# NONCOMMUTATIVE BISPECTRAL DARBOUX TRANSFORMATIONS

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Abstract. We prove a general theorem establishing the bispectrality of noncommutative Darboux transformations. It has a wide range of applications that establish bispectrality of such transformations for differential, difference and $q$-difference operators with values in all noncommutative algebras. All known bispectral Darboux transformations are special cases of the theorem. Using the methods of quasideterminants and the spectral theory of matrix polynomials, we explicitly classify the set of bispectral Darboux transformations from rank one differential operators and Airy operators with values in matrix algebras. These sets generalize the classical Calogero–Moser spaces and Wilson's adelic Grassmannian.

## 1. Introduction

**1.1.** The study of bispectral operators was initiated by Duistermaat and Grünbaum in [7] motivated by their applications to computer tomography and time-band limiting. Since 1985 the bispectral problem has attracted a lot of attention in pure mathematics and has been related to many diverse areas, such as integrable systems (KP hierarchy, Sato's Grassmannian and the Calogero-Moser system) [2, 7, 28, 31, 32], orthogonal polynomials [20, 19], representation theory ($W_{1+\infty}$, Virasoro and Kac–Moody algebras) [3, 12], ideal structure and automophisms of the first Weyl algebra [1, 4] and many others. Let $\Omega_1$ and $\Omega_2$ be two domains in $\mathbb{C}$. The continuous-continuous (scalar) bispectral problem asks for finding all analytic functions

$$\Psi \colon \Omega_1 \times \Omega_2 \to \mathbb{C}$$

for which there exist analytic differential operators $L(x, \partial_x)$ and $\Lambda(z, \partial_z)$ on $\Omega_1$ and $\Omega_2$, and two analytic functions $\theta \colon \Omega_1 \to \mathbb{C}$, $f \colon \Omega_2 \to \mathbb{C}$, such that

$$L(x, \partial_x)\Psi(x, z) = f(z)\Psi(x, z), \tag{1.1}$$

$$\theta(x)\Psi(x, z) = \Lambda(z, \partial_z)\Psi(x, z) \tag{1.2}$$

on $\Omega_1 \times \Omega_2$. There are discrete, $q$-difference and mixed versions of the problem [20, 19, 25], which have played a similarly important role.

The noncommutative version of the problem is stated for a (generally noncommutative) associative complex finite dimensional algebra $R$. It asks for classifying all $R$-valued analytic functions

$$\Psi \colon \Omega_1 \times \Omega_2 \to R$$

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for which there exist $R$-valued analytic differential operators $L(x,\partial_x)$ and $\Lambda(z,\partial_z)$ on $\Omega_1$ and $\Omega_2$, and $R$-valued analytic functions $\theta\colon \Omega_1 \to R$, $f\colon \Omega_2 \to R$, such that

$$L(x,\partial_x)\Psi(x,z) = f(z)\Psi(x,z), \tag{1.3}$$

$$\theta(x)\Psi(x,z) = \Psi(x,z)\Lambda(z,\partial_z) \tag{1.4}$$

on $\Omega_1 \times \Omega_2$. The right action of differential operators in the second equality is defined by

$$\Psi(x,z)\cdot(a(z)\partial_z^k) := (-1)^k\partial_z^k\left(\Psi(x,z)a(z)\right)$$

and linearity. It was observed in [8, 21, 29] that the right action was needed in order to obtain nontrivial examples which were absent if one insisted on left actions only. More precisely, if both actions are left, then they do not commute with each other because the algebra $R$ is noncommutative in general. The left action in $x$ and the right action in $z$ commute with each other, thus giving rise to a bimodule for two noncommutative algebras of differential operators.

The matrix bispectral problem is the problem in which one takes $R = M_n(\mathbb{C})$ in the above definition. It was first considered in [34, 35] with relation to the ZS-AKNS operators. In the last 20 years there has been a great deal of research on the continuous and discrete versions of the matrix bispectral problem and numerous relations were found with matrix orthogonal polynomials [9, 23, 30], spherical functions [22], and many other topics.

The construction and classification of noncommutative (or even just matrix) bispectral functions is significantly harder than the scalar problem. One of the major methods in the scalar case is the use of Darboux transformations for obtaining large families of complicated examples from simpler ones. In this approach, the first equation (1.1) for the transformed function is direct while the second one is highly nontrivial. This was carried out for second order operators in [7], all rank 1 bispectral operators [31] in Wilson's classification, Bessel functions [2], Airy functions [2, 28], orthogonal polynomials [20] and many other situations. Even though these results are of similar nature, their proofs relied on a variety of different algebraic, algebro-geometric and integrable systems techniques. A general theorem for bispectrality of scalar Darboux transformations was obtained in [1]. In some cases the algebras of dual bispectral operators were also classified [24, 26, 32].

**1.2.** In this paper we obtain two major sets of results. The first one is a very general theorem proving that Darboux transformations from noncommutative bispectral functions produce again noncommutative bispectral functions. All previous examples that we are aware of can be recovered as special cases of this theorem, for example all functions in [6, 18, 27, 34, 35]. All results on bispectrality of scalar Darboux transformations [1, 2, 7, 20, 25, 26, 28, 31] also arise as special cases of this theorem.

Our theorem establishes a general bispectrality type result for all Darboux transformations in bimodules of noncommutative algebras $B$. These noncommutative algebras $B$ can be taken to be the algebra of analytic differential operators with values in a finite dimensional algebra $R$, the algebras of difference or $q$-difference operators with values in $R$, and various other examples of a similar nature. For the sake of simplicity in the introduction we formulate only the special case of the theorem when the algebra $B$ is the first one in the list. We refer the reader to Theorem 2.1 for the general result for bispectral Darboux transformations in bimodules of noncommutative algebras.

Let $\Psi\colon \Omega_1 \times \Omega_2 \to R$ be a noncommutative bispectral function as in (1.3)–(1.4). We will assume that it is a nonsplit function in the two variables, in the sense that there are no nonzero analytic differential operators $L(x, \partial_x)$ and $\Lambda(z, \partial_z)$ that satisfy (1.3) or (1.4) with $f(z) \equiv 0$ or $\theta(x) \equiv 0$.

To formulate our result, we need to introduce several algebras associated to the function $\Psi(x, z)$. Let $B_1$ be the algebra of all $R$-valued analytic differential operators $P(x, \partial_x)$ on $\Omega_1$ for which there exits an $R$-valued analytic differential operator $S(z, \partial_z)$ on $\Omega_2$ such that

$$P(x, \partial_x)\Psi(x, z) = \Psi(x, z)S(z, \partial_z).$$

Let $B_2$ be the algebra consisting of all such operators $S(z, \partial_z)$. The nonsplit condition implies that the map

$$b\colon B_1 \to B_2, \quad \text{given by} \quad b(P(x, \partial_x)) := S(z, \partial_z),$$

is a well defined algebra isomorphism. Denote by $M$ the $B_1 - B_2$ bimodule of $R$-valued analytic functions on $\Omega_1 \times \Omega_2$. Let $K_i$ be the subalgebras of $B_i$ consisting of $R$-valued analytic functions, i.e. the differential operators in $B_i$ of order 0. Let

$$A_1 := b^{-1}(K_2), \quad A_2 := b(K_1).$$

These are the algebras of bispectral operators corresponding to $\Psi(x, z)$. They consist of the differential operators that have the properties (1.3)–(1.4) for some functions $f(z)$, $\theta(x)$. In particular,

$$L(x, \partial_x) \in A_1 \quad \text{and} \quad \Lambda(z, \partial_z) \in A_2.$$

**Theorem 1.1.** *In the above setting, for every factorization,*

$$L(x, \partial_x) = Q(x, \partial_x)g(x)^{-1}P(x, \partial_x)$$

*with $L(x, \partial_x) \in A_1$, $P(x, \partial_x), Q(x, \partial_x) \in B_1$, and a regular element $g(x) \in K_1$, the $R$-valued function*

$$\Phi(x, z) := P(x, \partial_x)\Psi(x, z)\colon \Omega_1 \times \Omega_2 \to R$$

*is bispectral, and, more precisely, satisfies*

$$\begin{aligned} &\left(P(x, \partial_x)Q(x, \partial_x)g(x)^{-1}\right)\Phi(x, z) = \Phi(x, z)b(L)(z) \\ &\qquad\qquad g(x)\Phi(x, z) = \Phi(x, z)\left(b(L)(z)^{-1}b(Q)(z, \partial_z)b(P)(z, \partial_z)\right). \end{aligned}$$

We recall that a regular element of an algebra is one that is not left or right zero divisor.

The general version of the above result (given in Theorem 2.1) deals with arbitrary noncommutative algebras $B_1, B_2$ and $B_1 - B_2$ bimodules $M$. It presents a general approach to establish bispectrality of Darboux transformations in bimodules of noncommutative algebras.

Previously, combinations of algebraic, algebro-geometric and integrable systems techniques were used to obtain bispectrality on a case by case basis. This theorem, not only proves all these results with a single method, but also has a wide range of new applications. The theorem directly relates the method of Darboux transformations from integrable systems to noncommutative algebra. We expect that it will be also helpful in relating bispectrality to matrix factorization [10].

**1.3.** The second set of results in the paper classifies the bispectral Darboux transformations in Theorem 1.1 in several important cases. The rank 1 bispectral functions of Wilson [31] have played a key role in the classification of scalar bispectral functions. They were first introduced in [31] in terms of wave functions for the KP hierarchy with unicursal spectral curves. In [2] it was proved that they are precisely the set of all bispectral Darboux transformations from the function $\Psi(x,z) = e^{xz}$. Our first result classifies all bispectral Darboux transformations from the function

$$\Psi(x,z) = e^{xz} I_n \quad \text{in the matrix case} \quad R = M_n(\mathbb{C}).$$

In this case $\Omega_1 = \Omega_2 = \mathbb{C}$, and $B_1$ and $B_2$ are the algebras of matrix differential operators with polynomial coefficients in the variables $x$ and $z$, respectively. The isomorphism $b$ is given by

$$b(\partial_x) = z,\ b(x) = -\partial_z, b(W) = W, \quad \forall\, W \in M_n(\mathbb{C}).$$

The algebras $K_1$ and $K_2$ are the algebras of matrix-valued polynomials in $x$ and $z$. The algebras $A_1$ and $A_2$ are the algebras of matrix differential operators in $x$ and $z$ with constant coefficients. The classification result is as follows:

**Theorem 1.2.** *The bispectral Darboux transformations from the function $e^{xz}I_n$ as in Theorem 1.1 for operators $L, Q, P$ with invertible leading terms are precisely the functions of the form*

$$\Phi(x,z) = W(x)P(x,\partial_x)(e^{xz}I_n),$$

*where $W(x)$ is an invertible matrix-valued rational function and $P(x,\partial_x)$ is a monic matrix differential operator of order $k$ that has a nondegenerate vector kernel with a basis of the form*

$$e^{\alpha_1 x}p_1(x), \dots, e^{\alpha_{kn} x}p_{kn}(x)$$

*for some (not necessarily distinct) $\alpha_j \in \mathbb{C}$ and vector valued polynomials $p_j(x) \in \mathbb{C}[x]^n$.*

The operators $P(x,\partial_x)$ are uniquely reconstructed from their vector kernels using quasideterminants, see Sect. 3.2. The restriction to operators $L, Q, P$ with invertible leading terms is necessary because of the standard pathological problems with leading terms that are zero divisors. See Definition 3.1 for the notion of nondegenerate space of vector-valued meromorphic functions.

In [31, 32], all rank 1 scalar bispectral functions were shown to be parametrized by the points of the adelic Grassmannian and, equivalently, the points of all Calogero–Moser spaces. The classification in Theorem 1.2 and Theorem 4.2 in [33] can be used to show that the above class coincides with the recently constructed classes of matrix bispectral functions by Wilson [33] and Bergvelt, Gekhtman and Kasman [5] in relation to certain solutions of the multicomponent KP hierarchy and the Gibbons–Hermsen system parametrized by the vector adelic Grassmannian. (More precisely, to match the two families, our bispectral functions should be multiplied by $W(x)^{-1}$ and by an invertible matrix-valued rational function in $z$ to make the leading terms of their asymptotic expansions at $\infty$ equal to $e^{xz}I_n$.) One should note that, even though the starting function $\Psi(x,z) = e^{xz}I_n$ is scalar matrix-valued, the set of bispectral matrix Darboux transformations from it is much more complicated than the set of bispectral Darboux transformation from $e^{xz}$, the latter being the union of all Calogero–Moser spaces [32].

