THE LAUNOIS–LENAGAN CONJECTURE

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Abstract. In this note we prove the Launois–Lenagan conjecture on the classification of the automorphism groups of the algebras of quantum matrices $R_q[M_n]$ of square shape for all positive integers n, base fields $\mathbb{K}$, and deformation parameters $q \in \mathbb{K}^*$ which are not roots of unity.

1. Introduction

The explicit description of automorphism groups of noncommutative algebras is a difficult problem even for algebras of low Gelfand–Kirillov dimension. Such was achieved in small number of examples. Andruskiewitsch–Dumas [2] and Launois–Lenagan [6] made two conjectures which predicted the explicit structure of the automorphism groups of important families of quantized universal enveloping algebras of nilpotent Lie algebras. The algebras in both families have arbitrarily large Gelfand–Kirillov dimensions. The former conjecture concerns the positive parts of the quantized universal enveloping algebras of all simple Lie algebras $\mathcal{U}_q^+(\mathfrak{g})$ and the latter one the algebras of quantum matrices of square shape $R_q[M_n]$. We proved the former conjecture in [12] by exhibiting a relationship to a certain type of "bi-finite unipotent" automorphisms of completed quantum tori and proving a rigidity for those. Here we use this method and results of Launois and Lenagan to settle the latter conjecture.

Let $\mathbb{K}$ be an arbitrary base field, $q \in \mathbb{K}^*$ an element which is not a root of unity, and n a positive integer. The algebra of quantum matrices $R_q[M_n]$ is the $\mathbb{K}$-algebra with generators x_{kl}, $1 \le k, l \le n$ and relations

$$\begin{aligned} x_{ij}x_{kj} &= q x_{kj}x_{ij}, \quad \text{for } i<k,\\ x_{ij}x_{il} &= q x_{il}x_{ij}, \quad \text{for } j<l,\\ x_{ij}x_{kl} &= x_{kl}x_{ij}, \quad \text{for } i<k, j>l,\\ x_{ij}x_{kl} - x_{kl}x_{ij} &= (q-q^{-1})x_{il}x_{kj}, \quad \text{for } i<k, j<l. \end{aligned}$$

The torus

$$\mathcal{H} := (\mathbb{K}^*)^{2n}/\{(c,\dots,c) \mid c \in \mathbb{K}^*\} \cong (\mathbb{K}^*)^{2n-1}$$

acts on $R_q[M_n]$ by algebra automorphisms by

$$(c_1,\dots,c_{2n})\cdot x_{kl} := c_k c_l^{-1} x_{kl}, \quad (c_1,\dots,c_{2n}) \in (\mathbb{K}^*)^{2n}. \tag{1.1}$$

One also has the transpose automorphism

$$\tau \in \operatorname{Aut} R_q[M_n] \text{ given by } \tau(x_{kl}) = x_{lk}.$$

Conjecture 1.1. (Launois–Lenagan, [6]) For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and integers $n > 1$ the group of algebra automorphisms of $R_q[M_n]$ is isomorphic to $\mathcal{H} \rtimes \mathbb{Z}_2$ via the identification $\mathbb{Z}_2 \cong \langle \tau \rangle$.

The case of $n = 2$ was proved by Alev and Chamarie [1] before this conjecture was stated. Recently, Launois and Lenagan proved it for $n = 3$ in [7]. Earlier they also classified in [6] the automorphism groups of quantum matrices of size $m \times n$ for $m \neq n$. The methods of [6] rely

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on specific properties of the height one prime ideals in that case and cannot be used to prove Conjecture 1.1. The conjecture was open for all $n > 3$.

The algebra $R_q[M_n]$ is a connected $\mathbb{N}$-graded algebra generated in degree one by assigning $\deg x_{kl} = 1$, $\forall 1 \leq k, l \leq n$. For $m \in \mathbb{N}$ denote by $R_q[M_n]^{\geq m}$ the sum of the components of $R_q[M_n]$ of degree $\geq m$. We call an automorphism Φ of $R_q[M_n]$ *unipotent* if

$$\Phi(x_{kl}) - x_{kl} \in R_q[M_n]^{\geq 2}.$$

Launois and Lenagan proved [7, §1.7] that Conjecture 1.1 follows from the following conjecture:

Conjecture 1.2. For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and $n \in \mathbb{Z}_+$ every unipotent automorphism of $R_q[M_n]$ is equal to the identity automorphism.

