STRATA OF PRIME IDEALS OF DE CONCINI–KAC–PROCESI ALGEBRAS AND POISSON GEOMETRY

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To Ken Goodearl on his 65th birthday

Abstract. To each simple Lie algebra $\mathfrak{g}$ and an element w of the corresponding Weyl group De Concini, Kac and Procesi associated a subalgebra $\mathcal{U}_-^w$ of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$, which is a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$ and a quantization of the coordinate ring of the Schubert cell corresponding to w. The torus invariant prime ideals of these algebras were classified by Mériaux and Cauchon [28], and the author [30]. These ideals were also explicitly described in [30]. They index the the Goodearl–Letzter strata of the stratification of the spectra of $\mathcal{U}_-^w$ into tori. In this paper we derive a formula for the dimensions of these strata and the transcendence degree of the field of rational Casimirs on any open Richardson variety with respect to the standard Poisson structure [17].

1. Introduction

Assume that A is a Noetherian $\mathbb{K}$-algebra equipped with a rational action of a $\mathbb{K}$-torus T by algebra automorphisms. Under very general assumptions Goodearl and Letzter [16] constructed a stratification of Spec$A$ into tori indexed by the T-primes of A. Previously Joseph [22] and Hodges–Levasseur–Toro [20] obtained such stratifications of the spectra of quantized coordinate rings of simple groups. The Goodearl–Letzter results showed that such stratifications of spectra of rings exist in much greater generality, in particular whenever A is an iterated skew polynomial extension under some natural assumptions relating the structure of A and the action of T. This generated a lot of research in ring theory targeted at the explicit description of the above stratification of Spec$R$ for concrete rings R. The Cauchon approach of deleted derivations [8] provides an iterative procedure to classify the T-primes of an iterated skew polynomial extension. The explicit realization of this procedure often leads to difficult combinatorial problems. After many specific rings were investigated, most notably the algebras of quantum matrices (see [8, 15, 25]), it was observed that almost all of them fit into the general class of quantized universal enveloping algebras of nilpotent (or slightly more generally solvable) Lie algebras.

The most general quantization of a class of nilpotent Lie algebras up to date was constructed by De Concini, Kac and Procesi [10] using the Lusztig root vectors of a universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$. The

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algebras are parametrized by elements w of the Weyl group $W_{\mathfrak{g}}$ of $\mathfrak{g}$. The corresponding algebra $\mathcal{U}^w_-$ is a deformation of the universal enveloping algebra of $\mathfrak{n}_- \cap w(\mathfrak{n}_+)$ where $\mathfrak{n}_\pm$ is a pair of opposite nilpotent subalgebras of $\mathfrak{g}$. It can be viewed [11, 30] as a quantization of the coordinate ring of the Schubert cell $B_+ w \cdot B_+$ in the full flag variety G/B_+ with respect to the standard Poisson structure [17]. Within the framework of quantum flag varieties, it was proved that the quantum Schubert cells of all quantum partial flag varieties are isomorphic to algebras of the form $\mathcal{U}^w_-$, see [31, Theorem 3.6], and this was used to classify the torus invariant prime ideals of all quantum flag varieties [31].

Denote by T the maximal torus of the connected simply connected algebraic group corresponding to $\mathfrak{g}$. In [28] Mériaux and Cauchon classified the T-primes of $\mathcal{U}^w_-$ for $q \in \mathbb{K}^*$ not a root of unity and arbitrary base field $\mathbb{K}$. There is no explicit description of the T-primes in this approach, in particular the inclusions between the T-primes in the Mériaux–Cauchon picture are currently unknown.

In [30] the author constructed a second realization of the algebras $\mathcal{U}^w_-$ and derived an explicit description of each T-prime of $\mathcal{U}^w_-$, based on works of Gorelik and Joseph [18, 21, 22]. This resulted in a new parametrization of the set of T-primes of $\mathcal{U}^w_-$ which also explicitly identified the poset structure of the T-spectrum with the inclusion relation with a particular Bruhat interval in the related Weyl group. The statement of these results are summarized in [30, Theorem 1.1]. Although these results were stated in [30] for fields $\mathbb{K}$ of characteristic 0 and deformation parameters q which are transcendetal over $\mathbb{Q}$, the proofs work in the general case without restrictions on $\mathbb{K}$ assuming only that q is not a root of unity. This will be addressed in a forthcoming preprint. The T-primes of $\mathcal{U}^w_-$ are indexed by the elements $y \in W_{\mathfrak{g}}$ such that $y \leq w$ under the Bruhat order in $W_{\mathfrak{g}}$. We denote by $I_w(y)$ the T-prime ideal of $\mathcal{U}^w_-$ corresponding to y as in [30, Theorem 1.1], and refer the reader to §2 for the explicit description of $I_w(y)$.

In this paper we derive a formula for the dimension of the Goodearl–Letzter stratum of $\mathcal{U}^w_-$ corresponding to each T-prime $I_w(y)$. This is done under the assumption that the base field $\mathbb{K}$ has characteristic 0 and the deformation parameter q is transcendental over $\mathbb{Q}$, see Theorem 4.2. The proof uses Poisson geometry.

Bell, Casteels, and Launois [2] obtained simultaneously and independently a formula for the dimensions of the Goodearl–Letzter stratum of $\mathcal{U}^w_-$ in the Mériaux–Cauchon parametrization [28] for an arbitrary base field $\mathbb{K}$, and $q \in \mathbb{K}^*$ is not a root of unity. They gave a second proof for the case of quantum matrices in [3]. Previously the dimension of the stratum for the special case of the 0 ideal of $\mathcal{U}^w_-$ was obtained by Bell and Launois in [5]. The two approaches of Mériaux and Cauchon [30] and the author [30] to the T spectrum of $\mathcal{U}^w_-$ are of very different nature and are not connected yet. In particular, one cannot transfer the explicit description of T-primes and their inclusions from the picture in [30] to the picture in [28], nor one can relate directly the dimension formulas from [2] to ours. In a forthcoming preprint we will describe a ring theoretic derivation of the dimension formulas for the Goodearl–Letzter strata of $\mathcal{U}^w_-$ in the picture in [30] which works in the more general case of an arbitrary base field $\mathbb{K}$ and q which is not a root of unity.