**1.4.** Our second classification result is for all bispectral matrix Darboux trasnformations from the Airy functions. The (generalized) Airy operator is the differential operator

$$M_{\mathrm{Ai}}(x,\partial_x) := \partial_x^N + \sum_{i=1}^{N-1} \alpha_i \partial_x^{N-i} + \alpha_0 x,$$

where $\alpha_0 \neq 0, \alpha_2, \ldots, \alpha_{N-1}$ are complex numbers. Its kernel is $N$-dimensional over $\mathbb{C}$. For any function $\psi(x)$ in the kernel of $M_{\mathrm{Ai}}(x,\partial_x)$, define the matrix Airy bispectral function

$$\Psi_{\mathrm{Ai}}(x,z) := \psi(x+z) I_n.$$

The bispectral Darboux transformations from it in the scalar case ($n = 1$) were classified in [2, 28] and played a key role in the treatment of the scalar bispectral problem. Here we obtain a classification of these transformations for all $n$.

In the Airy case, we again have $\Omega_1 = \Omega_2 = \mathbb{C}$. The algebras $B_1$ and $B_2$ are also the algebras of matrix-valued differential operators with polynomial coefficients in $x$ and $z$, but the isomorphism $b\colon B_1 \to B_2$ is given by

$$b(x) := M_{\mathrm{Ai}}(z,\partial_z), \;\; b(\partial_x) := -\partial_z, \;\; b(W) := W, \; \forall\, W \in M_n(\mathbb{C}).$$

Here and below $M_{\mathrm{Ai}}(x,\partial_x)$ is viewed as a scalar matrix-valued differential operator. The first two equations imply that $b(M_{\mathrm{Ai}}(x,\partial_x)) = z$. The definition of the isomorphism $b$ represents the bispectral equations for the matrix Airy function

$$M_{\mathrm{Ai}}(x,\partial_x)\Psi_{\mathrm{Ai}}(x,z) = \Psi_{\mathrm{Ai}}(x,z) z, \quad x\Psi_{\mathrm{Ai}}(x,z) = \Psi_{\mathrm{Ai}}(x,z) M_{\mathrm{Ai}}(z,\partial_z).$$

The algebras $K_1$ and $K_2$ associated to $\Psi_{\mathrm{Ai}}(x,z)$ are also the algebras of matrix-valued polynomial functions in $x$ and $z$, respectively. However, in contrast with §1.3, the algebras $A_1$ and $A_2$ consist of the operators of the form

$$q(M_{\mathrm{Ai}}(x,\partial_x)) \quad \text{and} \quad q(M_{\mathrm{Ai}}(z,\partial_z))$$

for matrix-valued polynomials $q(t)$. Denote by

$$\psi_1(x), \ldots, \psi_N(x)$$

a basis of the kernel of $M_{\mathrm{Ai}}(x,\partial_x)$. We have:

**Theorem 1.3.** *The bispectral Darboux transformations from the matrix Airy function $\Psi_{\mathrm{Ai}}(x,z)$ as in Theorem 1.1 for operators $L, Q, P$ with invertible leading terms are exactly the functions of the form*

$$\Phi(x,z) = W(x) P(x,\partial_x) \Psi_{\mathrm{Ai}}(x,z),$$

*where $W(x)$ is an invertible matrix-valued rational function and $P(x,\partial_x)$ is a monic matrix differential operator of order $dN$ whose vector kernel is a nondegenerate $ndN$-dimensional space with a basis which is a union of $N$-tuples of the form*

$$\sum_{j=0}^{N-1} \psi_i^{(j)}(x+\lambda) p_j(x), \quad i = 1, \ldots N$$

*for some matrix polynomials $p_j(x)$. (The matrix polynomials and complex numbers $\lambda \in \mathbb{C}$ can be different for different $N$-tuples of basis elements.)*

We expect that the families of bispectral Darboux transformations from the matrix Airy functions will have applications to the study of the geometry of quiver varieties and to tau-functions of integrable systems satisfying the string equation.

**1.5.** The paper is organized as follows. The proof of Theorem 1.1 and its general form for bispectral Darboux transformations in bimodules of noncommutative algebras is given in Sect. 2. This section also contains a review of the needed facts from noncommutative algebra. The classification results for matrix rank 1 Darboux transformations are in Sect. 4. The classification results for bispectral Darboux transformations from the matrix Airy function appear in Sect. 5. Sect. 6 contains examples illustrating the classification results.

The classification results rely on some facts on quasideterminants and the spectral theory of matrix polynomials. The former are reviewed in Sect. 3 and the latter in the appendix, Sect. 7.

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## 2. General Theorem

In this section we establish a theorem that proves bispectrality of noncommutative Darboux transformations in an abstract algebraic setting. After the statement of the theorem we explain how it specializes to the concrete settings of continuous and discrete matrix-valued bispectral operators.

**2.1.** Before we proceed with the statement of the theorem we recall the notion of Ore set and noncommutative localization. We refer the reader to [17, Ch. 6] for details. Let $R$ be a unital (in general noncommutative) ring. A subset $E$ of $R$ is called multiplicative if it is closed under multiplication and contains the identity element 1. A multiplicative subset $E$ of $R$ consisting of regular elements (non-zero divisors) is called a (left) Ore set if for all $e \in E$ and $r \in R$ there exist $g \in E$ and $t \in R$ such that $gr = te$. If this condition is satisfied, then one can form the localized (or quotient) ring $R[E^{-1}]$ consisting of left fractions $e^{-1}r$, where $e \in E$, $r \in R$ with a certain natural equivalence relation. The ring $R$ canonically embeds in the localization by $r \mapsto 1^{-1}r$.

For a ring $R$, we denote by $r(R)$ the multiplicative subset of its regular elements (i.e., elements that are not zero divisors).

**Theorem 2.1.** *Let $B_1$ and $B_2$ be two associative algebras and $M$ a $B_1 - B_2$ bimodule. Assume that there exists an algebra isomorphism $b\colon B_1 \to B_2$ and an element $\psi \in M$ such that*

$$P\psi = \psi b(P), \quad \forall P \in B_1.$$

*Assume that $A_i$ and $K_i$ are subalgebras of $B_i$ such that*

$$b(A_1) = K_2 \quad \text{and} \quad b(K_1) = A_2$$

*and for which the following conditions are satisfied:*

1. *$r(K_i)$ are Ore subsets of $B_i$ consisting of regular elements and*
2. *$\mathrm{Ann}_f(M) = 0$ for all $f \in r(K_i)$.*

*If $L = Qg^{-1}P$ for some $L \in A_1$, $P, Q \in B_1$, and $g \in r(K_1)$, then $\varphi := P\psi = \psi b(P)$ satisfies*

$$\left(PQg^{-1}\right)\varphi = \varphi b(L),$$
$$g\varphi = \varphi\left(b(L)^{-1}b(Q)b(P)\right).$$

Note that $b(L) \in K_2$. The two conditions ensure that the algebras $B_i$ embed in $B_i[r(K_i)^{-1}]$ and $M$ in $M[r(K_1)^{-1}][r(K_2)^{-1}]$, see e.g. [17, Ch. 10] for details. Both equalities take place in the localized module.

We prove the theorem in Sect. 2.4. In the next two subsections we describe how it is applied in the continuous–continuous and discrete-discrete noncommutative bispectral situations.

**2.2.** In the continuous–continuous noncommutative bispectral setting the theorem works as follows. Let $R$ be a complex finite dimensional algebra. For $i = 1, 2$, let $\Omega_i$ be two open subsets of $\mathbb{C}$. Let

$$\Psi\colon \Omega_1 \times \Omega_2 \to R$$

be an $R$-valued bispectral function. Recall from the introduction that this means that there exist $R$-valued analytic differential operators $L(x, \partial_x)$ and $\Lambda(z, \partial_z)$ on $\Omega_1$ and $\Omega_2$, and two $R$-valued analytic functions

$$\theta\colon \Omega_1 \to R, \quad f\colon \Omega_2 \to R,$$

such that

$$L(x, \partial_x)\Psi(x, z) = \Psi(x, z)f(z) \tag{2.1}$$

$$\theta(x)\Psi(x, z) = \Psi(x, z)\Lambda(z, \partial_z) \tag{2.2}$$

on $\Omega_1 \times \Omega_2$. We will furthermore assume that $\Psi(x, z)$ is a nonsplit function of $x$ and $z$ in the sense that it satisfies the condition

($*$) there are no nonzero analytic differential operators $L(x, \partial_x)$ and $\Lambda(z, \partial_z)$ that satisfy one of the above conditions with $f(z) \equiv 0$ or $\theta(x) \equiv 0$.

In this setting, denote by $B_1$ the algebra of all $R$-valued analytic differential operators $P(x, \partial_x)$ on $\Omega_1$ for which there exits an $R$-valued analytic differential operator $S(z, \partial_z)$ on $\Omega_2$ such that

$$P(x, \partial_x)\Psi(x, z) = \Psi(x, z)S(z, \partial_z).$$

Denote by $B_2$ the algebra of all such operators $S(z, \partial_z)$. It follows from the assumption ($*$) that the map $b\colon B_1 \to B_2$, given by $b(P(x, \partial_x)) := S(z, \partial_z)$, is a well defined algebra isomorphism. Let $M$ be the $B_1 - B_2$ bimodule consisting of $R$-valued analytic functions on $\Omega_1 \times \Omega_2$.

Let $K_i$ be the subalgebras of $B_i$ consisting of $R$-valued analytic functions. Set

$$A_1 := b^{-1}(K_2), \quad A_2 := b(K_1).$$

These are the algebras of bispectral operators corresponding to $\Psi(x, z)$. They consist of the differential operators that have the properties (2.1)–(2.2) for some functions $f(z)$, $\theta(x)$. In particular,

$$L(x, \partial_x) \in A_1 \quad \text{and} \quad \Lambda(z, \partial_z) \in A_2.$$

The sets of regular elements $r(K_i)$ consist of those $R$-valued analytic functions $\theta(x)$ and $f(z)$ that are nondegenerate (in the sense that their determinants are not identically zero on any connected component of $\Omega_i$). This immediately implies that the condition (2) in Theorem 2.1 is satisfied in this setting. By standard commutation arguments with differential operators one shows that the subsets $r(K_i)$ are Ore subsets of $B_i$.

Theorem 2.1 implies the following:

**Proposition 2.2.** *In the above setting, for every factorization,*

$$L(x, \partial_x) = Q(x, \partial_x)g(x)^{-1}P(x, \partial_x)$$

*for $L(x,\partial_x) \in A_1$, $P(x,\partial_x), Q(x,\partial_x) \in B_1$, $g(x) \in r(K_1)$, the $R$-valued function*

$$\Phi(x,z) := P(x,\partial_x)\Psi(x,z) \colon \Omega_1 \times \Omega_2 \to R$$

*is bispectral, and, more precisely, satisfies*

$$\begin{aligned}\left(P(x,\partial_x)Q(x,\partial_x)g(x)^{-1}\right)\Phi(x,z) &= \Phi(x,z)b(L)(z)\\ g(x)\Phi(x,z) &= \Phi(x,z)\left(b(L)(z)^{-1}b(Q)(z,\partial_z)b(P)(z,\partial_z)\right).\end{aligned}$$

**2.3.** In the discrete–discrete noncommutative bispectral setting Theorem 2.1 works as follows. As before, let $R$ be a complex finite dimensional algebra. For $i = 1, 2$, let $\Omega_i$ be two subsets of $\mathbb{C}$ which are invariant under the translation operator

$$T_x \colon x \mapsto x + 1$$

and its inverse. Denote by $M$ the space of $R$-valued functions on $\Omega_1 \times \Omega_2$. An $R$-valued difference operator on $\Omega_1$ is a finite sum of the form

$$\sum_{k\in\mathbb{Z}} c_k(x)T_x^k,$$

where $c_k$ are $R$-valued functions. The space $M$ is naturally equipped with the structure of a bimodule over the algebras of $R$-valued difference operators on $\Omega_1$ and $\Omega_2$, respectively, by setting

$$\left(\sum_{k\in\mathbb{Z}} c_k(x)T_x^k\right)\cdot\Psi(x,z) := \sum_{k\in\mathbb{Z}} c_k(x)\Psi(x+k,z),$$

$$\Psi(x,z)\cdot\left(\sum_{k\in\mathbb{Z}} c_k(z)T_z^k\right) := \sum_{k\in\mathbb{Z}} c_k(z-k)\Psi(x,z-k).$$

An $R$-valued discrete-discrete bispectral function is by definition an element of M (i.e., an $R$-valued function)

$$\Psi \colon \Omega_1 \times \Omega_2 \to R$$

for which there exist $R$-valued difference operators $L(x,T_x)$ and $\Lambda(z,T_z)$ on $\Omega_1$ and $\Omega_2$, and $R$-valued functions

$$\theta \colon \Omega_1 \to R, \quad f \colon \Omega_2 \to R,$$

such that

$$L(x,T_x)\Psi(x,z) = \Psi(x,z)f(z) \tag{2.3}$$

$$\theta(x)\Psi(x,z) = \Psi(x,z)\Lambda(z,T_z) \tag{2.4}$$

on $\Omega_1 \times \Omega_2$. Similarly to the continuous case, we will assume that $\Psi(x,z)$ is a nonsplit function of $x$ and $z$ in the sense that it satisfies the condition

($**$) there are no nonzero difference operators $L(x,T_x)$ and $\Lambda(z,T_z)$ that satisfy one of the above conditions with $f(z) \equiv 0$ or $\theta(x) \equiv 0$.

