going with $\xi' \in (V(\lambda')^*)_{\nu'}$ such that $c_{\xi'} \notin \mathcal{J}$. Applying the last relation twice gives

$$c_\xi c_{\xi'} \equiv q^{-(\nu+w\lambda,\nu'-w\lambda')-(\nu-w\lambda,\nu'+w\lambda')} c_{\xi'} c_\xi \mod \mathcal{J}.$$

Since $A_q(\mathfrak{n}_+(w))/\mathcal{J}$ is a prime ideal and the images of c_ξ, $c_{\xi'}$ are nonzero normal elements, they are regular. Therefore the power of q above must equal 0, and thus,

$$(\lambda, \lambda') - (\nu, \nu') = 0. \tag{6.16}$$

It follows from [21, Lemma A.1.17] that $\nu = u_\lambda(\lambda)$ for some $u_\lambda \in W$; that is $M_{\mathcal{J}}(\lambda) = \{u_\lambda \lambda\}$ (note that u_λ is non-unique for $\lambda \in P_+ \backslash P_{++}$). It follows from the inclusion relations for Demazure modules [21, Proposition 4.4.5] and the definition of J_w^+ that $u_\lambda \in W^{\leq w}$ for $\lambda \in P_{++}$. Applying one more time (6.16) gives that $u_\lambda = u_{\lambda'}$ for $\lambda, \lambda' \in P_{++}$ and that for $\lambda \in P_+$, $\lambda' \in P_{++}$, the element u_λ can be chosen so that $u_\lambda = u_{\lambda'}$.

Step 4. Completion of proof. By step 3,

$$\operatorname{Spec} A_q(\mathfrak{n}_+(w)) = \bigsqcup_{u\in W^{\le w}} \operatorname{Spec}'_u A_q(\mathfrak{n}_+(w)), \quad \text{where} \tag{6.17}$$
$$\operatorname{Spec}'_u A_q(\mathfrak{n}_+(w)) := \{\mathcal{J} \in \operatorname{Spec} A_q(\mathfrak{n}_+(w)) \mid M_{\mathcal{J}}(\lambda) = \{u\lambda\},\ \forall \lambda \in P_+\}.$$

Steps 1, 2 and 3 and the fact that $\dim V(\lambda)_{w\lambda} = 1$ imply the following:

(*) For all $u \in W^{\le w}$, we have $I_w(u) \in \operatorname{Spec}'_u A_q(\mathfrak{n}_+(w))$, all ideals in $\operatorname{Spec}'_u A_q(\mathfrak{n}_+(w))$ contain $I_w(u)$ and the stratum $\operatorname{Spec}'_u A_q(\mathfrak{n}_+(w))$ contains no other Q_+-graded prime ideals.

Therefore $\{I_u(w) \mid u \in W^{\le w}\}$ exhaust all Q_+-graded prime ideals of $A_q^+(\mathfrak{n}_+(w))$. For $u_1 \le u_2$ in $W^{\le w}$, we have $I_{u_1}(w) \subseteq I_{u_2}(w)$ because $V^-_{u_1}(\lambda) \supseteq V^-_{u_2}(\lambda)$. Step 2 and the inclusion relations between Demazure modules [21, Proposition 4.4.5] imply that there are no other inclusions between these ideals. This proves part (a).

All prime ideals of $A_q(\mathfrak{n}_+(w))$ are completely prime by [12, Theorem 2.3]. It follows from (*) and the definition of $M_{\mathcal{J}}(\lambda)$ that the stratum $\operatorname{Spec}_u A_q(\mathfrak{n}_+(w))$, defined in part (b) of the theorem, coincides with $\operatorname{Spec}'_u A_q(\mathfrak{n}_+(w))$ and equals

$$\{\mathcal{J} \in \operatorname{Spec} A_q(n_+(w)) \mid \mathcal{J} \supseteq I_w(u),\ \mathcal{J} \cap E_w(u) = \varnothing\}.$$

The second statement in part (b) follows from (6.17), or equivalently, from [3, §II.2.1].

The properties (*) imply that the ring $(A_q(\mathfrak{n}_+(w))/I_u(w))[E_u(w)^{-1}]$ is $\mathcal{H}$-simple since the stratum $\operatorname{Spec}'_u A_q(\mathfrak{n}_+(w))$ has a unique Q_+-graded ideal. This and step 2 prove part (c). Part (d) now follows from [3, Lemma II.3.7, Proposition II.3.8, Theorem II.6.4]. □

6.4. **The homogeneous prime elements of $A_q(\mathfrak{n}_+(w))$.** Denote the support of w:

$$\mathcal{S}(w) := \{i \in I \mid s_i \le w\} = \{i \in I \mid i = i_k \ \text{ for some } \ k \in [1, N]\}$$

where the second formula is in terms of a reduced expression (5.8).

