SPECTRA AND CATENARITY OF MULTIPARAMETER QUANTUM SCHUBERT CELLS

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Dedicated to Kenny Brown and Toby Stafford on the occasion of their 60th birthdays

Abstract. We study the ring theory of the multiparameter deformations of the quantum Schubert cell algebras obtained from 2-cocycle twists. This is a large family, which extends the Artin–Schelter–Tate algebras of twisted quantum matrices. We classify set theoretically the spectra of all such multiparameter quantum Schubert cell algebras, construct each of their prime ideals by contracting from explicit normal localizations, and prove formulas for the dimensions of their Goodearl–Letzter strata for base fields of arbitrary characteristic and all deformation parameters that are not roots of unity. Furthermore, we prove that the spectra of these algebras are normally separated and that all such algebras are catenary.

1. Introduction

The quantum Schubert cell algebras (or quantum nilpotent algebras) form a large family of iterated Ore extensions defined by De Concini, Kac, and Procesi [9], and Lusztig [30]. They are subalgebras of the negative part of a quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ defined over an arbitrary base field $\mathbb{K}$ and are indexed by the elements of the Weyl group W of $\mathfrak{g}$. The algebra corresponding to $w \in W$, to be denoted by $\mathcal{U}_-^w$, is a deformation of the universal enveloping algebra of the nilpotent Lie algebra $\mathfrak{n}_- \cap w(\mathfrak{n}_+)$, where $\mathfrak{n}_\pm$ are the niradicals of a pair of opposite Borel subalgebras of $\mathfrak{g}$. Furthermore, the algebras $\mathcal{U}_-^w$ are isomorphic [38] to certain localization of quotients of homogeneous coordinate rings of quantum partial flag varieties, similarly to the classical case of Schubert cells. They specialize [39] to the coordinate rings of Schubert cells equipped with the standard Poisson structure [5, 19].

There has been a great interest in the ring theoretic study of these algebras in an attempt to develop a somewhat general theory for iterated Ore extensions. The algebra $\mathcal{U}_-^w$ admits a natural rational action of the torus $\mathbb{T}^{|\mathcal{S}(w)|} = (\mathbb{K}^*)^{\times|\mathcal{S}(w)|}$ by algebra automorphisms, where $\mathcal{S}(w)$ is the support of w, see §2.4. The $\mathbb{T}^{|\mathcal{S}(w)|}$-prime spectrum of $\mathcal{U}_-^w$ was described in [33, 37], and the dimensions of the corresponding Goodearl–Letzter strata were computed in [3, 39]. The Goodearl–Lenagan conjecture [16] on polynormality of the torus invariant prime ideals of quantum matrices and more generally the algebras $\mathcal{U}_-^w$ was proved in [40], where catenarity and normal separation of the spectra of $\mathcal{U}_-^w$ was established too. The algebras $\mathcal{U}_-^w$ play an important role in various other contexts: Heckenberger–Schneider classified [22] the homogeneous coideal subalgebras of quantized universal enveloping algebras of Borel subalgebras in terms of $\mathcal{U}_-^w$ and Geiß–Leclerc–Schröer proved [12] that $\mathcal{U}_-^w$ are quantum cluster algebras for simply laced $\mathfrak{g}$, char $\mathbb{K} = 0$, and $q \in \mathbb{K}$ transcendental over $\mathbb{Q}$.

Artin, Schelter, and Tate proved [1] that multiparameter versions of the algebras of quantum matrices and the quantum linear groups can be obtained from the single parameter case by 2-cocycle twists. After this a number of authors investigated the effect of such twists on the spectra of graded algebras. The spectra of multiparameter quantum groups was studied by Hodges–Levasseur–Toro [21] and Costantini–Varagnolo [10]. The spectra of special cases of

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multiparameter quantum Schubert cell algebras were investigated by many authors: [15, 23, 35, 36], to name a few. Nowlin and Johnson [34, 25] proved that certain interesting classes of algebras defined in a completely independent way turn out to be isomorphic to special twists of quantum Schubert cell algebras for affine Kac–Moody algebras.

In this paper we carry out a general study of the ring theory of twisted quantum Schubert cell algebras when q is not a root of unity (without any assumptions on the base field $\mathbb{K}$). The algebra $\mathcal{U}_-^w$ is graded by the subgroup $\mathcal{Q}_{\mathcal{S}(w)}$ of the root lattice of $\mathfrak{g}$, generated by the simple roots in the support $\mathcal{S}(w)$ of $w \in W$, see §2.4. For $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)$ denote by $\mathcal{U}_{-,\mathbf{p}}^w$ the algebra obtained by a cocycle twist from $\mathcal{U}_-^w$ using $\mathbf{p}$. Firstly, we give an explicit classification of $\mathrm{Spec}\mathcal{U}_{-,\mathbf{p}}^w$. This is stated in Theorem 3.1. As in [38, 40] we use results of Joseph [26, 27] and Gorelik [20]. Most of those results were formulated in [27, 20] for $\mathbb{K} = k(q)$, $\mathrm{char}\, k = 0$. We show that the proofs of all such needed results work for an arbitrary $q \in \mathbb{K}^*$ which is not a root of unity and without restrictions on the characteristic of $\mathbb{K}$. At some steps we take shortcuts using results of Goodearl and Letzter [18], [4, §II.6]. Furthermore, Theorem 3.1 expresses each prime ideal of $\mathcal{U}_{-,\mathbf{p}}^w$ as a contraction from an explicit normal localization of a quotient of $\mathcal{U}_{-,\mathbf{p}}^w$ by a $\mathbb{T}^{|\mathcal{S}(w)|}$-prime. These localizations are smaller than the ones obtained via the Cauchon method of deleting derivations [7, 33].

The spectra $\mathrm{Spec}\mathcal{U}_{-,\mathbf{p}}^w$ are partitioned [18] into a union of Goodearl–Letzter strata, which are isomorphic to the spectra of Laurent polynomial rings. We prove an explicit formula for the dimensions of the latter in Theorem 4.5, which works for all 2-cocycles $\mathbf{p}$, $q \in \mathbb{K}^*$ not a root of unity, and arbitrary base fields $\mathbb{K}$. In the special one-parameter case this formula was obtained by Bell, Casteels, and Launois [3], and the author [39] when $\mathrm{char}\,\mathbb{K} = 0$ and q is transcendental over $\mathbb{Q}$. We give a (very short!) new proof of the one-parameter case in Proposition 4.1.

Furthermore, using results of [40] we prove in an explicit way that the $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant prime ideals of the algebras $\mathcal{U}_{-,\mathbf{p}}^w$ are equivariantly polynormal. In the special case of multiparameter quantum matrices this gives a constructive proof of a conjecture of Brown and Goodearl [4, Conjecture II.10.9]. Moreover, we show that $\mathrm{Spec}\mathcal{U}_{-,\mathbf{p}}^w$ are normally separated and that all algebras $\mathcal{U}_{-,\mathbf{p}}^w$ are catenary. This provides a very large class of iterated Ore extensions for which Gabber's theorem on catenarity of universal enveloping algebras of solvable Lie algebras can be extended. In a related direction, in a forthcoming publication we will prove another conjecture of Brown and Goodearl [4, Conjecture II.10.7] that all prime ideals of multiparameter quantum groups are completely prime under mild conditions of the cocycle twist.

2. Quantum groups, quantum Schubert cells, and their twists

2.1. Denote $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, \dots\}$. For $m \leq n \in \mathbb{Z}$ set $[m, n] = \{m, \dots, n\}$. Throughout the paper $\mathbb{K}$ will denote a base field (of arbitrary characteristic) and $q \in \mathbb{K}^*$ will denote an element which is not a root of unity. We fix a simple Lie algebra $\mathfrak{g}$ of rank r with Cartan matrix (c_{ij}). Let $\mathcal{U}_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{K}$ with deformation parameter q. Recall [24] that $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{K}$-algebra with generators

$$X_i^{\pm}, K_i^{\pm 1}, \ \ i \in [1, r]$$

and relations

$$K_i^{-1} K_i = K_i K_i^{-1} = 1, \, K_i K_j = K_j K_i, \, K_i X_j^{\pm} K_i^{-1} = q_i^{\pm c_{ij}} X_j^{\pm},$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1 - c_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0, \, i \neq j,$$

where $q_i = q^{d_i}$ and $\{d_i\}_{i=1}^r$ is the collection of positive relatively prime integers such that $(d_i c_{ij})$ is symmetric. Moreover $\mathcal{U}_q(\mathfrak{g})$ is a Hopf algebra with comultiplication, antipode and counit given by

$$\Delta(K_i) = K_i \otimes K_i,\ \Delta(X_i^+) = X_i^+ \otimes 1 + K_i \otimes X_i^+,\ \Delta(X_i^-) = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-$$

and

$$S(K_i) = K_i^{-1},\ S(X_i^+) = -K_i^{-1}X_i^+,\ S(X_i^-) = -X_i^- K_i,\ \epsilon(K_i) = 1,\ \epsilon(X_i^{\pm}) = 0.$$

The subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_1^{\pm}, \ldots, X_r^{\pm}\}$ will be denoted by $\mathcal{U}_{\pm}$. The sets of simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$ will be denoted by $\{\alpha_i\}_{i=1}^r$, $\{\alpha_i^\vee\}_{i=1}^r$, and $\{\omega_i\}_{i=1}^r$. The root and weight lattices of $\mathfrak{g}$ will be denoted by $\mathcal{Q}$ and $\mathcal{P}$. Let $\mathcal{Q}^+ = \mathbb{N}\alpha_1 + \ldots + \mathbb{N}\alpha_r$, $\mathcal{P}^+ = \mathbb{N}\omega_1 + \ldots + \mathbb{N}\omega_r$, and $\mathcal{P}^{++} = \mathbb{Z}_+\omega_1 + \ldots + \mathbb{Z}_+\omega_r$. Recall the standard partial order on $\mathcal{P}$: for $\nu_1, \nu_2 \in \mathcal{P}$ set $\nu_1 \geq \nu_2$ if $\nu_2 = \nu_1 - \gamma$ for some $\gamma \in \mathcal{Q}^+$. Let $\nu_1 > \nu_2$ if $\nu_1 \geq \nu_2$ and $\nu_1 \neq \nu_2$. If $\lambda = \sum_{i \in I} n_i \omega_i \in \mathcal{P}^+$, $n_i > 0$, $\forall i \in I$ we will say that the support of λ is I. Denote by $\langle ., . \rangle$ the symmetric bilinear form on $\mathrm{Span}_{\mathbb{Q}}\{\alpha_i\}_{i=1}^r$ such that $\langle \alpha_i, \alpha_j \rangle = d_i c_{ij}$, $\forall i, j \in [1, r]$. The q-weight spaces of a $\mathcal{U}_q(\mathfrak{g})$-module V are defined by

$$V_\nu = \{v \in V \mid K_i v = q^{\langle \nu, \alpha_i \rangle} v,\ \forall i \in [1, r]\},\ \nu \in \mathcal{P}.$$

A type one $\mathcal{U}_q(\mathfrak{g})$-module is a $\mathcal{U}_q(\mathfrak{g})$-module such that $V = \oplus_{\nu \in \mathcal{P}} V_\nu$. The category of (left) finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules is semisimple (see [24, Theorem 5.17] and the remark on p. 85 of [24] for the validity of this for general base fields $\mathbb{K}$ and $q \in \mathbb{K}^*$ not a root of unity). Furthermore, this category is closed under taking tensor products and duals, where the latter are defined as left $\mathcal{U}_q(\mathfrak{g})$-modules using the antipode of $\mathcal{U}_q(\mathfrak{g})$. The irreducible modules in this category are parametrized [24, Theorem 5.10] by the set of dominant integral weights $\mathcal{P}^+$. Let $V(\lambda)$ denote the irreducible type one $\mathcal{U}_q(\mathfrak{g})$-module of highest weight $\lambda \in \mathcal{P}^+$.

Let W and $\mathcal{B}_{\mathfrak{g}}$ be the Weyl and braid groups of $\mathfrak{g}$, and let $s_1, \ldots, s_r$ and $T_1, \ldots, T_r$ be their standard generating sets corresponding to the simple roots $\alpha_1, \ldots, \alpha_r$. Denote by $\ell \colon W \to \mathbb{N}$ the standard length function. The braid group $\mathcal{B}_{\mathfrak{g}}$ acts on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms by [24, eqs. 8.14 (2), (3), (7), and (8)] and on the finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules by [24, eq. 8.6 (2)]. These actions are compatible: $T_w(x.v) = (T_w x).(T_w v)$ for all $w \in W$, $x \in \mathcal{U}_q(\mathfrak{g})$, $v \in V(\lambda)$, $\lambda \in \mathcal{P}^+$.

2.2. If $\mathbb{K}$ is an algebraically closed field of characteristic 0, we will denote by G the connected simply connected algebraic group with Lie algebra $\mathfrak{g}$. For all base fields $\mathbb{K}$ and deformation parameters $q \in \mathbb{K}^*$ which are not roots of unity, one defines the quantum group $R_q[G]$ as the Hopf subalgebra of the restricted dual $(\mathcal{U}_q(\mathfrak{g}))^\circ$ equal to the span of the matrix coefficients of the modules $V(\lambda)$, $\lambda \in \mathcal{P}^+$:

$$c^\lambda_{\xi, v} \in (\mathcal{U}_q(\mathfrak{g}))^\circ, \quad c^\lambda_{\xi, v}(x) = \xi(xv), \quad v \in V(\lambda), \xi \in V(\lambda)^*, x \in \mathcal{U}_q(\mathfrak{g}). \tag{2.1}$$

Since we work with arbitrary base fields, G is merely a symbol. The canonical left and right actions of $\mathcal{U}_q(\mathfrak{g})$ on $\mathcal{U}_q(\mathfrak{g})^\circ$

$$x \rightharpoonup c = \sum c_{(2)}(x) c_{(1)},\ c \leftharpoonup x = \sum c_{(1)}(x) c_{(2)},\ x \in \mathcal{U}_q(\mathfrak{g}), c \in R_q[G] \tag{2.2}$$

preserve $R_q[G]$, where $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.