To describe in more concrete terms the results and methods of the paper we return to the general setting of a $\mathbb{K}$-algebra A with an action of a torus T by algebra automorphisms. Denote by $T - \mathrm{Spec} A$ the set of T-primes of A. For a T-prime I denote the Goodearl–Letzter stratum [16] of $\mathrm{Spec} A$:

$$\mathrm{Spec}_I A = \{J \in \mathrm{Spec} A \mid \cap_{t \in T} t \cdot J = I\}.$$

Following [16], consider the localization $(A/I)[\mathcal{E}_I^{-1}]$ by the set of all homogeneous nonzero elements $\mathcal{E}_I \subset A/I$. Goodearl and Letzter proved (under certain mild assumptions) that

$$\mathrm{Spec} A = \bigsqcup_{I \in H - \mathrm{Spec}} \mathrm{Spec}_I A,$$

$Z((A/I)[\mathcal{E}_I^{-1}])$ is isomorphic to a Laurent polynomial ring, and that $\mathrm{Spec}_I A$ is homeomorphic to the torus $\mathrm{Spec} Z((A/I)[\mathcal{E}_I^{-1}])$, see [14, Theorem 6.6]. This procedure was further developed in the book of Brown and Goodearl [6, Part II] and by Goodearl [14] who also showed that the same holds true for smaller sets $\mathcal{E}_I$ which match the original Joseph [21] and Hodges–Levasseur–Toro [20] methods, see [14, Theorem 5.3]. This partition has all topological properties required for a stratification, see [14, Lemma 3.4]. Brown and Goodearl also proved [6, Theorem II.6.4] that, if A is an iterated skew polynomial algebra satisfying certain general conditions which hold for the algebras $\mathcal{U}_-^w$, then the base field for all Laurent polynomial rings $Z((A/I)[\mathcal{E}_I^{-1}])$ is $\mathbb{K}$. Since all algebras $\mathcal{U}_\pm^w$ are iterated skew polynomial algebras, this fact applies to all of them.

In [30] we realized the algebras $\mathcal{U}_-^w$ as explicit quotients of Joseph's quantum translated Bruhat cell algebras, see [22, §10.4.8] and [18]. The latter are defined in terms of the quantized coordinate ring $\mathcal{O}_q(G)$ and satisfy similar commutation relations derived from the quantum R-matrix for $\mathfrak{g}$. In this paper, starting from this realization we construct enough elements in the center $Z((\mathcal{U}_-^w/I_w(y))[\mathcal{E}_{I_w(y)}^{-1}])$ and prove that this center is a Laurent polynomial ring in at least $\dim E_{-1}(w^{-1}y)$ variables. Here and below for a linear operator L acting on a vector space V and $c \in \mathbb{C}$ we denote by $E_c(L)$ the c-eigenspace of L. Next we pass to an integral form of $\mathcal{U}_-^w/I_w(y)$ and show that its specialization at $q = 1$ is isomorphic to the coordinate ring of the open Richardson variety

$$R_{y,w} = B^- y \cdot B^+ \cap B^+ w \cdot B^+ \subset G/B^+$$

with Poisson algebra structure coming from the standard Poisson structure on G/B^+, see [17] and §3. If the dimension of the stratum $\mathrm{Spec}_{I_w(y)} \mathcal{U}_-^w$ was strictly greater than $\dim E_{-1}(w^{-1}y)$, then the transcendence degree of the center of the Poisson field of rational functions on $R_{y,w}$ would be strictly greater than $\dim E_{-1}(w^{-1}y)$ which is shown to contradict with the dimension formulas for the symplectic leaves in $R_{y,w}$. This implies that both the dimension of the Goodearl–Letzter stratum $\mathrm{Spec}_{I_w(y)} \mathcal{U}_-^w$ and the transcendence degree of the center of the Poisson field of rational functions on the open Richardson variety $R_{y,w}$ are equal to $\dim E_{-1}(w^{-1}y)$.

In [32] we prove some further properties of the algebras $\mathcal{U}_-^w$. Firstly, for each torus invariant prime ideal $I_w(y)$ we construct efficient polynormal generating sets. In the special case of the algebras of quantum matrices this leads to an explicit proof of the Goodearl–Lenagan conjecture [15] that all torus invariant prime ideals of the algebras of quantum matrices have polynormal generating sets consisting of quantum minors. Furthermore, we prove that all Spec$\mathcal{U}_-^w$ are normally separated, and that each algebra $\mathcal{U}_-^w$ is catenary.

The paper is organized as follows. Sect. 2 contains background for quantized universal enveloping algebras, the algebras $\mathcal{U}_-^w$ and their spectra. Sect. 3 deals with the related Poisson structures on flag varieties. Sect. 4 carries out the connection between the two and contains the proofs of the main results.

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2. Quantized nilpotent algebras

Fix a base field $\mathbb{K}$ of characteristic 0 and $q \in \mathbb{K}$ which is transcendetal over $\mathbb{Q}$. Let $\mathfrak{g}$ be simple Lie algebra of rank r and Cartan matrix c_{ij}. Denote by $\mathcal{U}_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{K}$ with deformation parameter q. It is a Hopf algebra over $\mathbb{K}$ with generators

$$X_i^{\pm}, K_i^{\pm 1}, \ i = 1, \dots, r,$$

and relations

$$K_i^{-1}K_i = K_iK_i^{-1} = 1, \ K_iK_j = K_jK_i, \ K_iX_j^{\pm}K_i^{-1} = q_i^{\pm c_{ij}}X_j^{\pm},$$
$$X_i^+X_j^- - X_j^-X_i^+ = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0, \ i \neq j.$$

where d_i are positive relative prime integers such that (d_ic_{ij}) is a symmetric matrix and $q_i = q^{d_i}$. Denote by $\mathcal{U}_\pm$ the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_i^{\pm}\}_{i=1}^r$. Let H be the group generated by $\{K_i^{\pm 1}\}_{i=1}^r$ (i.e. the group of group like elements of $\mathcal{U}_q(\mathfrak{g})$).

Let P and P_+ be the sets of integral and dominant integral weights of $\mathfrak{g}$. The sets of simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$ will be denoted by $\{\alpha_i\}_{i=1}^r$, $\{\alpha_i^\vee\}_{i=1}^r$, and $\{\omega_i\}_{i=1}^r$, respectively. Let $\langle .,. \rangle$ be the nondegenerate invariant bilinear form on $\mathfrak{g}$ such that the square length of a short root is equal to 2.

The q-weight spaces of an H-module V are defined by

$$V_\lambda = \{v \in V \mid K_iv = q^{\langle \lambda, \alpha_i \rangle}v, \ \forall i = 1, \dots, r\}, \ \lambda \in P.$$

A $\mathcal{U}_q(\mathfrak{g})$-module is called a type 1 module if it is the sum of its q-weight spaces. Recall that all finite dimensional type 1 $\mathcal{U}_q(\mathfrak{g})$-modules are completely reducible and are parametrized by P_+, see [9, §10.1]. Let $V(\lambda)$ denote the irreducible weight $\mathcal{U}_q(\mathfrak{g})$ module of highest weight $\lambda \in P_+$. For each $\lambda \in P_+$ fix a highest weight vector v_λ of $V(\lambda)$ such that $\forall \lambda, \mu \in P_+$ $v_{\lambda+\mu} = v_\lambda \otimes v_\mu$ when $V(\lambda+\mu)$ is realized as a submodule of $V(\lambda) \otimes V(\mu)$.

All duals of finite dimensional $\mathcal{U}_q(\mathfrak{g})$ modules will be thought of as left modules using the antipode of $\mathcal{U}_q(\mathfrak{g})$.

