POISSON ORDERS ON LARGE QUANTUM GROUPS

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Abstract. We bring forward the notions of large quantum groups and their relatives. The starting point is the concept of distinguished pre-Nichols algebra [An3] belonging to a one-parameter family; we call such an object a *large* quantum unipotent subalgebra. By standard constructions we introduce *large* quantum groups and *large* quantum Borel subalgebras. We first show that each of these three large quantum algebras has a central Hopf subalgebra giving rise to a Poisson order in the sense of [BG]. We describe explicitly the underlying Poisson algebraic groups and Poisson homogeneous spaces in terms of Borel subgroups of complex semisimple algebraic groups of adjoint type. The geometry of the Poisson algebraic groups and Poisson homogeneous spaces that are involved and its applications to the irreducible representations of the algebras $U_{\mathfrak{q}} \supset U_{\mathfrak{q}}^{\geqslant} \supset U_{\mathfrak{q}}^{+}$ are also described. Multiparameter quantum super groups at roots of unity fit in ou context as well as quantizations in characteristic 0 of the 34-dimensional Kac-Weisfeler Lie algebras in characteristic 2 and the 10-dimensional Brown Lie algebras in characteristic 3. All steps of our approach are applicable in wider generality and are carried out using general constructions with restricted and non-restricted integral forms and Weyl groupoid actions. Our approach provides new proofs to results in the literature without reductions to rank two cases.

Contents

Key words and phrases. Hopf algebras, Nichols algebras, integral forms of quantum groups, Poisson orders, Poisson algebraic groups and homogeneous spaces.

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1. Introduction

1.1. **Quantum groups and Poisson orders.** Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra and let $\xi \in \mathbb{C}$ be a root of 1 with some restrictions on its order depending on $\mathfrak{g}$. In the papers [DK, DKP, DP] a quantized enveloping algebra $U_\xi(\mathfrak{g})$ at ξ was introduced and studied; it is a version of the Drinfeld-Jimbo quantized universal enveloping algebra different from the one defined in [L1, L2].

The algebra $U_\xi(\mathfrak{g})$ is module-finite over a central Hopf subalgebra $Z_\xi(\mathfrak{g})$ and the corresponding small quantum group of Lusztig [L1, L2] arises as the quotient $U_\xi(\mathfrak{g})/\!\!/Z_\xi(\mathfrak{g})$ in the sense of Hopf algebras. A geometric approach to the representation theory of $U_\xi(\mathfrak{g})$ was proposed in [DP], based on these facts. The key ingredients of this approach are:

- The existence of a Poisson structure on $Z_\xi(\mathfrak{g})$ so that the algebraic group M corresponding to this algebra is a Poisson algebraic group, whose Lie bialgebra is dual to the standard Lie bialgebra structure on $\mathfrak{g}$.
- The Hamiltonian vector fields on M extend to (explicit) derivations of $U_\xi(\mathfrak{g})$.

The approach consists in packing the irreducible finite-dimensional representations of $U_\xi(\mathfrak{g})$ along the symplectic leaves of M and predicting their dimensions. These ideas were distilled in the notion of Poisson order in [BG], see Section 2. The construction of a Poisson order structure on an algebra has substantial applications to the representation theory of the algebra: using this route the irreducible representations of quantum function algebras were studied in [DL], the Azumaya loci of symplectic reflection algebras was described in [BG], the irreducible representations of the 3 and 4-dimensional PI Sklyanin algebras were fully classified in [WWY1, WWY2], the Azumaya loci of the multiplicative quiver varieties and quantum character varieties were studied in [GJS]. See [BGo, Part III] for a comprehensive exposition of the applications the notion of Poisson order to the representation theory of quantum algebras at roots of unity.

1.2. **Large quantum groups and pre-Nichols algebras.** The main goal of this paper is to study by means of Poisson orders the representation theory of a larger class of Hopf algebras introduced by the second author in [An2] and studied in [An3]. The keystone of the definition of these Hopf algebras is the notion of *distinguished pre-Nichols algebra*.