We prove Conjecture 1.2 and thus obtain the validity of Conjecture 1.1. The key ingredient in the proof is the method of rigidity of quantum tori which we developed in [12]. The proof is in Section 3. Section 2 gathers the needed facts for quantum matrices and quantum tori.

2. Quantum matrices and quantum tori

2.1. Throughout the paper, for all integers $k, l \in \mathbb{Z}$, $k \leq l$, we will denote $[k, l] := \{k, \ldots, l\}$.

Let $N \in \mathbb{Z}_+$. A multiplicatively skew-symmetric matrix is a matrix $\mathbf{q} = (q_{ij}) \in M_N(\mathbb{K}^*)$ such that $q_{ii} = 1$ for all $i \in [1, N]$ and $q_{ij}q_{ji} = 1$, for all $1 \leq i < j \leq N$. To it one associates the *quantum torus* $\mathcal{T}_{\mathbf{q}}$ which is the $\mathbb{K}$-algebra with generators $Y_i^{\pm 1}$, $i \in [1, N]$ and relations

$$Y_i Y_j = q_{ij} Y_j Y_i, \ \ \forall 1 \leq i < j \leq N, \quad Y_i Y_i^{-1} = Y_i^{-1} Y_i = 1, \ \ \forall i \in [1, N].$$

The quantum torus $\mathcal{T}_{\mathbf{q}}$ is called *saturated* if for $u \in \mathcal{T}_{\mathbf{q}}$ and $k \in \mathbb{Z}_+$, $u^k \in Z(\mathcal{T}_{\mathbf{q}}) \Rightarrow u \in Z(\mathcal{T}_{\mathbf{q}})$. Define the multiplicative kernel of $\mathbf{q}$ by

$$\operatorname{Ker}(\mathbf{q}) = \Big\{ (m_1, \ldots, m_N) \in \mathbb{Z}^N \mid \prod_{j=1}^{N} q_{ij}^{m_j} = 1, \ \forall i \in [1, N] \Big\}.$$

The quantum torus $\mathcal{T}_{\mathbf{q}}$ is saturated if and only if

$$(\operatorname{Ker}(\mathbf{q})/k) \cap \mathbb{Z}^N = \operatorname{Ker}(\mathbf{q}), \ \ \forall k \in \mathbb{Z}_+,$$

see [12, §3.1]. In other words, for all $f \in \mathbb{Z}^N$ and $k \in \mathbb{Z}_+$

$$kf \in \operatorname{Ker}(\mathbf{q}) \Rightarrow f \in \operatorname{Ker}(\mathbf{q}).$$

It is clear that, if q_{ij}, $1 \leq i < j \leq N$ generate a torsion-free subgroup of $\mathbb{K}^*$, then $\mathcal{T}_{\mathbf{q}}$ is saturated. A vector $(d_1, \ldots, d_N) \in \mathbb{Z}_+^N$ will be called a *degree vector*. It gives rise to a $\mathbb{Z}$-grading on $\mathcal{T}_{\mathbf{q}}$ by setting $\deg Y_i^{\pm 1} = \pm d_i$. Denote by $\mathcal{T}_{\mathbf{q}}^r$ the subspace of $\mathcal{T}_{\mathbf{q}}$ of degree $r \in \mathbb{Z}$ and consider the completion

$$\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}} := \{ u_r + u_{r+1} + \ldots \mid r \in \mathbb{Z}, u_k \in \mathcal{T}_{\mathbf{q}}^k \text{ for } k \geq r \}.$$

For $r \in \mathbb{Z}$, let $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq r} := \{ u_r + u_{r+1} + \ldots \mid r \in \mathbb{Z}, u_k \in \mathcal{T}_{\mathbf{q}}^k \}$. We call a continuous automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ *unipotent* if

$$\phi(Y_i) - Y_i \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq d_i + 1}, \ \ \forall i \in [1, N].$$

Under this condition $\phi(Y_i) = (1 + u_i) Y_i$ for some $u_i \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq 1}$. Furthermore, $\phi(Y_i^{-1})$ is given by

$$\phi(Y_i^{-1}) = ((1 + u_i) Y_i)^{-1} = \sum_{r=0}^{\infty} (-1)^r Y_i^{-1} u_i^r \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq -d_i}$$

and all values of ϕ are explicitly expressed in terms of $u_1, \ldots, u_N$. We will say that a unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ is *bi-finite* if

$$\phi(Y_i), \phi^{-1}(Y_i) \in \mathcal{T}_{\mathbf{q}}, \ \ \forall i \in [1, N].$$

We refer the reader to [12, Section 3] for details on the above notions.