Corollary 6.6. *The homogeneous prime elements of $A_q(\mathfrak{n}_+(w))$ up to scalar multiples are*

$$D_{\varpi_i, w\varpi_i} \quad \textit{for} \quad i \in \mathcal{S}(w). \tag{6.18}$$

Proof. Theorem 6.5(i) implies that the height one Q_+-graded prime ideals of $A_q(\mathfrak{n}_+(w))$ are $I_w(s_i)$ for $i \in \mathcal{S}(w)$. Since $A_q(\mathfrak{n}_+(w)) \cong U_q(\mathfrak{n}_-(w))$ is a CGL extension (Lemma 5.4), it is an $\mathcal{H}$-UFD; thus, its height one Q_+-graded prime ideals are principal and their generators are precisely the homogeneous prime elements of $A_q(\mathfrak{n}_+(w))$. Applying Theorem 6.5(c) for $u = 1$ and taking into account that $I_w(1) = 0$ gives that $I_w(s_i) \cap E_w(1) \ne \varnothing$ for $i \in \mathcal{S}(w)$. However, $E_w(1)$ consists of monomials in the elements (6.18). Hence each of the (completely prime) ideals $I_w(s_i)$, $i \in \mathcal{S}(w)$ is generated by one of the elements in (6.18). The two sets have the same number of elements and $D_{\varpi_i, w\varpi_i} \in I_w(s_i)$. Hence,

$$I_w(s_i) = D_{\varpi_i, w\varpi_i} A_q(\mathfrak{n}_+(w)), \quad \forall i \in \mathcal{S}(w),$$

and the set (6.18) exhausts all homogeneous prime elements of $A_q(\mathfrak{n}_+(w))$ up to scalar multiples. □

7. Integral cluster structures on $A_q(\mathfrak{n}_+(w))$

7.1. **Statements of main results.** Recall the notation (1.1). Throughout the section, $\mathfrak{g}$ denotes an arbitrary symmetrizable Kac–Moody algebra and w a Weyl group element. We fix a reduced expression (5.8). Set

$$U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}^{1/2}} := U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}^{1/2}, \quad A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}} := A_q(\mathfrak{n}_+(w))_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}^{1/2}$$

and extend ι to an algebra isomorphism

$$\iota : U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}^{1/2}} \xrightarrow{\cong} A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}. \tag{7.1}$$

For $k \in [1, N]$, denote

$$x_k := q_{i_k}^{1/2} \iota(f^*_{\beta_k}) = q_{i_k}^{1/2}(q_{i_k}^{-1} - q_{i_k})\iota(f_{\beta_k}) \in A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}, \tag{7.2}$$

recall (5.11). For $j < k \in [1, N]$, set

$$a[j, k] := \|(w_{[j,k]} - 1)\varpi_{i_k}\|^2/4 \in \mathbb{Z}/2. \tag{7.3}$$

By applying ι to (5.18) and extending the scalars from $\mathbb{Q}(q)$ to $\mathbb{Q}(q^{1/2})$, we see that $A_q(\mathfrak{n}_+(w)) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q^{1/2})$ is a symmetric CGL extension on the generators $\iota(f^*_{\beta_1}), \dots, \iota(f^*_{\beta_N})$. It follows from Lemma 5.4(a) that the scalars λ_l, λ^*_k of the CGL extension are given by

$$\lambda_k = q_{i_k}^2, \quad \lambda^*_k = q_{i_k}^{-2}, \quad \forall k \in [1, N]. \tag{7.4}$$

Lemma 5.4(b) implies that $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ with the generators $x_1, \dots, x_N$ is an $\mathcal{A}^{1/2}$-form of the symmetric CGL extension $A_q(\mathfrak{n}_+(w)) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q^{1/2})$. It follows from (5.14) that the scalars

$$\nu_{kj} := q^{(\beta_k, \beta_j)/2}, \quad \forall 1 \le j < k \le N$$

satisfy Condition (A) in §3.4. Lemma 5.4(c) implies that the interval subalgebras of $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ are

$$\big(A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}\big)_{[j,k]} = T^{-1}_{w^{-1}_{\le j-1}}\big(A_q(\mathfrak{n}_+(w_{[j,k]}))_{\mathcal{A}^{1/2}}\big), \quad \forall j \le k \text{ in } [1, N]. \tag{7.5}$$

Our first main theorem on quantum Schubert cells is:

Theorem 7.1. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and $w \in W$ with a reduced expression* (5.8)*. Consider the $\mathcal{A}^{1/2}$-form $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ of the symmetric CGL extension $A_q(\mathfrak{n}_+(w)) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q^{1/2})$ with the generators $x_1, \dots, x_N$ given by* (7.2)*.*

(a) *The sequence of prime elements from Theorem* 3.2 *of $A_q(\mathfrak{n}_+(w)) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q^{1/2})$ with respect to the generators $x_1, \dots, x_N$ is*

$$y_k = q_{i_k}^{(O_-(k)+1)/2} D_{\varpi_{i_k}, w_{\le k}\varpi_{i_k}}, \quad k = 1, \dots, N.$$

The corresponding sequence of normalized prime elements is

$$\overline{y}_k = q^{a[1,k]} D_{\varpi_{i_k}, w_{\le k}\varpi_{i_k}}, \quad k = 1, \dots, N.$$

Moreover, $y_1, \dots, y_N, \overline{y}_1, \dots, \overline{y}_N \in A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$.