For each $\lambda \in \mathcal{P}^+$ we fix a highest weight vector v_λ of $V(\lambda)$ and denote

$$c^\lambda_\xi = c^\lambda_{\xi, v_\lambda} \in R_q[G].$$

The subspace

$$R^+ = \mathrm{Span}\{c^\lambda_\xi \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*\} \subset R_q[G]$$

is a subalgebra of $R_q[G]$. We will need the R-matrix commutation relations in R^+. Denote the canonical $\mathcal{Q}$-grading of $\mathcal{U}_q(\mathfrak{g})$:

$$\mathrm{wt}\, X_i^{\pm} = \pm\alpha_i,\ \mathrm{wt}\, K_i = 0, \quad i \in [1, r]. \tag{2.3}$$

For $\gamma \in \mathcal{Q}^+$, $\gamma \neq 0$ set $m(\gamma) = \dim(\mathcal{U}_+)_\gamma = \dim(\mathcal{U}_-)_{-\gamma}$. Denote by $\{u_{\gamma,i}\}_{i=1}^{m(\gamma)}$ and $\{u_{-\gamma,i}\}_{i=1}^{m(\gamma)}$ a pair of dual bases of $(\mathcal{U}_+)_\gamma$ and $(\mathcal{U}_-)_{-\gamma}$ with respect to the Rosso–Tanisaki form (see [24, Ch. 6] for a discussion of the properties of this form for arbitrary base fields $\mathbb{K}$). Then we have:

$$c^{\lambda_1}_{\xi_1} c^{\lambda_2}_{\xi_2} = q^{\langle \lambda_1, \lambda_2 \rangle - \langle \nu_1, \nu_2 \rangle} c^{\lambda_2}_{\xi_2} c^{\lambda_1}_{\xi_1} + \sum_{\gamma \in \mathcal{Q}^+, \gamma \neq 0} \sum_{i=1}^{m(\gamma)} q^{\langle \lambda_1, \lambda_2 \rangle - \langle \nu_1 - \gamma, \nu_2 + \gamma \rangle} c^{\lambda_2}_{S^{-1}(u_{\gamma,i})\xi_2} c^{\lambda_1}_{S^{-1}(u_{-\gamma,i})\xi_1}, \tag{2.4}$$

for all $\lambda_i \in \mathcal{P}^+$, $\nu_i \in \mathcal{P}$, and $\xi_i \in (V(\lambda_i)^*)_{\nu_i}$, see e.g. [4, Theorem I.8.15].

For $\lambda \in \mathcal{P}^+$ and $w \in W$ let $\xi_{w,\lambda} \in (V(\lambda)^*)_{-w\lambda}$ be such that $\langle \xi_{w,\lambda}, T_w v_\lambda \rangle = 1$. (Since $T_w(V(\lambda)_\lambda) = V(\lambda)_{w\lambda}$, $\dim V(\lambda)_{w\lambda} = 1$.) Define

$$e^\lambda_w = c^\lambda_{\xi_{w,\lambda}}, \quad \lambda \in \mathcal{P}^+, w \in W. \tag{2.5}$$

Then

$$e^{\lambda_1}_w e^{\lambda_2}_w = e^{\lambda_1+\lambda_2}_w = e^{\lambda_2}_w e^{\lambda_1}_w, \quad \forall \lambda_1, \lambda_2 \in \mathcal{P}^+, w \in W, \tag{2.6}$$

see [40, eq. (2.18)]. Denote the multiplicative subsets $E_w = \{e^\lambda_w \mid \lambda \in \mathcal{P}^+\} \subset R^+$.

Lemma 2.1. *(Joseph,* [27, Lemma 9.1.10]*) For all $w \in W$, E_w is an Ore subset in R^+*

In [27] this result was stated for fields $\mathbb{K}$ of characteristic 0 (see [27, §A.2.9]), but Joseph's proof works for all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity as we see below. By (2.4) the set E_{w_0} consists of normal elements of R^+. Therefore $\{e^n \mid n \in \mathbb{N}\}$ is an Ore subset of R^+ for all $e \in E_{w_0}$, in particular E_{w_0} is an Ore subset of R^+. Joseph proved iteratively that

$$\{e^n \mid n \in \mathbb{N}\} \text{ are Ore subsets of } R^+ \text{ for all } e \in E_w, w \in W \tag{2.7}$$

using the following procedure. Let σ be an automorphism of a ring A and ∂ be a locally nilpotent (right skew) σ-derivation of A (i.e. for all $a, b \in A$, $\partial(ab) = \partial(a)\sigma(b) + a\partial(b)$) such that $\sigma\partial\sigma^{-1}$ is a scalar multiple of ∂. The degree $\deg_\partial a$ of an element $a \in A \backslash \{0\}$ is defined as the minimal positive integer m such that $\partial^{m+1}(a) = 0$. Such a skew derivation ∂ is called right regular if for all σ-eigenvectors $a, b \in A$ of ∂-degree m and n, respectively, and $k \in [n, m+n]$

$$\partial^k(ab) = \sum_{i=0}^{m+n-k} s_i (\partial^{k-n+i} a)(\partial^{n-i} b)$$

for some $s_0, \ldots, s_{m+n-k} \in \mathbb{K}$, $s_0 \neq 0$. It is straightforward to check that, if $\sigma\partial\sigma^{-1} = q'\partial$ for some $q' \in \mathbb{K}^*$ which is not a root of unity, then ∂ is right regular, since in that case all coefficients s are products of q'-binomial coefficients and σ-eigenvalues. Joseph's iterative proof of Lemma 2.1 relies on the following fact [27, Lemma A.2.9]:

If $\{e^n \mid n \in \mathbb{N}\}$ is an Ore subset of A and $e \in A$ is a σ-eigenvector of degree $\deg_\partial e = m$, then $\{(\partial^m e)^n \mid n \in \mathbb{N}\}$ is an Ore subset of A.

This is applied to $A = R^+$ and $\sigma_i = (\leftharpoonup K_i^{-1})$, $\partial_i = (\leftharpoonup X_i^-)$, $i \in [1, r]$, recall (2.2). The skew derivations ∂_i are regular because $\sigma_i \partial_i \sigma_i^{-1} = q^{-2}\partial_i$, $\forall i \in [1, r]$. Let $w s_{i_1} \ldots s_{i_k} = w_0$ where $k = \ell(w_0) - \ell(w)$. Denote $w_j = w s_{i_1} \ldots s_{i_j}$, $j \in [0, k]$. It is easy to show by induction on j that $\deg_{\partial_{i_j}} e^\lambda_{w_j} = -\langle w_j \lambda, \alpha^\vee_{i_j} \rangle$, $e^\lambda_{w_{j-1}} = t_j \partial_{i_{j-1}}^{-\langle w_j \lambda, \alpha^\vee_{i_j} \rangle} (e^\lambda_{w_j})$ for some $t_j \in \mathbb{K}^*$, and that $\{(e^\lambda_{w_j})^n \mid n \in \mathbb{N}\}$ is an Ore subset of R^+. This implies (2.7) and Lemma 2.1.

Denote the algebras

$$R^w = R^+[E_w^{-1}], \quad w \in W$$

and their invariant subalgebras with respect to the action of $\leftharpoonup K_i$, $i \in [1, r]$, (cf. (2.2)):

$$R^w_0 = \{c^\lambda_\xi (e^\lambda_w)^{-1} \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*\}. \tag{2.8}$$

One does not need to take span in the right hand side of the above formula, see [20, eq. (2)]. For $\mu = \lambda_1 - \lambda_2 \in \mathcal{P}$, $\lambda_1, \lambda_2 \in \mathcal{P}^+$ set

$$e^\mu_w = e^{\lambda_1}_w (e^{\lambda_2}_w)^{-1} \in R^+[E_w^{-1}]. \tag{2.9}$$

It follows from (2.6) that this does not depend on the choice of λ_1, λ_2 and that $e_w^{\mu_1} e_w^{\mu_2} = e_w^{\mu_1+\mu_2}$ for all $\mu_1, \mu_2 \in \mathcal{P}$. For $y \in W$ define the quantum Schubert cell ideals of R^+

$$Q(y)^{\pm} = \mathrm{Span}\{c_\xi^\lambda \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*, \, \xi \perp \mathcal{U}_\pm T_y v_\lambda\}$$

and the ideals

$$Q(y)_w^{\pm} = \{c_\xi^\lambda e_w^{-\lambda} \mid \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*, \, \xi \perp \mathcal{U}_\pm T_y v_\lambda\} = Q(y)^{\pm}(R^+[E_w^{-1}]) \cap R_0^w \tag{2.10}$$

of R_0^w. Similarly to (2.8) one does not need to take a span in (2.10), see [20, 37].

2.3. The quantum Schubert cell algebras $\mathcal{U}_\pm^w$, $w \in W$ were defined by De Concini, Kac, and Procesi [9], and Lusztig [30, §40.2] as follows. Let

$$w = s_{i_1} \dots s_{i_l} \tag{2.11}$$

be a reduced expression of w (thus $l = \ell(w)$). Consider the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_l = s_{i_1} \dots s_{i_{l-1}}(\alpha_{i_l}) \tag{2.12}$$

and the Lusztig root vectors

$$X_{\beta_1}^{\pm} = X_{i_1}^{\pm}, X_{\beta_2}^{\pm} = T_{s_{i_1}}(X_{i_2}^{\pm}), \dots, X_{\beta_l}^{\pm} = T_{s_{i_1}} \dots T_{s_{i_{l-1}}}(X_{i_l}^{\pm}), \tag{2.13}$$

see [30, §39.3]. By [9, Proposition 2.2] and [30, Proposition 40.2.1] the subalgebras $\mathcal{U}_\pm^w$ of $\mathcal{U}_\pm$ generated by $X_{\beta_j}^{\pm}$, $j \in [1, l]$ do not depend on the choice of a reduced decomposition of w and have the PBW bases

$$(X_{\beta_l}^{\pm})^{n_l} \dots (X_{\beta_1}^{\pm})^{n_1}, \ \ n_1, \dots, n_l \in \mathbb{N}. \tag{2.14}$$

The grading (2.3) induces $\mathcal{Q}$-gradings on the subalgebras $\mathcal{U}_\pm^w$. The corresponding graded components will be denoted by $(\mathcal{U}_\pm^w)_\gamma$, $\gamma \in \mathcal{Q}$. The algebra R^+ is $\mathcal{P}$-graded by

$$\mathrm{wt}\, c_\xi^\lambda = \nu, \quad \lambda \in \mathcal{P}^+, \nu \in \mathcal{P}, \xi \in (V(\lambda)^*)_\nu. \tag{2.15}$$

This induces $\mathcal{P}$-gradings on the algebras R^w and $\mathcal{Q}$-gradings on the algebras R_0^w. The latter are given by

$$\mathrm{wt}\, c_\xi^\lambda e_w^{-\lambda} = \nu + w(\lambda), \quad \lambda \in \mathcal{P}^+, \nu \in \mathcal{P}, \xi \in (V(\lambda)^*)_\nu. \tag{2.16}$$

For a given $\gamma \in \mathcal{Q}^+ \backslash \{0\}$, let $m_w(\gamma) = \dim(\mathcal{U}_+^w)_\gamma = \dim(\mathcal{U}_-^w)_{-\gamma}$ and $\{u_{\gamma,i}\}_{i=1}^{m_w(\gamma)}$, $\{u_{-\gamma,i}\}_{i=1}^{m_w(\gamma)}$ be dual bases of $(\mathcal{U}_+^w)_\gamma$ and $(\mathcal{U}_-^w)_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [24, Ch. 6]. The quantum R matrix corresponding to w is given by

$$\mathcal{R}^w = 1 \otimes 1 + \sum_{\gamma \in \mathcal{Q}^+, \gamma \neq 0} \sum_{i=1}^{m_w(\gamma)} u_{\gamma,i} \otimes u_{-\gamma,i} \in \mathcal{U}_+ \widehat{\otimes} \mathcal{U}_-,$$

where $\mathcal{U}_+ \widehat{\otimes} \mathcal{U}_-$ is the completion of $\mathcal{U}_+ \otimes_{\mathbb{K}} \mathcal{U}_-$ with respect to the descending filtration [30, §4.1.1]. Recall that there is a unique algebra antiautomorphism τ of $\mathcal{U}_q(\mathfrak{g})$ given by

$$\tau(X_i^{\pm}) = X_i^{\pm}, \tau(K_i) = K_i^{-1}, \ \ i = 1, \dots, r, \tag{2.17}$$

see [24, Lemma 4.6(b)]. It is graded with respect to (2.3) and satisfies

$$\tau(T_w x) = T_{w^{-1}}^{-1}(\tau(x)), \ \ \forall w \in W, x \in \mathcal{U}_q(\mathfrak{g}), \tag{2.18}$$

see [24, eq. 8.18 (6)].

Proposition 2.2. [40, Theorem 2.6] *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, and $\mathfrak{g}$ is a simple Lie algebra. For all Weyl group elements $w \in W$ the maps*

$$\phi_w^+ \colon R_0^w \to \mathcal{U}_-^w, \quad \phi_w^+\big(c_\xi^\lambda e_w^{-\lambda}\big) = \big(c_{\xi, T_w v_\lambda}^\lambda \otimes \mathrm{id}\big)(\tau \otimes \mathrm{id})\mathcal{R}^w \ \ \textit{and}$$
$$\phi_w^- \colon R_0^w \to T_w(\mathcal{U}_+^{w^{-1}w_0}), \quad \phi_w^-\big(e_w^{-\lambda} c_\xi^\lambda\big) = \big(\mathrm{id} \otimes c_{\xi, T_w v_\lambda}^\lambda\big)\mathcal{R}^w,$$

where $\lambda \in \mathcal{P}^+$, $\xi \in V(\lambda)^*$, are well defined surjective graded algebra antihomomorphisms in the plus case and well defined surjective graded algebra homomorphisms in the minus case (with respect to the $\mathcal{Q}$-gradings (2.3) *and* (2.16)*), recall* (2.1)*. Their kernels are* $\ker \phi^{\pm}_w = Q(w)^{\pm}_w$*.*

The plus case of this proposition was proved in [40, Theorem 2.6]. The minus case is analogous. In another form the plus case was obtained in [37, Theorem 3.7] for a version of $\mathcal{U}_q(\mathfrak{g})$ equpped with the opposite comultiplication, and different braid group action and Lustig's root vectors. We note that in the plus and minus cases the term $e_w^{-\lambda}$ appears on different sides. This is because of the difference in the formulas for the coproducts of $X_i^{\pm}$. Moreover, the plus case requires the use of the antiautomorphism τ, while the minus case does not. This is due to the fact that one constructs dual bases of $\mathcal{U}_\pm$ with respect to the Rosso–Tanisaki form by multiplying Lusztig's root vectors in the opposite order, see e.g. [24, eqs. 8.30 (1) and (2)].