Denote the Weyl and braid groups of $\mathfrak{g}$ by $W_\mathfrak{g}$ and $\mathcal{B}_\mathfrak{g}$, respectively. Let $s_1, \ldots, s_r$ be the simple reflections of $W_\mathfrak{g}$ corresponding to the roots $\alpha_1, \ldots, \alpha_r$, and $T_1, \ldots, T_r$ be the standard generators of $\mathcal{B}_\mathfrak{g}$. Recall that one has a section $W_\mathfrak{g} \to \mathcal{B}_\mathfrak{g}$, $w \mapsto T_w$, of the canonical projection $\mathcal{B}_\mathfrak{g} \to W_\mathfrak{g}$. Given a a reduced expression $w = s_{i_1} \ldots s_{i_k}$ one sets $T_w = T_{i_1} \ldots T_{i_k}$. The latter does not depend of the choice of a reduced expression of w.

There are natural actions of $\mathcal{B}_\mathfrak{g}$ on $\mathcal{U}_q(\mathfrak{g})$ and the modules $V(\lambda)$, see [26, §5.2 and §37.1] for details. They have the properties that $T_w(x.v) = (T_w x).(T_w v)$, $T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu}$ for all $w \in W_\mathfrak{g}$, $x \in \mathcal{U}_q(\mathfrak{g})$, $\lambda \in P_+$, $v \in V(\lambda)$, $\mu \in P$.

For a reduced decomposition

$$w = s_{i_1} \ldots s_{i_k} \tag{2.1}$$

of an element $w \in W_\mathfrak{g}$, define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \ldots, \beta_k = s_{i_1} \ldots s_{i_{k-1}} \alpha_{i_k} \tag{2.2}$$

and Lusztig's root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{s_{i_1}} X^\pm_{i_2}, \ldots, X^\pm_{\beta_k} = T_{s_{i_1} \ldots s_{i_{k-1}}} X^\pm_{i_k}, \tag{2.3}$$

see [26, §39.3]. De Concini, Kac and Procesi defined [10] the subalgebras $\mathcal{U}^w_\pm$ of $\mathcal{U}_\pm$ generated by $X^\pm_{\beta_j}$, $j = 1, \ldots, k$ and proved:

Theorem 2.1. *(De Concini, Kac, Procesi)* [10, Proposition 2.2] *The algebras $\mathcal{U}^w_\pm$ do not depend on the choice of a reduced decomposition of w and have the PBW basis*

$$(X^\pm_{\beta_k})^{n_k} \ldots (X^\pm_{\beta_1})^{n_1}, \ \ n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}. \tag{2.4}$$

The fact that the space spanned by the monomials (2.4) does not depend on the choice of a reduced decomposition of w was independently obtained by Lusztig [26, Proposition 40.2.1].

The quantized coordinate ring $R_q[G]$ of the split, connected, simply connected algebraic group G corresponding to $\mathfrak{g}$ is the Hopf subalgebra of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by all matrix entries $c^\lambda_{\xi,v}$, $\lambda \in P_+$, $v \in V(\lambda), \xi \in V(\lambda)^*$: $c^\lambda_{\xi,v}(x) = \langle \xi, xv \rangle$ for $x \in \mathcal{U}_q(\mathfrak{g})$. Denote by R^+ the subalgebra of $R_q[G]$ spanned by all matrix entries $c^\lambda_{\xi,v_\lambda}$ where $\lambda \in P_+$, $\xi \in V(\lambda)^*$ and v_λ is the fixed highest weight vector of $V(\lambda)$. The group H acts on $R_q[G]$ on the left and right by

$$x \rightharpoonup c = \sum c_{(2)}(x) c_{(1)}, \ c \leftharpoonup x = \sum c_{(1)}(x) c_{(2)}, \ x \in H, c \in R_q[G] \tag{2.5}$$

in terms of the standard notation for the comultiplication $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.

For all $\lambda \in P_+$ and $w \in W_\mathfrak{g}$ define $\xi_{w,\lambda} \in (V(\lambda)^*)_{-w\lambda}$ such that $\langle \xi_{w,\lambda}, T_w v_\lambda \rangle = 1$. Let

$$c^\lambda_w = c^\lambda_{\xi_{w,\lambda}, v_\lambda}. \tag{2.6}$$

Then $c^\lambda_w c^\mu_w = c^{\lambda+\mu}_w = c^\mu_w c^\lambda_w$, $\forall \lambda, \mu \in P_+$. Set

$$c_w = \{c^\lambda_w \mid \lambda \in P_+\}.$$

It is [22, Lemma 9.1.10] an Ore subset of R^+.

Denote the localization

$$R^w = R^+[c_w^{-1}]$$

(see [22, 18]) and observe that (2.5) induces H-actions on R^w. The invariant subalgebra with respect to the left action (2.5) of H is denoted by R^w_0 and is called the quantum translated Bruhat cell [22, §10.4.8]. It was studied in great detail by Gorelik in [18]. For $\lambda \in P_+$ set $c^{-\lambda}_w = (c^\lambda_w)^{-1} \in R^w$. We have that

$$R^w_0 = \{c^{-\lambda}_w c^\lambda_{\xi, v_\lambda} \mid \lambda \in P_+, \xi \in V(\lambda)^*\} \tag{2.7}$$

since

(2.8)

$$\forall \lambda, \mu \in P_+,\ \xi \in V(\lambda)^*, \quad \exists \xi' \in V(\lambda+\mu)^* \text{ such that } c^{-\lambda}_w c^\lambda_{\xi, v_\lambda} = c^{-\lambda-\mu}_w c^{\lambda+\mu}_{\xi', v_{\lambda+\mu}}.$$

For $y \in W_\mathfrak{g}$ define the ideals

$$Q(y)^\pm_w = \{c^{-\lambda}_w c^\lambda_{\xi, v_\lambda} \mid \xi \in V(\lambda)^*,\ \xi \perp \mathcal{U}_\pm T_y v_\lambda\} \tag{2.9}$$

of R^w_0. We do not need to take span in the right hand side of (2.9) because of (2.8). The ideals (2.9) are nontrivial if and only if $y \geq w$ in the plus case and $y \leq w$ in the minus case, see [18], and are completely prime and H-invariant (with respect to the right action (2.5)). Gorelik proved in [18] that all H-invariant, prime ideals of R^w_0 are of the form $Q(y_-)^-_w + Q(y_+)^+_w$ for some $y_\pm \in W_\mathfrak{g}$, $y_- \leq w \leq y_+$.

Recall that the quantum R-matrix associated to $w \in W_\mathfrak{g}$ is given by

$$\mathcal{R}^w = \prod_{j=k,\dots,1} \exp_{q_{i_j}} \Big((1 - q_{i_j}^{-2}) X^+_{\beta_j} \otimes X^-_{\beta_j} \Big) \tag{2.10}$$

where

$$\exp_{q_i}(y) = \sum_{n=0}^\infty q_i^{n(n+1)/2} \frac{y^n}{[n]_{q_i}!}.$$

and $q_i = q^{2/\langle \alpha_i, \alpha_i \rangle}$. In (2.10) the terms are multiplied in the order $j = k, \dots, 1$. The R-matrix $\mathcal{R}^w$ belongs to a certain completion [26, §4.1.1] of $\mathcal{U}^w_+ \otimes \mathcal{U}^w_-$ and does not depend on the choice of a reduced decomposition of w.