(b) *The η-function $\eta : [1, N] \to \mathbb{Z}$ of $A_q(\mathfrak{n}_+(w)) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q^{1/2})$ from Theorem 3.2 is given by*

$$\eta(k) := i_k, \quad \forall k \in [1, N]. \tag{7.6}$$

(c) *The normalized interval prime elements of $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ are*

$$\overline{y}_{[j,k]} = q^{a[j,k]} D_{w_{\le j-1}\varpi_{i_k}, w_{\le k}\varpi_{i_k}} = q^{a[j,k]} T_{w_{\le k-1}} D_{\varpi_{i_k}, w_{[j,k]}\varpi_{i_k}}$$

for all $j < k$ in $[1, N]$ such that $i_j = i_k$.

In the rest of this section we will use the notation (3.4) for the predecessor and successor functions p, s and the notation (3.5) for the functions $O_\pm$ associated to the η-function (7.6). Eq. (6.9) and Theorem 7.1 imply that for $k > j$,

$$D_{\varpi_{i_k}, w_{\le k}\varpi_{i_k}} D_{\varpi_{i_j}, w_{\le j}\varpi_{i_j}} = q^{-((w_{\le k}+1)\varpi_{i_k}, (w_{\le j}-1)\varpi_{i_j})} D_{\varpi_{i_j}, w_{\le j}\varpi_{i_j}} D_{\varpi_{i_k}, w_{\le k}\varpi_{i_k}},$$

and thus there is a unique toric frame $M^w\colon \mathbb{Z}^N \to \mathrm{Fract}(A_q(\mathfrak{n}_+(w))^\vee_{\mathcal{A}^{1/2}})$ with cluster variables

$$M^w(e_k) = q^{a[1,k]}\, D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}, \quad \forall k \in [1,N]$$

and matrix $\mathbf{r}^w$ with

$$(\mathbf{r}^w)_{kj} := q^{-((w_{\leq k}+1)\varpi_{i_k}, (w_{\leq j}-1)\varpi_{i_j})/2}, \quad \forall\, 1 \leq j < k \leq N. \tag{7.7}$$

We will use a quantum cluster algebra in which the exchangeable variables are

$$\mathbf{ex}(w) := \{k \in [1,N] \mid s(k) \neq \infty\}. \tag{7.8}$$

The number of elements of this set is $N - |\mathcal{S}(w)|$. We will index the columns of the exchange matrices of this quantum cluster algebra (which have sizes $N \times (N - |\mathcal{S}(w)|)$) by the elements of the set $\mathbf{ex}(w)$.

Proposition 7.2. *The matrix $\widetilde{B}^w$ of size $N \times (N - |\mathcal{S}(w)|)$ with entries*

$$(\widetilde{B}^w)_{jk} = \begin{cases} 1, & \text{if } j = p(k) \\ -1, & \text{if } j = s(k) \\ a_{i_j i_k}, & \text{if } j < k < s(j) < s(k) \\ -a_{i_j i_k}, & \text{if } k < j < s(k) < s(j) \\ 0, & \text{otherwise} \end{cases}$$

is compatible with $\mathbf{r}^w$, and more precisely, its columns $(\widetilde{B}^w)^k$, $k \in \mathbf{ex}(w)$ satisfy

$$\Omega_{\mathbf{r}^w}((\widetilde{B}^w)^k, e_l) = q_{i_k}^{-\delta_{kl}} = (\lambda_k^*)^{\delta_{kl}/2} \quad \text{and} \quad \sum_j (\widetilde{B}^w)_{jk}(w_{\leq j} - 1)\varpi_{i_j} = 0 \tag{7.9}$$

for all $k \in \mathbf{ex}(w)$, $l \in [1,N]$, recall (7.4).

The next theorem relates the integral quantum cluster algebra and upper quantum cluster algebra with initial seed $(M^w, \widetilde{B}^w)$ (both defined over $\mathcal{A}^{1/2}$) to the algebra $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$.

Theorem 7.3. *In the setting of Theorem* 7.1 *the following hold:*

(a) $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}} = \mathbf{A}(M^w, \widetilde{B}^w, \varnothing)_{\mathcal{A}^{1/2}} = \mathbf{U}(M^w, \widetilde{B}^w, \varnothing)_{\mathcal{A}^{1/2}}$.