2.4. Let C be an (additive) abelian group and R be a C-graded $\mathbb{K}$-algebra. Following Artin, Schelter, and Tate [1], for a 2-cocycle $\mathbf{p} \in Z^2(C, \mathbb{K}^*)$, define a new algebra structure on R by twisting the multiplication of R by

$$b_1 * b_2 = \mathbf{p}(\alpha_1, \alpha_2) b_1 b_2, \quad \alpha_i \in C, b_i \in R_{\alpha_i}, i = 1, 2.$$

This algebra will be denoted by $R_{\mathbf{p}}$. Up to an isomorphism $R_{\mathbf{p}}$ depends only on the cohomology class of $\mathbf{p}$, see [1, §3]. Thus we can assume that the 2-cocycle $\mathbf{p}$ is normalized by $\mathbf{p}(0,0) = 1$. The group of such cocycles will be denoted by $Z^2(C, \mathbb{K}^*)_n$. The normalization $\mathbf{p}(0,0) = 1$ implies that for all $\alpha \in C$, $\mathbf{p}(\alpha, 0) = \mathbf{p}(0, \alpha) = 1$. Therefore the multiplications $R_0 \times R_\alpha \to R_\alpha$, $R_\alpha \times R_0 \to R_\alpha$ remain undeformed for all $\alpha \in C$ and the multiplicative identity in R is an identity of $R_{\mathbf{p}}$. Define

$$\mathbf{r}(\alpha, \beta) = \mathbf{p}(\alpha, \beta) \mathbf{p}(\beta, \alpha)^{-1}, \ \ \alpha, \beta \in C. \tag{2.19}$$

By [1, Proposition 1], if C is a free abelian group, then

$$\mathbf{r} \colon C \times C \to \mathbb{K}^* \ \text{ is a bicharacter,} \tag{2.20}$$

which is clearly multiplicatively skew symmetric in the sense that $\mathbf{r}(\beta, \alpha) = \mathbf{r}(\alpha, \beta)^{-1}$, $\forall \alpha, \beta \in C$.

For $w \in W$ denote the support of w:

$$\mathcal{S}(w) = \{ i \in [1, r] \mid s_i \leq w \} = \{ i \in [1, r] \mid s_i \ \text{appears in a reduced expression of} \ w \}.$$

For $I \subseteq [1, r]$ denote $\mathcal{Q}_I = \oplus_{i \in I} \mathbb{Z} \alpha_i$, $\mathcal{P}_I = \oplus_{i \in I} \mathbb{Z} \omega_i$, $\mathcal{Q}_I^+ = \mathcal{Q}_I \cap \mathcal{Q}^+$, and $\mathcal{P}_I^+ = \mathcal{P}_I \cap \mathcal{P}^+$. Then

$$[1, r] \backslash \mathcal{S}(w) = \{ i \in [1, r] \mid w(\omega_i) = \omega_i \}, \tag{2.21}$$

see [40, Lemma 3.2 and eq. (3.2)]. We have:

Lemma 2.3. [40, Lemma 3.2 (ii), eq. (2.43)] *The subgroup of $\mathcal{Q}$ generated by all γ such that $(\mathcal{U}^w_\pm)_\gamma \neq 0$ is equal to $\mathcal{Q}_{\mathcal{S}(w)}$.*

For $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$ define the multiparameter quantum Schubert cell algebra

$$\mathcal{U}^w_{\pm, \mathbf{p}} = (\mathcal{U}^w_\pm)_{\mathbf{p}}.$$

For $\gamma \in \mathcal{Q}_{\mathcal{S}(w)}$ define the character of $\mathbb{T}^{|\mathcal{S}(w)|}$

$$t \mapsto t^\gamma = \prod_{i \in \mathcal{S}(w)} t_i^{\langle \gamma, \omega_i \rangle}, \quad t = (t_i)_{i \in \mathcal{S}(w)} \in \mathbb{T}^{|\mathcal{S}(w)|} \tag{2.22}$$

and consider the rational $\mathbb{T}^{|\mathcal{S}(w)|}$-action on $\mathcal{U}^w_\pm$ by algebra automorphisms

$$t.x = t^\gamma x, \quad t \in \mathbb{T}^{|\mathcal{S}(w)|}, x \in (\mathcal{U}^w_\pm)_\gamma, \gamma \in \mathcal{Q}_{\mathcal{S}(w)}. \tag{2.23}$$

Since the twists $\mathcal{U}^w_{\pm, \mathbf{p}}$ are graded, for all $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$, (2.23) induces a rational $\mathbb{T}^{|\mathcal{S}(w)|}$-action on $\mathcal{U}^w_{\pm, \mathbf{p}}$ by algebra automorphisms.

For simplicity of the notation define

$$b^{\lambda}_{w,\xi} = \phi^+_w(c^{\lambda}_{\xi} e^{-\lambda}_w), \ a^{\lambda}_{y,w} = \phi^+_w(e^{\lambda}_y e^{-\lambda}_w) \in \mathcal{U}^w_-, \quad y \leq w \in W, \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^* \tag{2.24}$$

and denote by $xy = x.y$ the multiplication in $\mathcal{U}^w_{\pm,\mathbf{p}}$. Eq. (2.21) implies that $w(\mu) = y(\mu) = \mu$ for all $\mu \in \mathcal{P}_{[1,r]\backslash\mathcal{S}(w)}$, $y \leq w \in W$ and from (2.6) we obtain that

$$a^{\lambda+\lambda'}_{y,w} = s a^{\lambda}_{y,w}, \quad \forall \lambda \in \mathcal{P}^+_{\mathcal{S}(w)}, \lambda' \in \mathcal{P}^+_{[1,r]\backslash\mathcal{S}(w)}, y \leq w \in W, \tag{2.25}$$

for some $s \in \mathbb{K}^*$. Recall Proposition 2.2 and (2.24). Eq. (2.4) implies the following commutation relation in $\mathcal{U}^w_{-,\mathbf{p}}$:

(2.26)

$$\begin{aligned}
&\mathbf{p}(w\lambda_2+\nu_2, w\lambda_1+\nu_1)^{-1} b^{\lambda_2}_{w,\xi_2} b^{\lambda_1}_{w,\xi_1} = \mathbf{p}(w\lambda_1+\nu_1, w\lambda_2+\nu_2)^{-1} q^{\langle w\lambda_1-\nu_1, w\lambda_2+\nu_2\rangle} b^{\lambda_1}_{w,\xi_1} b^{\lambda_2}_{w,\xi_2} + \\
&+ \sum_{\gamma\in\mathcal{Q}^+,\gamma\neq 0} \sum_{i=1}^{m(\gamma)} \mathbf{p}(w\lambda_1+\nu_1-\gamma, w\lambda_2+\nu_2+\gamma)^{-1} q^{\langle w\lambda_1-\nu_1+\gamma, w\lambda_2+\nu_2+\gamma\rangle - \langle\gamma, w\lambda_2\rangle} \times \\
&\qquad \times b^{\lambda_1}_{w,S^{-1}(u_{-\gamma,i})\xi_1} b^{\lambda_2}_{w,S^{-1}(u_{\gamma,i})\xi_2}
\end{aligned}$$

for all $\lambda_i \in \mathcal{P}^+$, $\nu_i \in -w\lambda_i + \mathcal{Q}_{\mathcal{S}(w)}$, and $\xi_i \in (V(\lambda_i)^*)_{\nu_i}$. The elements $u_{\pm\gamma,i} \in (\mathcal{U}_\pm)_{\pm\gamma}$ are as in (2.4). For the values of $\gamma \in \mathcal{Q}^+$ for which $w\lambda_2+\nu_2+\gamma \notin \mathcal{Q}(\mathcal{S}(w))$ or $w\lambda_1+\nu_1-\gamma \notin \mathcal{Q}(\mathcal{S}(w))$, i.e. $\mathbf{p}(w\lambda_2+\nu_2+\gamma, w\lambda_1+\nu_1-\gamma)$ is not defined, the term $b^{\lambda_1}_{w,S^{-1}(u_{-\gamma,i})\xi_1} b^{\lambda_2}_{w,S^{-1}(u_{\gamma,i})\xi_2}$ vanishes by Proposition 2.2 and Lemma 2.3. The related product in (2.26) is set to be equal to 0.

The Levendorskii–Soibelman straightening law (see for instance [4, Proposition I.6.10] and [40, eq. (2.38)]) is the following straightening law in $\mathcal{U}^w_{-,\mathbf{p}}$:

$$\begin{aligned}
X^-_{\beta_j} X^-_{\beta_i} &- \mathbf{r}(\beta_j,\beta_i) q^{-\langle\beta_j,\beta_i\rangle} X^-_{\beta_i} X^-_{\beta_j} \\
&= \sum_{\mathbf{k}=(k_{i+1},\ldots,k_{j-1})\in\mathbb{N}^{\times(j-i-2)}} s_{\mathbf{k}} (X^-_{\beta_{j-1}})^{k_{j-1}} \ldots (X^-_{\beta_{i+1}})^{k_{i+1}}, \quad s_{\mathbf{k}} \in \mathbb{K},
\end{aligned} \tag{2.27}$$

for all $i < j$, recall (2.19). Fix a reduced expression of w as in (2.11). For $j \in [0,l]$ denote $w_j = s_{i_1} \ldots s_{i_j}$, (thus $w_0 = 1$ and $w_l = w$). By abuse of notation we will denote by the same symbols the restrictions of $\mathbf{p}$ and $\mathbf{r}$ to $\mathcal{Q}_{\mathcal{S}(w_j)} \times \mathcal{Q}_{\mathcal{S}(w_j)}$. Because of (2.20), for all $j \in [1,l]$, there exists a unique $t_j \in \mathbb{T}^{|\mathcal{S}(w_j)|}$ such that

$$t_j^{\beta_i} = \mathbf{r}(\beta_j,\beta_i) q^{-\langle\beta_j,\beta_i\rangle}, \ \ \forall i \leq j, \tag{2.28}$$

recall (2.22). The following lemma is a direct consequence of (2.14), (2.27), and (2.28).

Lemma 2.4. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, Weyl group elements $w \in W$, reduced expressions* (2.11)*, and 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$ we have:*

(a) The subalgebra of $\mathcal{U}^w_{-,\mathbf{p}}$ generated by $X^-_{\beta_1}, \ldots, X^-_{\beta_j}$ is isomorphic to $\mathcal{U}^{w_j}_{-,\mathbf{p}}$, $j \in [1,l]$.

(b) The algebra $\mathcal{U}^{w_j}_{-,\mathbf{p}}$ is isomorphic to the Ore extension $\mathcal{U}^{w_{j-1}}_{-,\mathbf{p}}[x, (t_j\cdot), \delta_j]$ for some (left) $(t_j\cdot)$-skew derivation δ_j of $\mathcal{U}^{w_{j-1}}_{-,\mathbf{p}}$ satisfying $(t_j\cdot)\delta_j = q^{-\langle\beta_j,\beta_j\rangle}\delta_j(t_j\cdot)$, $j \in [1,l]$. Here $(t_j\cdot)$ denotes the restriction to $\mathcal{U}^{w_{j-1}}_{-,\mathbf{p}}$ of the automorphism of $\mathcal{U}^{w_j}_{-,\mathbf{p}}$ induced from the action (2.23)*, and $\mathcal{U}^1_{-,\mathbf{p}} = \mathbb{K}$.*

(c) The eigenvalues $t_j.X^-_{\beta_j} = q^{-\langle\beta_j,\beta_j\rangle} X^-_{\beta_j}$ are not roots of unity.

Since the algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are iterated Ore extensions they are noetherian, and the Brown–Goodearl theorem [4, Proposition II.2.9] implies that each $\mathbb{T}^{|\mathcal{S}(w)|}$-prime ideal of $\mathcal{U}^w_{-,\mathbf{p}}$ with respect to the action (2.23) is prime.

3. A description of the spectra of $\mathcal{U}^w_{-,\mathbf{p}}$

3.1. In this section we describe an explicit stratification of Spec$\mathcal{U}^w_{-,\mathbf{p}}$ by the spectra of commutative Laurent polynomial rings over $\mathbb{K}$ and the structure of the poset of $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant prime ideals of $\mathcal{U}^w_{-,\mathbf{p}}$ for the action (2.23). For $y \in W^{\leq w}$ define the ideals

$$I_w(y) = \phi^+_w(Q(w)^+_w + Q(y)^-_w) = \phi^+_w(Q(y)^-_w)$$

of $\mathcal{U}^w_{-,\mathbf{p}}$ and denote

$$A_{y,w} = \mathbb{K}^*\{a^\lambda_{y,w} \mid \lambda \in \mathcal{P}^+\} = \mathbb{K}^*\{a^\lambda_{y,w} \mid \lambda \in \mathcal{P}^+_{\mathcal{S}(w)}\},$$

cf. (2.24)–(2.25). By (2.6) and (2.26), $A_{y,w}$ is a multiplicative subset of $\mathcal{U}^w_{-,\mathbf{p}}$ consisting of elements which are normal modulo $I_w(y)$:

$$a^\lambda_{y,w} x = \mathbf{r}((w-y)\lambda, \gamma) q^{-\langle (w+y)\lambda, \gamma\rangle} x a^\lambda_{y,w} \mod I_w(y), \tag{3.1}$$

$\forall \lambda \in \mathcal{P}^+, \gamma \in \mathcal{Q}_{\mathcal{S}(w)}, x \in (\mathcal{U}_{-,\mathbf{p}})_\gamma$. The following theorem describes Spec$\mathcal{U}^w_{-,\mathbf{p}}$ set-theoretically and provides some information for its Zariski topology.

Theorem 3.1. *Assume that $\mathbb{K}$ is an arbitrary field, $q \in \mathbb{K}^*$ is not a root of unity, $\mathfrak{g}$ is a simple Lie algebra, w is an element of the corresponding Weyl group W, and $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$ is a normalized cocycle. Then:*

(a) The $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant prime ideals of $\mathcal{U}^w_{-,\mathbf{p}}$ are the ideals $I_w(y)$ for all $y \in W^{\leq w}$. The map $y \in W^{\leq w} \mapsto I_w(y) \in \mathbb{T}^{|\mathcal{S}(w)|} - \mathrm{Spec}\mathcal{U}^w_{-\mathbf{p}}$ is an isomorphism of posets with respect to the Bruhat order and the inclusion order on ideals. In particular, the ideals $I_w(y)$ are distinct.