We will need the following result from [12], see Theorem 1.2 there.

Theorem 2.1. *Fix an arbitrary base field $\mathbb{K}$, a multiplicatively skew-symmetric matrix $\mathbf{q} \in M_N(\mathbb{K}^*)$ for which the quantum torus $\mathcal{T}_{\mathbf{q}}$ is saturated, and a degree vector $\mathbf{d} \in \mathbb{Z}_+^N$. For every bi-finite unipotent automorphism ϕ of the completed quantum torus $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, there exists an N-tuple*

$$(u_1, u_2, \ldots, u_N) \ \text{ of elements of } \ Z(\mathcal{T}_{\mathbf{q}})^{\geq 1}$$

such that $\phi(Y_i) = (1+u_i)Y_i$ for all $i \in [1, N]$, where $Z(\mathcal{T}_{\mathbf{q}})^{\geq 1} := Z(\mathcal{T}_{\mathbf{q}}) \cap \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq 1}$.

2.2. Returning to the algebras of quantum matrices $R_q[M_n]$, fix a positive integer n and denote for brevity

$$N := n^2. \tag{2.1}$$

Define $X_1, \ldots, X_N \in R_q[M_n]$ by

$$X_{k+(l-1)n} := x_{kl} \ \text{ for } \ k, l \in [1, n].$$

The form of the commutation relations for x_{kl} produces an iterated Ore extension presentation for $R_q[M_n]$

$$R_q[M_n] = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \ldots [X_N; \sigma_N, \delta_N]. \tag{2.2}$$

Denote by R_j the j-th algebra in the chain, i.e., the subalgebra of $R_q[M_n]$ generated by $X_1, \ldots, X_j$. For all $j \in [1, N]$, σ_j is an automorphism of R_{j-1} such that

$$\sigma_j(X_i) = q^{a_{ji}} X_i, \ \ \forall i < j \tag{2.3}$$

for some integers a_{ji}, the explicit form of which will not play any role. Moreover, δ_j is a locally nilpotent σ_j-derivation of R_{j-1} and $\sigma_j \delta_j = q^{-2} \delta_j \sigma_j$, $\forall\, 2 \leq j \leq N$. Cauchon's procedure of deleting derivations [3] constructs a sequence of N-tuples $(X_1^{(k)}, \ldots, X_N^{(k)})$ of the division ring of fractions $\operatorname{Fract}(R_q[M_n])$ for $k = N+1, \ldots, 2$. The first one is given by

$$(X_1^{(N+1)}, \ldots, X_N^{(N+1)}) := (X_1, \ldots, X_N)$$

and the others are constructed inductively by

$$X_j^{(k)} := \begin{cases} X_j^{(k+1)}, & \text{if } \ j \geq k \\ \sum_{r=0}^{\infty} \frac{(1-q^{-2})^{-r}}{[r]_{q^{-2}}!} \left[\delta_k^r \sigma_k^{-r} \left(x_j^{(k+1)} \right) \right] \left(x_j^{(j+1)} \right)^{-r}, & \text{if } \ j < k \end{cases} \tag{2.4}$$

for $k = N, \ldots, 2$ (in terms of the standard notation for q-integers and factorials $[0]_q = 1$, $[r]_q = (1-q^r)/(1-q)$ for $r > 0$, and $[r]_q! = [0]_q \ldots [r]_q$ for $r \geq 0$). Denote the final N-tuple

$$\overline{X}_1 := X_1^{(2)}, \ldots, \overline{X}_N := X_N^{(2)} \in \operatorname{Fract}(R_q[M_n]) \tag{2.5}$$

and the subalgebra

$$\mathcal{T} = \langle \overline{X}_1^{\pm 1}, \ldots, \overline{X}_N^{\pm 1} \rangle \subset \operatorname{Fract}(R_q[M_n]).$$

Define the multiplicatively skew-symmetric matrix

$$\mathbf{q} = (q_{ij}) \in M_N(\mathbb{K}^*) \ \text{ such that } \ q_{ji} = q^{a_{ji}}, \ \ \forall\, 1 \leq i < j \leq N \tag{2.6}$$

for the integers a_{ji} defined in (2.3). Cauchon proved in [3] the $\mathbb{K}$-algebra isomorphism

$$\mathcal{T}_{\mathbf{q}} \cong \mathcal{T}, \ \text{ where } \ Y_j \mapsto \overline{X}_j, \ 1 \leq j \leq N. \tag{2.7}$$