(b) *For each $\sigma \in \Xi_N \subset S_N$, the quantum cluster algebra $\mathbf{A}(M^w, \widetilde{B}^w, \varnothing)_{\mathcal{A}^{1/2}}$ has a seed with cluster variables*

$$M^w_\sigma(e_l) = q^{a[j,k]}\, D_{w_{\leq j-1}\varpi_{i_j}, w_{\leq k}\varpi_{i_j}} = q^{a[j,k]}\, T_{w_{\leq j-1}} D_{\varpi_{i_j}, w_{[j,k]}\varpi_{i_j}}$$

for $j := \min \sigma([1,l])$ and $k := \max\{m \in \sigma([1,l]) \mid i_m = i_j\}$. The initial seed $(M^w, \widetilde{B}^w)$ equals the seed corresponding to $\sigma = \mathrm{id}_N \in \Xi_N$.

(c) *The seeds in* (b) *are linked by sequences of one-step mutations of the following kind:*

Suppose that $\sigma, \sigma' \in \Xi_N$ are such that $\sigma' = (\sigma(k), \sigma(k+1)) \circ \sigma = \sigma \circ (k, k+1)$ for some $k \in [1, N-1]$. If $\eta(\sigma(k)) \neq \eta(\sigma(k+1))$, then $M^w_{\sigma'} = M^w_\sigma \cdot (k, k+1)$ in terms of the action (3.29). *If $\eta(\sigma(k)) = \eta(\sigma(k+1))$, then $M^w_{\sigma'} = \mu_k(M^w_\sigma) \cdot (k, k+1)$.*

We illustrate Theorem 7.3 and the constructions in §§5.4, 6.1 and 7.1 with two examples of quantum unipotent cells in non-symmetric Kac–Moody algebras $\mathfrak{g}$: a finite dimensional one and an affine one.

Example 7.4. Let $\mathfrak{g}$ be of type B_2 and w be the longest Weyl group element $s_1 s_2 s_1 s_2$. The corresponding root sequence (5.9) is

$$\beta_1 = \alpha_1, \;\; \beta_2 = s_1(\alpha_2) = \alpha_1 + \alpha_2, \;\; \beta_3 = s_1 s_2(\alpha_1) = \alpha_1 + 2\alpha_2, \;\; \text{and} \;\; \beta_4 = s_1 s_2 s_1(\alpha_2) = \alpha_2.$$

The root vectors f_{β_k}, $1 \le k \le 4$, satisfy

$$f_{\beta_2} f_{\beta_1} = q^2 f_{\beta_1} f_{\beta_2}, \qquad f_{\beta_3} f_{\beta_1} = f_{\beta_1} f_{\beta_3} + \frac{1-q^{-2}}{q^{-1}+q} f_{\beta_2}^2, \qquad f_{\beta_3} f_{\beta_2} = q^2 f_{\beta_2} f_{\beta_3},$$
$$f_{\beta_4} f_{\beta_1} = q^{-2} f_{\beta_1} f_{\beta_4} - q^{-2} f_{\beta_2}, \quad f_{\beta_4} f_{\beta_2} = f_{\beta_2} f_{\beta_4} - (q^{-1}+q) f_{\beta_3}, \qquad f_{\beta_4} f_{\beta_3} = q^2 f_{\beta_3} f_{\beta_4}.$$

Note that the scalar in the right hand side of the second equation is not in $\mathcal{A}$. The CGL extension $U_q(\mathfrak{n}_-(w)) = U_q(\mathfrak{n}_-)$ is the $\mathbb{C}(q)$-algebra with these generators and relations. Its η-function from Theorem 3.2 is given by $\eta(1) = \eta(3) = 1$, $\eta(2) = \eta(4) = 2$. The generators of the integral form $U_q(\mathfrak{n}_-)^\vee_{\mathcal{A}}$ of the CGL extension $U_q(\mathfrak{n}_-)$ (cf. (5.11) and Lemma 5.4(b)) are

$$f^*_{\beta_k} = c_k f_{\beta_k}, \quad \text{where } c_1 = c_3 = q^{-2} - q^2, \ c_2 = c_4 = q^{-1} - q.$$

They satisfy

$$f^*_{\beta_2} f^*_{\beta_1} = q^2 f_{\beta_1} f_{\beta_2}, \qquad f^*_{\beta_3} f^*_{\beta_1} = f^*_{\beta_1} f^*_{\beta_3} - q^{-1}(q^{-2}-q^2)(f^*_{\beta_2})^2,$$
$$f^*_{\beta_3} f^*_{\beta_2} = q^2 f^*_{\beta_2} f^*_{\beta_3}, \qquad f^*_{\beta_4} f^*_{\beta_1} = q^{-2} f^*_{\beta_1} f^*_{\beta_4} - q^{-2}(q^{-2}-q^2) f^*_{\beta_2},$$
$$f^*_{\beta_4} f^*_{\beta_2} = f^*_{\beta_2} f^*_{\beta_4} - (q^{-1}-q) f^*_{\beta_3}, \qquad f^*_{\beta_4} f^*_{\beta_3} = q^2 f^*_{\beta_3} f^*_{\beta_4}.$$

Recall the isomorphism (7.1). The rescaled generators of $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}} = A_q(\mathfrak{n}_+)_{\mathcal{A}^{1/2}}$ are

$$x_k = c'_k \iota(f^*_{\beta_k}), \quad \text{where } c'_1 = c'_3 = q, \ c'_2 = c'_4 = q^{1/2}.$$