Remark 2.2. The group H acts on $\mathcal{U}^w_-$ by $K_i.x = K_i x K_i^{-1}$, $x \in \mathcal{U}^w_-$. The torus $(\mathbb{K}^*)^{\times r}$ acts on $\mathcal{U}^w_-$ by

$$(a_1, \dots, a_r) \cdot x = (\prod_{i=1}^r a_i^{\langle \lambda, \alpha_i^\vee \rangle}) x, \quad \forall x \in (\mathcal{U}^w_-)_\lambda, \lambda \in Q_+,$$

where Q_+ denotes the positive part $\mathbb{Z}_{\geq 0}\alpha_1 + \ldots + \mathbb{Z}_{\geq 0}\alpha_r$ of the root lattice of $\mathfrak{g}$. A subspace of $\mathcal{U}^w_-$ is H-invariant if and only if it is $(\mathbb{K}^*)^{\times r}$-invariant. In particular the set of H-primes and $(\mathbb{K}^*)^{\times r}$-primes of $\mathcal{U}^w_-$ are the same. Although $(\mathbb{K}^*)^{\times r}$-invariance fits directly to the scheme in [16], we will use the H-action since it is more intrinsic in term of the Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$.

In [30] we proved:

Theorem 2.3. [30, Theorem 3.7] *The map*

$$\phi_w \colon R_0^w \to \mathcal{U}_-^w, \ \ \phi_w(c_w^{-\lambda} c^\lambda_{\xi, v_\lambda}) = (c^\lambda_{\xi, T_w v_\lambda} \otimes \mathrm{id})(\mathcal{R}^w), \ \lambda \in P_+, \xi \in V(\lambda)^*$$

is a (well defined) surjective algebra homomorhism. It is H-equivariant with respect to the right action (2.5) *of H on R_0^w. The kernel of ϕ_w is $Q(w)^+_w$.*

This isomorphism is similar to one previously investigated by De Concini and Procesi in [11].

Using Gorelik's description [18] of the H spectrum of R_0^w leads to:

Theorem 2.4. [30, Theorem 1.1] *Fix $w \in W_{\mathfrak{g}}$. For each $y \in W^{\leq w}$ define*

$$\begin{aligned}(2.11) \quad I_w(y) &= \phi_w(Q(y)^-_w + Q(w)^+_w) = \phi_w(Q(y)^-_w) \\ &= \{(c^{w,\lambda}_\eta \otimes \mathrm{id})(\mathcal{R}^w) \mid \lambda \in P_+, \eta \in (V_w(\lambda) \cap \mathcal{U}_- T_y v_\lambda)^\perp\}.\end{aligned}$$

Then:

(a) $I_w(y)$ is an H-invariant completely prime ideal of $\mathcal{U}_-^w$ and all H-invariant prime ideals of $\mathcal{U}_-^w$ are of this form.

(b) The correspondence $y \in W^{\leq w} \mapsto I_w(y)$ is an isomorphism from the poset $W^{\leq w}$ to the poset of H-invariant prime ideals of $\mathcal{U}_-^w$ ordered under inclusion; that is $I_w(y) \subseteq I_w(y')$ for $y, y' \in W^{\leq w}$ if and only if $y \leq y'$.

Here and below for $w \in W_{\mathfrak{g}}$ we denote by $W_{\mathfrak{g}}^{\leq w}$ the set of all $y \in W_{\mathfrak{g}}$, $y \leq w$.

3. Poisson structures on flag varieties

Denote by $\mathfrak{g}_{\mathbb{C}}$ the complex simple Lie algebra corresponding to $\mathfrak{g}$ and by $G_{\mathbb{C}}$ the connected, simply connected algebraic group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $B^\pm_{\mathbb{C}}$ be a pair of opposite Borel subgroups. Set $T_{\mathbb{C}} = B^+_{\mathbb{C}} \cap B^-_{\mathbb{C}}$. Denote by Δ_+ the set of positive roots of $\mathfrak{g}_{\mathbb{C}}$. Fix two sets of root vectors $\{x^+_\alpha\}_{\alpha \in \Delta_+}$, $\{x^-_\alpha\}_{\alpha \in \Delta_+}$ with respect to Lie $T_{\mathbb{C}}$ ($x^\pm_\alpha \in \mathfrak{g}^{\pm\alpha}_{\mathbb{C}}$) normalized by $\langle x^+_\alpha, x^-_\alpha \rangle = 1$, where $\langle .,. \rangle$ denotes the nondegenerate invariant bilinear form on $\mathfrak{g}_{\mathbb{C}}$ such that $\langle \alpha, \alpha \rangle = 2$ for a long root α.

The standard Poisson structure [17] on the flag variety $G_{\mathbb{C}}/B^+_{\mathbb{C}}$ is defined

$$\pi = \sum_{\alpha \in \Delta_+} \chi(x^+_\alpha) \wedge \chi(x^-_\alpha) \tag{3.1}$$

where $\chi \colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{Vect}(G_{\mathbb{C}}/B^+_{\mathbb{C}})$ denotes the infinitesimal action of $G_{\mathbb{C}}$ on $G_{\mathbb{C}}/B^+_{\mathbb{C}}$. The action of the torus $T_{\mathbb{C}}$ on $(G_{\mathbb{C}}/B^+_{\mathbb{C}}, \pi)$ is Poisson.

The open Richardson varieties are the intersections of opposite Schubert cells in the flag variety $G_{\mathbb{C}}/B^+_{\mathbb{C}}$

$$R_{y_-, y_+} = B^-_{\mathbb{C}} y_- \cdot B^+_{\mathbb{C}} \cap B^+_{\mathbb{C}} y_+ \cdot B^+_{\mathbb{C}} \subset G_{\mathbb{C}}/B^+_{\mathbb{C}}, \quad y_\pm \in W. \tag{3.2}$$

They are nonempty if and only if $y_- \leq y_+$ in the Bruhat order on $W_{\mathfrak{g}}$. In recent years Richardson varieties played an important role in various algebro-geometric problems for flag varieties [5, 24, 4] and the study of the totally nonnegative parts of flag varieties [27, 29].

Theorem 3.1. *(1) The $T_{\mathbb{C}}$-orbits of symplectic leaves of $(G_{\mathbb{C}}/B^+_{\mathbb{C}}, \pi)$ are precisely the open Richardson varieties R_{y_-,y_+}, $y_\pm \in W, y_- \leq y_+$. In particular, all open Richardson varieties are regular Poisson submanifolds of $(G_{\mathbb{C}}/B^+_{\mathbb{C}}, \pi)$.*

(2) The codimension of a symplectic leaf in R_{y_-,y_+} is

$$\dim \ker(1 + y_+^{-1} y_-) = \dim E_{-1}(y_+^{-1} y_-).$$

Proof. Part (1) is a special case of [17, Theorem 0.4], see [7, Theorem 0.4] in the case of Grassmannians. It also follows from [12, Example 4.9].