Define the map

$$\mu \colon [1, N] \to [1, 2n-1] \ \text{ by } \ \mu(k + (l-1)n) := n + l - k, \ \ \forall k, l \in [1, n]. \tag{2.8}$$

The meaning of this map is as follows. The algebra of quantum matrices $R_q[M_n]$ is isomorphic to the quantum Schubert cell subalgebras $\mathcal{U}_q^{\pm}[w]$ of $\mathcal{U}_q(\mathfrak{sl}_{2n})$ for the following choice of Weyl group element

$$w = (s_n \ldots s_1) \ldots (s_{2n-1} \ldots s_n) \in S_{2n} \tag{2.9}$$

which is in reduced form. Here $s_1, \dots, s_{2n-1}$ denote the simple reflections of the symmetric group S_{2n}. We refer the reader to [9, §2.1] and [10, Section 4] for details. Within this framework the generators $X_1, \dots, X_N$ of $R_q[M_n]$ are matched with the simple reflections in the expression (2.9) read from left to right. The map μ simply encodes this matching. Define the following successor function $s\colon [1,N] \to [1,N] \cup \{\infty\}$ attached to μ:

$$s(i) = \min\{j > i \mid \text{such that } \ \mu(j) = \mu(i)\} \ \text{ if such a } j \text{ exits, otherwise } \ s(i) = \infty.$$

Given two subsets $I = \{k_1 < \ldots < k_r\}$ and $J = \{l_1 < \ldots < l_r\}$ of $[1,n]$ having the same cardinality, define the quantum minor

$$[I|J] = \sum_{w \in S_r} (-q)^{\ell(w)} x_{k_1 l_{w(1)}} \ldots x_{k_r l_{w(r)}} \in R_q[M_n].$$

We will use the following special quantum minors $\Delta_1, \dots, \Delta_N \in R_q[M_n]$ (which combinatorially are the ones that contain consecutive rows and columns, and either touch the right side or the bottom of an $n \times n$ matrix). Given $i \in [1,N]$, we represent it as $i = k + (l-1)n$ for some $k, l \in [1,n]$ and define

$$\Delta_i = \begin{cases} [\{k, \dots, n\}|\{l, \dots, n+l-k\}], & \text{if } \ k \geq l \\ [\{k, \dots, n-l+k\}|\{l, \dots, n\}], & \text{if } \ k < l. \end{cases}$$

The following theorem is due to Cauchon [4, Proposition 5.2.1]. The above setting and the statement of this result follows the framework of [5, Theorem 1.3] which establishes such a fact for all quantum Schubert cell algebras. The theorem below also follows from it and [10, Lemma 4.3].

Theorem 2.2. *For all $i \in [1,N]$ (recall (2.1)) the final N-tuple $(\overline{X}_1, \dots, \overline{X}_N)$ of Cauchon's elements of* $\mathrm{Fract}(R_q[M_n])$ *is given by*

$$\overline{X}_i = \begin{cases} \Delta_i \Delta_{s(i)}^{-1}, & \text{if } \ s(i) \neq \infty \\ \Delta_i, & \text{if } \ s(i) = \infty. \end{cases} \tag{2.10}$$

3. Automorphisms of square quantum matrices

3.1. In this section we prove Conjecture 1.2 which implies the validity of the Launois–Lenagan conjecture. The proof of the following theorem is given in §3.3.

Theorem 3.1. *Let $\mathbb{K}$ be an arbitrary base field, $q \in \mathbb{K}^*$ an element which is not a root of unity, and $n \in \mathbb{Z}_+$. Every unipotent automorphism of $R_q[M_n]$ equals the identity automorphism.*

For $n \in \mathbb{Z}_+$, we have the group embedding

$$\eta\colon \mathcal{H} \rtimes \mathbb{Z}_2 \hookrightarrow \mathrm{Aut}\, R_q[M_n], \ \ \eta(h, \overline{k})(u) = h \cdot (\tau^k(u)), \ \ h \in \mathcal{H}, k = 0, 1$$

in terms of the notation from the introduction. Launois and Lenagan proved in [7, Proposition 1.9] that every automorphism of $R_q[M_n]$ is a composition of an automorphism in the image of η and a unipotent automorphism. Hence Theorem 3.1 implies the validity of Conjecture 1.1.