The algebra $A_q(\mathfrak{n}_+)_{\mathcal{A}^{1/2}}$ is the $\mathcal{A}^{1/2}$-algebra with generators $x_1, \dots, x_4$ and relations

$$x_2 x_1 = q^2 x_1 x_2, \qquad x_3 x_1 = x_1 x_3 - (q^{-2}-q^2) x_2^2, \qquad x_3 x_2 = q^2 x_2 x_3,$$
$$x_4 x_1 = q^{-2} x_1 x_4 - q^{-1}(q^{-2}-q^2) x_2, \quad x_4 x_2 = x_2 x_4 - (q^{-1}-q) x_3, \qquad x_4 x_3 = q^2 x_3 x_4.$$

By Theorem 7.3, $A_q(\mathfrak{n}_+)_{\mathcal{A}^{1/2}}$ has the structure of a quantum cluster algebra over $\mathcal{A}^{1/2}$ with initial cluster variables

$$\overline{y}_1 = x_1, \quad \overline{y}_2 = x_2, \quad \overline{y}_3 = x_1 x_3 - q^{-2} x_2^2, \quad \overline{y}_4 = x_2 x_4 - q^{-1} x_3$$

(where the 3rd and 4th variables are frozen) and mutation matrix

$$\widetilde{B} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that this is the quantum cluster algebra of type B_2 with principal coefficients. □

Example 7.5. Let $\mathfrak{g}$ be the twisted affine Kac–Moody algebra of type $A_2^{(2)}$, whose Dynkin diagram is

$$\underset{\alpha_0}{\circ} \Leftarrow \underset{\alpha_1}{\circ}$$

Following the standard convention [22, Ch. 6,8], we label its simple roots by $\{0, 1\}$ instead of $\{1, 2\}$. We have $(\alpha_0, \alpha_0) = 2$, $(\alpha_1, \alpha_1) = 8$, $d_0 = 1$, $d_1 = 4$ and $q_0 = q$, $q_1 = q^4$. Consider the Weyl group element $w = s_0 s_1 s_0 s_1 s_0$. The corresponding root sequence (5.9) is

$$\beta_1 = \alpha_0, \qquad \beta_2 = 4\alpha_0 + \alpha_1, \qquad \beta_3 = 3\alpha_0 + \alpha_1,$$
$$\beta_4 = 8\alpha_0 + 3\alpha_1, \qquad \beta_5 = 5\alpha_0 + 2\alpha_1.$$

The root vectors f_{β_k}, $1 \le k \le 5$, given by (5.9), satisfy the relations

$$\begin{aligned}
&z_2z_1 = q^4 z_1z_2, && z_3z_1 = q^2 z_1z_3 + az_2, && z_3z_2 = q^4 z_2z_3,\\
&z_4z_1 = q^4 z_1z_4 + \frac{ab}{q^4-q^2}z_3^3, && z_4z_2 = q^8 z_2z_4 + bz_3^4 && z_4z_3 = q^4 z_3z_4,\\
&z_5z_1 = q^2 z_1z_5 + \frac{abc(q^6-1)}{q^2(q^2-1)^2(q^8-1)}z_3^2, && z_5z_2 = q^4 z_2z_5 + \frac{bc}{q^4-q^2}z_3^3,\\
&z_5z_3 = q^2 z_3z_5 + cz_4, && z_5z_4 = q^4 z_4z_5
\end{aligned} \tag{7.10}$$

with $a = c = -q^2[4]_q$ and $b = -q^2(q^{-1}-q)^3/[4]_q$, where $[n]_q = (q^n - q^{-n})/(q-q^{-1})$. Note that $b \notin \mathcal{A}$. The CGL extension $U_q(\mathfrak{n}_-(w))$ is the $\mathbb{C}(q)$-algebra with these generators and relations. Its η-function from Theorem 3.2 is given by $\eta(1) = \eta(3) = \eta(5) = 0$, $\eta(2) = \eta(4) = 1$. The generators of the integral form $U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}}$ of the CGL extension $U_q(\mathfrak{n}_-(w))$ (cf. (5.11) and Lemma 5.4(b)) are

$$f^*_{\beta_k} = c_k f_{\beta_k}, \quad \text{where } c_1 = c_3 = c_5 = q^{-1} - q, \ c_2 = c_4 = q^{-4} - q^4.$$

They satisfy the relations (7.10) for $a = c = q(q^2-1)$, $b = q^{-2}(q^8-1) \in \mathcal{A}$, and furthermore, $U_q(\mathfrak{n}_-(w))^\vee_{\mathcal{A}}$ is the $\mathcal{A}$-algebra with these generators and relations. Recall the isomorphism (7.1). The rescaled generators of $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ are

$$x_k = c'_k \iota(f^*_{\beta_k}), \quad \text{where } c'_1 = c'_3 = c'_5 = q^{1/2}, \ c'_2 = c'_4 = q^2.$$