Nichols algebras of diagonal type are essential for various classification problems of Hopf algebras. Those with finite dimension were classified in the celebrated paper [H2] while the defining relations were provided in [An1, An2]. Let $\mathfrak{q}$ be a braiding matrix as in the list of [H2] and let $\mathcal{B}_\mathfrak{q}$ be the corresponding finite-dimensional Nichols algebra of diagonal type. The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_\mathfrak{q}$ of $\mathcal{B}_\mathfrak{q}$ is a covering of the latter defined by excluding the powers of the root vectors of Cartan type from its defining ideal. The Hopf algebras dealt with in the present paper are Drinfeld doubles of the bosonizations of the distinguished pre-Nichols algebras; they are denoted $U_\mathfrak{q}$, see §4.3 and are module-finite over the central Hopf subalgebra $Z_\mathfrak{q}$ defined in [An3], see §4.5. On the other hand, the graded dual of $\widetilde{\mathcal{B}}_\mathfrak{q}$ gives rise to a Lie algebra $\mathfrak{n}_\mathfrak{q}$ which is either 0 or the nilpotent part of a semisimple Lie algebra $\mathfrak{g}_\mathfrak{q}$ that is explicitly determined [AAR3].

We focus on Hopf algebras $U_\mathfrak{q}$ with a further restriction: the related Nichols algebra $\mathcal{B}_\mathfrak{q}$ is deformable, i.e. belongs to a one-parameter family of Nichols algebras. We call them *large quantum groups*. By inspection, the matrix $\mathfrak{q}$ is of one of three types:

(a) Cartan type (multiparameter versions of the quantum groups from [DKP] without restrictions on ξ);

(b) super type (multiparameter quantum groups associated to finite dimensional simple contragredient Lie superalgebras at roots of unity);

(c) modular types $\mathtt{wk}(4)$ or $\mathtt{br}(2)$ (quantizations at root of unity of some simple Lie algebras in characteristics 2 and 3 respectively).

But it stems from the list in [H2] that there are finite-dimensional Nichols algebras of diagonal type that do not belong to such one-parameter families.

Remark 1.1. To be precise we need three technical assumptions:

(i) The base field is $\mathbb{C}$ to have on hand symplectic leaves, although this is not essential, see for details [BG].

(ii) Condition (4.23) is needed for the centrality of $Z_{\mathfrak{q}}$ in $U_{\mathfrak{q}}$.

(iii) The Non-degeneracy Assumption 7.5 is used to identify some dual vector spaces in order to compute some Lie bialgebras.

See the Appendix A and the survey [AA] for full details on these algebras. We consider the chain of subalgebras $U_{\mathfrak{q}}^{+} \subset U_{\mathfrak{q}}^{\geqslant} \subset U_{\mathfrak{q}}$ where

- $U_{\mathfrak{q}}^{\geqslant}$, the *large quantum Borel subalgebra*, is identified with the bosonization of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$;
- $U_{\mathfrak{q}}^{+}$, the *large quantum unipotent subalgebra*, is identified with $\widetilde{\mathcal{B}}_{\mathfrak{q}}$.

Intersecting the central subalgebra $Z_{\mathfrak{q}}$ of $U_{\mathfrak{q}}$ gives the chain of central Hopf subalgebras

$$Z_{\mathfrak{q}}^{+} \subset Z_{\mathfrak{q}}^{\geqslant} \subset Z_{\mathfrak{q}}. \tag{1.1}$$

Each of these central Hopf subalgebras is actually isomorphic to a tensor product of a polynomial algebra and a Laurent polynomial algebra. The maximal spectra of the Hopf algebras $Z_{\mathfrak{q}}$, $Z_{\mathfrak{q}}^{\geqslant}$ and $Z_{\mathfrak{q}}^{+}$ are the complex algebraic groups $M_{\mathfrak{q}}$, $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{+}$, respectively. We shall also need the opposite Borel $U_{\mathfrak{q}}^{\leqslant}$ and its central Hopf subalgebra $Z_{\mathfrak{q}}^{\leqslant}$ with maximal spectrum $M_{\mathfrak{q}}^{\leqslant}$ and correspondingly $U_{\mathfrak{q}}^{-}$, $Z_{\mathfrak{q}}^{-}$ and $M_{\mathfrak{q}}^{-}$.