Theorem 3.2. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and integers $n > 1$, the map $\eta\colon \mathbb{K} \rtimes \mathbb{Z}_2 \to \mathrm{Aut}\, R_q[M_n]$ is a group isomorphism.*

For completeness, we note that $R_q[M_1] = \mathbb{K}[x_{11}]$ and $\mathrm{Aut}\, R_q[M_n] \cong \mathbb{K} \rtimes \mathbb{K}^*$.

3.2. Recall the definition (2.7) of the quantum torus $\mathcal{T}$ and the notation $N := n^2$. Eq. (2.4) implies that the $\mathbb{N}$-grading of $R_q[M_n]$ from §2.1 extends to a $\mathbb{Z}$-grading of $\mathcal{T}$ by defining $\deg \overline{X}_i^{\pm 1} = \pm 1$ for all $i \in [1, N]$. Furthermore, for all intermediate steps of the Cauchon procedure

$$\deg X_i^{(3)} = \ldots = \deg X_i^{(N)} = \deg X_i = 1, \ \ \forall i \in [1, N].$$

Denote the corresponding graded subspaces of $\mathcal{T}$ by $\mathcal{T}^r$, $r \in \mathbb{Z}$. Theorem 2.2 implies that $\{\Delta_1, \ldots, \Delta_N\}$ is another set of independent generators of $\mathcal{T}$. For this set of generators of $\mathcal{T}$ consider the degree vector $\mathbf{d} = (d_1, \ldots, d_N)$ where d_i is the size of the quantum minor Δ_i. If $i = k + (l-1)n \in [1, N]$ for $k, l \in [1, n]$, then

$$d_i = n + 1 - \max\{k, l\}. \tag{3.1}$$

By Theorem 2.2 the above $\mathbb{Z}$-grading of $\mathcal{T}$ coincides with the $\mathbb{Z}$-grading from §2.1 associated to this degree vector $\mathbf{d}$. Denote by $\widehat{\mathcal{T}}$ the corresponding completion as defined in §2.1. Let $\widehat{\mathcal{T}}^{\geq r} := \{u_r + u_{r+1} + \ldots \mid u_k \in \mathcal{T}^k \ \text{ for } \ k \geq r\}$, $r \in \mathbb{Z}$. For every subalgebra R of $\widehat{\mathcal{T}}$ and $r \in \mathbb{Z}$ denote $R^{\geq r} := R \cap \widehat{\mathcal{T}}^{\geq r}$ and $R^r := R \cap \mathcal{T}^r$.

It follows from Theorem 2.2 that for all $i \in [1, N]$

$$\Delta_i = \overline{X}_i \ldots \overline{X}_{s^{n(i)}(i)},$$

where $n(i)$ is the largest natural number such that $s^{n(i)}(i) \neq \infty$. In particular, the generators $\Delta_1, \ldots, \Delta_N$ of $\mathcal{T}$ satisfy the relations

$$\Delta_i \Delta_j = q'_{ij} \Delta_j \Delta_i,$$

where

$$q'_{ij} := \prod_{k=0}^{n(i)} \prod_{l=0}^{n(j)} q_{s^k(i), s^l(j)}.$$

Analogously to [12, §4.2] we define an injective map from the group of unipotent automorphisms of $R_q[M_n]$ to the set of bi-finite unipotent automorphisms of $\widehat{\mathcal{T}}$. Here is the general form of this correspondence:

Proposition 3.3. *Assume that $\mathcal{T}_{\mathbf{q}}$ is a quantum torus with generators $Y_1, \ldots, Y_N$ as in §2.1 and $\mathbf{d} \in \mathbb{Z}_+^N$ a degree vector. Let R be a connected $\mathbb{N}$-graded subalgebra of $\mathcal{T}_{\mathbf{q}}$ (equipped with the $\mathbb{Z}$-grading associated to $\mathbf{d}$) such that $Y_i \in R$, $\forall i \in [1, N]$. For each automorphisms Φ of R satisfying*

$$\Phi(u) - u \in R^{\geq r+1}, \ \ \forall u \in R^r, r \in \mathbb{N}, \tag{3.2}$$

there exists a unique bi-finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ such that $\phi(Y_i) = \Phi(Y_i)$, $\forall i \in [1, N]$. The assignment $\Phi \mapsto \phi$ defines an injective homomorphism from the group of automorphisms Φ of R satisfying (3.2) into the group of unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. The image of this homomorphism is contained in the set of bi-finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$.