(b) For all $y \in W^{\leq w}$, $I_w(y) \cap A_{y,w} = \emptyset$ and the quotient ring $R_{y,w} = (\mathcal{U}^w_{-,\mathbf{p}}/I_w(y))[A^{-1}_{y,w}]$ is a $\mathbb{T}^{|\mathcal{S}(w)|}$-simple domain.

(c) For every prime ideal J of $\mathcal{U}^w_{-,\mathbf{p}}$ there exists a unique $y \in W^{\leq w}$ such that

$$J \supseteq I_w(y) \quad \text{and} \quad J \cap A_{y,w} = \emptyset.$$

Denote the corresponding subset of Spec$\mathcal{U}^w_{-,\mathbf{p}}$ by $\mathrm{Spec}_{I_w(y)}\mathcal{U}^w_{-,\mathbf{p}}$, so

$$\mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}} = \bigsqcup_{y \in W^{\leq w}} \mathrm{Spec}_{I_w(y)}\mathcal{U}^w_{-,\mathbf{p}}. \tag{3.2}$$

(d) The center $Z(R_{y,w})$ is a Laurent polynomial ring over $\mathbb{K}$ and we have a homeomorphism

$$\iota_y \colon \mathrm{Spec}Z(R_{y,w}) \to \mathrm{Spec}_{I_w(y)}\mathcal{U}^w_{-,\mathbf{p}},$$

where for $J_0 \in \mathrm{Spec}Z(R_{y,w})$, $\iota_y(J_0)$ is the ideal of $\mathcal{U}^w_{-,\mathbf{p}}$ containing $I_w(y)$ such that $\iota_y(J_0)/I_w(y) = J_0 R_{y,w} \cap (\mathcal{U}^w_{-,\mathbf{p}}/I_w(y))$. The Zariski closures of the strata $\mathrm{Spec}_{I_w(y)}\mathcal{U}^w_{-,\mathbf{p}}$ are given by

$$\overline{\mathrm{Spec}_{I_w(y)}\mathcal{U}^w_{-,\mathbf{p}}} = \bigsqcup_{y' \in W^{\leq y}} \mathrm{Spec}_{I_w(y')}\mathcal{U}^w_{-,\mathbf{p}}. \tag{3.3}$$

Gorelik described in [20] the torus invariant prime ideals of the algebras R^w_0 and the resulting stratification of SpecR^w_0 as in Theorem 3.1 under the condition that $\mathbb{K} = k(q)$ for a field k of characteristic 0. Via the antihomomorphism from Proposition 2.2 this establishes Theorem 3.1 in the untwisted case under those assumptions on $\mathbb{K}$ and q. We show that Gorelik's arguments work under the weaker assumptions on $\mathbb{K}$ and q, taking shortcuts at some steps using results of Goodearl and Letzter [18, 13]. The proof of Theorem 3.1 is given in §3.4 and §3.2–3.3 contain some preparatory results. It follows from Theorem 3.1 (c)-(d) that $\mathrm{Spec}_{I_w(y)}\mathcal{U}^w_{-,\mathbf{p}}$ are precisely the Goodearl–Letzter strata [18] of Spec$\mathcal{U}^w_{-,\mathbf{p}}$. Namely, we have that

$$\mathrm{Spec}_{I_w(y)}\mathcal{U}^w_{-,\mathbf{p}} = \{J \in \mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}} \mid \cap_{t \in \mathbb{T}^{|\mathcal{S}(w)|}} t.J = I_w(y)\}. \tag{3.4}$$

Remark 3.2. For $I \subseteq [1, r]$

$$R_I^+ = \mathrm{Span}\{c_\xi^\lambda \mid \lambda \in \mathcal{P}_I^+, \xi \in V(\lambda)^*\} \subseteq R^+$$

is a subalgebra of R^+ (called the quantum partial flag variety associated to I), see [38]. Joseph's argument for the proof of Lemma 2.1 shows that $E_{\mathcal{S}(w),w} = \{e_w^\lambda \mid \lambda \in \mathcal{P}^+_{\mathcal{S}(w)}\}$ is an Ore subset of $R^+_{\mathcal{S}(w)}$. Define the subalgebra

$$R^w_{\mathcal{S}(w),0} = \{c_\xi^\lambda (e_w^\lambda)^{-1} \mid \lambda \in \mathcal{P}^+_{\mathcal{S}(w)}, \xi \in V(\lambda)^*\} \subset R^+_{\mathcal{S}(w)}[E^{-1}_{\mathcal{S}(w),w}]$$

and its ideals

$$Q(y)^\pm_{\mathcal{S}(w),w} = \{c^\lambda_{w,\xi}(e_w^\lambda)^{-1} \mid \lambda \in \mathcal{P}^+_{\mathcal{S}(w)}, \xi \in V(\lambda)^*,\ \xi \perp \mathcal{U}_\pm T_y v_\lambda\}. \tag{3.5}$$

The argument of the proof of [40, Theorem 2.6] gives that the restriction

$$\phi_+^w|_{R^w_{\mathcal{S}(w),0}} \colon R^w_{\mathcal{S}(w),0} \to \mathcal{U}_-^w$$

is a surjective algebra antihomomorphism with kernel $Q(w)^+_{\mathcal{S}(w),w}$, see also [38, Theorem 3.6]. The proof of Theorem 3.1 works in the same way if the map ϕ_w^+ is substituted with its restriction $\phi_w^+|_{R^w_{\mathcal{S}(w),0}}$ and the $\mathbb{T}^{|\mathcal{S}(w)|}$-primes of $\mathcal{U}_-^w$ are also given by

$$I_w(y) = \phi_+^w(Q(y)^-_{\mathcal{S}(w),w}), \quad \forall y \in W^{\leq w}. \tag{3.6}$$

We finish with noting that [17, Theorem 2.3] of Goodearl and Letzter, and Lemma 2.4 imply at once:

Proposition 3.3. *Every prime ideal of $\mathcal{U}^w_{-,\mathbf{p}}$ is completely prime for all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, Weyl group elements $w \in W$, and 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$ such that the subgroup of $\mathbb{K}^*$ generated by*

$$\mathbf{r}(\beta_i, \beta_j) q^{\langle \beta_i, \beta_j \rangle} = \Big(\mathbf{r}(\beta_j, \beta_i) q^{-\langle \beta_j, \beta_i \rangle} \Big)^{-1}, \quad 1 \leq i < j \leq \ell(w)$$

is torsion free, recall (2.12).

3.2. We start with a couple of auxiliary results for the proof of Theorem 3.1. The following lemma was proved by Joseph for $\mathbb{K} = \mathbb{C}$, $q \in \mathbb{C}^*$ not a root of unity [27, Proposition 10.1.8] and for $\mathbb{K} = k(q)$, $\mathrm{char}\, k = 0$, [26, Proposition 7.3]. It is easy to verify that Joseph's proof works in the general case. We provide a second proof based on Proposition 2.2.

Lemma 3.4. *Let $\mathbb{K}$ be an arbitrary field, $q \in \mathbb{K}^*$ not a root of unity. The ideals $Q(w)^\pm$ and $Q(y)^\pm_w$ of R^+ and R_0^w are completely prime for all $w \in W$, $y \in W^{\leq w}$ in the plus case and $y \in W^{\geq w}$ in the minus case.*

Proof. By Lemma 2.4 (b) $\mathcal{U}^w_\pm$ is a domain, $\forall w \in W$. Proposition 2.2 implies that $Q(w)^\pm_w = \ker \phi^\pm_w$ is a completely prime ideal of R_0^w. It is straightforward to verify that the map

$$(R_0^w / Q(w)^\pm_w) \# \mathbb{K}[\mathcal{P}] \to (R^+[E_w^{-1}] / (Q(w)^\pm R^+[E_w^{-1}]),$$
$$(c_\xi^\lambda e_w^{-\lambda} + Q(w)^\pm_w) \# \mu \mapsto c_\xi^\lambda e_w^{-\lambda+\mu} + Q(w)^\pm R^+[E_w^{-1}], \quad \mu \in \mathcal{P}, \lambda \in \mathcal{P}^+, \xi \in V(\lambda)^*$$

is (a well defined) algebra isomorphism, where the smash product is defined via the action

$$\mu \cdot (r + Q(w)^\pm_w) = q^{\mp \langle w\mu, \gamma \rangle} (r + Q(w)^\pm_w), \quad \mu \in \mathcal{P}, r \in (R_0^w)_\gamma, \gamma \in \mathcal{Q},$$

cf. (2.4) and (2.16). Therefore $Q(w)^\pm R^+[E_w^{-1}]$ is a completely prime ideal of $R^+[E_w^{-1}]$. It follows that $Q(w)^\pm = Q(w)^\pm R^+[E_w^{-1}] \cap R^+$ is a completely prime ideal of R^+, $\forall w \in W$.

We have $T_w v_\lambda \in \mathcal{U}_\pm T_y v_\lambda$, for all $\lambda \in \mathcal{P}^+$ and $y \in W^{\geq w}$ in the plus case, $y \in W^{\leq w}$ in the minus case, see [27, Lemma 4.4.3]. Therefore in those cases $Q(y)^\pm \cap E_w = \emptyset$ and thus $Q(y)^\pm R^+[E_w^{-1}]$ is a completely prime ideal of $R^+[E_w^{-1}]$. So $Q(y)^\pm_w = Q(y)^\pm R^+[E_w^{-1}] \cap R_0^w$ is a completely prime ideal of R_0^w. $\square$

Lemma 3.5. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, Weyl group elements $y \leq w$, and 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$:*

(a) $Q(w)^+_w + Q(y)^-_w$ is a completely prime ideal of R^w_0 and

$$(Q(w)^+_w + Q(y)^-_w) \cap \{e^\lambda_y e^{-\lambda}_w \mid \lambda \in \mathcal{P}^+_{\mathcal{S}(w)}\} = \emptyset, \tag{3.7}$$

(b) $I_w(y) = \phi^+_w(Q(w)^+_w + Q(y)^-_w)$ is a completely prime ideal of $\mathcal{U}^w_{-,\mathbf{p}}$ and $I_w(y) \cap A_{y,w} = \emptyset$.

Assume in the setting of §2.4 that C is a finite rank free abelian group. Then for a graded subspace I of R, I is a completely prime ideal of $R_{\mathbf{p}}$ if and only if I is a completely prime ideal of R, see the proof of [15, Theorem 4.1]. Therefore it is sufficient to prove only the untwisted case of Lemma 3.5 (a). Taking into account Proposition 2.2, we see that part (b) of Lemma 3.5 follows from the first part of the lemma.

Gorelik [20, Lemma 6.6] stated the untwisted case of Lemma 3.5 (a) for $\mathbb{K} = k(q)$, $\operatorname{char} k = 0$. Her proof works under the above more general assumptions on $\mathbb{K}$ and q. We review the key steps of her proof below to show this. For $\mu \in \mathcal{P}$ denote the automorphism

$$\psi^\mu_w(r) = c^{-\mu}_w r c^\mu_w, \quad r \in R^w$$

of R^w. It is obvious that R^w_0 is stable under it. Recall the skew derivations $\partial_i = (\leftharpoondown X^-_i)$ of R^+ from §2.2. By a direct commutation argument one shows that for all $w \in W$, $i \in [1, r]$ such that $s_i w < w$, $\lambda \in \mathcal{P}^+$, $\mu, \nu \in \mathcal{P}$, $\xi \in (V(\lambda)^*)_\nu$, and t in the algebraic closure of $\mathbb{K}$:

$$\begin{aligned} &\exists k \in \mathbb{Z}_+ \text{ such that } (\psi^\mu_w - t.\mathrm{id})^k c^\lambda_\xi = 0 \Rightarrow \\ &\qquad \exists k \in \mathbb{Z}_+ \text{ such that } (\psi^\mu_w - t q^{-\langle \nu, w\mu\rangle + \langle \nu - m\alpha_i, s_i w \mu\rangle} \mathrm{id})^k (\partial^m_i c^\lambda_\xi) = 0, \end{aligned} \tag{3.8}$$

where $m \in \mathbb{N}$ denotes the ∂_i-degree of c^λ_ξ. For $r \in R^w$ set

$$\mathrm{wt}_w(r) = \gamma \in \mathcal{Q} \text{ if } (\psi^\mu_w - q^{\langle \gamma, \mu \rangle}\mathrm{id})^k r = 0 \text{ for some } k \in \mathbb{Z}_+ \text{ and all } \mu \in \mathcal{P}.$$

We say that $\xi \in V(\lambda)^*$ is homogeneous if $\xi \in (V(\lambda)^*)_\nu$ for some $\nu \in \mathcal{P}$. For a reduced expression $v = s_{i_1} \ldots s_{i_k} \in W$ and a homogeneous element $\xi \in V(\lambda)^*$ denote

$$\partial^*_v(c^\lambda_\xi) := \partial^{m_1}_{i_1} \ldots \partial^{m_k}_{i_k}(c^\lambda_\xi), \tag{3.9}$$

where for $j = k, \ldots, 1$, m_j is the ∂_{i_j}-degree of $\partial^{m_{j+1}}_{i_{j+1}} \ldots \partial^{m_{i_k}}_{i_k}(c^\lambda_\xi)$, see §2.2. By induction (3.8) implies that for all homogeneous $\xi \in V(\lambda)$, if $\mathrm{wt}_w(c^\lambda_\xi e^{-\lambda}_w)$ exists, then

$$\begin{aligned} &\mathrm{wt}_w(c^\lambda_\xi e^{-\lambda}_w) - w^{-1}\,\mathrm{wt}(c^\lambda_\xi e^{-\lambda}_w) \\ &\qquad = \mathrm{wt}_1((\partial^*_{w^{-1}}(c^\lambda_\xi))e^{-\lambda}_1) - \mathrm{wt}((\partial^*_{w^{-1}}(c^\lambda_\xi))e^{-\lambda}_1) = -2\,\mathrm{wt}((\partial^*_{w^{-1}}(c^\lambda_\xi))e^{-\lambda}_1), \end{aligned} \tag{3.10}$$

recall (2.6) and (2.16). The last equality follows from $e^{-\mu}_1 c^\lambda_\xi e^\mu_1 = q^{-\langle \mu, \lambda + \nu \rangle} c^\lambda_\xi$, $\forall \lambda \in \mathcal{P}^+$, $\mu, \nu \in \mathcal{P}$, $\xi \in (V(\lambda)^*)_\nu$, which is a special case of (2.4). The automorphisms ψ^μ_w, $\mu \in \mathcal{P}$ of R^w_0 commute, because of (2.6) and act locally finitely, since they preserve the grading (2.16). Therefore R^w_0 is a direct sum of the common generalized ψ^μ_w-eigenspaces, $\mu \in \mathcal{P}$. Eqs. (3.8) and (3.10), and the fact that for each $\gamma \in \mathcal{Q}$ there exists $\lambda \in \mathcal{P}^+$ such that $(R^w_0)_\gamma = \{c^\lambda_\xi e^{-\lambda}_w \mid \xi \in (V(\lambda)^*)_{\gamma - w\lambda}\}$ imply that

$$R^w_0 = \bigoplus_{\gamma \in -2\mathcal{Q}^+} R^w_0[\gamma], \tag{3.11}$$

where $R^w_0[\gamma]$ is the span of all homogeneous elements $r \in R^w_0$ for which $\mathrm{wt}_w(r)$ exists and

$$\mathrm{wt}_w(r) - w^{-1}\,\mathrm{wt}(r) = \gamma, \;\; \gamma \in -2\mathcal{Q}^+.$$