1.3. **Main results.** As said this paper deals with the geometry of the Poisson algebraic group $M_{\mathfrak{q}}$ towards understanding the representation theory of large quantum groups. This last question contain the description of the irreducible representations of quantum supergroups at roots of unity, an important problem which is wide open even in the simplest case of $U_q(\mathfrak{sl}(m|n))$. We present a foundation for a thorough investigation of these representations. We first summarize the main results in the following statement.

Theorem A. *Let $U_{\mathfrak{q}}$ be a large quantum group as above. Then*

(a) *The pair $(U_{\mathfrak{q}}, Z_{\mathfrak{q}})$ has the structure of a Poisson order in the sense of* [BG].

(b) *The algebraic Poisson group $M_{\mathfrak{q}}$ is* solvable. *The Lie bialgebra of $M_{\mathfrak{q}}$ is dual to the standard Lie bialgebra structure on $\mathfrak{g}_{\mathfrak{q}}$; hence the symplectic leaves of $M_{\mathfrak{q}}$ are known.*

(c) *Every $z \in M_{\mathfrak{q}}$ with corresponding maximal ideal $\mathfrak{M}_z$ gives rise to a finite-dimensional algebra $\mathcal{H}_z = U_{\mathfrak{q}}/U_{\mathfrak{q}}\mathfrak{M}_z$. Then $\mathcal{H}_z \simeq \mathcal{H}_{z'}$ whenever z and z' belong to the same symplectic leave S. By abuse of notation we set $\mathcal{H}_S = \mathcal{H}_z$. Every irreducible representation of $U_{\mathfrak{q}}$ is finite-dimensional and*

$$\operatorname{Irr} U_{\mathfrak{q}} = \bigcup_{S \text{ symplectic leave of } M} \operatorname{Irr} \mathcal{H}_S.$$

Furthermore, we have analogous results for the pairs $(U_{\mathfrak{q}}^{\geqslant}, Z_{\mathfrak{q}}^{\geqslant})$ and $(U_{\mathfrak{q}}^{+}, Z_{\mathfrak{q}}^{+})$. Next we make more precise the claims of Theorem A. We fix a large quantum group $U_{\mathfrak{q}}$.

1.3.1. *Poisson orders.* We denote by $\mathcal{Z}(A)$ the center of an algebra A. Because of the assumption that $\mathcal{B}_{\mathfrak{q}}$ is deformable in the class of Nichols algebras as mentioned above, we get Poisson order structures on the pairs $(U_{\mathfrak{q}}, \mathcal{Z}(U_{\mathfrak{q}}))$, $(U_{\mathfrak{q}}^{\geqslant}, \mathcal{Z}(U_{\mathfrak{q}}^{\geqslant}))$ and $(U_{\mathfrak{q}}^{+}, \mathcal{Z}(U_{\mathfrak{q}}^{+}))$ by specialization. As these centers are singular, it is more convenient to look at the central subalgebras in (1.1). Part (a) of Theorem A is included in the following result, see Theorem 6.2.

Theorem B. *The pairs* $(U_{\mathfrak{q}}, Z_{\mathfrak{q}})$, $(U_{\mathfrak{q}}^{\geqslant}, Z_{\mathfrak{q}}^{\geqslant})$ *and* $(U_{\mathfrak{q}}^{+}, Z_{\mathfrak{q}}^{+})$ *have Poisson order structures in the sense of* [BG] *obtained from specialization.*

Presently it is not known whether for the remaining braiding matrices $\mathfrak{q}$ in the list of [H2] the pair $(U_{\mathfrak{q}}, Z_{\mathfrak{q}})$ has the structure of a Poisson order. Indeed the other Nichols algebras of diagonal type with arithmetic root system in the classification given in [H2] do not admit such a one-parameter family and for instance our proof of Theorem 6.2 does not generalize to them.