They satisfy the relations (7.10) for $a = c = q^2-1$, $b = q^8-1 \in \mathcal{A} \subset \mathcal{A}^{1/2}$, and furthermore, $A_q(\mathfrak{n}_+)_{\mathcal{A}^{1/2}}$ is the $\mathcal{A}^{1/2}$-algebra with these generators and relations. By Theorem 7.3, $A_q(\mathfrak{n}_+)_{\mathcal{A}^{1/2}}$ has the structure of a quantum cluster algebra over $\mathcal{A}^{1/2}$ with initial cluster variables

$$\begin{aligned}
&\overline{y}_1 = x_1, \quad \overline{y}_2 = x_2, \quad \overline{y}_3 = qx_1x_3 - q^{-1}x_2, \quad \overline{y}_4 = q^4x_2x_4 - q^{-4}x_3^4,\\
&\overline{y}_5 = q^3x_1x_3x_5 - qx_2x_5 - q^{-1}[3]_q x_3^3 - qx_1x_4.
\end{aligned}$$

(where the 4th and 5th variables are frozen) and mutation matrix

$$\widetilde{B} = \begin{bmatrix} 0 & -4 & 1 \\ 1 & 0 & -1 \\ -1 & 4 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

□

7.2. **Proof of Theorem 7.1.** If $u_1, u_2 \in W$ are such that $\ell(u_1u_2) = \ell(u_1) + \ell(u_2)$, then we have the decomposition

$$A_q(\mathfrak{n}_+(u_1u_2))_{\mathcal{A}} = A_q(\mathfrak{n}_+(u_1))_{\mathcal{A}}\, T^{-1}_{u_1^{-1}}(A_q(\mathfrak{n}_+(u_2)))_{\mathcal{A}}.$$

This follows by applying the isomorphism ι to the dual PBW basis (5.13) of $U_q(\mathfrak{n}_-(u_1u_2))^\vee_{\mathcal{A}}$. The next lemma shows the equality of the unipotent quantum minors in Theorem 7.1(c) and that they belong to the correct integral forms.

Lemma 7.6. *If $u_1, u_2 \in W$ are such that $\ell(u_1u_2) = \ell(u_1) + \ell(u_2)$, then*

$$D_{u_1\mu, u_1u_2\mu} = T^{-1}_{u_1^{-1}} D_{\mu, u_2\mu} \in T^{-1}_{u_1^{-1}} A_q(\mathfrak{n}_+(u_2))_{\mathcal{A}} \subset A_q(\mathfrak{n}_+(u_1u_2)), \quad \forall \mu \in P_+. \tag{7.11}$$

Proof. It was proved in [9, Proposition 6.3] that $\Psi^{-1}(D_{\mu,u_2\mu}) \in t\cdot\varphi^*(\mathbf{B}^{\mathrm{up}})$ in the notation of Remarks 5.3 and 6.1. Theorem 5.2 and the commutative diagram in Remark 6.1 imply that $D_{\mu,u_2\mu} \in \iota(\mathbf{B}^{\mathrm{up}}) \subset A_q(\mathfrak{n}_+(u_1))_{\mathcal{A}}$.

The equality (7.11) can be derived from [9, Proposition 7.1] and Remark 6.1, but it also has a direct proof as follows. For all $y_k \in U_q(\mathfrak{n}_+(u_k))$, $k=1,2$ and $h \in P^\vee$, we have

$$\begin{aligned}
&\langle D_{u_1\mu,u_1u_2\mu}, y_1T^{-1}_{u_1^{-1}}(y_2)q^h\rangle = \langle \xi_{u_1\mu}, y_1T^{-1}_{u_1^{-1}}(y_2)v_{u_1u_2\mu}\rangle = \langle \xi_\mu, T_{u_1^{-1}}(y_1)y_2v_{u_2\mu}\rangle\\
&= \langle \xi_\mu, y_2v_{u_2\mu}\rangle\epsilon(y_1) = \langle D_{\mu,u_2\mu}, y_2\rangle\epsilon(y_1) = (\iota^{-1}(D_{\mu,u_2\mu}), y_2)_{RT}\,\epsilon(y_1)\\
&= (T^{-1}_{u_1^{-1}}\iota^{-1}(D_{\mu,u_2\mu}), T^{-1}_{u_1^{-1}}y_2)_{RT}\,\epsilon(y_1) = \langle T^{-1}_{u_1^{-1}}D_{\mu,u_2\mu}, y_1T^{-1}_{u_1^{-1}}(y_2)q^h\rangle,
\end{aligned}$$

where the sixth equality uses (5.12). □

Proof of Theorem 7.1. We have

$$e_{\beta_k}v_{w_{\le k-1}\varpi_{i_k}} = T^{-1}_{w^{-1}_{\le k-1}}(e_{i_k}T^{-1}_{i_k}v_{\varpi_{i_k}}) = T^{-1}_{w^{-1}_{\le k-1}}v_{\varpi_{i_k}} = v_{w_{\le p(k)}\varpi_{i_k}}$$