Let $B_1$ be the algebra of all $R$-valued difference operators $P(x,T_x)$ on $\Omega_1$ for which there exits an $R$-valued difference operator $S(z,T_z)$ on $\Omega_2$ such that

$$P(x,T_x)\Psi(x,z) = \Psi(x,z)S(z,T_z).$$

Let $B_2$ be the algebra of all such operators $S(z,\partial_z)$. The assumption ($**$) implies that the map $b \colon B_1 \to B_2$, given by $b(P(x,\partial_x)) := S(z,\partial_z)$ is a well defined algebra isomorphism. Denote by $K_i$ be the subalgebras of $B_i$ consisting of $R$-valued functions. The algebras

$$A_1 := b^{-1}(K_2), \quad A_2 := b(K_1)$$

consist of the bispectral difference operators corresponding to $\Psi(x,z)$ (i.e., difference operators in $x$ and $z$ that have the properties (2.3)–(2.4)). The sets $r(K_i)$ of regular elements of the algebras $K_i$ consist of the nonvanishing functions $\theta(x)$ and $f(z)$. It is easy to verify that the second condition in Theorem 2.1 is satisfied in this situation and that $r(K_i)$ are Ore subsets of $B_i$. We leave the details to the reader.

Theorem 2.1 implies the following:

**Proposition 2.3.** *In the above setting, for every factorization*

$$L(x,T_x) = Q(x,T_x)g(x)^{-1}P(x,T_x)$$

*with* $L(x,T_x) \in A_1$, $P(x,T_x), Q(x,T_x) \in B_1$, $g(x) \in r(K_1)$, *the function*

$$\Phi(x,z) := P(x,T_x)\Psi(x,z)$$

*is an* $R$*-valued discrete-discrete bispectral function on* $\Omega_1 \times \Omega_2$. *More precisely, it satisfies*

$$\left(P(x,T_x)Q(x,T_x)g(x)^{-1}\right)\Phi(x,z) = \Phi(x,z)b(L)(z),$$
$$g(x)\Phi(x,z) = \Phi(x,z)\left(b(L)(z)^{-1}b(Q)(z,T_z)b(P)(z,T_z)\right).$$

Analogously, Theorem 2.1 applies to noncommutative bispectral functions for $q$-difference operators and to every mixed situation, where the operators on the two sides are differential, difference, or $q$-difference of different type. We leave to the reader the formulations of those results, which are analogous to the ones above.

The bispectral functions in [6, 18, 27, 34] are special cases of the continuous-continuous version of the theorem and the ones in [8, 9, 21, 22, 23, 30] are special cases of the discrete-continuous version.

**2.4.** We finish this section with the proof of Theorem 2.1.

*Proof of Theorem 2.1.* The first statement is straightforward,

$$\left(PQg^{-1}\right)\varphi = \left(PQg^{-1}\right)P\psi = PL\psi = P\psi b(L) = \varphi b(L).$$

The key point of the theorem is the second statement. To prove it, first note that since $b\colon B_1 \to B_2$ is an isomorphism, the subalgebras $A_i$ of $B_i$ also satisfy the conditions (1)–(2). The Ore subset assumptions imply that

$$g = PL^{-1}Q \quad \text{and thus} \quad b(g) = b(P)b(L)^{-1}b(Q).$$

Therefore,

$$g\varphi = g\psi b(P) = \psi b(P)b(L)^{-1}b(Q)b(P) = \varphi\left(b(L)^{-1}b(Q)b(P)\right).$$

□

## 3. Quasideterminants, kernels and factorizations of matrix differential operators

In this section we gather some facts relating the factorizations of matrix-valued differential operators and their kernels, which will be needed in the following sections. These results are minor variations of those obtained by Etingof, Gelfand and Retakh in [11]. One difference in our treatment is that we focus on vector kernels rather than matrix kernels (considered in [11]) because our classification results for bispectral Darboux transformations from the next 2 sections are naturally formulated in terms of the former.

**3.1.** We first recall the notion of quasideterminants introduced and studied by Gelfand and Retakh [13, 14]. Let $R$ be an associative algebra. Let $X = (x_{ij})$ be a matrix with elements in $R$. For each pair of indices $i, j$ denote by $r_i(X)$ the $i$-th row of $X$ and by $c_j(X)$ the $j$-th column of $X$. Denote by $X^{ij}$ the matrix obtained by removing the $i$-th row and the $j$-th column of $X$. For each row $r$ and index $j$, define the vector $r^{(j)}$ obtained by removing the $j$-th entry. Similarly, for a column $c$, denote by $c^{(i)}$ the column vector obtained by its $i$-th entry. Finally, we assume that the matrix $X^{ij}$ is invertible. The quasideterminant $|X|_{ij}$ is defined by

$$|X|_{ij} := x_{ij} - r_i(X)^{(j)}(X^{ij})^{-1}c_j(X)^{(i)} \in R.$$

Let $\Omega \subseteq \mathbb{C}$ be a domain and $R$ be the algebra of meromorphic functions on $\Omega$ with values in $M_n(\mathbb{C})$.

For any sets of matrix-valued meromorphic functions $F_1, \dots, F_k$ on $\Omega$ and vector-valued meromorphic functions $f_1, \dots, f_{nk}$ on $\Omega$, define the Wronski matrices

(3.1)

$$W(F_1,\dots,F_k) = \begin{pmatrix} F_1 & \dots & F_k \\ F_1' & \dots & F_k' \\ \dots & \dots & \dots \\ F_1^{(k-1)} & \dots & F_k^{(k-1)} \end{pmatrix}, W(f_1,\dots,f_{nk}) = \begin{pmatrix} f_1 & \dots & f_{nk} \\ f_1' & \dots & f_{nk}' \\ \dots & \dots & \dots \\ f_1^{(nk-1)} & \dots & f_{nk}^{(nk-1)} \end{pmatrix}.$$

Such a set of functions $F_1, \dots, F_k$ is called *nondegerate* if $W(F_1, \dots, F_k)$ is an invertible matrix-valued meromorphic function on $\Omega$. The invertibility condition is independent of whether (3.1) is considered as an $n \times n$ matrix with values in $R$ or an $nk \times nk$ matrix whose entries are meromorphic functions on $\Omega$. This is equivalent to saying that the determinant of the Wronski $nk \times nk$ matrix is a nonzero meromorphic function on $\Omega$.

**Definition 3.1.** A subspace of dimension $nk$ of the space of vector-valued meromorphic functions on $\Omega$ will be called *nondegenerate* if it has a basis $f_1, \dots, f_{nk}$ such that the Wronski matrix $W(f_1, \dots, f_{nk})$ is an invertible matrix-valued meromorphic function on $\Omega$.

It is clear that the condition in the definition does not depend on the choice of basis.

**Remark 3.2.** For every basis of a nondegenerate subspace of the space of vector-valued meromorphic functions on $\Omega$, there exists an arrangement of the basis elements into the columns of a set of meromorphic functions $F_1, \dots, F_k \colon \Omega \to M_n(\mathbb{C})$ with the property that for all $m \leq k$ the set $F_1, \dots, F_m$ is nondegenerate on $\Omega$. This follows from the fact that, for every nondegenerate square matrix $M$, there exists a permutation matrix $\Sigma$ such that all leading principal minors of $M\Sigma$ are nondegenerate (the fact is applied for $M$ equal to the Wronski matrix of the basis elements).

**3.2.** Define the vector kernel of a differential operator $P(x, \partial_x)$ with matrix-valued meromorphic coefficients on $\Omega$ by

$$\operatorname{vker} P(x,\partial_x) := \{\text{meromorphic functions } f(x) \colon \Omega \to \mathbb{C}^n \mid P(x,\partial_x)f(x) = 0\}. \tag{3.2}$$

By the standard theorem for existence and uniqueness of solutions of ordinary differential equations, $\dim \operatorname{vker} P(x, \partial_x) \leq kn$, and each $a \in \Omega$ at which $P(x, \partial_x)$ is regular has a neighborhood $\mathcal{O}_a \subseteq \Omega$ such that the restriction of $P(x, \partial_x)$ to $\mathcal{O}_a$ satisfies $\dim \operatorname{vker} P(x, \partial_x) = kn$.

**Proposition 3.3.** *Let $\Omega \subseteq \mathbb{C}$ be a domain and $n, k$ be positive integers.*

*(a) Assume that $P(x, \partial_x)$ is an $M_n(\mathbb{C})$-valued meromorphic differential operator on $\Omega$ of order $k$ that has $nk$-dimensional vector kernel. Then its kernel is nondegenerate in the sense of Definition 3.1.*

*(b) Assume that $V$ is a nondegenerate subspace of the space of vector-valued meromorphic functions on $\Omega$ of dimension $nk$. Then there is a unique monic differential operator $P(x, \partial_x)$ of order $k$ with matrix-valued meromorphic coefficients on $\Omega$ whose vector kernel equals $V$. For every basis of $V$ choose an arrangement of the basis elements onto the columns of a set of meromorphic functions $F_1, \ldots, F_k \colon \Omega \to M_n(\mathbb{C})$ with the property that for all $m \leq k$ the set $F_1, \ldots, F_m$ is nondegenerate on $\Omega$ (cf. Remark 3.2). The operator $P(x, \partial_x)$ is given by*

$$P(x, \partial_x)F(x) = |W(F_1, \ldots, F_k, F)|_{k+1,k+1} \tag{3.3}$$

*for each matrix-valued meromorphic function $F(x)$ and by*

$$P(x, \partial_x) = (\partial_x - b_k) \ldots (\partial_x - b_1), \tag{3.4}$$

*where*

$$b_j := W_j' W_j^{-1}, \ \ W_j := |W(F_1, \ldots, F_j)|_{jj}. \tag{3.5}$$

*Proof.* (a) Denote by $\Omega^\circ$ the open dense subset of $\Omega$ on which $P(x, \partial_x)$ is analytic. Choose a basis $f_1(x), \ldots, f_{nk}(x)$ of $\operatorname{vker} P(x, \partial_x)$. Let $c_0, \ldots, c_{nk} \in \mathbb{C}$. The statement of this part of the proposition follows from the fact that if

$$W(f_1, \ldots, f_{nk})(x_0) \cdot (c_1, \ldots c_{nk})^t = 0$$

for some $x_0 \in \Omega^\circ$, then this holds for all $x_0 \in \Omega^\circ$, where $(.)^t$ denotes the transpose of a matrix. (This is nothing but the uniqueness statement for solutions of the initial value problem for ordinary differential equations.)

(b) By [11, Theorem 1.1] there exists a unique monic differential operator $P(x, \partial_x)$ of order $k$ such that

$$P(x, \partial_x)F_i(x) = 0, \quad \forall\, 1 \leq i \leq k$$

and this operator is given by (3.3) and (3.4). Since $\operatorname{vker} P \supseteq V$, $nk \geq \dim \operatorname{vker} P$ and $\dim V = nk$, we have $\operatorname{vker} P = V$. □

**3.3.** The factorizations of matrix-valued differential operators are described by the next proposition.

**Proposition 3.4.** *Let $L(x, \partial_x)$ be an $M_n(\mathbb{C})$-valued monic meromorphic differential operator on a domain $\Omega \subseteq \mathbb{C}$ of order $l$ whose vector kernel has dimension $nl$. All factorizations*

$$L(x, \partial_x) = Q(x, \partial_x)P(x, \partial_x) \tag{3.6}$$

*into $M_n(\mathbb{C})$-valued monic meromorphic differential operators of orders $l - k$ and $k$ are in bijection with the nondegenerate $nk$-dimensional subspaces of vker$L$. This bijection is given by $P \mapsto \operatorname{vker} P$. The inverse bijection from nondegenerate subspaces of* $\operatorname{vker} L$ *to monic differential operators $P$ satisfying* (3.6) *is given by Proposition 3.3 (b).*

This proposition is similar to [16, Theorem 1], but the latter omits the condition for nondegeneracy of $\operatorname{vker} P$.

*Proof.* Assume that we have a factorization as in (3.6). Obviously, $\operatorname{vker} P \subseteq \operatorname{vker} L$. We will show that $\dim \operatorname{vker} P = nk$. Then by Proposition 3.3 (a), $\operatorname{vker} P$ is nondegenerate.