To deduce part (2), consider the double flag variety $G_{\mathbb{C}}/B^+_{\mathbb{C}} \times G_{\mathbb{C}}/B^-_{\mathbb{C}}$ with the Poisson structure

$$\pi^d = \sum_{\alpha \in \Delta_+} \big(\chi_1(x^+_\alpha) \wedge (\chi_1(x^-_\alpha) + \chi_2(x^-_\alpha)) \\ - \chi_2(x^-_\alpha) \wedge (\chi_1(x^+_\alpha) + \chi_2(x^+_\alpha)) \big) + \sum_j \chi_1(h_j) \wedge \chi_2(h_j).$$

Here $\{h_j\}$ denotes an orthonormal basis of Lie $T_{\mathbb{C}}$ with respect to the restriction of $\langle ., . \rangle$ and $\chi_i \colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{Vect}(G_{\mathbb{C}}/B^+_{\mathbb{C}} \times G_{\mathbb{C}}/B^-_{\mathbb{C}})$, $i = 1, 2$, denote the infinitesimal actions of $G_{\mathbb{C}}$ derived from the actions of $G_{\mathbb{C}}$ on the first and second factor of the Cartesian product.

It is easy to verify that the embedding of the single flag variety in the double flag variety $\eta \colon G_{\mathbb{C}}/B^+_{\mathbb{C}} \to G_{\mathbb{C}}/B^+_{\mathbb{C}} \times G_{\mathbb{C}}/B^-_{\mathbb{C}}$ given by $\eta(g \cdot B^+_{\mathbb{C}}) = (g \cdot B^+_{\mathbb{C}}, B^-_{\mathbb{C}})$ is Poisson with respect to π and π^d. Obviously

$$\eta(R_{y_-,y_+}) = \big((B^+_{\mathbb{C}} \times B^-_{\mathbb{C}}) \cdot (y_+ \cdot B^+_{\mathbb{C}}, B^-_{\mathbb{C}}) \big) \cap \big(\Delta(G_{\mathbb{C}}) \cdot (y_- \cdot B^+_{\mathbb{C}}, B^-_{\mathbb{C}}) \big) \tag{3.3}$$

where $\Delta(G_{\mathbb{C}})$ denotes the diagonal subgroup of $G_{\mathbb{C}} \times G_{\mathbb{C}}$. Part (2) now follows from the fact [12, Example 4.9] that the codimension of the symplectic leaves of the restriction of π^d to the right hand side of (3.3) is $\dim E_{-1}(y_+^{-1} y_-)$. □

The field $\mathbb{C}(R_{y_-,y_+})$ of rational functions on each open Richardson variety R_{y_-,y_+} becomes a Poisson field under the Poisson bracket induced from (3.1). Denote its center by $Z_\pi(\mathbb{C}(R_{y_-,y_+}))$. The latter is called the field of rational Casimir functions on (R_{y_-,y_+}, π). Each such function f should be constant on the intersection of its domain with a generic symplectic leaf of (R_{y_-,y_+}, π). It is easy to see that the symplectic leaves of π are smooth locally closed subvarieties of $G_{\mathbb{C}}/B^+_{\mathbb{C}}$, e.g. by applying [7, Theorem 1.9]. Analogously to [13, Proposition 4.2] we obtain:

Lemma 3.2. *For all $y_\pm \in W_{\mathfrak{g}}$, $y_- \leq y_+$, the transcendence degree of the field $Z_\pi(\mathbb{C}(R_{y_-,y_+}))$ is less than or equal to $\dim E_{-1}(y_+^{-1} y_-)$.*

Another way to prove Lemma 3.2 is as follows. Denote by $\pi^\sharp$ the bundle map $T^* R_{y_-,y_+} \to T R_{y_-,y_+}$ given by $\pi^\sharp_m(\eta_m) = (\eta_m \otimes \mathrm{id}) \pi_m$ for $m \in R_{y_-,y_+}$, $\eta_m \in T^*_m R_{y_-,y_+}$. Then

$$Z_\pi(\mathbb{C}(R_{y_-,y_+})) = \{ f \in \mathbb{C}(R_{y_-,y_+}) \mid \pi^\sharp(\eta)(f) = 0, \ \forall \eta \in \Gamma(R_{y_-,y_+}, T^*_m R_{y_-,y_+}) \}.$$

Since the codimension of all symplectic leaves of (R_{y_-,y_+}, π) is $\dim E_{-1}(y_+^{-1} y_-)$ there exist $k = \dim R_{y_-,y_+} - \dim E_{-1}(y_+^{-1} y_-)$ generically linearly independent regular vector fields $X_1, \dots, X_k$ on R_{y_-,y_+} such that

$$X_j f = 0, \quad \forall f \in Z_\pi(\mathbb{C}(R_{y_-,y_+})), j = 1, \dots, k.$$

This implies that the transcendence degree of $Z_\pi(\mathbb{C}(R_{y_-,y_+}))$ is less than or equal to $\dim R_{y_-,y_+} - k = \dim E_{-1}(y_+^{-1} y_-)$.

In Theorem 4.2 we will prove an inverse inequality which will show that the transcendence degree of $Z_\pi(\mathbb{C}(R_{y_-,y_+}))$ is equal to $\dim E_{-1}(y_+^{-1} y_-)$. This can be also obtained directly by constructing enough functions in the Poisson center without going to quantized algebras of functions.

4. Dimensions of the Goodearl–Letzter strata

Fix $w \in W_{\mathfrak{g}}$ and $y \in W_{\mathfrak{g}}^{\leq w}$. By Theorems 2.3 and 2.4

$$\mathcal{U}_-^w / I_w(y) \cong R_0^w / (Q(y)_w^- + Q(w)_w^+). \tag{4.1}$$

Since $Q(y)_w^- + Q(w)_w^+$ is a completely prime ideal, the quotient rings in (4.1) are domains. Denote by $L_{y,w}$ the localization of $R_0^w/(Q(y)_w^- + Q(w)_w^+)$ by all nonzero homogeneous elements with respect to the right H-action (2.5). The Goodearl–Letzter results [16, Theorem 6.6] and the Brown–Goodearl result on strong rationality [6, Theorem II.6.4] imply that the center $Z(L_{y,w})$ of $L_{y,w}$ is a Laurent polynomial ring over $\mathbb{K}$. Denote by $n_{y,w}$ the number of independent variables in this ring. Then [16, Theorem 6.6] implies that $\mathrm{Spec}_{I_w(y)} \mathcal{U}_-^w$ is homeomorphic to the spectrum of a Laurent polynomial ring in $n_{y,w}$ variables. In this section we prove that $n_{y,w} = \dim E_{-1}(w^{-1} y)$.