and $e^m_{\beta_k}v_{w_{\le k-1}\varpi_i} = 0$ for $m>1$. Hence

$$\langle D_{\varpi_{i_k},w_{\le k}\varpi_{i_k}}, y_1e^m_{\beta_k}\rangle = \delta_{m1}\langle D_{\varpi_{i_k},w_{\le p(k)}\varpi_{i_k}}, y_1\rangle$$

for all $y_1 \in U_q(\mathfrak{n}_+(w_{\le k-1}))$, $m \in \mathbb{Z}_{\ge 0}$. It follows from (5.12) and (7.5) that in $A_q(\mathfrak{n}_+(w_{\le k}))_{\mathcal{A}} = \big(A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}\big)_{[1,k]} \subset A_q(\mathfrak{n}_+(w))_{\mathcal{A}}$ we have

$$D_{\varpi_{i_k},w_{\le k}\varpi_{i_k}} \equiv D_{\varpi_{i_k},w_{\le p(k)}\varpi_{i_k}}\iota(f^*_{\beta_k}) \mod A_q(\mathfrak{n}_+(w_{\le k-1}))_{\mathcal{A}}. \tag{7.12}$$

Therefore,

$$q_{i_k}^{(O_-(k)+1)/2}D_{\varpi_{i_k},w_{\le k}\varpi_{i_k}} \equiv q_{i_k}^{(O_-(p(k))+1)/2}D_{\varpi_{i_k},w_{\le p(k)}\varpi_{i_k}}x_k \mod A_q(\mathfrak{n}_+(w_{\le k-1}))_{\mathcal{A}^{1/2}}$$

for all $k \in [1,N]$. Part (b) and the first statement in part (a) now follow from Corollary 6.6.

We have $w_{\le k}\varpi_{i_k} = w_{\le k-1}(\varpi_{i_k} - \alpha_{i_k}) = w_{\le p(k)}\varpi_{i_k} - \beta_k$. Iterating this gives

$$\begin{aligned}
a[1,k] &= \|w_{\le k}\varpi_{i_k} - \varpi_{i_k}\|^2/4 = \|\beta_{p^{O_-(k)}(k)} + \cdots + \beta_k\|^2\\
&= (O_-(k)+1)\|\alpha_{i_k}\|^2/4 + \sum_{0\le l\le m\le O_-(k)} (\beta_{p^l(k)}, \beta_{p^m(k)})/2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{y}_k &= \Big(\prod_{0\le l\le m\le O_-(k)} \nu^{-1}_{p^l(k)p^m(k)}\Big)y_k\\
&= \Big(\prod_{0\le l\le m\le O_-(k)} q^{(\beta_{p^l(k)},\beta_{p^m(k)})/2}\Big)q^{(O_-(k)+1)\|\alpha_{i_k}\|^2/4}D_{\varpi_{i_k},w_{\le k}} = q^{a[1,k]}D_{\varpi_{i_k},w_{\le k}}
\end{aligned}$$

which proves the second statement in part (a).

It follows from Lemma 7.6 that $y_1,\dots,y_N,\overline{y}_1,\dots,\overline{y}_N \in A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$. Part (c) follows from (7.5) and part (b). □

7.3. **Proof of Theorem 7.3.**

Proof of Proposition 7.2. Extend $\widetilde{B}^w$ to an $(N+r)\times(N-\mathcal{S}(w))$ matrix whose rows are indexed by $[-r,-1]\sqcup[1,N]$ and columns by $\mathbf{ex}(w)$ by setting

$$(\widetilde{B}^w)_{-i,k} := \begin{cases} 1, & \text{if } i_k = i \text{ and } p(k) = -\infty\\ 0, & \text{otherwise}\end{cases}$$

for $i \in [1,r]$, $k \in \mathbf{ex}(w)$.

Denote for simplicity $b_{jk} := (\widetilde{B})_{jk}$. We apply [2, Theorem 8.3 and §10.1] to the double word $1,\dots,r,-i_1,\dots,i_N$, which gives

$$\sum_{j=1}^{N} b_{jk}\,\mathrm{sign}(j-l)\big((w_{\le j}\varpi_{i_j}, w_{\le l}\varpi_{i_l}) - (\varpi_{i_j},\varpi_{i_l})\big) + \sum_{i=1}^{r} b_{-i,k}((w_{\le j}-1)\varpi_{i_j},\varpi_i) = 2\delta_{kl}d_k \tag{7.13}$$

for all $k \in \mathbf{ex}(w)$, $l \in [1,N]$. The graded nature of the seed corresponding to the double word (cf. [2, Definition 6.5]) means that

$$\sum_{j=1}^{N} b_{jk} w_{\le j}\varpi_{i_j} + \sum_{i=1}^{r} b_{-i,k}\varpi_i = 0, \tag{7.14}$$

$$\sum_{j=1}^{N} b_{jk}\varpi_{i_j} + \sum_{i=1}^{r} b_{-i,k}\varpi_i = 0 \tag{7.15}$$