Fix a point $a \in \Omega$ at which $L(x, \partial_x)$ is regular and a neighborhood $\mathcal{O}_a$ of it on which $Q$ and $P$ are analytic and the restriction of $P$ has $nk$-dimensional kernel. For this restriction $\operatorname{vker} P \subseteq \operatorname{vker} L$, i.e., there exists an $nk$-dimensional subspace $V$ of the vector kernel of $L(x, \partial_x)$ such that the vector kernel of the restriction of $P$ to $\mathcal{O}_a$ is $V|_{\mathcal{O}_a}$. By applying Proposition 3.3 (b), we construct a differential operator $P'(x, \partial_x)$ on $\Omega$ with meromorphic coefficients such that $\operatorname{vker} P' = V$. The restrictions of $P$ and $P'$ to $\mathcal{O}_a$ coincide by Proposition 3.3 (b) and the two differential operators are meromorphic on $\Omega$. Since $\Omega$ is connected, $P = P'$. So, $\operatorname{vker} P = V$, and, in particular, $\dim \operatorname{vker} P = nk$.

In the other direction, for an $nk$-dimensional subspace $V$ of the vector kernel of $L$ on $\Omega$ and a basis of it, we can form meromorphic functions $F_1, \ldots, F_k \colon \Omega \to M_n(\mathbb{C})$ from a basis of $V$ as in Remark 3.2. Furthermore, one can complement the basis of $V$ to a basis of $vkerL$ and form a set of meromorphic functions $F_{k+1}, \ldots, F_l \colon \Omega \to M_n(\mathbb{C})$ from them, such that $F_1, \ldots F_m$ is a nondegenerate set for all $m \leq l$. Then, by (3.4),

$$L(x, \partial_x) = Q(x, \partial_x) P(x, \partial_x)$$

for

$$Q(x, \partial_x) := (\partial_x - b_l) \ldots (\partial_x - b_{k+1}), \quad P(x, \partial_x) := (\partial_x - b_k) \ldots (\partial_x - b_1),$$

where $b_j$ are given by (3.5). □

In the setting of the above proposition the meromorphic operator $PQ$ is called a matrix Darboux transformation from $L = QP$. For an eigenfuction $\Psi(x, z)$ of $L(x, \partial_x)$ (satisfying $L(x, \partial_x)\Psi(x, z) = \Psi(x, z) f(z)$) the function $P(x, \partial_x)\Psi(x, z)$ is also called a Darboux transformation from $\Psi(x, z)$.

**Remark 3.5.** Some authors consider the following form of Darboux transformations. For a given matrix-valued monic meromorphic differential operator $L(x, \partial_x)$, consider all matrix-valued meromorphic operators $\overline{L}(x, \partial_x)$ that have the property

$$P(x, \partial_x) L(x, \partial_x) = \overline{L}(x, \partial) P(x, \partial_x) \tag{3.7}$$

for some matrix-valued monic meromorphic operators $P(x, \partial_x)$. As before, we call the function $P(x, \partial_x)\Psi(x, z)$ a Darboux transformation from $\Psi(x, z)$.

A priori it appears that this form of Darboux transformations is more general than the ones we consider. However, this is not the case, i.e., Theorem 2.1 also covers these transformations. To see this, one first proves that (3.7) implies $L(\operatorname{vker} P) \subseteq \operatorname{vker} P$. Since $\dim \operatorname{vker} P < \infty$, $\operatorname{vker} P \subseteq \operatorname{vker} h(L)$ for some polynomial $h(z)$. Therefore $h(L) = QP$ for some matrix-valued meromorphic differential operator $Q$ on $\Omega$ and we have the Darboux transformation

$$h(L) = QP \mapsto PQ = h(\overline{L})$$

which falls within the class treated in Theorem 2.1. On the level of wave functions it represents the Darboux transformation (3.7)

$$\Psi(x, z) \mapsto P(x, \partial_x)\Psi(x, z).$$

## 4. Classification of matrix bispectral Darboux transformations in the rank 1 case

All bispectral algebras of rank 1 scalar ordinary differential operators were classified by Wilson [31]. In [2] it was proved that the corresponding wave functions are bispectral Darboux transformations from the function $\Psi(x, z) = e^{xz}$ in the sense of Theorem 2.1. At the time the result was phrased in different terms, but it is easy to convert the statements of [2] to the ones in Theorem 2.1, and more precisely, to the case of Proposition 2.2 when the size of the matrices is $1 \times 1$.

In this section we classify explicitly all bispectral Darboux transformations of the corresponding matrix analog in full generality, i.e., we obtain a full classification of all matrix rank 1 bispectral Darboux transformations. This is achieved in Theorem 4.3. The only restriction we impose is that the leading terms of the differential operators $L, Q$ and $P$ in Proposition 2.2 be invertible matrix-valued meromorphic functions. Without loss of generality this situation is the same as the one when $L, Q$ and $P$ are monic. (The classifications of the factorizations of differential operators with noninvertible leading terms are related to well known pathological problems from the zero divisors of the algebra $M_n(\mathbb{C})$ and will be considered elsewhere.)

**4.1.** In the setting of the previous section, let $\Omega_1 = \Omega_2 := \mathbb{C}$. Define the function

$$\Psi(x,z) = e^{xz} I_n$$

on $\mathbb{C} \times \mathbb{C}$, where $I_n$ is the identity matrix. Let $B_1$ and $B_2$ be both equal to the algebra of matrix-valued differential operators with polynomial coefficients. We will use the variables $x$ and $z$ for the first and second algebra, respectively. Let $b \colon B_1 \to B_2$ be the isomorphism given by

$$b(\partial_x) := z, \ \ b(x) = -\partial_z, \ \ b(W) := W, \ \forall\, W \in M_n(\mathbb{C}).$$

We are in the bispectral situation of Proposition 2.2:

$$P(x,\partial_x)\Psi(x,z) = \Psi(x,z)(bP)(z,\partial_z), \ \ \forall P(x,\partial_x) \in B_1.$$

Let $K_1$ and $K_2$ be the algebras of matrix-valued polynomial functions and $x$ and $z$ respectively. Denote by $A_1$ and $A_2$ the algebras of matrix-valued differential operators (in $x$ and $z$) with constant coefficients. Then

$$b(A_1) = K_2, \ \ b(K_2) = A_1.$$

The regular elements of the algebras $K_1$ and $K_2$ are the matrix-valued polynomial functions $\theta(x)$ and $f(z)$ that are nondegenerate in the sense that their determinants are nonzero polynomials. By standard commutation arguments for matrix differential operators the sets $r(K_i)$ are Ore subsets of $B_i$. The localizations $B_1[r(K_1)^{-1}]$ and $B_2[r(K_2)^{-1}]$ are nothing but the algebras of differential operators with rational coefficients in the variables $x$ and $z$. Theorem 2.1 implies the following:

**Theorem 4.1.** *If $L(x,\partial_x)$ is a matrix-valued differential operator with constant coefficients, and $Q(x,\partial_x)$ and $P(x,\partial_x)$ are matrix-valued differential operators with rational coefficients, such that*

$$L(x,\partial_x) = Q(x,\partial_x)P(x,\partial_x), \tag{4.1}$$

*then*

$$\Phi(x,z) := P(x,\partial_x)(e^{xz} I_n)$$

*is a matrix bispectral function.*

*More precisely, if*

$$L(x,\partial_x) = Q'(x,\partial_x)g(x)^{-1}P'(x,\partial_x) \tag{4.2}$$

*for matrix-valued differential operators with polynomial coefficients $Q'(x,\partial_x)$, $P'(x,\partial_x)$ and a nondegenerate matrix-valued polynomial function $g(x)$, then $\Phi(x,z) := P'(x,\partial_x)(e^{xz} I_n)$*

*satisfies*

$$\left(P'(x,\partial_x)Q'(x,\partial_x)g(x)^{-1}\right)\Phi(x,z)=\Phi(x,z)(bL)(z),$$
$$g(x)\Phi(x,z)=\Phi(x,z)\left((bL)(z)^{-1}(bQ')(z,\partial_z)(bP')(z,\partial_z)\right).$$

**4.2.** Our goal below is to classify all wave functions $\Phi(x,z)$ that can be obtained from Theorem 4.1 for operators $L(x,\partial_x)$ with nondegenerate leading term. This is done in Theorem 4.3. Denote the vector space of vector-valued quasipolynomial functions

$$\mathcal{QP}_n := \oplus_{\alpha\in\mathbb{C}} e^{\alpha x}\mathbb{C}[x]^n, \tag{4.3}$$

where $\mathbb{C}[x]$ denotes the space of polynomials and $\mathbb{C}[x]^n$ the space of vector-valued polynomial functions. The elements of $\mathcal{QP}_n$ have the form $\sum_\alpha e^{\alpha x}p_{\alpha,i}(x)$, where the sum is finite and $p_{\alpha,i}(x)\in\mathbb{C}[x]^n$.

By converting $n$-th order vector differential equations to first order ones and using Proposition 3.4, one easily obtains the following lemma. We leave its proof to the reader.

**Lemma 4.2.** *(a) Assume that $L(x,\partial_x)$ is a monic matrix-valued differential operator with constant coefficients. If $Q(x,\partial_x)$ and $P(x,\partial_x)$ are monic matrix-valued differential operators with meromorphic coefficients in $\mathbb{C}$ which satisfy* (4.1)*, then* vker $P(x,\partial_x)$ *is a nondegenare subspace of $\mathcal{QP}_n$.*

*(b) For every nondegenerate finite dimensional subspace $V$ of $\mathcal{QP}_n$ whose dimension is a multiple of $n$, there exist monic matrix-valued differential operators $L$, $Q$ and $P$ satisfying the properties in part (a) such that* vker $P(x,\partial_x)=V$.

The next theorem is our classification result of the matrix rank 1 bispectral Darboux transformations.

**Theorem 4.3.** *Assume that $L(x,\partial_x)$ is a monic matrix-valued differential operator with constant coefficients, and $Q(x,\partial_x)$ and $P(x,\partial_x)$ are monic matrix-valued differential operators whose coefficients are rational functions. If* (4.1) *is satisfied, then* vker $P(x,\partial_x)$ *is a nondegenerate subspace of $\mathcal{QP}_n$ that has a basis of the form*

$$\{e^{\alpha_j x}p_j(x)\mid 1\le j\le kn\}, \tag{4.4}$$

*where $\alpha_1,\dots,\alpha_{kn}$ are not necessarily distinct complex numbers and $p_j(x)\in\mathbb{C}[x]^n$. Furthermore, for every such subspace of $\mathcal{QP}_n$ there exists a triple of operators $L$, $Q$ and $P$ with the above properties such that $v\ker P(x,\partial_x)$ equals this subspace.*

**Corollary 4.4.** *The bispectral Darboux transformations from the function $e^{xz}I_n$ as in Theorem 4.1 for operators $L,Q,P$ with nondegenerate leading terms are precisely the functions of the form*

$$\Phi(x,z)=W(x)P(x,\partial_x)(e^{xz}I_n),$$

*where $P(x,\partial_x)$ is a monic matrix differential operator of order $k$ with vector kernel having a basis of the form* (4.4) *and $W(x)$ is a matrix-valued nondegenerate rational function. The operator $P(x,\partial_x)$ is given by* (3.3)*, where $F_1(x),\dots,F_k(x)$ are the matrix-valued functions with columns*

$$e^{\alpha_1 x}p_1(x),\dots,e^{\alpha_{kn}x}p_{kn}(x),$$

*$\alpha_j\in\mathbb{C}$, $p_j(x)\in\mathbb{C}[x]^n$.*

The bispectral functions in this family were related to wave functions of solutions of the $n$-KP hierarchy by Wilson [33] and Bergvelt, Gekhtman, and Kasman [27]. The fact that the family in Corollary 4.4 is the same as the one in [5, 33] (up to a multiplication on the left by $W(x)^{-1}$ and by an invertible matrix-valued rational function in $z$) follows from Theorem 4.2 in [33] and Corollary 4.4. Since the proof of Theorem 4.2 in [33] is only sketched there, we note that the reference to it is not used in proofs here, but only to relate the bispectral functions in this section to those in [5, 33]. The bispectrality of the special elements of this family obtained by transformations from $L(x, \partial_x) = q(\partial_x)D$ for a diagonal matrix $D$ and a (scalar) polynomial $q(t)$ was previously obtained in [6]. Applications of Theorem 2.1 to generalized bispectrality in relation to the results of Kasman [27] are described in §4.4.