It follows from (2.4) that

$$\psi^\mu_w(r) = q^{\langle \mu, w^{-1}\gamma' \rangle} r \mod Q(w)^+_w, \;\; \forall \gamma' \in \mathcal{Q}, r \in (R^w_0)_{\gamma'}.$$

Thus $\oplus_{\gamma\in -2\mathcal{Q}^+\backslash\{0\}} R_0^w[\gamma] \subseteq Q(w)_w^+$. Assume that $R_0^w[0]\cap Q(w)_w^+ \neq 0$. Then there exist $\lambda\in\mathcal{P}^+$ and a homogeneous element $\xi\in V(\lambda)^*$, $\xi\neq 0$ such that $c_\xi^\lambda e_w^{-\lambda}\in R_0^w[0]$ and $\xi(\mathcal{U}^+T_w v_\lambda) = 0$. Because of (3.10), $\mathrm{wt}((\partial_{w^{-1}}(c_\xi^\lambda))e_1^{-\lambda}) = 0$, so $\partial_{w^{-1}}(c_\xi^\lambda) = s e_1^\lambda$ for some $s\in\mathbb{K}^*$. Thus $\xi((X_{i_1}^-)^{m_1}\dots(X_{i_l}^-)^{m_l}v_\lambda)\neq 0$ for some $m_1,\dots,m_l\in\mathbb{N}$ where $i_1,\dots,i_l$ are the indices of the reduced decomposition (2.11) of w. Yet $(X_{i_1}^-)^{m_1}\dots(X_{i_l}^-)^{m_l}v_\lambda\in\mathcal{U}^+T_w v_\lambda$ by [27, Lemma 4.4.3 (v)], so $\xi((X_{i_1}^-)^{m_1}\dots(X_{i_l}^-)^{m_l}v_\lambda) = 0$ because $\xi(\mathcal{U}_+T_w v_\lambda) = 0$. Thus the assumption is not correct and

$$Q(w)_w^+ = \oplus_{\gamma\in -2\mathcal{Q}^+\backslash\{0\}} R_0^w[\gamma]. \tag{3.12}$$

Moreover $R_0^w[0]$ is a subalgebra of R_0^w. The ideal $Q(y)_w^-$ is ψ_w^μ-stable and homogeneous with respect to the grading (2.16), thus $Q(y)_w^- = (Q(y)_w^-\cap R_0^w[0])\oplus(Q(y)_w^-\cap(\oplus_{\gamma\in -2\mathcal{Q}^+\backslash\{0\}} R_0^w[\gamma]))$ and

$$R_0^w/(Q(y)_w^- + Q(w)_w^+)\cong R_0^w[0]/(Q(y)_w^-\cap R_0^w[0])\cong (Q(y)_w^- + R_0^w[0])/Q(y)_w^- \hookrightarrow R_0^w/Q(y)_w^-.$$

Therefore $Q(y)_w^- + Q(w)_w^+$ is a completely prime ideal of R_0^w, since $R_0^w/Q(y)_w^-$ is a domain by Lemma 3.4.

Next we go over the key steps of Gorelik's proof of (3.7) to show that it works for all base fields $\mathbb{K}$, $q\in\mathbb{K}^*$ not a root of unity. We have that $e_y^\lambda e_w^{-\lambda}\in Q(w)_w^+ + Q(y)_w^-$ if and only if $e_y^\lambda e_w^{\lambda'}\in Q(w)^+ + Q(y)^-$ for some $\lambda'\in\mathcal{P}^+$. (For some of the arguments below λ' should be chosen sufficiently large.) Define $s_i\star w = \max(s_i w, w)$. Let $s_i y > y$ and $y\leq w$. Then $s_i y\leq s_i\star w$, [27, Proposition A.1.7]. Gorelik proves that

$$e_y^\lambda e_w^{\lambda'}\in Q(w)^+ + Q(y)^- \quad\Rightarrow\quad e_{s_i y}^\lambda e_{s_i\star w}^{\lambda'}\in Q(s_i\star w)^+ + Q(s_i y)^- \tag{3.13}$$

as described below. Since (3.7) is obvious for $y = w = w_0$, (3.7) follows from (3.13) by induction.

Define the automorphisms $\overline{\sigma}_i = (\leftharpoondown K_i)$ of R^+ and the (left skew) derivations $\overline{\partial}_i = (\leftharpoondown X_i^+)$, $i\in[1,r]$, cf. §2.2. For a homogeneous element $r\in R^+$ with respect to the grading (2.15) in analogy to (3.9) define $\overline{\partial}_i^*(r) = \overline{\partial}_i^n(r)$ where $n = \deg_{\overline{\partial}_i} r$. Since $\mathbb{K}[X_i^+]\mathcal{U}_- T_{s_i y}v_\lambda\subseteq\mathcal{U}_- T_y v_\lambda$, $\forall\lambda\in\mathcal{P}^+$,

$$\overline{\partial}_i^n Q(y)^-\subseteq Q(s_i y)^-,\ \ \forall n\in\mathbb{N}. \tag{3.14}$$

If $s_i w < w$, then $\deg_{\overline{\partial}_i} e_w^{\lambda'} = 0$ and $e_{s_i y}^\lambda e_w^{\lambda'} = t\overline{\partial}_i^*(e_y^\lambda e_w^{\lambda'})\in Q(w)^+ + Q(s_i y)^-$ for some $t\in\mathbb{K}^*$, because $Q(w)^+$ is $\overline{\partial}_i$-invariant. This proves (3.13) in the case $s_i w < w$. The case $s_i w > w$ of (3.13) is more complicated. First one decomposes according to (3.11):

$$e_y^\lambda e_w^{-\lambda} = b_0 + \ldots + b_m \ \text{ for }\ b_0\in Q(y)_w^-\cap R_0^w[0], b_j\in R_0^w[-2\gamma_j], \gamma_j\in\mathcal{Q}^+\backslash\{0\}, j\in[1,m],$$

$\gamma_j\neq\gamma_{j'}$ for $j\neq j'$. Then

$$\mathrm{wt}(b_j) = \mathrm{wt}(e_y^\lambda e_w^{-\lambda}) = (w-y)\lambda,\ \ \forall j\in[0,m]. \tag{3.15}$$

Denote $n := \deg_{\overline{\partial}_i^*}(e_y^\lambda e_w^{\lambda'})$. Since the elements $b_0,\dots,b_m$ are linearly independent, cf. (3.11),

$$\deg_{\overline{\partial}_i^*}(b_j e_w^{\lambda+\lambda'})\leq n,\ \forall j\in[0,m] \ \text{ and }\ e_{s_i y}^\lambda e_{s_i w}^{\lambda'} = t\overline{\partial}_i^*(e_y^\lambda e_w^{\lambda'}) = \sum_{j\in M} t\overline{\partial}_i^*(b_j e_w^{\lambda+\lambda'}),\ t\in\mathbb{K}^*, \tag{3.16}$$

where $M = \{j\in[0,m] \mid \deg_{\overline{\partial}_i^*}(b_j e_w^{\lambda+\lambda'}) = n\}$. Analogously to (3.8) one shows that for $s_i w > w$, if $r\in R^+$ is homogeneous and $\mathrm{wt}_w(r)$ exists, then $\mathrm{wt}_{s_i w}(\overline{\partial}_i r)$ exists and

$$\mathrm{wt}_w(r) + w^{-1}\,\mathrm{wt}(r) = \mathrm{wt}_{s_i w}(\overline{\partial}_i r) + (s_i w)^{-1}\,\mathrm{wt}(\overline{\partial}_i r). \tag{3.17}$$

Eqs. (3.15) and (3.16) imply that $\mathrm{wt}(\overline{\partial}_i^*(b_j e_w^{\lambda+\lambda'})) = \mathrm{wt}(e_{s_i y}^\lambda e_{s_i w}^{\lambda'})$, for all $j\in M$. It follows from this, (3.17), and (3.12) that $\sum_{j\in M\backslash\{0\}}\overline{\partial}_i^*(b_j e_w^{\lambda+\lambda'})\in Q(s_i w)^+$. Eq. (3.14) implies that $\overline{\partial}_i^*(b_0 e_w^{\lambda+\lambda'})\in Q(s_i y)^-$, which completes the proof of (3.13) in the case $s_i w > w$.

3.3. For $J \in \mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}}$ and $\lambda \in \mathcal{P}^+$ denote

$$C_J(\lambda) = \{\nu \in \mathcal{P} \mid \exists \xi \in (V(\lambda)^*)_\nu, b^\lambda_{w,\xi} \notin J\}.$$

For all $\lambda \in \mathcal{P}^+$, we have $-w\lambda \in C_J(\lambda)$ since $a^\lambda_{w,w} = 1 \notin J$. Denote by $D_J(\lambda)$ the set of minimal elements of the set $C_J(\lambda)$.

Lemma 3.6. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, $w \in W$, $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$, and $I \in \mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}}$, there exists a unique $y \in W^{\leq w}$ such that $D_J(\lambda) = \{-y\lambda\}$ for all $\lambda \in \mathcal{P}^+$.*

Gorelik's analogous result [20, §5.2.1] was formulated under the assumption $\mathbb{K} = k(q)$ for a field k, $\mathrm{char}\, k = 0$. This proof works under the more general assumptions on $\mathbb{K}$ and q. We sketch this below.

Proof of Lemma 3.6. Fix $J \in \mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}}$. Let $\lambda \in \mathcal{P}^+$, $\nu \in D_J(\lambda)$, $\xi \in (V(\lambda)^*)_\nu, b^\lambda_{w,\xi} \notin J$. Applying (2.26), we obtain that $b^\lambda_{w,\xi}$ defines a nonzero normal element of $\mathcal{U}^w_{-,\mathbf{p}}/J$:

$$b^\lambda_{w,\xi} x = \mathbf{r}(\nu + w\lambda, \gamma) q^{\langle \nu - w\lambda, \gamma\rangle} x b^\lambda_{w,\xi} \mod J, \ \forall x \in (\mathcal{U}^w_{-,\mathbf{p}})_\gamma, \gamma \in \mathcal{Q}_{\mathcal{S}(w)}.$$

Applying this twice and using that $\mathbf{r}(.,.)$ is multiplicatively skew symmetric (see §2.4), we obtain that for all $\lambda_i \in \mathcal{P}^+$, $\nu_i \in D_J(\lambda_i)$, $\xi_i \in (V(\lambda_i)^*)_{\nu_i}, b^{\lambda_i}_{w,\xi_i} \notin J$, $i = 1, 2$,

$$b^{\lambda_1}_{w,\xi_1} b^{\lambda_2}_{w,\xi_2} = q^{\langle \nu_1 - w\lambda_1, \nu_2 + w\lambda_2\rangle + \langle \nu_2 - w\lambda_2, \nu_1 + w\lambda_1\rangle} b^{\lambda_1}_{w,\xi_1} b^{\lambda_2}_{w,\xi_2} \mod J.$$

Since J is prime and the images of $b^{\lambda_i}_{w,\xi_i}$ in $\mathcal{U}^w_{-,\mathbf{p}}/J$ are nonzero normal elements, they are regular. Using the fact that $q \in \mathbb{K}^*$ is not a root of unity, we obtain

$$\langle \nu_1 - w\lambda_1, \nu_2 + w\lambda_2\rangle + \langle \nu_2 - w\lambda_2, \nu_1 + w\lambda_1\rangle = 0, \text{ so } \langle \nu_1, \nu_2\rangle = \langle \lambda_1, \lambda_2\rangle.$$

By [27, Lemma A.1.17], if $\lambda_1, \lambda_2 \in \mathcal{P}^{++}$, then $\nu_i = -y\lambda_i$, $i = 1, 2$, for some $y \in W$. Therefore there exists $y \in W$ such that $D_J(\lambda) = \{-y\lambda\}$, $\forall \lambda \in \mathcal{P}^{++}$. Let $\lambda \in \mathcal{P}^{++}$. Since $b^\lambda_{w,\xi} \neq 0$ for some $\xi \in (V(\lambda)^*)_{-y\lambda}$ and $\ker \phi^+_w = Q(w)^+_w$ (see Proposition 2.2), we have $(\mathcal{U}_+ T_w v_\lambda)_{y\lambda} \neq 0$. By [27, Proposition 4.4.5], $y \leq w$. Analogously to the proof of [27, Proposition 9.3.8] one shows that $D_J(\lambda) = \{-y\lambda\}$, $\forall \lambda \in \mathcal{P}^+$ for the same Weyl group element y. □

3.4. We proceed with the proof of Theorem 3.1. For $y \in W^{\leq w}$ denote

$$\mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}} = \{J \in \mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}} \mid D_J(\lambda) = \{-y\lambda\}, \ \forall \lambda \in \mathcal{P}^+\}.$$

The definition of the sets $D_J(\lambda)$ and Lemma 3.6 imply

$$\mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}} = \bigsqcup_{y \in W^{\leq w}} \mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}}. \tag{3.18}$$

Proof of Theorem 3.1. By Lemma 3.5 (b), $I_w(y) \in \mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}}$, $\forall y \in W^{\leq w}$. Since $\dim(V(\lambda))_{y\lambda} = 1$, $\forall \lambda \in \mathcal{P}^+, y \in W$, each stratum $\mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}}$ contains a unique $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant prime ideal, which would have to be precisely the ideal $I_w(y)$. This implies part (a) of the theorem except the statement for the poset structure of $\mathbb{T}^{|\mathcal{S}(w)|} - \mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}}$. It also implies

$$I_w(y) = \cap_{t \in \mathbb{T}^{|\mathcal{S}(w)|}} t.J, \ \ \forall J \in \mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}}, \tag{3.19}$$

recall (2.23). Furthermore, we have

$$\mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}} = \{J \in \mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}} \mid J \supseteq I_w(y) \text{ and } J \cap A_{y,w} = \emptyset\}. \tag{3.20}$$

The left inclusion in (3.20) follows from (3.19) and the definition of the sets $D_J(\lambda)$. The equality follows from (3.18).