For $\lambda \in P_+$ set

$$a_\lambda = c_w^{-\lambda} c_y^\lambda, \tag{4.2}$$

recall (2.6). Applying the standard R-matrix commutation relations [22, Theorem I.8.16] in $R_q[G]$ gives

$$a_\lambda \big(c_w^{-\mu} c_{\xi, v_\mu}^\mu \big) - q^{-\langle (y+w)\lambda, \nu + w\mu \rangle} \big(c_w^{-\mu} c_{\xi, v_\mu}^\mu \big) a_\lambda \in Q(y)_w^- + Q(w)_w^+, \quad \forall \xi \in V(\mu)_\nu^*, \mu \in P_+, \nu \in P. \tag{4.3}$$

Denote the image of a_λ in $R_0^w/(Q(y)_w^- + Q(w)_w^+)$ by $\overline{a}_\lambda$. By (4.3) all $\overline{a}_\lambda$ are normal elements. They are all nonzero since for $w_1, w_2 \in W_{\mathfrak{g}}$, $\lambda \in P_+$, one has $T_{w_1} v_\lambda \in \mathcal{U}_+ T_{w_2} v_\lambda$ if and only if $w_1 \geq w_2$, see [22, Proposition 4.4.5].

Recall that $L_{y,w}$ denotes the localization of $R_0^w/(Q(y)_w^- + Q(w)_w^+)$ by the set of all nonzero homogeneous elements. Represent $\lambda \in P$ as $\lambda = \lambda_+ - \lambda_-$, for $\lambda_\pm \in P_+$ with non-intersecting support and set

$$\overline{a}_\lambda = (\overline{a}_{\lambda_-})^{-1} \overline{a}_{\lambda_+} \in L_{y,w}.$$

Then $\overline{a}_\lambda$ are normal elements of $L_{y,w}$ for all $\lambda \in P$ and

$$\overline{a}_\lambda \big(c_w^{-\mu} c_{\xi, v_\mu}^\mu + Q(y)_w^- + Q(w)_w^+ \big) = q^{-\langle (y+w)\lambda, \nu + w\mu \rangle} \big(c_w^{-\mu} c_{\xi, v_\mu}^\mu + Q(y)_w^- + Q(w)_w^+ \big) \overline{a}_\lambda \tag{4.4}$$

for all $\xi \in V(\mu)_\nu^*, \mu \in P_+, \nu \in P$.

Denote

$$P_{y,w} = \{ \lambda \in P \mid (y+w)\lambda = 0 \}.$$

Obviously $P_{y,w}$ is a lattice of rank $\dim E_{-1}(w^{-1}y) = \dim \ker(y+w)$. Denote by $\lambda_1, \ldots, \lambda_{\dim E_{-1}(w^{-1}y)}$ a basis of $P_{y,w}$. Set

$$Z_{y,w} = \mathrm{Span}\{\overline{a}_\lambda \mid \lambda \in P_{y,w}\}.$$

Lemma 4.1. *For all $y \leq w$, $Z_{y,w}$ is a subring of $Z(L_{y,w})$ which is isomorphic to a Laurent polynomial ring in $\dim E_{-1}(w^{-1}y)$ variables over $\mathbb{K}$. In particular, $n_{y,w} \geq E_{-1}(w^{-1}y)$.*

Proof. The fact that $Z_{y,w} \subseteq Z(L_{y,w})$ follows from (4.3). Applying [30, eq. (3.14)], one gets that for all $\lambda, \mu \in P$, $\overline{a}_\lambda \overline{a}_\mu$ is a nonzero scalar multiple of $\overline{a}_{\lambda+\mu}$. In particular $Z_{y,w}$ is a ring generated by $(\overline{a}_{\lambda_i})^{\pm 1}$, $i = 1, \ldots, \dim E_{-1}(w^{-1}y)$. It is isomorphic to a Laurent polynomial ring in $\dim E_{-1}(w^{-1}y)$ variables since the elements $\overline{a}_\lambda$ are linearly independent for different λ's. This is so because the elements $\overline{a}_\lambda$ belong to different homogeneous components of $L_{y,w}$ with respect to right action (2.5) of H:

$$\overline{a}_\lambda \leftharpoonup K_i = q^{\langle (y-w)\lambda, \alpha_i \rangle} \overline{a}_\lambda = q^{2\langle y\lambda, \alpha_i\rangle} \overline{a}_\lambda$$

for all $\lambda \in P_{y,w}$. $\square$

The main result of the paper is:

Theorem 4.2. *Assume that the base field $\mathbb{K}$ has characteristic) and that the deformation parameter q is transcendental over $\mathbb{Q}$.*

For all $w \in W_{\mathfrak{g}}$, $y \in W^{\leq w}$ the Goodearl–Letzter stratum $\mathrm{Spec}_{I_w(y)} \mathcal{U}^w_-$ is homeomorphic to the spectrum of a Laurent polynomial ring over $\mathbb{K}$ in $\dim E_{-1}(w^{-1}y)$ variables and the transcendence degree of $Z_\pi(\mathbb{C}(R_{y,w}))$ is equal to $\dim E_{-1}(w^{-1}y)$.

As we noted earlier, [5, Theorem II.6.4] implies that the stratum $\mathrm{Spec}_{I_w(y)} \mathcal{U}^w_-$ is homeomorphic to the spectrum of a Laurent polynomial ring over $\mathbb{K}$. It is sufficient to prove the first part of Theorem 4.2 for one choice of the base field $\mathbb{K}$ since all algebras $R_q[G], R^w_0, Q(y)^-_w + Q(w)^+_w, L_{y,w}, Z(L_{y,w})$ behave appropriately under extensions of the base field from $\mathbb{Q}(q)$ to $\mathbb{K}$. We will prove Theorem 4.2 for $\mathbb{K} = \mathbb{C}(q)$ and from now on we will assume that $\mathbb{K} = \mathbb{C}(q)$.

Before we proceed with the proof, we will recall some facts about integral forms. Set $\mathcal{A} = \mathbb{C}[q, q^{-1}]$. Denote by $\mathcal{U}^{\mathrm{res}}_{\mathcal{A}}$ the restricted integral form of $\mathcal{U}_q(\mathfrak{g})$ over $\mathcal{A}$, see [9, Chapter 9] for detail. (Usually integral forms of quantized universal enveloping algebras are defined over $\mathbb{Z}[q, q^{-1}]$, but this will not be needed for our purposes.) Denote $V(\lambda)^{\mathrm{res}}_{\mathcal{A}} = \mathcal{U}^{\mathrm{res}}_{\mathcal{A}} v_\lambda \subset V(\lambda)$, cf. [9, §10.1]. Note that $\mathcal{U}^{\mathrm{res}}_{\mathcal{A}}$ and $V(\lambda)^{\mathrm{res}}_{\mathcal{A}}$ are stable under the action of the braid group $\mathcal{B}_{\mathfrak{g}}$ for all $\lambda \in P_+$. In particular, $V(\lambda)^{\mathrm{res}}_{\mathcal{A}}$ and $(V(\lambda)^*)^{\mathrm{res}}_{\mathcal{A}}$ are also generated by lowest weight vectors: $V(\lambda)^{\mathrm{res}}_{\mathcal{A}} = \mathcal{U}^{\mathrm{res}}_{\mathcal{A}} T_{w_\circ} v_\lambda$, $(V(\lambda)^*)^{\mathrm{res}}_{\mathcal{A}} = \mathcal{U}^{\mathrm{res}}_{\mathcal{A}} \xi_\lambda$. Denote by $R^+_{\mathcal{A}}$ the $\mathcal{A}$-subalegbra of R^+ consisting of all sums of elements of the form $c^\lambda_{\xi, v_\lambda}$, for $\lambda \in P_+$, $\xi \in (V(\lambda)^*)^{\mathrm{res}}_{\mathcal{A}} = \mathcal{U}^{\mathrm{res}}_{\mathcal{A}} \xi_\lambda$. Note that $c_w \subset R^w_{\mathcal{A}}$ and denote $R^w_{\mathcal{A}} = R^+_{\mathcal{A}}[c_w^{-1}]$. The group H acts on $R^w_{\mathcal{A}}$ on the left and right by (2.5). The invariant subalgebra with respect to the left action will be denoted by $(R^w_0)_{\mathcal{A}}$. Clearly,

$$(R^w_0)_{\mathcal{A}} = \{c_w^{-\lambda} c^\lambda_{\xi, v_\lambda} \mid \lambda \in P_+, \xi \in (V(\lambda)^*)^{\mathrm{res}}_{\mathcal{A}}\}. \tag{4.5}$$