**4.3.** We will break the proof of the Theorem 4.3 into two lemmas.

**Lemma 4.5.** *Let, for* $1 \leq j \leq kn$, $\alpha_j \in \mathbb{C}$ *and* $p_j(x)$ *be vector-valued rational functions for which*

$$e^{\alpha_1 x} p_1(x), \ldots, e^{\alpha_{kn} x} p_{kn}(x) \tag{4.5}$$

*form a nondegenerate (and thus, linearly independent) set. Then the coefficients of the unique monic matrix-valued differential operator with kernel spanned by* (4.5) *are rational functions.*

*Proof.* We argue by induction on $k$. In the case $k = 1$ the differential operator equals

$$\partial_x - F'(x)F^{-1}(x),$$

where $F(x)$ is the matrix function with columns $e^{\alpha_1 x} p_1(x), \ldots, e^{\alpha_n x} p_n(x)$, and (e.g. by Cramer's rule) $F'(x)F^{-1}(x)$ has rational coefficients.

In the general case, by (3.4), the differential operator has the form

$$P(x, \partial_x) = \overline{P}(x, \partial_x)(\partial_x - F'(x)F^{-1}(x))$$

with $F(x)$ being nondegenerate and with columns as in (4.5). Therefore, $\partial_x - F'(x)F^{-1}(x)$ has rational coefficients and the operator $\overline{P}(x, \partial_x)$ has vector kernel spanned by

$$(\partial_x - F'(x)F^{-1}(x))(e^{\alpha_j x} p_j(x)), \quad n < j \leq kn.$$

By the inductive assumption, applied to $\overline{P}(x, \partial_x)$, the operator $\overline{P}(x, \partial_x)$ has rational coefficients, which implies the same property for $P(x, \partial_x)$. □

**Lemma 4.6.** *Assume that* $P$ *is matrix-valued differential operator with coefficients that are rational functions. If*

$$\sum_{s=1}^{r} e^{\lambda_s x} p_s(x) \in \operatorname{vker} P$$

*for some vector-valued polynomial functions* $p_s(x)$ *and distinct complex numbers* $\lambda_1, \ldots, \lambda_r$, *then*

$$e^{\lambda_s x} p_s(x) \in \operatorname{vker} P, \quad \forall 1 \leq s \leq r.$$

*Proof.* We have

$$\sum_{s=1}^{r} e^{\lambda_s x} \left( e^{-\lambda_s x} P(x, \partial_x)(e^{\lambda_s x} p_s(x)) \right) = 0.$$

Since the functions $e^{-\lambda_s x} P(x, \partial_x)(e^{\lambda_s x} p_s(x))$ are rational, the nonzero functions in the sum are linearly independent because $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct. Thus all terms in the sum vanish. □

*Proof of Theorem 4.3.* The first part of the theorem follows from Lemmas 4.6 and 4.2 (a). The second part of the theorem follows from Lemmas 4.5 and 4.2 (b). □

**4.4.** There are generalized versions of the differential bispectral problem from (1.3)–(1.4). In one of them, one keeps the differential eigenvalue problem in (1.3) but replaces the operator $\Lambda(z, \partial_z)$ in the right hand side of (1.4) with a mixed differential-translation operator. In this subsection, we briefly describe an application of Theorem 2.1 that gives a second proof of the results in [27] on the construction of such generalized bispectral functions.

Firstly, for an analytic function $\Psi \colon \Omega_1 \times \Omega_2 \to M_n(\mathbb{C})$ with an expansion

$$\Psi(x,z) = \sum_{j,k=0}^{\infty} (x-x_0)^j a_{j,k} (z-z_0)^k, \quad a_{j,k} \in M_n(\mathbb{C}), x_0 \in \Omega_1, z_0 \in \Omega_2$$

and a matrix $W \in M_n(\mathbb{C})$, define the translation action

$$\Psi(x,z) \cdot (T_z^W) := \sum_{j,k=0}^{\infty} (x-x_0)^j a_{j,k} (z-W-z_0)^k$$

whenever the series converges. A mixed differential-translation operator is an operator obtained by composing differential operators with translation operators of this kind.

For an invertible matrix $H \in M_n(\mathbb{C})$, denote by $Z(H) = \{W \in M_n(\mathbb{C}) \mid WH = HW\}$ its centralizer. Consider the analytic function

$$\Psi_H(x,z) := e^{xzH}.$$

Let $B_1$ be the algebra of differential operators in $x$ with coefficients that are quasipolynomial functions with values in $Z(H)$ (i.e., functions with values in $Z(H)$ whose matrix entries are finite sums of terms of the form $x^j e^{\alpha x}$, $j \in \mathbb{N}$, $\alpha \in \mathbb{C}$). Let $B_2$ be the algebra consisting of mixed differential-translation operators of the form

$$\sum_{j \in \mathbb{N}, c \in \mathbb{C}} a_{j,c}(z) \partial_z^j T_z^{cH}$$

given by finite sums and terms $a_{j,c}(z)$ that are polynomials with values in $Z(H)$. The following define an isomorphism $b \colon B_1 \to B_2$

$$b(x) = -\partial_z, \; b(\partial_x) = z, \; b(e^{ax}) = T_z^{-aH^{-1}}, \; b(W) = W, \;\; \forall W \in Z(H)$$

and the function $\Psi_H(x,z)$ satisfies

$$P \cdot \Psi_H(x,z) = \Psi_H \cdot b(P), \quad \forall P \in B_1. \tag{4.6}$$

Let $A_1$ be the subalgebra of $B_1$ consisting of constant differential operators in $x$ with values in $Z(H)$ and $K_2$ be the subalgebra of $B_2$ consisting of polynomial in $z$ with values in $Z(H)$. The identities in (4.6) for $P \in A_1$ are the spectral differential equations in $x$

$$L(x, \partial_x) \Psi_H(x,z) = \Psi_H(x,z) b(L)(z), \quad L(x, \partial_x) \in A_1.$$

Let $A_2$ be the subalgebra of $B_2$ spanned by the differential-translation operators of the form $\partial_z^j T_z^{cH^{-1}} W$, $j \in \mathbb{N}$, $c \in \mathbb{C}$, $W \in Z(H)$. Let $K_1$ be the subalgebra of $B_1$ consisting of quasipolynomials in $z$ with values in $Z(H)$. The identities in (4.6) for $P \in K_1$ are the spectral differential-translation equations in $z$

$$b^{-1}(\Lambda)(x) \Psi_H(x,z) = \Psi_H(x,z) \cdot \Lambda, \quad \Lambda \in A_2.$$

We are in the situation of Theorem 2.1 and can apply it to obtain functions that satisfy spectral differential equations in $x$ and spectral differential-translation equations in $z$ by Darboux transformations from $\Psi_H(x,z)$. This gives a second proof of the results of Kasman [27].

**Theorem 4.7.** *Let $H \in M_n(\mathbb{C})$ be an invertible matrix. If $L(x, \partial_x)$ is a $Z(H)$-valued differential operator with constant coefficients, and $Q(x, \partial_x)$ and $P(x, \partial_x)$ are $Z(H)$-valued differential operators such that*

$$L(x, \partial_x) = Q(x, \partial_x)P(x, \partial_x), \tag{4.7}$$

*then*

$$\Phi(x, z) := P(x, \partial_x)\Psi_H(x, z)$$

*is a generalized bispectral function, satisfying a spectral differential equation in $x$ and a spectral differential-translation equation in $z$.*

*More precisely, if*

$$L(x, \partial_x) = Q'(x, \partial_x)g(x)^{-1}P'(x, \partial_x)$$

*for $Z(H)$-valued differential operators with quasipolynomial coefficients $Q'(x, \partial_x)$, $P'(x, \partial_x)$ and a nondegenerate $Z(H)$-valued quasipolynomial function $g(x)$, then the function $\Phi(x, z) := P'(x, \partial_x)e^{xz}I_n$ satisfies*

$$\begin{aligned}\left(P'(x, \partial_x)Q'(x, \partial_x)g(x)^{-1}\right)\Phi(x, z) &= \Phi(x, z)(bL)(z),\\ g(x)\Phi(x, z) &= \Phi(x, z)\left((bL)(z)^{-1}b(Q')b(P')\right).\end{aligned}$$

The first equation is a spectral differential equation in $x$. The second one is a spectral differential-translation equation in $z$; the point here is that $b(Q')$ and $b(P')$ are differential-translation operators.

The classification of the factorizations of the form (4.7) for monic differential operators $L$, $Q$ and $P$ is simpler than the classification in Theorem 4.3. The point is that the adelic type condition is not needed in this case since the operators $P(x, \partial_x)$ and $Q(x, \partial_x)$ are not required to have rational coefficients. (For each such factorization $P(x, \partial_x)$ and $Q(x, \partial_x)$ have quasirational coefficients, recall Lemma 4.2 (a).) Since $L(x, \partial_x)$ and $Q(x, \partial_x)$ are $Z(H)$-valued, their vector kernels are nondegenerate $H$-invariant subspaces of $\mathcal{QP}_n$, recall (4.3). One easily shows that every nondegenerate $H$-invariant subspace of $\mathcal{QP}_n$ of dimension divisible by $n$ is the vector kernel of a unique monic differential operator $P(x, \partial_x)$ satisfying (4.7) for some differential operators $L(x, \partial_x)$ and $Q(x, \partial_x)$ with the stated properties. Thus we obtain the following:

*The generalized bispectral Darboux transformations in Theorem 4.7 for monic differential operators $L$, $Q$, and $P$ are classified by the nondegenerate $H$-invariant subspaces of $\mathcal{QP}_n$ of dimension divisible by $n$. For such a subspace, one constructs the unique monic differential operator $P(x, \partial_x)$ with that vector kernel and the corresponding generalized bispectral function $\Phi(x, z)$; this differential operator $P(x, \partial_x)$ is $Z(H)$-valued with quasipolynomial coefficients.*

## 5. Bispectral Darboux transformations from matrix Airy functions

In this section we give an explicit classification of all bispectral Darboux transformations from the matrix Airy functions. We often make use of the spectral theory of matrix polynomials reviewed in the appendix.

**5.1.** We recall the setting of the (generalized) scalar Airy bispectral functions from [2]. Fix complex numbers $\alpha_0 \neq 0, \alpha_2, \dots, \alpha_{N-1}$. Consider the scalar differential operator

$$M_{\mathrm{Ai}}(x, \partial_x) := \partial_x^N + \sum_{i=1}^{N-1} \alpha_i \partial_x^{N-i} + \alpha_0 x,$$

called a (generalized) Airy operator. For $N = 2$ this is the classical Airy operator up to normalization. It is well known that the operator $M_{\mathrm{Ai}}(x, \partial_x)$ has $N$-dimensional vector kernel over $\mathbb{C}$ consisting of entire functions. Choose a basis of the kernel, $\psi_1(x), \ldots, \psi_N(x)$. For each function $\psi(x) \in \operatorname{vker} M$, define

$$\psi(x, z) := \psi(x + z). \tag{5.1}$$

The latter satisfies the bispectral equations

$$M_{\mathrm{Ai}}(x, \partial_x)\psi = z\psi \quad \text{and} \quad M_{\mathrm{Ai}}(z, \partial_z)\psi = x\psi.$$

The bispectral Darboux transformations from it (in the sense of Theorem 2.1) were classified in [2] and played an important role in the overall classification of scalar bispectral operators.

Now we move to the matrix situation. As in the matrix rank 1 case from Section 4, we take $\Omega_1 = \Omega_2 := \mathbb{C}$. Define the function

$$\Psi_{\mathrm{Ai}}(x, z) = \psi(x, z) I_n$$

on $\mathbb{C} \times \mathbb{C}$, where $I_n$ is the identity matrix. Let $B_1$ and $B_2$ be both equal to the algebra of matrix-valued differential operators with polynomial coefficients. We will use the variables $x$ and $z$ for the first and second algebra, respectively. Let $b\colon B_1 \to B_2$ be the isomorphism given by

$$b(x) := M_{\mathrm{Ai}}(z, \partial_z), \;\; b(\partial_x) := -\partial_z, \;\; b(W) := W, \; \forall\, W \in M_n(\mathbb{C}).$$

The first two equations imply that $b(M_{\mathrm{Ai}}(x, \partial_x)) = z$. We are in the bispectral situation of Proposition 2.2,

$$P(x, \partial_x)\Psi_{\mathrm{Ai}}(x, z) = \Psi_{\mathrm{Ai}}(x, z)(bP)(z, \partial_z), \;\; \forall P(x, \partial_x) \in B_1.$$

Let $K_1$ and $K_2$ be the algebras of matrix-valued polynomial functions in $x$ and $z$, respectively. Denote by $A_1$ and $A_2$ the algebras consisting of operators of the form

$$q(M_{\mathrm{Ai}}(x, \partial_x)) \quad \text{and} \quad q(M_{\mathrm{Ai}}(z, \partial_z))$$

for matrix-valued polynomials $q(t)$. Then

$$b(A_1) = K_2, \;\; b(K_2) = A_1.$$

The regular elements of the algebras $K_1$ and $K_2$ are the matrix-valued polynomial function $\theta(x)$ and $f(z)$ that are nondegenerate in the sense that their determinants are nonzero polynomials. By standard commutation arguments for matrix differential operators the sets $r(K_i)$ of regular elements are Ore subsets of $B_i$. The localizations $B_1[r(K_1)^{-1}]$ and $B_2[r(K_2)^{-1}]$ are isomorphic to the algebras of differential operators with rational coefficients in the variables $x$ and $z$.