To complete the proof of part (a) note that $y_1 \leq y_2 \in W^{\leq w}$ implies $\mathcal{U}_- T_{y_1} v_\lambda \supseteq \mathcal{U}_- T_{y_2} v_\lambda$, see [27, Lemma 4.4.3], therefore $Q(y_1)^-_w \subseteq Q(y_2)^-_w$ and $I_w(y_1) = \phi^+_w(Q(y_1)^-_w) \subseteq \phi^+_w(Q(y_2)^-_w) = I_w(y_2)$. Now assume that $I_w(y_1) \subseteq I_w(y_2)$ for some $y_1, y_2 \in W^{\leq w}$. Since $a^\lambda_{y_2,w} \notin I_w(y_2)$ by Lemma 3.5, $a^\lambda_{y_2,w} \notin I_w(y_1)$, $\forall \lambda \in \mathcal{P}^+$. Therefore $T_{y_2} v_\lambda \in \mathcal{U}_- T_{y_1} v_\lambda$ (because otherwise we would get $c^{\lambda}_{y_2} e^{-\lambda}_w \in Q(y_1)^-_w$ and $a^{\lambda}_{y_2,w} \in I_w(y_1)$). Applying this to any $\lambda \in \mathcal{P}^{++}$ gives $y_1 \leq y_2$ by [27, Proposition 4.4.5].

Part (c) follows from (3.18) and (3.20). The rings $R_{y,w}$ are $\mathbb{T}^{|\mathcal{S}(w)|}$-simple, since otherwise a $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant maximal ideal of $R_{y,w}$ will contract to a $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant prime ideal in the stratum $\mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}}$, which properly contains $I_w(y)$. This will contradict with the fact that $I_w(y)$ is the only $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant prime ideal in $\mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}}$. This and Lemma 3.5 (b) prove part (b) of the theorem.

It follows from Lemma 2.4 that Goodearl's result [4, Theorem II.6.4] is applicable to the algebras $\mathcal{U}^w_{-,\mathbf{p}}$ and as a consequence of this all of their $\mathbb{T}^{|\mathcal{S}(w)|}$-prime ideals are strongly rational. This means that the zero components of the centers $R_{y,w}$ with respect to the grading (2.3) reduce to scalars: $(Z(R_{y,w}))_0 = \mathbb{K}$, $\forall y \leq w$. Next we apply two results of Goodearl and Letzter: [18, Lemma 6.3 (c)] implies that

$$\dim(Z(R_{y,w}))_\gamma = 0 \text{ or } 1, \quad \text{for all } \gamma \in \mathcal{Q}_{\mathcal{S}(w)} \tag{3.21}$$

and that $Z(R_{y,w})$ are Laurent polynomial rings over $\mathbb{K}$. Furthermore, [18, Corollary 6.5] gives that contraction and extension provide mutually inverse homeomorphisms between $\mathrm{Spec} Z(R_{y,w})$ and $\mathrm{Spec} R_{y,w}$, for all $y \leq w$. Finally, because of (3.20) and general localization facts $J \mapsto (J/I_w(y))R_{y,w}$ is a homeomorphism between $\mathrm{Spec}_y \mathcal{U}^w_{-,\mathbf{p}}$ and $\mathrm{Spec} R_{y,w}$. Eq. (3.3) follows from part (a) and (3.19). This completes the proof of part (d). □

4. Dimensions of the Goodearl–Letzter strata of $\mathcal{U}^w_{-,p}$

4.1. The spectra of the algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are partitioned (3.2) into disjoint unions of the Goodearl–Letzter strata $\mathrm{Spec}_{I_w(y)} \mathcal{U}^w_{-,\mathbf{p}}$, cf. (3.4). Each of them is homeomorphic to the spectrum of a Laurent polynomial ring over $\mathbb{K}$, namely $Z(R_{y,w})$, see Theorem 3.1 (d). To determine $\mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}}$ set theoretically, one needs to solve the problem for computing the dimensions of these Laurent polynomial rings. In this section we obtain an explicit formula for those dimensions. This is done in Theorem 4.5 and §4.2–4.3 contain some preparatory results.

Let $\mu \in \mathcal{P}$. If $\mu = \lambda_1 - \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathcal{P}^+$ with disjoint support, define

$$a^{\mu}_{y,w} = a^{\lambda_1}_{y,w} (a^{\lambda_2}_{y,w})^{-1} \in R_{y,w}. \tag{4.1}$$

Since $\mathbf{r}(.,.)$ is a bicharacter, see (2.20), eq. (3.1) implies that

$$a^{\mu}_{y,w} x = \mathbf{r}((w-y)\mu, \gamma) q^{-\langle (w+y)\mu, \gamma \rangle} x a^{\mu}_{y,w}, \quad \forall \mu \in \mathcal{P}, \gamma \in \mathcal{Q}_{\mathcal{S}(w)}, x \in (R_{y,w})_\gamma. \tag{4.2}$$

Recall (3.21). For $y \in W^{\leq w}$ denote the sublattice of $\mathcal{Q}_{\mathcal{S}(w)}$

$$\mathcal{Z}_{y,w} = \{ \gamma \in \mathcal{Q}_{\mathcal{S}(w)} \mid (Z(R_{y,w}))_\gamma \neq 0 \}. \tag{4.3}$$

By Theorem 3.1 (d)

(4.4) $\quad Z(R_{y,w})$ is a Laurent polynomial ring over $\mathbb{K}$ of dimension equal to $\mathrm{rank} \mathcal{Z}_{y,w}$.

The special case for untwisted quantum Schubert cell algebras of the general dimension result in the below stated Theorem 4.5 was previously obtained by Bell, Casteels, and Launois in [3], and the author in [39] under the additional condition that $\mathrm{char}\, \mathbb{K} = 0$ and q is transcendental over $\mathbb{Q}$. We give a very simple new derivation of this formula:

Proposition 4.1. [3, 39] *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and Weyl group elements $y \leq w$, the Goodearl–Letzter strata $\mathrm{Spec}_{I_w(y)} \mathcal{U}^w_-$ are homeomorphic to the spectra Laurent polynomial rings over $\mathbb{K}$ of dimension equal to* $\dim \mathrm{Ker}(w+y)$.

Proof. If $\mu \in \mathrm{Ker}_{\mathcal{P}}(w+y) = \{ \mu \in \mathcal{P} \mid (w+y)\mu = 0 \}$, then by (4.2), $a^{\mu}_{y,w} \in (Z(R_{y,w}))_{(w-y)\mu} \backslash \{0\}$. Therefore $\mathcal{Z}_{y,w} \supseteq (w-y) \mathrm{Ker}_{\mathcal{P}}(w+y) = 2w \,\mathrm{Ker}_{\mathcal{P}}(w+y)$. So $\mathrm{rank} \mathcal{Z}_{y,w} \geq \dim \ker(w+y)$.

Let $z \in (Z(R_{y,w}))_\gamma \backslash \{0\}$, $\gamma \in \mathcal{Q}_{\mathcal{S}(w)}$. The central property of z and (4.2) imply

$$a^{\mu}_{y,w} z = q^{-\langle (w+y)\mu, \gamma \rangle} z a^{\mu}_{y,w} = q^{-\langle (w+y)\mu, \gamma \rangle} a^{\mu}_{y,w} z$$

for all $\mu \in \mathcal{P}$. Since $q \in \mathbb{K}^*$ is not a root of unity, $\gamma \in ((w+y)\mathcal{P})^\perp \cap \mathcal{Q}$, i.e. $\mathcal{Z}_{y,w} \subseteq ((w+y)\mathcal{P})^\perp \cap \mathcal{Q}$. Because $\mathrm{rank}((w+y)\mathcal{P})^\perp \cap \mathcal{Q} = \dim\ker(w+y)$, $\mathrm{rank}\mathcal{Z}_{y,w} \leq \dim\ker(w+y)$. Therefore $\mathrm{rank}\mathcal{Z}_{y,w} = \dim\ker(w+y)$, and the proposition follows from Theorem 3.1 (d), cf. (4.4). □

4.2. Fix a reduced expression

$$s_{i_1} \dots s_{i_l} \tag{4.5}$$

of $w \in W$ and recall (2.12). We will identify the subexpressions of (4.5) with the subsets $D \subseteq [1,l]$. For $D \subset [1,l]$ set $s_{i_j}^D = s_{i_j}$, if $j \in D$ and $s_{i_j}^D = 1$ otherwise. Define $w_{(j)} = s_{i_1} \dots s_{i_j}$, $w_{(j)}^D = s_{i_1}^D \dots s_{i_j}^D$, $w^D = w_{(l)}^D$ and $\overline{w}_{(j)}^D = s_{i_{j+1}}^D \dots s_{i_l}^D$, $j \in [1,l]$. A subexpression $D \subseteq [1,l]$ is called a Cauchon diagram if for all $j \in [1, l-1]$, $s_{i_j} \overline{w}_{(j)}^D > \overline{w}_{(j)}^D$. Taking inverses establishes a bijection between the set of those and the set of the positive subexpressions of Marsh and Rietsch [32] of the reverse expression of (4.5). By [32, Lemma 3.5] for each $y \in W^{\leq w}$, there exists a unique Cauchon diagram $D \subseteq [1,l]$ such that $w^D = y$. Using it, define the lattice

$$\mathcal{Q}_{y,w} = \sum_{j \in [1,l] \backslash D} \mathbb{Z} w_{(j-1)}^D(\alpha_j). \tag{4.6}$$

We will see shortly that this does not depend on the choice of a reduced expression of w. The following lemma provides a second characterization of $\mathcal{Q}_{y,w}$.

Lemma 4.2. *Let $y \leq w \in W$. For a reduced expression (4.5) of w, let $D \subseteq [1,l]$ be the Cauchon diagram such that $w^D = y$. In the notation (2.12):*

$$\mathcal{Q}_{y,w} = \sum_{j \in [1,l] \backslash D} \mathbb{Z}\beta_{i_j}.$$

By [39, Lemma 3.2 (ii)], $\mathcal{Q}_{\mathcal{S}(w)} = \mathbb{Z}\beta_1 + \ldots + \mathbb{Z}\beta_l$. Thus $\mathcal{Q}_{y,w}$ is a sublattice of $\mathcal{Q}_{\mathcal{S}(w)}$. Lemma 4.2 follows at once form the fact that

$$s_{i_1} \dots s_{i_j}(\alpha_{i_{j+1}}) = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_{j+1}}) \mod (\mathbb{Z} s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j})), \quad \forall j \in [2, l-1].$$

For $y \in W^{\leq w}$ denote by $\mathrm{Supp}(\mathcal{U}_{-,\mathbf{p}}^w / I_w(y))$ the subgroup of $\mathcal{Q}_{\mathcal{S}(w)}$ generated by $\gamma \in \mathcal{Q}_{\mathcal{S}(w)}^+$ such that $(\mathcal{U}_{-,\mathbf{p}}^w / I_w(y))_\gamma \neq 0$, which is the same as the subgroup generated by $\gamma \in \mathcal{Q}_{\mathcal{S}(w)}$ such that $(R_{y,w})_\gamma \neq 0$. Obviously, these supports do not depend on $\mathbf{p}$. The next result describes them. (As a side result it also implies that $\mathcal{Q}_{y,w}$ is independent of the choice of a reduced expression of w.)

Theorem 4.3. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, Weyl group elements $y \leq w$, and 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$*

$$\mathrm{Supp}(\mathcal{U}_{-,\mathbf{p}}^w / I_w(y)) = \mathcal{Q}_{y,w}.$$

Proof. We will first show that

$$\mathrm{Supp}(\mathcal{U}_{-}^w / I_w(y)) \supseteq \mathcal{Q}_{y,w}. \tag{4.7}$$

We will use the fact that, if $y', y'' \in W$ and $i \in [1,r]$ are such that $y' s_i > y'$ and $s_i y'' > y''$, then $y' s_i y'' > y' y''$. In terms of the reduced expression (4.5), let $D \subseteq [1,l]$ be the Cauchon diagram such that $y = w^D$. Denote $[1,l] \backslash D = \{j_1 < \ldots < j_{l-|D|}\}$. For $m \in [0, l-|D|]$ define

$$D_m = D \sqcup \{j_1, \dots j_m\} = D \cup [1, j_m], \ y_m = w^{D_m}.$$

Then $y_0 = y$, $y_{l-|D|} = w$, and $y_m = w_{(j_m)} \overline{w}_{(j_m)}^D = w_{(j_m-1)} s_{i_{j_m}} \overline{w}_{(j_m)}^D$, $\forall m = 1, \dots, l-|D|$. Moreover, $y_{m-1} = w_{(j_m-1)} \overline{w}_{(j_m)}^D$, $\forall m = 1, \dots, l-|D|$. Since (4.5) is a reduced expression $w_{(j_m-1)} s_{i_{j_m}} > w_{(j_m-1)}$. Because D is a Cauchon diagram, $s_{i_{j_m}} \overline{w}_{(j_m)}^D > \overline{w}_{(j_m)}^D$. By the above mentioned fact $y = y_0 < y_1 < \ldots < y_{l-|D|} = w$. By Theorem 3.1 (a)-(b), $a_{y_m,w}^\lambda \notin I_w(y)$, $\forall m \in [1, l-|D|]$. Therefore $a_{y_m,w}^\lambda \in (\mathcal{U}_-^w / I_w(y))_{(w-y_m)\lambda}$ and $a_{y_{m-1},w}^\lambda \in (\mathcal{U}_-^w / I_w(y))_{(w-y_{m-1})\lambda}$.

By an easy computation one obtains that $y_m\lambda = y_{m-1}\lambda - \langle \overline{w}^D_{(j_m)}\lambda, \alpha^\vee_{i_{j_m}}\rangle \beta_{j_m}$ in terms of the notation (2.12). Thus

$$\langle \overline{w}^D_{(j_m)}\lambda, \alpha^\vee_{i_{j_m}}\rangle \beta_{j_m} \in \mathrm{Supp}(\mathcal{U}^w_-/I_w(y)), \quad \forall \lambda \in \mathcal{P}^+, m \in [1, l-|D|].$$

Since $\gcd\{\langle \overline{d}^D_{(j_m)}\lambda, \alpha^\vee_{i_{j_m}}\rangle \mid \lambda \in \mathcal{P}^+\} = 1$,

$$\beta_j \in \mathrm{Supp}(\mathcal{U}^w_-/I_w(y)), \quad j \in [1,l]\backslash D.$$

Now (4.7) follows from Lemma 4.2.