Analogously to (2.7) one does not need to take a sum in the right hand side of (4.5). Similarly to (2.9), for $u \in W_{\mathfrak{g}}$ define the ideals

$$(Q(u)^{\pm}_{w})_{\mathcal{A}} = \{c_w^{-\lambda} c^{\lambda}_{\xi, v_\lambda} \mid \xi \in (V(\lambda)^*)^{\mathrm{res}}_{\mathcal{A}},\, \xi \perp \mathcal{U}_{\pm} T_y v_\lambda\}.$$

It is clear that the natural embeddings

$$R^{+}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(q) \to R^{+},\;\; (R^{w}_{0})_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}(q) \to R^{w}_{0},\;\; (Q(u)^{\pm}_{w})_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(q) \to Q(u)^{\pm}_{w},$$

and

$$[(R^{w}_{0})_{\mathcal{A}}/((Q(y)^{-}_{w})_{\mathcal{A}} + (Q(w)^{+}_{w})_{\mathcal{A}})] \otimes_{\mathcal{A}} \mathbb{C}(q) \to R^{w}_{0}/(Q(y)^{-}_{w} + Q(w)^{+}_{w})$$

are isomorphisms. Denote the localization $\mathcal{A}_1 = (\mathcal{A})_{(q-1)}$ of $\mathcal{A}$ at $(q-1)$. Let $(L_{y,w})_{\mathcal{A}_1}$ denote the localization of $(R^{w}_{0})_{\mathcal{A}}/((Q(y)^{-}_{w})_{\mathcal{A}} + (Q(w)^{+}_{w})_{\mathcal{A}}))$ by all homogeneous elements which do not belong to

$$(q-1)(R^{w}_{0})_{\mathcal{A}}/((Q(y)^{-}_{w})_{\mathcal{A}} + (Q(w)^{+}_{w})_{\mathcal{A}}).$$

Then $(L_{y,w})_{\mathcal{A}_1}$ has a canonical structure of an $\mathcal{A}_1$ algebra and

$$(L_{y,w})_{\mathcal{A}_1} \otimes_{\mathcal{A}_1} \mathbb{C}(q) \cong L_{y,z} \;\text{ and }\; Z((L_{y,w})_{\mathcal{A}_1}) \otimes_{\mathcal{A}_1} \mathbb{C}(q) \cong Z(L_{y,z}). \tag{4.6}$$

Next, we recall some general facts about specializations. Assume that C is an $\mathcal{A}$ algebra. One calls the $\mathbb{C}$-algebra

$$\overline{C} = C/(q-1)C \cong C \otimes_{\mathcal{A}} \mathbb{C} \tag{4.7}$$

the specialization of C at 1. The tensor product is defined via the homomorphism $\kappa \colon \mathcal{A} \to \mathbb{C}$, $\kappa(q) = 1$. If $\overline{C}$ is a commutative algebra then it has a canonical Poisson algebra structure defined as follows. Denote the quotient map $\eta \colon C \to C/(q-1)C = \overline{C}$ For $a, b \in \overline{C}$ let a', b' be two preimages of a, b under η. Define

$$\{a, b\} = \eta((a'b' - b'a')/(q-1)). \tag{4.8}$$

It is well known that the above bracket does not depend on the choice of a', b' and that it turns $\overline{C}$ into a Poisson algebra. Obviously

$$\eta(Z(C)) \subseteq Z_{\{.,.\}}(\overline{C}) \tag{4.9}$$

where the left hand side denotes the center of the Poisson algebra $(C, \{.,.\})$.

The same construction defines the specialization $\overline{C}$ of an $\mathcal{A}_1$ algebra C at 1. Provided that $C/(q-1)C$ is commutative, one defines a Poisson algebra structure on $\overline{C}$ by (4.8).

It is well known that

$$\overline{R^{+}} \cong \mathbb{C}[G_{\mathbb{C}}/U^{+}_{\mathbb{C}}] \tag{4.10}$$

where $U^{+}_{\mathbb{C}}$ denotes the unipotent radical of $B^{+}_{\mathbb{C}}$. This is implicit in [21, §9.1.6] and appears in the proof in [23, Théorème 3]. Eq. (4.9) and Theorem 4.6 (both) from [30] imply at once that

$$\overline{R^{w}_{0}} \cong \mathbb{C}[wB^{-}_{\mathbb{C}} \cdot B^{+}_{\mathbb{C}}] \tag{4.11}$$

(where $wB^{-}_{\mathbb{C}} \cdot B^{+}_{\mathbb{C}} \subset G_{\mathbb{C}}/B^{+}_{\mathbb{C}}$ is the translated one Schubert cell of $G_{\mathbb{C}}/B^{+}_{\mathbb{C}}$, see [30, §4.2]) and

$$\overline{(R^{w}_{0}/(Q(y)^{-}_{w} + Q(w)^{+}_{w}))_{\mathcal{A}}} \cong \mathbb{C}[R_{y,w}]. \tag{4.12}$$

Note that in the setting of [30, Theorem 4.6], the variety $S_w(y)$ is isomorphic to $R_{y,w}$ via the map (4.10). We claim that the induced Poisson structure on

$\mathbb{C}[R_{y,w}]$ is exactly the one coming from $-\pi$, recall (3.1). The above argument proves that it is sufficient to show that the induced from (4.11) Poisson structure on $\mathbb{C}[wB^-_{\mathbb{C}} \cdot B^+_{\mathbb{C}}]$ is equal to the one coming from $-\pi$ which follows from the following lemma.