Theorem 2.1 implies the following:

**Theorem 5.1.** *Let $q(t)$ be a matrix-valued polynomial. Let $Q(x, \partial_x)$ and $P(x, \partial_x)$ be matrix-valued differential operators with rational coefficients, such that*

$$q(M_{\mathrm{Ai}}(x, \partial_x)) = Q(x, \partial_x)P(x, \partial_x). \tag{5.2}$$

*Then*

$$\Phi(x, z) := P(x, \partial_x)\Psi_{\mathrm{Ai}}(x, z)$$

*is a matrix bispectral function. More precisely, if*

$$L(x, \partial_x) = Q'(x, \partial_x)g(x)^{-1}P'(x, \partial_x)$$

*for matrix-valued differential operators with polynomial coefficients* $Q'(x,\partial_x)$*,* $P'(x,\partial_x)$ *and a nondegenerate matrix-valued polynomial* $g(x)$*, then* $\Phi(x,z) := P'(x,\partial_x)\Psi_{\mathrm{Ai}}(x,z)$ *satisfies*

$$\left(P'(x,\partial_x)Q'(x,\partial_x)g(x)^{-1}\right)\Phi(x,z) = \Phi(x,z)q(z),$$
$$g(x)\Phi(x,z) = \Phi(x,z)\left(q(z)^{-1}(bQ')(z,\partial_z)(bP')(z,\partial_z)\right).$$

In Theorem 5.4 below we classify all bispectral Darboux functions $\Phi(x,z)$ that enter in Theorem 5.1 corresponding to $q(t)$ and $P(x,\partial_x)$ with nondegenerate leading terms.

**5.2.** First, we describe the kernels of the differential operators $L(x,\partial_x)$ in Theorem 5.1. In the notation (5.1),

$$\psi_i^{(j)}(x,\lambda) = (\partial_z^j\psi_i(x,z))|_{z=\lambda} = \partial_x^j\psi_i(x,\lambda), \quad \forall\,\lambda\in\mathbb{C}, 1\le i\le N, j\in\mathbb{N}.$$

Define the space

$$\mathcal{QA}_n := \bigoplus_{\lambda\in\mathbb{C},1\le i\le N,j\in\mathbb{N}} \psi_i^{(j)}(x,\lambda)\mathbb{C}^n.$$

Denote the standard basis of $\mathbb{C}^n$ by $\{e_1,\dots,e_n\}$. We have

$$\left(M_{\mathrm{Ai}}(x,\partial_x)-\lambda\right)\psi_i^{(j)}(x,\lambda) = \psi_i^{(j-1)}(x,\lambda),$$

and

$$\{\psi_i^{(j)}(x,\lambda) \mid 1\le i\le N, 0\le j\le k-1\}$$

forms a basis of the kernel of $(M_{\mathrm{Ai}}(x,\partial_x)-\lambda)^k$ for all $\lambda\in\mathbb{C}$ and $k\in\mathbb{Z}_+$, see [2]. One easily obtains from this that

$$\{\psi_i^{(j)}(x,\lambda)e_l \mid \lambda\in\mathbb{C}, 1\le i\le N, j\in\mathbb{N}, 1\le l\le n\} \text{ is a basis of } \mathcal{QA}_n. \tag{5.3}$$

The fact that $\psi_i(x)$ are in the kernel of $M_{\mathrm{Ai}}(x,\partial_x)$ implies that

$$\mathcal{QA}_n = \bigoplus_{\lambda\in\mathbb{C},1\le i\le N,0\le j\le N} \psi_i^{(j)}(x,\lambda)\mathbb{C}[x]^n$$

and that

$$\{\psi_i^{(j)}(x,\lambda)x^k e_l \mid \lambda\in\mathbb{C}, 1\le i\le N, 0\le j\le N-1, k\in\mathbb{N}, 1\le l\le n\} \text{ is a basis of } \mathcal{QA}_n. \tag{5.4}$$

Fix a monic matrix-valued polynomial

$$q(t) = \sum_{j=0}^d a_j t^j, \quad \text{where} \quad a_j\in M_n(\mathbb{C}),\ a_d = I_n.$$

**Proposition 5.2.** *For each matrix polynomial* $q(t)$ *as above, the vector kernel of the differential operator* $q(M_{\mathrm{Ai}}(x,\partial_x))$ *over* $\mathbb{C}$ *is* $ndN$*-dimensional and consists of entire functions. More precisely, for each root* $\lambda$ *of* $\det(q(t))$ *and a choice of Jordan chains of* $q(t)$ *corresponding to* $\lambda$ *as in Theorem 7.1* $\{v_{0,l},\dots v_{k_l,l} \mid 1\le l\le s\}$*, the elements*

$$\sum_{r=0}^j \psi_i^{(r)}(x,\lambda)\frac{v_{j-r,l}}{r!}, \quad 1\le i\le N, 0\le j\le k_l$$

*belong to* $\operatorname{vker} q(M_{\mathrm{Ai}}(x,\partial_x))$*. The set of all such elements for all roots* $\lambda$ *of* $\det(q(t))$ *and a corresponding set of Jordan chains as in Theorem 7.1 forms a basis of* $\operatorname{vker} q(M_{\mathrm{Ai}}(x,\partial_x))$*.*

*Proof.* Using the bispectrality of the functions $\psi_i(x,z)$ and the definition of Jordan chains (7.2), we obtain

$$M_{\mathrm{Ai}}(x,\partial_x)\left(\sum_{r=0}^{j}\psi_i^{(r)}(x,\lambda)\frac{v_{j-r,l}}{r!}\right)=\sum_{r=0}^{j}\partial_z^j(M_{\mathrm{Ai}}(x,\partial_x)\psi_i(x,z))|_{z=\lambda}\frac{v_{j-r,l}}{r!}$$
$$=\sum_{r=0}^{j}\partial_z^j(\psi_i(x,z)q(z))|_{z=\lambda}\frac{v_{j-r,l}}{r!}=\sum_{r=0}^{j}\sum_{c=0}^{r}\binom{r}{c}\psi_i^{(c)}(x,\lambda)q^{(r-c)}(\lambda)\frac{v_{j-r,l}}{r!}=$$
$$=\sum_{c=0}^{j}\left(\frac{\psi_i^{(c)}(x,\lambda)}{c!}\sum_{r=c}^{j}\frac{q^{(r-c)}(\lambda)}{(r-c)!}v_{(j-c)-(r-c),l}=0\right).$$

To prove that the set in Proposition 5.2 gives a basis of $\operatorname{vker} q(M_{\mathrm{Ai}}(x,\partial_x))$, first note that by (7.3) the cardinality of this set equals $N\deg(\det(q(t)))=ndN$. Because $\dim\operatorname{vker} q(M_{\mathrm{Ai}}(x,\partial_x))\le ndN$, all we need to show is that these elements are linearly independent. This follows from (5.3) and the linear independence in Theorem 7.1. □

A direct argument with matrix polynomials leads to the following corollary of Proposition 5.2.

**Corollary 5.3.** *(a) Let $q(t)$ be a monic matrix polynomial. If $Q(x,\partial_x)$ and $P(x,\partial_x)$ are monic matrix-valued differential operators with meromorphic coefficients in $\mathbb{C}$ which satisfy* (5.2), *then* $\operatorname{vker} P(x,\partial_x)$ *is a nondegenrate subspace of $\mathcal{QA}_n$.*

*(b) For every nondenerate finite dimensional subspace $V$ of $\mathcal{QA}_n$ whose dimension is a multiple of $nN$, there exist a monic matrix polynomial $q(t)$ and monic matrix-valued differential operators $Q$ and $P$ satisfying the properties in part (a) such that* $\operatorname{vker} P(x,\partial_x)=V$.

Introduce a $\mathbb{Z}_N$-action on $\mathcal{QA}_n$, where the generator $\sigma$ of $\mathbb{Z}_N$ acts by

$$\sigma(\psi_i^{(j)}(x,\lambda)v):=\psi_{i+1}^{(j)}(x,\lambda)v,\quad \forall\, j\in\mathbb{N},1\le i\le N,v\in\mathbb{C}^n$$

and the $i+1$ term in the right hand side is taken modulo $N$ (i.e., $N+1:=1$). Here we use the fact that the set in (5.3) is a basis of $\mathcal{QA}_n$. In terms of the basis elements from (5.4), this action is given by

$$\sigma(\psi_i^{(j)}(x,\lambda)p(x)):=\psi_{i+1}^{(j)}(x,\lambda)p(x),\quad \forall\, 0\le j\le N-1,1\le i\le N,p(x)\in\mathbb{C}[x]^n.$$

This follows at once from the fact that the functions $\psi_i(x)$ are in the kernel of $M_{\mathrm{Ai}}(x,\partial_x)$.

The next theorem is our classification result for all bispectral Darboux transformations from the matrix Airy functions.

**Theorem 5.4.** *Assume that $q(t)$ is a monic matrix-valued polynomial, and $Q(x,\partial_x)$ and $P(x,\partial_x)$ are monic matrix-valued differential operators whose coefficients are rational functions. If* (5.2) *is satisfied, then* $\operatorname{vker} P(x,\partial_x)$ *is a nondegenerate $\mathbb{Z}_N$-invariant subspace of $\mathcal{QA}_n$ that has a basis consisting of elements of the form*

$$\sum_{j=0}^{N-1}\psi_i^{(j)}(x,\lambda)p_j(x), \tag{5.5}$$

*where $p_j(x)\in\mathbb{C}[x]^n$. The indices $i=1,\dots,N$ and the complex numbers $\lambda\in\mathbb{C}$ can be different for different basis elements.*

*Furthermore, for every such subspace of* $\mathcal{QA}_n$ *of dimension that is a multiple of* $nN$*, there exists a matrix polynomial* $q(t)$ *and a pair of operators* $Q$ *and* $P$ *with the above properties such that* $\operatorname{vker} P(x, \partial_x)$ *equals this subspace.*

**Corollary 5.5.** *The bispectral Darboux transformations from the matrix Airy function* $\Psi_{\mathrm{Ai}}(x, z)$ *as in Theorem 5.1 for* $q(t)$ *and* $P(x, \partial_x)$ *with nondegenerate leading terms are exactly the functions of the form*

$$\Phi(x, z) = W(x)P(x, \partial_x)\Psi_{\mathrm{Ai}}(x, z),$$

*where* $P(x, \partial_x)$ *is a monic matrix differential operator of order* $dN$ *with* $ndN$*-dimensional vector kernel which is a* $\mathbb{Z}_N$*-invariant subspace of* $\mathcal{QA}_n$ *possessing a basis consisting of elements of the form* (5.5) *and* $W(x)$ *is a matrix-valued nondegenerate rational function. The operator* $P(x, \partial_x)$ *is uniquely reconstructed from the basis using the second part of Proposition 3.4.*

**5.3.** We proceed with the proof of Theorem 5.4. The first part of the theorem follows from the following proposition and Corollary 5.3 (a).