1.3.2. *Poisson algebraic groups and Lie bialgebras.* Recall the semisimple Lie algebra $\mathfrak{g}_{\mathfrak{q}}$ determined in [AAR3] and fix a Cartan subalgebra $\mathfrak{h}_{\mathfrak{q}}$. We consider on $\mathfrak{g}_{\mathfrak{q}}$ the Lie bialgebra structure with the standard Belavin–Drinfeld triple [BD] that we extend trivially to $\mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$, see [ES, §4.4]. Let $\mathfrak{m}_{\mathfrak{q}}$ be the Lie bialgebra dual to $\mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$.

Let $G_{\mathfrak{q}}$ be the semisimple algebraic group of adjoint type with $\operatorname{Lie} G_{\mathfrak{q}} \simeq \mathfrak{g}_{\mathfrak{q}}$. For instance, when $U_{\mathfrak{q}} = U_q(\mathfrak{sl}(m|n))$, $G_{\mathfrak{q}} \simeq \mathrm{PSL}_m(\mathbb{C}) \times \mathrm{PSL}_n(\mathbb{C})$. Let $B_{\mathfrak{q}}^{+}$ be a Borel subgroup of $G_{\mathfrak{q}}$, $T_{\mathfrak{q}} \leqslant B_{\mathfrak{q}}^{+}$ a maximal torus and $N_{\mathfrak{q}}^{+} \leqslant B_{\mathfrak{q}}^{+}$ the unipotent radical; we identify $N_{\mathfrak{q}}^{+} \simeq B_{\mathfrak{q}}^{+}/T_{\mathfrak{q}}$. Also $B_{\mathfrak{q}}^{-}$ is the opposite Borel subgroup and $N_{\mathfrak{q}}^{-} \leqslant B_{\mathfrak{q}}^{-}$ is its unipotent radical.

Here is a more precise statement of Theorem A Part (b), see Theorems 7.10 and 8.2.

Theorem C. (a) *The Poisson algebraic group* $M_{\mathfrak{q}}$ *is isomorphic to the product of two Borel subgroups of* $G_{\mathfrak{q}}$ *and* $\operatorname{Lie} M_{\mathfrak{q}} \simeq \mathfrak{m}_{\mathfrak{q}}$ *as Lie bialgebras.*

(b) *The symplectic leaves of* $M_{\mathfrak{q}}$ *are in bijective correspondence with the coadjoint orbits of* $G_{\mathfrak{q}}$*; each leaf is isomorphic to an open dense subset of the corresponding coadjoint orbit.*

Here are the promised versions for $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{+}$.

Theorem D. (a) *The Poisson algebraic group* $M_{\mathfrak{q}}^{\geqslant}$ *is isomorphic to the Borel subgroup* $B_{\mathfrak{q}}^{+}$*. The Poisson structure is invariant under the left and right actions of* $T_{\mathfrak{q}}$.

(b) *The torus orbits of symplectic leaves of* $M_{\mathfrak{q}}^{\geqslant}$ *are the double Bruhat cells of* $G_{\mathfrak{q}}$ *that lie in* $B_{\mathfrak{q}}^{+}$.

(c) *The algebraic group* $M_{\mathfrak{q}}^{+}$ *is isomorphic to the unipotent radical* $N_{\mathfrak{q}}^{+}$ *of* $B_{\mathfrak{q}}^{+}$*. It has a Poisson structure arising from the identification* $N_{\mathfrak{q}}^{+} \simeq B_{\mathfrak{q}}^{+}/T_{\mathfrak{q}}$ *which is invariant under the left action of* $T_{\mathfrak{q}}$ *and is a reduction of the Poisson structure on* $B_{\mathfrak{q}}^{+}$ *from (a) under the right action of* $T_{\mathfrak{q}}$.

(d) *The torus orbits of symplectic leaves of* $M_{\mathfrak{q}}^{+}$ *are the open Richardson varieties of the flag variety* $G_{\mathfrak{q}}/B_{\mathfrak{q}}^{+}$ *that lie inside an open Schubert cell identified with* $N_{\mathfrak{q}}^{+}$.

See Theorems 8.4 and 8.7. we refer to [FZ, KLS] for information on double Bruhat cells and open Richardson varieties respectively.