for all $k \in \mathbf{ex}(w)$. Subtracting (7.14) from (7.13) gives the second identity in (7.9). The linear combination (7.13) + ((7.14), ϖ_{i_l}) − ((7.15), $w_{\le l}\varpi_{i_l}$) yields the identity

$$\sum_{j=1}^{N} b_{jk}\,\mathrm{sign}(j-l)\big((w_{\le j}+1)\varpi_{i_j}, (w_{\le l}-1)\varpi_{i_l}\big) = 2\delta_{kl}d_k$$

for all $k \in \mathbf{ex}(w)$, $l \in [1,N]$, which is precisely the first identity in (7.9) in view of (7.7). □

Proposition 7.7. *In the setting of Theorem* 7.1*, the* $\mathcal{A}^{1/2}$*-form* $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ *of the symmetric CGL extension* $A_q(\mathfrak{n}_+(w)) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q^{1/2})$ *with the generators* $x_1,\dots,x_N$ *from* (7.2) *satisfies all conditions in Theorem* 4.8.

Proof. The scalars ν_{kl} are integral powers of $q^{1/2}$ and thus are units of $\mathcal{A}^{1/2}$. Obviously condition (A) is satisfied for the base field $\mathbb{K} = \mathbb{Q}(q^{1/2})$. Recall from Lemma 5.4(a) and (7.4) that

$$\lambda_k = q_{i_k}^2 = q^{2d_{i_k}} = q^{\|\alpha_{i_k}\|^2} \quad \text{for} \quad k \in [1,N],$$

and from Theorem 7.1(b) that $\eta(k) = i_k$ for $k \in [1,N]$. Therefore, Condition (B) is satisfied for the positive integers $\{d_i \mid i \in I\}$ from (5.1).

The homogenous prime elements $y_1,\dots,y_N$ belong to $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ by Theorem 7.1(a).

It remains to show that the condition (3.28) holds. Because of (7.5) and Lemma 7.6 it is sufficient to consider the case when $i = 1$ and $s(i) = N$. Since the η-function of the CGL extension $A_q(\mathfrak{n}_+(w))$ is given by Theorem 7.1(b), this means that $i_1 = i_N = i$ and $i_k \ne i$ for $k \in [2,N-1]$. It is well known that for $\mathfrak{g} = \mathfrak{sl}_2$ and $l \ge n \in \mathbb{Z}_{>0}$

$$e_1^{(l)} \cdot T_1^{-1} v_{n\varpi_1} = \delta_{ln} v_{n\varpi_1}.$$

For $k \in [2,N-1]$, $T_i^{-1}v_{\varpi_i}$ is a highest weight vector for the copy of $U_q(\mathfrak{sl}_2)$ inside $U_q(\mathfrak{g})$ generated by $e_{i_k}, f_{i_k}, h_{i_k}$ of highest weight $\langle s_i\varpi_i, h_{i_k}\rangle = -a_{i_k i}\varpi_{i_k}$. Hence, for $l \ge -a_{i_k i}$,

$$e_{\beta_k}^{(l)} \cdot T_{w_{\le k}^{-1}}^{-1} T_i^{-1} v_{\varpi_i} = T_{w_{\le k-1}^{-1}}^{-1} (e_{i_k}^{(l)} \cdot T_{i_k}^{-1} T_i^{-1} v_{\varpi_i}) = \delta_{l,-a_{i_k i}} T_i^{-1} v_{\varpi_i}.$$

Set $a := (-a_{i_1 i}, \dots, -a_{i_{N-2} i}) \in \mathbb{Z}_{\geq 0}^{N-2}$. Iterating this and using (7.12) and the identity $T_i^{-2} v_{\varpi_i} = -q_i^{-1} v_{\varpi_i}$ gives

$$\big\langle D_{\varpi_i, w\varpi_i} - q_i^{-1} x_1 x_N, e_{\beta_2}^{(l_2)} \dots e_{\beta_{N-1}}^{(l_{N-1})} \big\rangle = \begin{cases} -q_i^{-1}, & \text{if } a = (l_2, \dots, l_{N-1}) \\ 0, & \text{if } a \not\succeq (l_2, \dots, l_{N-1}) \end{cases}$$

with respect to the the reverse lexicographic order (3.9). It follows from (5.12) and (7.2) that

$$\mathrm{lt}(D_{\varpi_i, w\varpi_i} - q_i^{-1} x_1 x_N) = -q_i^{-1} \Big(\prod_{k=2}^{N-1} q_{i_k}^{-(a_{i_k i}^2 + a_{i_k i} + 1)/2} \Big) x_2^{-a_{i_2 i}} \dots x_{N-1}^{-a_{i_{N-1} i}}.$$

By a straightforward calculation with powers of q, one obtains from this that the condition (3.28) is satisfied. □

Theorem 7.3 follows by combining Theorems 4.8 and 7.1 and Propositions 7.2 and 7.7.

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