Mériaux and Cauchon [33] gave another classification of $\mathbb{T}^{|\mathcal{S}(w)|} - \mathrm{Spec}\mathcal{U}^w_-$, associating to each $y \in W^{\leq w}$, an ideal $J_w(y) \in \mathbb{T}^{|\mathcal{S}(w)|} - \mathrm{Spec}\mathcal{U}^w_-$. It is based no the Cauchon method of deleting derivations [7], which is very different from the one in [37]. It is not known yet whether $J_w(y) = I_w(y)$, $\forall y \in W^{\leq w}$. Mériaux and Cauchon [33] proved that a certain localization of $\mathcal{U}^w_-/J_w(y)$ by an Ore subset of homogeneous elements is isomorphic to a quantum torus with generators of weight $\{\beta_{i_n} \mid n \in [1,l]\backslash D\}$, where D is the Cauchon diagram such that $y = w^D$. Combining this with Lemma 4.2 implies that $\mathrm{Supp}(\mathcal{U}^w_-/J_w(y)) = \mathcal{Q}_{y,w}$, $\forall y \in W^{\leq w}$. Since $\{I_w(y) \mid y \in W^{\leq w}\} = \{J_w(y) \mid y \in W^{\leq w}\}$ and $\mathrm{Supp}(\mathcal{U}^w_-/I_w(y)) \supseteq \mathcal{Q}_{y,w}$, $\forall y \in W^{\leq w}$ we have

$$\mathrm{Supp}(\mathcal{U}^w_-/I_w(y)) = \mathcal{Q}_{y,w}, \quad \forall y \in W^{\leq w}.$$

□

4.3. Assume that R is a $\mathbb{K}$-algebra graded by a group C. We will say that $u \in R$ is a *diagonal normal element* if $u \in R_{\gamma_0}$ for some $\gamma_0 \in C$ and for all $\gamma \in C$ there exists $t_\gamma \in \mathbb{K}^*$ such that $ur = t_\gamma ru$, $\forall r \in R_\gamma$. If R is a prime ring and $u \neq 0$, then $t \colon C \to \mathbb{K}^*$ is a character.

For all $y \leq w \in W$ there exists $n_{y,w} \in \mathbb{Z}_+$ such that all homomorphisms $\mathcal{Q}_{y,w} \to \mathbb{Z}$ have the form $\gamma \in \mathcal{Q}_{y,w} \mapsto \langle \lambda, \gamma\rangle$ for some $\lambda \in (1/n_{y,w})\mathcal{P}$.

Proposition 4.4. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, Weyl group elements $y \leq w$, 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$, and nonzero diagonal normal elements $u \in R_{y,w}$, there exits $\mu \in (1/2n_{y,w})\mathcal{P}$ such that $(w-y)\mu \in \mathcal{Q}_{\mathcal{S}(w)}$, $u \in (R_{y,w})_{(w-y)\mu}$, $\langle (w+y)\mu, \mathcal{Q}_{y,w}\rangle \subseteq \mathbb{Z}$, and*

$$ux = \mathbf{r}((w-y)\mu, \gamma) q^{-\langle (w+y)\mu, \gamma\rangle} xu, \quad \forall \gamma \in \mathcal{Q}_{y,w}, x \in (R_{y,w})_\gamma$$

in terms of $n_{y,w} \in \mathbb{Z}_+$ defined above.

Proof. The set of diagonal normal elements of an algebra graded by an abelian group is invariant under twisting. Because of this, one only needs to prove the proposition for the algebras $\mathcal{U}^w_-$, i.e. when $\mathbf{p}$ is trivial.

Let $u \in \mathcal{U}^w_-/I_w(y)$ be such that $u(a^\lambda_{y,w})^{-1} \in R_{y,w}$ is a nonzero diagonal normal element for some $\lambda \in \mathcal{P}^+$. Then (4.2) implies that u is a nonzero diagonal normal element of $\mathcal{U}^w_-/I_w(y)$. By the same reasoning, if we establish the proposition for u, then its validity for $u(a^\lambda_{y,w})^{-1}$ will follow. Recall (4.5) and (2.12). By Cauchon's method of deleting derivations [7] and the nature of the iterated Ore extension $\mathcal{U}^w_-$ given by Lemma 2.4, (see [33]), it follows that there exists a localization of $\mathcal{U}^w_-/I_w(y)$ by an Ore set of homogeneous elements, which is graded isomorphic to the $\mathbb{K}$-quantum torus $\mathcal{T}$ with generators $z_j^{\pm 1}$ (of weights β_{i_j}), $j \in D'$ for some subset $D' \subseteq [1,l]$ and relations $z_j z_{j'} = q^{-\langle \beta_{i_j}, \beta_{i_{j'}}\rangle} z_{j'} z_j$, $\forall j > j' \in D'$. Since $\mathcal{T}$ is obtained from $\mathcal{U}^w_-/I_w(y)$ by a localization by homogeneous elements, u is a diagonal normal element of $\mathcal{T}$. Let $D' = \{j_1 < \ldots < j_k\}$. Write u as a sum of monomials $t z_{j_1}^{m_1} \ldots z_{j_k}^{m_k}$, $t \in \mathbb{K}^*$, $m_1, \ldots, m_k \in \mathbb{Z}$. Let $t z_{j_1}^{m_1} \ldots z_{j_k}^{m_k}$ be one such monomial that occurs in u. By Theorem 4.3 the lattice generated by β_{i_j}, $j \in D'$ is equal to $\mathcal{Q}_{y,w}$. Since u is a diagonal normal element

$$u z_n = q^{\langle m_1 \beta_{i_{j_1}} + \ldots + m_{n-1}\beta_{i_{j_{n-1}}} - m_{n+1}\beta_{i_{j_{n+1}}} - \ldots - m_k \beta_{i_{j_k}}, \beta_{i_{j_n}}\rangle} z_n u, \quad \forall n \in [1,k]$$

and

$$\beta_{i_{j_n}} \in \mathcal{Q}_{y,w} \mapsto \langle m_1 \beta_{i_{j_1}} + \ldots + m_{n-1}\beta_{i_{j_{n-1}}} - m_{n+1}\beta_{i_{j_{n+1}}} - \ldots - m_k \beta_{i_{j_k}}, \beta_{i_{j_n}}\rangle \in \mathbb{Z}, \;\; n \in [1,k]$$

defines a group homomorphism. Therefore there exists $\mu_0 \in (1/n_{y,w})\mathcal{P}$ such that

$$ux = q^{\langle \mu_0, \gamma \rangle} xu, \ \ \forall x \in \mathcal{T}_\gamma, \gamma \in \mathcal{Q}_{y,w}. \tag{4.8}$$

This implies that (4.8) holds for all $x \in (\mathcal{U}_-^w/I_w(y))_\gamma$ and thus for all $x \in (R_{y,w})_\gamma$. Let $u \in (\mathcal{U}_-^w/I_w(y))_{\gamma_0}$, $\gamma_0 \in \mathcal{Q}_{y,w}$. Using (4.2) we obtain that for all $\lambda \in \mathcal{P}$

$$ua_{y,w}^\lambda = q^{\langle \mu_0, (w-y)\lambda \rangle} a_{y,w}^\lambda u = q^{\langle \mu_0, (w-y)\lambda \rangle} q^{-\langle (w+y)\lambda, \gamma_0 \rangle} u a_{y,w}^\lambda.$$

Since $q \in \mathbb{K}^*$ is not a root of unity $\langle \mu_0, (w-y)\lambda \rangle - \langle (w+y)\lambda, \gamma_0 \rangle = 0$, i.e.

$$\langle \lambda, w^{-1}(\mu_0 - \gamma_0) \rangle = \langle \lambda, y^{-1}(\mu_0 + \gamma_0) \rangle, \quad \forall \lambda \in \mathcal{P}.$$

Thus $w^{-1}(\mu_0 - \gamma_0) = y^{-1}(\mu_0 + \gamma_0)$ and

$$(wy^{-1} - 1)(-\mu_0) = (wy^{-1} + 1)\gamma_0.$$

By standard linear algebra for Cayley transforms there exits $\mu \in \mathrm{Span}_\mathbb{Q}\{\alpha_i\}_{i=1}^r$ such that

$$\gamma_0 = (wy^{-1} - 1)y\mu = (w - y)\mu \ \text{ and } \ -\mu_0 = (wy^{-1} + 1)y\mu = (w + y)\mu$$

(see e.g. the proof of [40, Theorem 3.6]). We have $w\mu = (\gamma_0 - \mu_0)/2 \in (1/2n_{y,w})\mathcal{P}$, so $\mu \in (1/2n_{y,w})\mathcal{P}$. Furthermore $(w-y)\mu = \gamma_0 \in \mathcal{Q}_{\mathcal{S}(w)}$ and $\langle (w+y)\mu, \mathcal{Q}_{y,w} \rangle = \langle -\mu_0, \mathcal{Q}_{y,w} \rangle \subseteq \mathbb{Z}$, so μ satisfies all required properties. This completes the proof of the proposition. □

By developing further the arguments of the proof of Proposition 4.4 we will classify all normal elements of the localizations $R_{y,w}$ in a forthcoming publication.

4.4. Denote the lattice

$$\mathcal{L}_{y,w,\mathbf{p}} = \{(w-y)\mu \mid \mu \in \mathcal{P}, \ \mathbf{r}((w-y)\mu, \gamma) q^{-\langle (w+y)\mu, \gamma \rangle} = 1, \ \forall \gamma \in \mathcal{Q}_{y,w}\}, \tag{4.9}$$

recall (4.6). One can equivalently use only $\mu \in \mathcal{P}_{\mathcal{S}(w)}$ in (4.9) because of (2.21).

The following theorem provides an explicit formula for the dimensions of the Goodearl–Letzter strata $\mathrm{Spec}_{I_w(y)} \mathcal{U}_{-,\mathbf{p}}^w$ and completes the set theoretic description of the spectra of all multiparameter quantum Schubert cell algebras $\mathcal{U}_{-,\mathbf{p}}^w$.

Theorem 4.5. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, Weyl group elements $y \leq w$, and 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$, the Goodearl–Letzter strata $\mathrm{Spec}_{I_w(y)} \mathcal{U}_{-,\mathbf{p}}^w$ are homeomorphic to the spectra of Laurent polynomial rings over $\mathbb{K}$ of dimension equal to the rank of the lattice $\mathcal{L}_{y,w,\mathbf{p}}^u$, see* (4.9).

Proof. Recall the definition of $n_{y,w} \in \mathbb{Z}_+$ from §4.3 and define the lattice

$$\mathcal{L}'_{y,w,\mathbf{p}} = \{(w-y)\mu \mid \mu \in (1/2n_{y,w})\mathcal{P}, \ (w-y)\mu \in \mathcal{Q}_{\mathcal{S}(w)}, \ \langle (w+y)\mu, \mathcal{Q}_{y,w} \rangle \subseteq \mathbb{Z},$$
$$\mathbf{r}((w-y)\mu, \gamma) q^{-\langle (w+y)\mu, \gamma \rangle} = 1, \ \forall \gamma \in \mathcal{Q}_{y,w}\}.$$

It follows from (4.2) that $a_{y,w}^\mu \in (\mathcal{Z}_{y,w})_{(w-y)\mu} \backslash \{0\}$ for all $\mu \in \mathcal{P}$ that satisfy the equation in (4.9). Therefore $\mathcal{Z}_{y,w} \supseteq \mathcal{L}_{y,w,\mathbf{p}}$. Since every central element of a graded ring is a diagonal normal element, Proposition 4.4 implies that $\mathcal{Z}_{y,w} \subseteq \mathcal{L}'_{y,w,\mathbf{p}}$. Therefore

$$\mathcal{L}_{y,w,\mathbf{p}} \subseteq \mathcal{Z}_{y,w} \subseteq \mathcal{L}'_{y,w,\mathbf{p}}.$$

One easily shows that the index $[\mathcal{L}'_{y,w,\mathbf{p}} : \mathcal{L}_{y,w,\mathbf{p}}]$ is finite, starting from $[\mathcal{P} : (1/2n_{y,w})\mathcal{P}] < \infty$. Therefore

$$\mathrm{rank}\mathcal{L}_{y,w,\mathbf{p}} = \mathrm{rank}\mathcal{Z}_{y,w} = \mathrm{rank}\mathcal{L}'_{y,w,\mathbf{p}}.$$

The theorem follows from the fact (4.4) that $Z(R_{y,w})$ is a Laurent polynomial ring of dimension $\mathrm{rank}\mathcal{Z}_{y,w}$ and Theorem 3.1 (d). □

Remark 4.6. Cauchon [7] proved that for a CGL extensions R and a torus invariant prime ideal I of R, R/I admits a localization, which is isomorphic to a quantum torus. Computing the dimension of the corresponding Goodearl–Letzter stratum amounts to computing the dimension of its center. Bell and Launois applied this to obtain a formula in [2, Proposition 3.3] for uniparameter CGL extensions, which correspond to cases when the Cauchon localization is a special kind of quantum torus with exchange relations involving powers of q. For cocycles $\mathbf{p}$ taking values in the cyclic subgroup of $\mathbb{K}^*$ generated by q, this leads to a formula for the Goodearl–Letzter strata of $\mathcal{U}^w_{-,\mathbf{p}}$ in terms of the dimensions of the kernels of large square matrices of size $\ell(w) - \ell(y)$, which is less efficient than Theorem 4.5.

5. Equivariant polynormality, normal separation, and catenarity of $\mathcal{U}^w_{-,\mathbf{p}}$

5.1. In this section, using results of our previous paper [41], we prove that all $\mathbb{T}^{|\mathcal{S}(w)|}$-prime ideals of $\mathcal{U}^w_{-,\mathbf{p}}$ are equivariantly polynormal, that the spectra $\mathrm{Spec}\mathcal{U}^w_{-,\mathbf{p}}$ are normally separated, and that all algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are catenary.