Proof. Let $\mathbf{q} = (q_{ij})$ and $\mathbf{d} = (d_1, \ldots, d_N)$. We have $\Phi(Y_i) = Y_i + u'_i$, where $u'_i \in R^{\geq d_i + 1} \subset \mathcal{T}_{\mathbf{q}}^{\geq d_i+1}$, $i \in [1, N]$. Write $u'_i = u_i Y_i$ for some $u_i \in \mathcal{T}_{\mathbf{q}}^{\geq 1}$. Then $\Phi(Y_i) = (1 + u_i) Y_i$ and $(1 + u_i) Y_i (1 + u_j) Y_j = q_{ij} (1 + u_j) Y_j (1 + u_i) Y_i$, $\forall i, j \in [1, N]$, because Φ is an automorphism of R. By [12, Lemma 3.4] there exists a unique unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ given by

$$\phi(Y_i) := (1 + u_i) Y_i = \Phi(Y_i). \tag{3.3}$$

Since $\mathcal{T}_{\mathbf{q}}$ is a domain, $\phi|_R = \Phi$. Denote by ψ the unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ associated by this construction to Φ^{-1}. We have $(\psi\phi)|_R = (\phi\psi)|_R = \mathrm{id}$. Hence $\psi = \phi^{-1}$ because $Y_1, \ldots, Y_N \in R$. This and Eq. (3.3) imply that ϕ is a bi-finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. The map $\Phi \mapsto \phi$ is injective since $\Phi = \phi|_R$.

Assume that Φ_1 and Φ_2 are two automorphisms of R satisfying (3.2). Let ϕ_1 and ϕ_2 be the corresponding bi-finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. Then $\phi_1\phi_2(Y_i) = \phi_1\Phi_2(Y_i) = \Phi_1\Phi_2(Y_i) \in R^{\geq d_i+1} \subset \mathcal{T}_{\mathbf{q}}^{\geq d_i+1}$, $\forall i \in [1,N]$ because $\phi_1|_R = \Phi_1$ and $\Phi_2(Y_i) \in R$. Therefore $\phi_1\phi_2$ is also a bi-finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and is precisely the automorphism associated to $\Phi_1\Phi_2$. □

We apply Proposition 3.3 to the quantum torus $\mathcal{T}$ with generating set $\{\Delta_1, \ldots, \Delta_N\}$ and the degree vector $\mathbf{d}$ from Eq. (3.1). The algebra $R_q[M_n]$ satisfies the conditions of the lemma since $\Delta_i \in R_q[M_n]$ and the $\mathbb{N}$-grading of $R_q[M_n]$ is the restriction of the $\mathbb{Z}$-grading of $\mathcal{T}$. Each unipotent automorphism Φ of $R_q[M_n]$ satisfies

$$\Phi(u) - u \in R_q[M_n]^{\geq r+1}, \ \ \forall u \in R_q[M_n]^r, r \in \mathbb{N}$$

because $R_q[M_n]$ is generated by $X_1, \ldots, X_N$. Thus, to each unipotent automorphism Φ of $R_q[M_n]$ Proposition 3.3 associates a bi-finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}$ such that $\phi(\Delta_i) = \Phi(\Delta_i)$ for all $i \in [1,n]$.

Proposition 3.4. *Let Φ be a unipotent automorphism of $R_q[M_n]$, where $q \in \mathbb{K}^*$ is not a root of unity. Then there exist $u_1, \ldots, u_N \in Z(\mathcal{T})^{\geq 1}$ such that*

$$\Phi(\Delta_i) = (1 + u_i)\Delta_i, \ \ \forall i \in [1,N].$$

Proof. Recall from [12, §3.1] that the property of a quantum torus being saturated does not depend on the choice of set of generators. The entries q_{ij} of the the multiplicatively skew-symmetric matrix $\mathbf{q}$ given by (2.6) are powers of q. They generate a torsion-free subgroup of $\mathbb{K}^*$ because q is not a root of unity. Thus $\mathcal{T}_{\mathbf{q}}$ is saturated, and due to the isomorphism (2.7), $\mathcal{T}$ is also saturated.