**Proposition 5.6.** *Assume that* $P(x, \partial_x)$ *is a matrix-valued differential operator with rational coefficients. If*

$$\sum_{\lambda \in \mathbb{C}} \sum_{i=1}^{N} \sum_{j=0}^{N-1} \psi_i^{(j)}(x, \lambda) p_{j,i}^{\lambda}(x) \in \operatorname{vker} P(x, \partial_x) \tag{5.6}$$

*for some vector-valued polynomials* $p_{j,i}(x)$*, then*

$$\sigma^k \left( \sum_{j=0}^{N-1} \psi_i^{(j)}(x, \lambda) p_{j,i}^{\lambda}(x) \right) \in \operatorname{vker} P(x, \partial_x), \quad \forall\, \lambda \in \mathbb{C}, 1 \le i \le N, 0 \le j, k \le N-1.$$

*Proof.* It follows from (5.4) that

$$\{\psi_i^{(j)}(x, \lambda) e_l \mid \lambda \in \mathbb{C}, 1 \le i \le N, 0 \le j \le N-1, 1 \le l \le n\}$$

is a linearly independent set over $\mathbb{C}(x)$. Thus, (5.6) and the assumption that $P(x, \partial_x)$ has rational coefficients imply that

$$\sum_{j=0}^{N-1} \psi_i^{(j)}(x, \lambda) p_{j,i}^{\lambda}(x) \in \operatorname{vker} P(x, \partial_x), \quad \forall\, \lambda \in \mathbb{C}, 1 \le i \le N. \tag{5.7}$$

Using the facts that each such element belongs to $\operatorname{vker} P(x, \partial_x)$ and $P(x, \partial_x)$ has rational coefficients, and recursively expressing

$$\psi_i^{(N)}(x, \lambda) = -\sum_{j=0}^{N-1} \alpha_i \psi_i^{(j)}(x, \lambda) - \alpha_0 (x + \lambda) \psi_i(x, \lambda),$$

produces a $\mathbb{C}[x]$-linear combination of

$$\{\psi_i^{(j)}(x, \lambda) e_l \mid 0 \le j \le N-1, 1 \le l \le n\}$$

that equals 0. In view of (5.4), this is only possible if each coefficient of the combination equals 0. However, exactly the same thing would happen if we apply $P(x, \partial_x)$ to

$$\sum_{j=0}^{N-1} \psi_{i+k}^{(j)}(x, \lambda) p_{j,i}^{\lambda}(x) = \sigma^k \left( \sum_{j=0}^{N-1} \psi_i^{(j)}(x, \lambda) p_{j,i}^{\lambda}(x) \right)$$

instead of (5.7). Thus, for all $0 \le k \le N-1$, these elements belong to $\operatorname{vker} P(x, \partial_x)$ which completes the proof of the proposition. □

The second part of Theorem 5.4 follows from the following proposition and Corollary 5.3 (b).

**Proposition 5.7.** *Assume that $V$ is a nondegenerate $ndN$-dimensional $\mathbb{Z}_N$-invariant subspace of $\mathcal{QA}_n$ having a basis consisting of elements of the form (5.5), where $p_j(x) \in \mathbb{C}[x]^n$. (The indices $i = 1, \dots, N$ and the complex numbers $\lambda \in \mathbb{C}$ can be different for different basis elements.) Let $P(x, \partial_x)$ be the unique monic matrix-valued differential operator on $\mathbb{C}$ of order $dN$ whose vector kernel kernel equals $V$ (recall Proposition 3.3 (b)). Then $P(x, \partial_x)$ has rational coefficients.*

*Proof.* Denote the operator

$$P(x, \partial_x) = \partial_x^{dN} + \sum_{l=0}^{dN-1} a_l(x) \partial_x^l,$$

where $a_l(x)$ are matrix-valued meromorphic functions on $\mathbb{C}$.

The definition of the $\mathbb{Z}_N$-action on $\mathcal{QA}_n$ implies that we can assume that the basis of $V$, consisting of elements of the form (5.5), is itself $\mathbb{Z}_N$-invariant. Let

$$\sigma^k \left( \sum_{j=0}^{N-1} \psi_i^{(j)}(x, \lambda) p_j(x) \right) = \sum_{j=0}^{N-1} \psi_{i+k}^{(j)}(x, \lambda) p_j(x), \quad 1 \le k \le N-1 \tag{5.8}$$

be $N$ elements in the vector kernel of $P(x, \partial_x)$. Using the condition that they belong to $\operatorname{vker} P(x, \partial_x)$ and expressing

$$\psi_i^{(N)}(x, \lambda) = -\sum_{j=1}^{N-1} \alpha_i \psi_i^{(j)}(x, \lambda) - \alpha_0 (x + \lambda) \psi_i(x, \lambda),$$

produces a $\mathbb{C}[x]$-linear combination of

$$\{\psi_i^{(j)}(x, \lambda) e_l \mid 0 \le j \le N-1, 1 \le l \le n\}$$

that equals 0. Once again (5.4) implies that each coefficient of the combination should equal 0. Thus, the fact that one of the elements (5.8) belongs to $\operatorname{vker} P(x, \partial_x)$ is equivalent to imposing $N$ conditions of the form

$$\sum_{l=0}^{dN-1} a_l(x) c_{l,j}(x) = 0, \quad \forall\, 0 \le j \le N-1,$$

where $c_{l,j}(x)$ are some vector-valued polynomial functions uniquely determined from $p_0(x), \dots, p_{N-1}(x)$. However, the condition that every element in (5.8) belongs to $\operatorname{vker} P(x, \partial_x)$ is equivalent to imposing exactly the same $N$ conditions because $\psi_i(x, \lambda)$ satisfy the same Airy equation.

So, the conditions on the operator $P(x, \partial_x)$ (having vector kernel equal to $V$) are equivalent to imposing $dN$ linear vector conditions on the matrix meromorphic functions $\{a_l(x) \mid 0 \le l \le dN-1\}$. Because the coefficients of these conditions are vector-valued polynomial functions and there is a unique monic differential operator $P(x, \partial_x)$ with the stated properties, all its coefficients $a_l(x)$ should be rational. □

## 6. Examples

In this section we present several examples illustrating the classification results from the previous two sections.

**6.1.** First, we give three examples of matrix rank 1 bispectral Darboux transformations, classified in Theorem 4.3 and Corollary 4.4.

**Example 6.1.** In the case of $2 \times 2$ matrices, consider the differential operator

$$L(x, \partial_x) := \partial_x^2 I_2 \in A_1.$$

(Below all instances of the identity matrix $I_n$ which are clear from the setting will be suppressed.) The vector kernel of $L$ consists of all vector linear functions. In the setting of Theorem 4.3, take the subspace $V$ of $\mathcal{QP}_2$ with basis

$$f_1(x) := (x, 0)^t, \quad f_2(x) = (a, x)^t$$

for an arbitrary $a \in \mathbb{C}$. Set

$$F_1(x) := (f_1(x), f_2(x)) = \begin{pmatrix} x & a \\ 0 & x \end{pmatrix}.$$

Consider the corresponding operator $P(x, \partial_x)$ given by (3.3), which in this case simplifies to

$$P(x, \partial_x) = \partial_x - F_1'(x) F_1^{-1}(x) = \partial_x - \begin{pmatrix} x^{-1} & -ax^{-2} \\ 0 & x^{-1} \end{pmatrix}.$$

By Theorem 4.3, we have the factorization $L(x, \partial_x) = Q(x, \partial_x) P(x, \partial_x)$ from which the operator $Q$ is computed to be

$$Q(x, \partial_x) = \partial_x + \begin{pmatrix} x^{-1} & -ax^{-2} \\ 0 & x^{-1} \end{pmatrix}.$$

This gives rise to the bispectral Darboux transformation

$$\Phi(x, z) := P(x, \partial_x)(e^{xz} I_2) = e^{xz} \begin{pmatrix} z - x^{-1} & ax^{-2} \\ 0 & z - x^{-1} \end{pmatrix}.$$

It satisfies

$$\widetilde{L}(x, \partial_x) \Phi(x, z) = \Phi(x, z) z^2$$

where

$$\widetilde{L}(x, \partial_x) := P(x, \partial_x) Q(x, \partial_x) = \partial^2 - \begin{pmatrix} 2x^{-2} & -4ax^{-3} \\ 0 & 2x^{-2} \end{pmatrix}.$$

For the dual spectral equation, given by Theorem 4.1, we need to represent $L(x, \partial_x)$ in the form (4.2):

$$\partial_x^2 I_2 = Q'(x, \partial_x) \frac{1}{x^4} P'(x, \partial_x) \text{ with } Q' := \partial_x x^2 + \begin{pmatrix} x & -a \\ 0 & x \end{pmatrix}, \ P' := x^2 \partial_x - \begin{pmatrix} x & -a \\ 0 & x \end{pmatrix} \in B_1.$$

Now, by Theorem 4.1, the function $\Phi(x, z)$ satisfies the dual spectral equation

$$\Phi(x, z) \widetilde{\Lambda}(z, \partial_z) = x^4 \Phi(x, z)$$

where

$$\begin{aligned}\widetilde{\Lambda}(z,\partial_z) &:= \frac{1}{z^2} b\left[\partial_x x^2 + \begin{pmatrix} x & -a \\ 0 & x \end{pmatrix}\right] \circ b\left[x^2\partial_x - \begin{pmatrix} x & -a \\ 0 & x \end{pmatrix}\right] \\ &= \frac{1}{z^2}\left[z\partial_z^2 - \partial_z + \begin{pmatrix} 0 & -a \\ 0 & 0 \end{pmatrix}\right] \circ \left[\partial_z^2 z + \partial_z + \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right] \\ &= z^{-2}(z\partial_z^2 - \partial_z)(\partial_z^2 z + \partial_z)I_2 + \begin{pmatrix} 0 & -2az^{-2}\partial_z \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

When $a = 1$, this is the first example in [6] and [18], where a (slightly more complicated) third order dual spectral equation was considered.

**Example 6.2.** For $3 \times 3$ matrices, consider the operator $L = \partial_x^2 I_3 \in A_1$. Its vector kernel consists of vector linear functions. Let $V$ be the subspace of $\mathcal{QP}_3$ with a basis consisting of the columns of the matrix

$$F_1(x) := \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix}.$$

The corresponding operator $P(x,\partial_x)$, given by (3.3), is

$$P(x,\partial_x) := \partial_x - F_1'(x)F_1(x)^{-1} = \partial_x - (x^{-1}I_3 - x^{-2}J + x^{-3}J^2)$$

where $J$ is the super-diagonal matrix

$$J := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 4.3, we have the factorization $L(x,\partial_x) = Q(x,\partial_x)P(x,\partial_x)$ from which the operator $Q$ is computed to be

$$Q(x,\partial_x) = \partial_x + (x^{-1}I_3 - x^{-2}J + x^{-3}J^2).$$

The bispectral Darboux transformation

$$\Phi(x,z) := P(x,\partial_x)(e^{xz}I_3) = e^{xz}\begin{pmatrix} z - x^{-1} & x^{-2} & -x^{-3} \\ 0 & z - x^{-1} & x^{-2} \\ 0 & 0 & z - x^{-1} \end{pmatrix}$$

satisfies

$$\widetilde{L}(x,\partial_x)\Phi(x,z) = \Phi(x,z)z^2$$

where

$$\widetilde{L}(x,\partial_x) := P(x,\partial_x)Q(x,\partial_x) = \partial_x^2 - 2x^{-2}I_3 + 4x^{-3}J - 6x^{-4}J^2.$$

To write the dual spectral equation in Theorem 4.1, we represent $L(x,\partial_x)$ in the form (4.2), $\partial_x^2 I_2 = Q'(x,\partial_x)x^{-3}P'(x,\partial_x)$ with

$$Q'(x,\partial_x) := \partial_x x^3 + (x^2I_3 - xJ + J^2), \quad P'(x,\partial_x) := x^3\partial_x - (x^2I_3 - xJ + J^2) \in B_1.$$

Theorem 4.1 implies that $\Phi(x,z)$ satisfies the dual spectral equation

$$\Phi(x,z)\widetilde{\Lambda}(z,\partial_z) = x^4\Phi(x,z)$$

with

$$\begin{aligned}\widetilde{\Lambda}(z,\partial_z) &:= z^{-2}b(Q')(z,\partial_z)b(P')(z,\partial_z) \\ &= z^{-2}(z\partial_z^3 I_3 - (\partial_z^2 I_3 + \partial_z J + J^2))(\partial_z^3 zI_3 + (\partial_z^2 I_3 + \partial_z J + J^2)).\end{aligned}$$

This bispectral function is the second example in [18], where a conjecture is stated about the full algebra of dual differential operators with eigenfunction $\Phi(x,z)$. It contains lower order operators, e.g., the degenerate operator

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \partial_z^2 + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2z^{-1} & 1 & 0 \end{pmatrix} \partial_z + \begin{pmatrix} 1 & 0 & 0 \\ -2z^{-1} & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Example 6.3.** For $3\times 3$ matrices, consider the operator

$$L(x,\partial_x) = \partial_x^2 I_3 + \partial_x J + J^2 = \begin{pmatrix} \partial_x^2 & \partial_x & 1 \\ 0 & \partial_x^2 & \partial_x \\ 0 & 0 & \partial_x^2 \end{pmatrix} \in A_1.$$

A basis of $\operatorname{vker} L(x,\partial_x)$ is given by

$$(1,0,0)^t, (x,0,0)^t, (0,1,0)^t, (-x^2/2, x, 0)^t, (0, -x^2/2, x)^t, (-x^2/2, 0, 1)^t.$$

Consider the subspace of $\mathcal{QP}_3$ with a basis consisting of the columns of the matrix function

$$F_1(x) = \begin{pmatrix} x & -x^2/2 & -x^2/2 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding operator $P(x,\partial_x)$, given by (3.3), is $P(x,\partial_x) := \partial_x - F_1'(x)F_1(x)^{-1}$, or explicitly,

$$P(x,\partial_x) = \partial_x - \begin{pmatrix} x^{-1} & -3/2 & -x/2 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = x^{-1}\Big\{ x\partial_x - \begin{pmatrix} 1 & -3x/2 & -x^2/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big\} = x^{-1}P'.$$

By Theorem 4.3, we have the factorization $L(x,\partial_x) = Q(x,\partial_x)P(x,\partial_x)$ from which the operator $Q$ is computed to be

$$Q = \partial_x + \begin{pmatrix} x^{-1} & -1/2 & -x/2 \\ 0 & x^{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix} = \Big\{ \partial_x x + \begin{pmatrix} 1 & -x/2 & -x^2/2 \\ 0 & 1 & x \\ 0 & 0 & 0 \end{pmatrix} \Big\} x^{-1} = Q'x^{-1}.$$

The operators $P'(x,\partial_x)$ and $Q'(x,\partial_x)$ will be needed for the dual bispectral equation.