1.3.3. *Representations.* Since $U_{\mathfrak{q}}$ is a free $Z_{\mathfrak{q}}$-module of finite rank, it is a PI-algebra. Let V be an irreducible representation of $U_{\mathfrak{q}}$; by the preceding V is finite-dimensional and by the Schur Lemma, $Z_{\mathfrak{q}}$ acts on V by some $z \in M_{\mathfrak{q}}$ (a central character) with corresponding maximal ideal $\mathfrak{M}_z$. Now the algebra $\mathcal{H}_z = U_{\mathfrak{q}}/U_{\mathfrak{q}}\mathfrak{M}_z$ is non-zero and finite-dimensional and V becomes a $\mathcal{H}_z$-module. In other words the irreps of $U_{\mathfrak{q}}$ with central character z are in bijective correspondence with the irreps of $\mathcal{H}_z$. Thus

$$\operatorname{Irr} U_{\mathfrak{q}} = \bigcup_{z \in M_{\mathfrak{q}}} \operatorname{Irr} \mathcal{H}_z.$$

This circle of ideas is already present in [DP]. In this way, Part (c) of Theorem A boils down to the following statement.

Theorem E. *For every two points z, z' in the same symplectic leaf of $M_{\mathfrak{q}}$, the algebras $\mathcal{H}_z$ and $\mathcal{H}_{z'}$ are isomorphic. In particular there is a dimension preserving bijection between the irreps of $U_{\mathfrak{q}}$ with central characters z and z'.*

See Theorem 8.2. For instance, let $z = e$ be the identity of $M_{\mathfrak{q}}$. Then its symplectic leaf is $S = \{e\}$ and $\mathcal{H}_S = \mathcal{H}_e$ is the Drinfeld double of a suitable bosonization of the Nichols algebra $\mathcal{B}_{\mathfrak{q}}$. Assume that the matrix $\mathfrak{q}$ is of Cartan type. Then $\mathcal{H}_e$ is a variation of the small quantum group of Lusztig (with an extra copy of the finite torus), with a notoriously difficult representation theory treated intensively in the literature. Also, arguing as in [DP] one concludes that $U_{\mathfrak{q}}$ is a maximal order, hence for generic z, $\mathcal{H}_z$ is semisimple. But for super and modular types, the representation theory of $\mathcal{H}_e$ is largely unknown, except for the somewhat standard fact that simple modules are classified by highest weights (but there is not even a conjecture for their characters). Also, $U_{\mathfrak{q}}$ is not a maximal order because it has nilpotent elements.

We next write down the corresponding formulations for $M_{\mathfrak{q}}^{\geqslant}$, $M_{\mathfrak{q}}^{\leqslant}$, $M_{\mathfrak{q}}^{+}$ and $M_{\mathfrak{q}}^{-}$. Let $\star \in \{\geqslant, \leqslant, +, -\}$. If $z \in M_{\mathfrak{q}}^{\star}$, then we denote by $\mathfrak{M}_z^{\star}$ its maximal ideal in $Z_{\mathfrak{q}}^{\star}$ and

$$\mathcal{H}_z^{\star} = U_{\mathfrak{q}}^{\star}/U_{\mathfrak{q}}^{\star}\left(\mathfrak{M}_z^{\star}\right). \tag{1.2}$$

Clearly these are finite-dimensional algebras.

Theorem F. (a) *For every z, z' in the same double Bruhat cell of $B_{\mathfrak{q}}^{+}$, the algebras $\mathcal{H}_z^{\geqslant}$ and $\mathcal{H}_{z'}^{\geqslant}$ are isomorphic. Analogously for $\mathcal{H}_z^{\leqslant}$ and $\mathcal{H}_{z'}^{\leqslant}$.*

(b) *For every z, z' in the same open Richardson variety, the algebras $\mathcal{H}_z^{+}$ and $\mathcal{H}_{z'}^{+}$ are isomorphic. Analogously for $\mathcal{H}_z^{-}$ and $\mathcal{H}_{z'}^{-}$.*

See Theorems 8.4 and 8.7.