Lemma 4.3. *The induced from* (4.10) *Poisson structure on* $\mathbb{C}[G/U^+_{\mathbb{C}}]$ *is equal to the one coming from following bivector field on* $G/U^+_{\mathbb{C}}$

$$-\sum_{\alpha\in\Delta_+} \chi(x^+_\alpha)\wedge\chi(x^-_\alpha),$$

where $\chi\colon \mathfrak{g}_{\mathbb{C}} \to G_{\mathbb{C}}/U^+_{\mathbb{C}}$ *is the infinitesimal action of* $G_{\mathbb{C}}$ *on* $G_{\mathbb{C}}/U^+_{\mathbb{C}}$*.*

Proof. Denote by w_0 the longest element of $W_{\mathfrak{g}}$. The algebras $\mathcal{U}_\pm$ are Q graded by $\deg X^\pm_i = \pm\alpha_i$. Denote by $(\mathcal{U}_\pm)_{\pm\gamma}$, $\gamma \in Q_+$ their graded components and by $m(\gamma) = \dim(\mathcal{U}_+)_\gamma = \dim(\mathcal{U}_-)_{-\gamma}$ their dimensions. For each $\gamma\in Q_+$ fix a pair of dual bases $\{u_{\gamma,i}\}_{i=1}^{m(\gamma)}$ and $\{u_{-\gamma,i}\}_{i=1}^{m(\gamma)}$ of $(\mathcal{U}_+)_\gamma$ and $(\mathcal{U}_-)_{-\gamma}$ with respect to the Rosso–Tanisaki form. Then

$$\mathcal{R}^{w_0} = 1 + \sum_{\gamma\in Q_+,\gamma\neq 0}\sum_{i=1}^{m(\gamma)} u_{\gamma,i}\otimes u_{-\gamma,i} \tag{4.13}$$

and we have the R-matrix commutation relation in R^+:

$$\begin{aligned} c^{\lambda_1}_{\xi_1,v_{\lambda_1}} c^{\lambda_2}_{\xi_2,v_{\lambda_2}} = q^{\langle\nu_1,\nu_2\rangle-\langle\lambda_1,\lambda_2\rangle}\Big(& c^{\lambda_2}_{\xi_2,v_{\lambda_2}} c^{\lambda_1}_{\xi_1,v_{\lambda_1}} \\ & + \sum_{\gamma\in Q_+,\gamma\neq 0}\sum_{i=1}^{m(\gamma)} c^{\lambda_2}_{S^{-1}(u_{\gamma,i})\xi_2,v_{\lambda_2}} c^{\lambda_1}_{S^{-1}(u_{-\gamma,i})\xi_1,v_{\lambda_1}}\Big), \end{aligned} \tag{4.14}$$

for $\lambda_i\in P_+$, $\nu_i \in P$, $\xi_i \in V(\lambda_i)^*_{\nu_i}$, $i=1,2$. Eq. (2.10) implies:

$$\overline{(R^{w_0}-1)/(q-1)} = 2\sum_{\alpha\in\Delta_+} x^+_\alpha\otimes x^-_\alpha.$$

Therefore in terms of the notation from (4.14)

$$\begin{aligned} \Big\{\overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}, \overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}}\Big\} = (\langle\nu_1,\nu_2\rangle - \langle\lambda_1,\lambda_2\rangle)\overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}\,\overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}} \\ + 2\sum_{\alpha\in\Delta_+}\Big(\chi(x^-_\alpha)\overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}\Big)\Big(\chi(x^+_\alpha)\overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}}\Big) \end{aligned}$$

and

$$\begin{aligned} &\Big\{\overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}, \overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}}\Big\} = \frac{1}{2}\Big\{\overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}, \overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}}\Big\} - \frac{1}{2}\Big\{\overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}}, \overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}\Big\} \\ &= -\sum_{\alpha\in\Delta_+}\Big(\chi(x^+_\alpha)\overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}\Big)\Big(\chi(x^-_\alpha)\overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}}\Big) + \sum_{\alpha\in\Delta_+}\Big(\chi(x^-_\alpha)\overline{c^{\lambda_1}_{\xi_1,v_{\lambda_1}}}\Big)\Big(\chi(x^+_\alpha)\overline{c^{\lambda_2}_{\xi_2,v_{\lambda_2}}}\Big) \end{aligned}$$

□

The above proves that $\overline{(L_{y,w})_{\mathcal{A}_1}}$ is isomorphic to a subring of $\mathbb{C}(R_{y,w})$ which contains $\mathbb{C}[R_{y,w}]$ and the induced Poisson bracket on $\overline{(L_{y,w})_{\mathcal{A}_1}}$ coincides with the one coming from $-\pi$.

Proof of Theorem 4.2. Assume that the first statement is not true. Lemma 4.1 implies that $n_{y,w} \geq E_{-1}(w^{-1}y) + 1$. From (4.6) one obtains that the Krull dimension of $Z((L_{y,w})_{\mathcal{A}_1})$ is at least $\dim E_{-1}(w^{-1}y) + 2$. On the other hand (4.9) and the discussion before the proof of the theorem imply that

$$Z((L_{y,w})_{\mathcal{A}_1})/[(q-1)Z((L_{y,w})_{\mathcal{A}_1})]$$

is isomorphic to a subring of $Z_\pi(\mathbb{C}(R_{y_-,y_+}))$. It follows from Lemma 3.2 that the Krull dimension of $Z((L_{y,w})_{\mathcal{A}_1})$ is at most $\dim E_{-1}(w^{-1}y) + 1$ which is a contradiction. This proves the first part of the theorem.

The second part of the theorem follows from the first. From Lemma 3.2 we know that the transcendence degree of $Z_\pi(\mathbb{C}(R_{y,w}))$ is less than or equal to $\dim E_{-1}(w^{-1}y)$. If the inequality is strict, then working the above argument backwards leads to $n_{y,w} < \dim E_{-1}(w^{-1}y)$ which is a contradiction. This completes the proof of the theorem. □

Remark 4.4. Due to Joseph [21] and Hodges–Levasseur–Toro [20] the H-prime ideals of the quantized coordinate ring $\mathcal{O}_q(G)$ are parametrized by pairs $(y,w) \in W_\mathfrak{g} \times W_\mathfrak{g}$ and the corresponding stratum of Spec$\mathcal{O}_q(G)$ has dimension equal to $\dim E_1(w^{-1}y)$. On the Poisson side the $T_\mathbb{C}$-orbits of symplectic leaves of the Poisson–Lie group $G_\mathbb{C}$ equipped with the standard Poisson structure are the double Bruhat cells $G_\mathbb{C}^{y,w} = B_\mathbb{C}^- y B_\mathbb{C}^- \cap B_\mathbb{C}^+ w B_\mathbb{C}^+$ and the codimension of a symplectic leaf of $G_\mathbb{C}^{y,w}$ in $G_\mathbb{C}^{y,w}$ is equal [19, Theorem A.2.1 and Proposition A.2.2] to $\dim E_1(w^{-1}y)$. It is not hard to show that the tanscendence degree of the center of the Poisson field $\mathbb{C}(G_\mathbb{C}^{y,w})$ is equal to $\dim E_1(w^{-1}y)$.

It would be very interesting to understand the relation between these facts and the dimension formulas in Theorem 4.2. (Note the difference of ± 1 eigenspaces.) We believe that there exists a ring theoretic counterpart of the construction of weak splitting of surjective Poisson submersions in [17, Sect. 3] which relates the H-stratifications of Spec$\mathcal{O}_q(G)$ and Spec$\mathcal{U}_-^w$, and eventually the dimension formulas.

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