The bispectral Darboux transformation

$$\Phi(x,z) := P(x,\partial_x)(e^{xz}I_3) = \begin{pmatrix} z - x^{-1} & 3/2 & x/2 \\ 0 & z - x^{-1} & 0 \\ 0 & 0 & z \end{pmatrix}$$

satisfies

$$\widetilde{L}(x,\partial_x)\Phi(x,z) = \Phi(x,z) \begin{pmatrix} z^2 & z & 1 \\ 0 & z^2 & z \\ 0 & 0 & z^2 \end{pmatrix}$$

where the differential operator $\widetilde{L}(x,\partial_x)$ is given by

$$\widetilde{L}(x,\partial_x) := P(x,\partial_x)Q(x,\partial_x),$$

omitting the detailed expansion for the sake of brevity. By Theorem 4.1, the function $\Phi(x,z)$ satisfies the dual spectral equation

$$x^2\Phi(x,z) = \Phi(x,z)\widetilde{\Lambda}(x,\partial_x)$$

where
$\widetilde{\Lambda}(z,\partial_z) = (z^2 I_3 + zJ + J^2)^{-1} b(Q')(z,\partial_x) b(P')(z,\partial_z) =$

$$\begin{pmatrix} z^{-2} & -z^{-3} & 0 \\ 0 & z^{-2} & -z^{-3} \\ 0 & 0 & z^{-2} \end{pmatrix} \Big\{ z\partial_z - \begin{pmatrix} 1 & \partial_z/2 & -\partial_z^2/2 \\ 0 & 1 & -\partial_z \\ 0 & 0 & 0 \end{pmatrix} \Big\} \Big\{ \partial_z z + \begin{pmatrix} 1 & 3\partial_z/2 & -\partial_z^2/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big\}.$$

**6.2.** We finish with an example illustrating the classification results for all bispectral Darboux transformations from the matrix Airy functions.

**Example 6.4.** Consider the matrix polynomial

$$q(t) = (I_2 t - J)^2 = I_2 t^2 - 2Jt$$

where

$$J := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For the second order Airy operator $M_{\mathrm{Ai}}(x,\partial_x) = \partial_x^2 - x$, consider the operator

$$L(x,\partial_x) := q(M_{\mathrm{Ai}}(x,\partial_x)) = (I_2\partial_x^2 - I_2 x - J)^2 \in B_1.$$

It satisfies $b(L(x,\partial_x)) = q(z) = z^2 I_2 - 2zJ$, i.e.,

$$L(x,\partial_x)\Psi_{\mathrm{Ai}}(x,z) = \Psi_{\mathrm{Ai}}(x,z)(z^2 I_2 - 2zJ).$$

According to our scheme (from Proposition 5.2, Theorem 5.4 and Corollary 5.5), we first need to find the Jordan chains of the monic matrix polynomial $q(t)$. Its characteristic polynomial $\chi(t) = \det(q(t)) = t^4$ has only one root 0 and

$$q(0) = 0, \quad q'(0) = -2J, \quad q''(0) = 2I_2.$$

Denote by $e_1, e_2$ the standard basis of $\mathbb{C}^2$. Then

$$\begin{aligned} & e_2, \\ & e_1, (1/2)e_2, e_1 \end{aligned}$$

is a maximal family of Jordan chains as in Theorem 7.1. Recall from §5.1 that $\psi_1(x), \psi_2(x)$ denotes a basis of $\operatorname{vker} M_{\mathrm{Ai}}(x,\partial_x)$. Proposition 5.2 produces the following basis of $\operatorname{vker} L(x,\partial_x)$ :

$$\begin{pmatrix} 0 \\ \psi_i(x) \end{pmatrix}, \begin{pmatrix} \psi_i(x) \\ 0 \end{pmatrix}, \begin{pmatrix} \psi_i'(x) \\ \psi_i(x)/2 \end{pmatrix}, \begin{pmatrix} \psi_i(x) + \psi_i''(x)/2 \\ \psi_i'(x)/2 \end{pmatrix}, \quad i = 1,2.$$

(One can further simplify it by taking linear combinations.) Consider the 4-dimensional subspace $V$ of $\mathcal{QA}_2$ with basis

$$\begin{pmatrix} \psi_i'(x) + \psi_i(x) \\ \psi_i(x) \end{pmatrix}, \begin{pmatrix} \psi_i''(x) \\ \psi_i'(x) \end{pmatrix}, \quad i = 1,2.$$

It satisfies the conditions in Theorem 5.4, in particular $V$ is $\mathbb{Z}_2$-invariant. It can be computed by using quasideterminants. However, more effectively, it can be computed by setting $P(x,\partial_x) = \partial_x^2 + b(x)\partial_x + c(x)$ and then solving for $b(x)$ and $c(x)$ from the equations $P(x,\partial_x)f(x) = 0$ for the 4 basis elements $f(x)$ of $V$. The concrete solution of the equations for $b(x)$ and $c(x)$ is done by expressing all derivatives $\psi_i^{(k)}(x)$ in terms of $\psi_i(x)$ and $\psi_i'(x)$ using the Airy equation, and then setting the coefficients in front of $\psi_i(x)$ and $\psi_i'(x)$ in the equations $P(x,\partial_x)f(x) = 0$ to be equal to 0. This gives that

$$P(x,\partial_x) = \partial_x^2 + \frac{1}{x-1}\begin{pmatrix} 0 & 1-x \\ 1 & -1 \end{pmatrix}\partial_x - \frac{1}{x-1}\begin{pmatrix} (x-1)^2 & 2x-2 \\ 0 & x^2 \end{pmatrix}.$$

By Theorem 4.3, $L(x,\partial_x) = Q(x,\partial_x)P(x,\partial_x)$ for some second order monic differential operator $Q(x,\partial_x)$. By comparing coefficients one obtains

$$Q(x,\partial_x) = \partial_x^2 - \partial_x \frac{1}{x-1}\begin{pmatrix} 0 & 1-x \\ 1 & -1 \end{pmatrix} + \frac{1}{(x-1)}\begin{pmatrix} -x^2 & 1 \\ 0 & -(x-1)^2 \end{pmatrix}.$$

The bispectral Darboux transformation

$$\Phi(x,z) := P(x,\partial_x)\Psi_{\mathrm{Ai}}(x,z)$$

satisfies

$$\widetilde{L}(x,\partial_x)\Phi(x,z) = \Phi(x,z)(z^2 I_2 - 2J)$$

where

$$\widetilde{L}(x,\partial_x) := P(x,\partial_x)Q(x,\partial_x).$$

For the sake of brevity we leave the detailed computation of $\widetilde{L}(x,\partial_x)$ to the reader.

To write the dual spectral equation from Theorem 5.1, we represent $q(M_{\mathrm{Ai}}(x,\partial_x))$ in the form (4.2),

$$q(M_{\mathrm{Ai}}(x,\partial_x)) = Q'(x,\partial_x)\frac{1}{(x-1)^2}P'(x,\partial_x)$$

with

$$Q'(x,\partial_x) := Q(x,\partial_x)(x-1), \quad P'(x,\partial_x) := (x-1)P(x,\partial_x) \in B_1.$$

Explicitly we have

$$\begin{aligned} P'(x,\partial_x) &= (x-1)\partial_x^2 + \begin{pmatrix} 0 & 1-x \\ 1 & -1 \end{pmatrix}\partial_x - \begin{pmatrix} (x-1)^2 & 2x-2 \\ 0 & x^2 \end{pmatrix}, \\ Q'(x,\partial_x) &= \partial_x^2(x-1) - \partial_x \begin{pmatrix} 0 & 1-x \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -x^2 & 1 \\ 0 & -(x-1)^2 \end{pmatrix}. \end{aligned}$$

Theorem 5.1 implies that $\Phi(x,z)$ satisfies the dual spectral equation

$$\Phi(x,z)\widetilde{\Lambda}(z,\partial_z) = (x-1)^3\Phi(x,z)$$

with

$$\begin{aligned} \widetilde{\Lambda}(z,\partial_z) &:= (zI_2 - J)^{-2} b(Q')(z,\partial_z) b(P')(z,\partial_z) \\ &= (z^{-2} + 2z^{-1}J) b(Q')(z,\partial_z) b(P')(z,\partial_z) \end{aligned}$$

where

$$\begin{aligned} b(P')(z,\partial_z) &= (M_z - 1)\partial_z^2 - \begin{pmatrix} 0 & 1-M_z \\ 1 & -1 \end{pmatrix}\partial_z - \begin{pmatrix} (M_z-1)^2 & 2M_z - 2 \\ 0 & M_z^2 \end{pmatrix}, \\ b(Q')(z,\partial_z) &= \partial_z^2 (M_z - 1)^2 + \partial_z \begin{pmatrix} 0 & 1-M_z \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -M_z^2 & 1 \\ 0 & -(M_z-1)^2 \end{pmatrix} \end{aligned}$$

and we abbreviated $M_x := M_{\mathrm{Ai}}(x,\partial_x)$, $M_z := M_{\mathrm{Ai}}(z,\partial_z)$.

## 7. Appendix. Matrix polynomials

Here we collect some facts about the spectral theory of matrix polynomials, used in the paper. We follow [15] and refer the reader to [15, Sections 1.4-1.6 and S1.5] for details.

Consider a monic matrix polynomial

$$q(t) = \sum_{j=0}^{d} a_j t^j, \quad \text{where} \quad a_j \in M_n(\mathbb{C}),\ a_d = I_n.$$

Define the monic polynomial

$$\chi(t) := \det(q(t))$$

of order $nd$. It equals the characteristic polynomial of the companion matrix

$$C = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{d-1} \end{pmatrix} \in M_{nd}(\mathbb{C}), \tag{7.1}$$

see [15, Theorem 1.1].

A sequence of vectors $v_0, v_1 \dots, v_k \in \mathbb{C}^n$, with $v_0 \neq 0$ is called a *Jordan chain* of $q(t)$ of length $k+1$ corresponding to $\lambda \in \mathbb{C}$ if the following equalities holds:

$$\sum_{r=0}^{j} \frac{1}{r!} q^{(r)}(\lambda) v_{j-r} = 0,\ \forall\, 0 \leq j \leq k. \tag{7.2}$$

Such a chain exists only if $\lambda$ is a root of $\chi(t)$. Its leading term $v_0$ is an eigenvector of $q(\lambda)$. The condition (7.2) is equivalent to saying that the function

$$u(x) := \left( \frac{x^k}{k!} v_0 + \frac{x^{k-1}}{(k-1)!} v_1 + \cdots + v_k \right) e^{\lambda x}$$

is a solution of the ODE $q(\partial_x)u(x) = 0$. When $q(t)$ is the characteristic polynomial of a square matrix $A$, this notion recovers the classical notion of a Jordan chain for $A$.

The following theorem is a generalization of the well known Jordan normal form theorem for square matrices, see [15, Theorem 1.12] for details.

**Theorem 7.1.** *Let $q(t)$ be a monic matrix polynomial and $\lambda$ be a root of $\chi(t)$ of multiplicity $m$. Then there exist positive integers $k_1, \dots, k_s$ and Jordan chains*

$$v_{0,l}, \dots, v_{k_l,l} \in \mathbb{C}^n$$

*(for $1 \leq l \leq s$) of $q(t)$ with respect to $\lambda$ such that $\{v_{0,1}, \dots, v_{0,s}\}$ is a basis of $\ker q(\lambda)$ and*

$$(k_1 + 1) + \cdots + (k_s + 1) = m. \tag{7.3}$$

The lengths of the Jordan chains in the theorem can be read off from the local Smith form of $q(t)$ in terms of the partial multiplicities of $q(t)$ at $\lambda$, see [15, Section S1.5].

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