We start with a brief review of equivariant polynormality. Assume that a ring R is equipped with an action of a group Γ by algebra automorphism. We say that an element $u \in R$ is Γ-normal if it is a Γ-eigenvector and if there exists $g \in \Gamma$ such that $ur = (g.r)u$ for all $r \in R$. Sometimes Γ-normality is defined requiring only the second condition, see [13], but this is not sufficient to extend the definition to Γ-polynormality for Γ-stable ideals of R. Given a Γ-stable ideal I of R, we say that a sequence $u_1, \ldots, u_N \in R$ is a Γ-polynormal generating sequence of I if $\{u_1, \ldots, u_N\}$ generates I and for all $i = 1, \ldots, N$ the element u_i is a Γ-normal element of R modulo the ideal generated by $u_1, \ldots, u_{i-1}$. Note that it follows from the the conditions on the elements $u_1, \ldots, u_{i-1}$ that the two-sided ideal of R generated by them is Γ-stable and equals $Ru_1 + \ldots + Ru_i$. In particular, R is generated by $u_1, \ldots, u_N$ as a one-sided (left or right) ideal.

For all $\mu \in \mathcal{P}_{\mathcal{S}(w)}$, $\vartheta \in \mathcal{Q}_{\mathcal{S}(w)}$ there exists a unique $t_{\mu,\vartheta} \in \mathbb{T}^{|\mathcal{S}(w)|}$ such that

$$(t_{\mu,\vartheta})^\gamma = \mathbf{r}(\vartheta, \gamma) q^{\langle \vartheta - 2w\mu, \gamma \rangle}, \ \ \forall \gamma \in \mathcal{Q}_{\mathcal{S}(w)},$$

cf. (2.22) and (2.26). It is clear that the map $(\mu, \vartheta) \in \mathcal{P}_{\mathcal{S}(w)} \times \mathcal{Q}_{\mathcal{S}(w)} \mapsto t_{\mu,\vartheta} \in \mathbb{T}^{|\mathcal{S}(w)|}$ is a group homomorphism. Thus its image

$$\mathcal{P}\mathcal{Q}_{w,\mathbf{p}} = \{ t_{\mu,\vartheta} \mid \mu \in \mathcal{P}_{\mathcal{S}(w)}, \vartheta \in \mathcal{Q}_{\mathcal{S}(w)} \} \tag{5.1}$$

is a subgroup of $\mathbb{T}^{|\mathcal{S}(w)|}$, which is a quotient of $\mathcal{P}_{\mathcal{S}(w)} \times \mathcal{Q}_{\mathcal{S}(w)}$.

For every $\lambda \in \mathcal{P}^+_{\mathcal{S}(w)}$ and $y \leq w \in W$ fix a finite subset $\mathcal{V}_{y,w}(\lambda)^* \subset V(\lambda)^*$ consisting of homogeneous elements, such that the restriction map

$$\xi \in V(\lambda)^* \mapsto \xi|_{\mathcal{U}_+ T_w v_\lambda} \in (\mathcal{U}_+ T_w v_\lambda)^*$$

sends bijectively $\mathcal{V}_{y,w}(\lambda)^*$ to a basis of $(\mathcal{U}_- T_y v_\lambda \cap \mathcal{U}_+ T_w v_\lambda)^\perp \subset (\mathcal{U}_+ T_w v_\lambda)^*$. Given $\Omega \subset \mathcal{P}^+_{\mathcal{S}(w)}$ denote

$$\mathcal{V}_{y,w}(\Omega) = \bigsqcup_{\lambda \in \Omega} \mathcal{V}_{y,w}(\lambda).$$

Define the maps

$$\mathrm{hw} \colon \mathcal{V}_{y,w}(\Omega) \to \Omega, \mathrm{wt} \colon \mathcal{V}_{y,w}(\omega) \to \mathcal{P}, \quad \mathrm{hw}(\xi) = \lambda, \mathrm{wt}(\xi) = \nu, \ \text{ if } \ \xi \in (V(\lambda)^*)_\nu.$$

Denote the partial ordering on $\mathcal{V}_{y,w}(\Omega)$:

$$\xi \preceq \xi', \ \text{ if } \ \mathrm{hw}(\xi) = \mathrm{hw}(\xi') \ \text{ and } \ \mathrm{wt}(\xi) \leq \mathrm{wt}(\xi'). \tag{5.2}$$

Theorem 5.1. *Assume that $\mathbb{K}$ is an arbitrary base field, $q \in \mathbb{K}^*$ is not a root of unity, $\mathfrak{g}$ is an arbitrary simple Lie algebra, $w \in W$, and $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$ is a normalized 2-cocycle.*

(a) Since $\mathcal{U}^w_{-,\mathbf{p}}$ is noetherian, for each $y \in W^{\leq w}$ there exists a finite subset Ω of $\mathcal{P}^+_{\mathcal{S}(w)}$ such that the ideal $I_w(y)$ is generated by $b^\lambda_{w,\xi}$ for $\lambda \in \Omega$, $\xi \in V(\lambda)^*$, recall (3.6). Fix any linear ordering $\xi_1 < \ldots < \xi_N$ on $\mathcal{V}_{y,w}(\Omega)$, which is a refinement of the partial ordering (5.2). Then

$$b^{\mathrm{hw}(\xi_1)}_{w,\xi_1}, \ldots, b^{\mathrm{hw}(\xi_N)}_{w,\xi_N}$$

is a $\mathbb{T}^{|\mathcal{S}(w)|}$-polynormal generating sequence of the ideal $I_w(y)$. More precisely

$$b^{\mathrm{hw}(\xi_i)}_{w,\xi_i} x = (t_{\mathrm{hw}(\xi_i),\mathrm{wt}(\xi_i)+w(\mathrm{hw}(\xi_i))}.x) x b^{\mathrm{hw}(\xi)}_{w,\xi_i} \mod \left(\mathcal{U}^w_{-,\mathbf{p}} b^{\mathrm{hw}(\xi_1)}_{w,\xi_1} + \ldots + \mathcal{U}^w_{-,\mathbf{p}} b^{\mathrm{hw}(\xi_{i-1})}_{w,\xi_{i-1}} \right)$$

for all $x \in \mathcal{U}^w_{-,\mathbf{p}}$, $i = 1, \ldots, N$. All automorphism related to the above normal elements come from the subgroup $\mathcal{PQ}_{w,\mathbf{p}}$ of $\mathbb{T}^{|\mathcal{S}(w)|}$, cf. (5.1).

(b) If the base field $\mathbb{K}$ has characteristic 0 and q is transcendental over $\mathbb{Q}$, then the conclusion of part (a) is valid for $\Omega = \{\omega_i \mid i \in \mathcal{S}(w)\}$.

Proof. The untwisted cases of parts (a) and (b) of Theorem 5.1 were proved in [41, Theorem 3.6] and [41, Theorem 3.4], respectively. The theorem follows from this, since all elements of the polynormal generating sets are homogeneous with respect to the grading (2.3). □

A special case of Theorem 5.1 gives a constructive proof of a conjecture of Brown and Goodearl, [4, Conjecture II.10.9], that the torus invariant prime ideals of the multiparameter algebras of quantum matrices have polynormal generating sets consisting of quantum minors. (This is an extension of the Goodearl–Lenagan conjecture [16] on polynormality in single parameter quantum matrix algebras, which we proved in [41].) Artin, Schelter, and Tate showed [1, p. 889] (see also [15, Lemma 3.6]) that the multiparameter algebras of quantum matrices $\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$ are obtained by twists from the single parameter ones if λ is a square root of q in $\mathbb{K}$ (provided that such exists). In [33, Proposition 2.1.1] (and [41, Lemma 4.1]) the single parameter algebras of quantum matrices were realized as special cases of the algebras $\mathcal{U}^w_+$ (and $\mathcal{U}^w_-$) in a way that matches the corresponding gradings by free abelian groups. Therefore the isomorphism of [41, Lemma 4.1] realizes $\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$ as a special case of the algebras $\mathcal{U}^w_{-,\mathbf{p}}$. In this case by [41, Lemma 4.3], if $\operatorname{char} \mathbb{K} = 0$ and q is transcendental over $\mathbb{Q}$, the generating sets from Theorem 5.1 (b) consist of quantum minors, which are explicitly listed in [41, Theorem 4.4]. More generally, for all simple Lie algebras $\mathfrak{g}$ and Weyl group elements w, if $\operatorname{char} \mathbb{K} = 0$ and q is transcendental over $\mathbb{Q}$, then in Theorem 5.1 one can choose $\Omega = \{\omega_1, \ldots, \omega_r\}$ by [41, Theoerm 3.4]. The last two facts rely on a theorem of Joseph [28, Théorème 3] and need the stronger assumption on $\mathbb{K}$ and q because of a specialization argument.

5.2. Finally we establish normal separation of $\operatorname{Spec} \mathcal{U}^w_{-,\mathbf{p}}$ and catenarity of $\mathcal{U}^w_{-,\mathbf{p}}$.

Theorem 5.2. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, simple Lie algebras $\mathfrak{g}$, $w \in W$, and 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$, the spectra of the algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are normally separated.*

Proof. Using Goodearl's result [13, Corollary 4.6] it suffices to show that $\mathcal{U}^w_{-,\mathbf{p}}$ has graded normal separation. This can be proved in two different ways. Firstly, assume that $I_1 \subsetneq I_2$ are two graded (i.e. $\mathbb{T}^{|\mathcal{S}(w)|}$-invariant) prime ideals of $\mathcal{U}^w_{-,\mathbf{p}}$. If $x_1, \ldots, x_n$ is a $\mathbb{T}^{|\mathcal{S}(w)|}$-polynormal generating sequence of I_2 as in Theorem 5.1 (a), then the first x_i which does not belong to I_1 produces a graded normal separating element for the pair of ideals I_1, I_2. Secondly, by Theorem 3.1 (a) for every pair $I_1 \subsetneq I_2$ of graded prime ideals of $\mathcal{U}^w_{-,\mathbf{p}}$, there exist $y_1 < y_2 \in W^{\leq w}$ such that $I_1 = I_w(y_1)$, $I_2 = I_w(y_2)$. Let $\lambda \in \mathcal{P}^{++}$. Then $y_1\lambda > y_2\lambda$, so $a^\lambda_{y_1,w} \in I_w(y_2)$. Theorem 3.1 (c) implies that $a^\lambda_{y_1,w} \notin I_w(y_1)$ and eq. (3.1) gives that $a^\lambda_{y_1,w}$ is normal modulo $I_w(y_1)$. □

Recall that a ring R is Auslander–Gorenstein if the injective dimension of R (as both right and left R-module) is finite, and for all integers $0 \leq i < j$ and finitely generated (right or left) R-modules M, we have $\operatorname{Ext}^i_R(N, R) = 0$ for all R-submodules N of $\operatorname{Ext}^j_R(M, R)$. A ring R is

Auslander regular if, in addition, the global dimension of R is finite. The grade of a finitely generated R-module M is given by

$$j(M) = \inf\{i \geq 0 \mid \mathrm{Ext}^i_R(M,R) \neq 0\}.$$

An algebra R is Cohen–Macauley if

$$j(M) + \mathrm{GK}\dim M = \mathrm{GK}\dim R$$

for all finitely generated R-modules M. We will need the following two results:

Theorem 5.3. *(Goodearl–Lenagan,* [14]*) Assume that A is an affine, noetherian, Auslander–Gorenstein and Cohen–Macaulay algebra over a field, with finite Gelfand–Kirillov dimension. If* Spec$A$ *is normally separated, then A is catenary. If, in addition, A is a prime ring, then Tauvel's height formula holds.*

Tauvel's height formula holds for a ring R if for all prime ideals J of R, the height of J is equal to

$$\mathrm{GK}\dim R - \mathrm{GK}\dim(R/J).$$

Theorem 5.4. *(Ekström, Levasseur–Stafford,* [11, 29]*) Assume R is a noetherian, Auslander regular ring. Let $S = R[x;\sigma,\delta]$ be an Ore extension of R. Then:*

(a) S is Auslander regular.

(b) If $R = \oplus_{k\geq 0} R_k$ is a connected graded Cohen–Macauley $\mathbb{K}$-algebra over a field $\mathbb{K}$ such that $\sigma(R_k) \subseteq R_k$ for all $k \geq 0$, then S is Cohen–Macauley.

The next theorem proves that all algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are catenary.

Theorem 5.5. *For all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, simple Lie algebras $\mathfrak{g}$, $w \in W$, and and 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}_{\mathcal{S}(w)}, \mathbb{K}^*)_n$, the algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are Auslander regular, Cohen–Macauley, catenary, and Tauvel's height formula holds for them. For all $y \in W^{\leq w}$ the height of prime ideal $I_w(y)$ is equal to $\ell(y)$ and*

$$\mathrm{GK}\dim(\mathcal{U}^w_{-,\mathbf{p}}/I_w(y)) = \ell(w) - \ell(y).$$

Proof. Lemma 2.4 (a)-(b) and Theorem 5.4 (a) imply that $\mathcal{U}^w_{-,\mathbf{p}}$ is Auslander regular. Given $\lambda \in \mathcal{P}^{++}$, we can specialize the $-\mathcal{Q}^+_{\mathcal{S}(w)}$-grading of $\mathcal{U}^w_{-,\mathbf{p}}$ from (2.3) to an $\mathbb{N}$-grading by

$$(\mathcal{U}^w_{-,\mathbf{p}})_n = \{x \in (\mathcal{U}^w_{-,\mathbf{p}})_{-\gamma} \mid \gamma \in \mathcal{Q}^+_{\mathcal{S}(w)}, \langle \lambda, \gamma \rangle = n\}, \ \ n \in \mathbb{N}.$$

Obviously the $\mathbb{T}^{|\mathcal{S}(w)|}$-action (2.23) preserves each graded component $(\mathcal{U}^w_{-,\mathbf{p}})_n$ and $\mathcal{U}^w_{-,\mathbf{p}}$ is connected. Lemma 2.4 (a)-(b) and Theorem 5.4 (b) imply that $\mathcal{U}^w_{-,\mathbf{p}}$ is Cohen–Macauley. It also follows from Lemma 2.4 (a)-(b) that the algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are affine, noetherian domains with $\mathrm{GK}\dim \mathcal{U}^w_{-,\mathbf{p}} = \ell(w)$. Theorem 5.3 implies that all algebras $\mathcal{U}^w_{-,\mathbf{p}}$ are catenary and satisfy Tauvel's height formula.

Since the Gelfand–Kirillov dimension of a graded algebra does not change under twisting, the last statement of the theorem follows from [41, Theorem 5.8]. □

Some special cases of the untwisted case of Theorems 5.1, 5.2, and 5.5 were established by Caldero [6], Cauchon [8], Goodearl–Lenagan [14], and Malliavin [31]. The general untwisted case of Theorems 5.1, 5.2, and 5.5 was proved by the author in [41].

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