Consider the bi-finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}$ associated to Φ by the above construction. Applying Theorem 2.1, we obtain

$$\phi(\Delta_i) = (1 + u_i)\Delta_i \ \text{ for some } \ u_i \in Z(\mathcal{T})^{\geq 1}, \ \ \forall i \in [1,N].$$

Taking into account that $\Phi(\Delta_i) = \phi(\Delta_i)$ implies the statement of the proposition. □

3.3. Recall the definition of the surjective map $\mu \colon [1,N] \to [1,2n-1]$ from Eq. (2.8). The height one $\mathcal{H}$-prime ideals of $R_q[M_n]$ are encoded in it as follows. For $i \in [1,2n-1]$ denote

$$f(i) := \min \mu^{-1}(i).$$

In other words $f(i)$ records the first element of the preimage $\mu^{-1}(i)$. Launois, Lenagan, and Rigal proved [8, Proposition 4.2] that, in this framework, the height one $\mathcal{H}$-prime ideals of $R_q[M_n]$ are

$$R_q[M_n]\Delta_{f(i)} \ \text{ for } \ i \in [1,2n-1]. \tag{3.4}$$

(In particular, all $\Delta_{f(i)}$ are normal elements of $R_q[M_n]$.) This is again a general property of all quantum Schubert cell algebras: all of their height one torus-invariant prime ideals are generated by a quantum minor which is a normal element [11, Proposition 6.9]. Launois and Lenagan proved [7, Proposition 1.10] that every unipotent automorphism Φ of $R_q[M_n]$ fixes the height one prime ideals (3.4), i.e., fixes the elements $\Delta_{f(i)}$,

$$\Phi(\Delta_{f(i)}) = \Delta_{f(i)}, \ \ \forall i \in [1,2n-1]. \tag{3.5}$$

Proof of Theorem 3.1. The idea of the proof is similar to the proof of [12, Proposition 3.14].

Let Φ be a unipotent automorphism of $R_q[M_n]$ and ϕ the corresponding bi-finite unipotent automorphism of $\widehat{\mathcal{T}}$. Proposition 3.3 implies that

$$\Phi(\Delta_i) = (1 + u_i)\Delta_i \ \text{ for some } \ u_i \in Z(\mathcal{T})^{\geq 1}, \ \ \forall i \in [1,N].$$

The center of $\mathcal{T}$ was described by Launois and Lenagan in [6, Theorem 3.4]:

$$Z(\mathcal{T}) = \mathbb{K}[(\Delta_{f(1)}\Delta_{f(2n-1)}^{-1})^{\pm 1}, \ldots, (\Delta_{f(n-1)}\Delta_{f(n+1)}^{-1})^{\pm 1}, \Delta_{f(n)}^{\pm 1}].$$

Eq. (3.5) and the fact that $\phi(\Delta_i) = \Phi(\Delta_i)$, $\forall i \in [1, N]$ imply

$$\phi(u) = u, \forall u \in Z(\mathcal{T}). \tag{3.6}$$

Denote by ψ the bi-finite unipotent automorphism of $\widehat{\mathcal{T}}$ corresponding to Φ^{-1}. By Propositions 3.3 and 3.4, $\psi = \phi^{-1}$ and

$$\psi(\Delta_i) = \Phi^{-1}(\Delta_i) = (1 + v_i)\Delta_i \text{ for some } v_i \in Z(\mathcal{T})^{\geq 1}, \ \forall i \in [1, N].$$

Applying Eq. (3.6) we obtain

$$\Delta_i = \phi\psi(\Delta_i) = \phi((1 + v_i)\Delta_i) = \phi(1 + v_i)\phi(\Delta_i) = (1 + v_i)(1 + u_i)\Delta_i, \ \forall i \in [1, N].$$

Because $\mathcal{T}$ is a domain, we have

$$(1 + v_i)(1 + u_i) = 1, \ \forall i \in [1, N]. \tag{3.7}$$

The condition $u_i, v_i \in Z(\mathcal{T})^{\geq 1}$ implies $u_i = v_i = 0$ for all $i \in [1, N]$. Indeed, if the highest terms of $1 + u_i$ and $1 + v_i$ are in degrees $r_i, t_i \in \mathbb{N}$, then the highest term of the left hand side of (3.7) is in degree $r_i + t_i$ because $\mathcal{T}$ is a domain. The right hand side of (3.7) is in degree 0. Hence $r_i = t_i = 0$ and all elements u_i and v_i vanish. Therefore $\phi = \mathrm{id}$ and $\Phi = \mathrm{id}$. □

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