Notice also that $\mathcal{H}_z$ is a Hopf-Galois $\mathcal{H}_e$-object since $U_{\mathfrak{q}}$ is a cleft $\mathcal{H}_{\varepsilon}$-comodule algebra, see §3.1. Analogously, $\mathcal{H}_z^{\star}$ is a Hopf-Galois $\mathcal{H}_e^{\star}$-object for $\star \in \{\geqslant, \leqslant, +, -\}$.

1.4. **Strategy and organization.** Our proofs of Theorems C–F follow a different strategy from that of [DK, DKP, DP]. These papers rely on direct computations of Poisson brackets in terms of coordinates coming from Cartesian products of one-parameter unipotent groups and subsequent reductions to the rank 2 case. This approach does not work in the more general context of §1.2 for several reasons, the simplest of which is that the quantum Serre relations for quantum supergroups or for quantum groups at -1 involve more than two Chevalley generators.

Instead our approach is based on intrinsic properties of pairings between restricted and non-restricted integral forms of Hopf algebras. It does not rely on reduction to low rank cases. In particular, this approach provides new proofs of results in [DK, DKP, DP]. We expect that these ideas could be applied to other situations not covered in this paper.

Next we overview briefly the main steps of the strategy:

Step 1. Let $\mathbb{C}(\nu)$ be the field of rational functions on q and A the subalgebra defined in (5.1). Since $\mathfrak{q}$ belongs to a family, there exists a chain of $\mathbb{C}(\nu)$-algebras

$$U_{\mathbf{q}}^{+} \subset U_{\mathbf{q}}^{\geqslant} \subset U_{\mathbf{q}}$$

and non-restricted integral forms over A

$$U_{\mathbf{q},\mathsf{A}}^{+} \subset U_{\mathbf{q},\mathsf{A}}^{\geqslant} \subset U_{\mathbf{q},\mathsf{A}}$$

such that the algebras $U_{\mathfrak{q}}^{+} \subset U_{\mathfrak{q}}^{\geqslant} \subset U_{\mathfrak{q}}$ arise as specializations from these integral forms. This provides Poisson order structures on the pairs $(U_{\mathfrak{q}}^{+}, \mathcal{Z}(U_{\mathfrak{q}}^{+}))$, $(U_{\mathfrak{q}}^{\geqslant}, \mathcal{Z}(U_{\mathfrak{q}}^{\geqslant}))$ and $(U_{\mathfrak{q}}, \mathcal{Z}(U_{\mathfrak{q}}))$. This step is carried out in Section 5 in the framework of [DP, BG] evoked in Section 2.

Step 2. We use Theorem 2.3 (on the restriction of Poisson order structures obtained from specialization to central subalgebras) to prove that the Poisson order structures on $(U_{\mathfrak{q}}^{+}, \mathcal{Z}(U_{\mathfrak{q}}^{+}))$ and $(U_{\mathfrak{q}}^{\geqslant}, \mathcal{Z}(U_{\mathfrak{q}}^{\geqslant}))$ restrict to $(U_{\mathfrak{q}}^{+}, Z_{\mathfrak{q}}^{+})$ and $(U_{\mathfrak{q}}^{\geqslant}, Z_{\mathfrak{q}}^{\geqslant})$. To get a Poisson order structure on $(U_{\mathfrak{q}}, Z_{\mathfrak{q}})$ by restriction from $(U_{\mathfrak{q}}, \mathcal{Z}(U_{\mathfrak{q}}))$, we need first to establish in Theorem 4.7 that the Weyl groupoid action preserves the central subalgebras $Z_{\mathfrak{q}}$. Along the way we also obtain that these Poisson structures on the algebras $Z_{\mathfrak{q}}$ are equivariant under the Weyl groupoid. This step is carried out in Section 6.

Step 3. This is the matter of Section 7 We introduce in §5.4 non-restricted integral forms $U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,\pm}$ of $U_{\mathbf{q}}^{\pm}$ and A-linear perfect pairings $U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,\pm} \times U_{\mathbf{q},\mathsf{A}}^{\mp} \to \mathsf{A}$. We prove

(i) the specializations of $U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,\pm}$ are isomorphic to the Lusztig algebras defined in [AAR1], see Proposition 5.9;

(ii) the cobrackets of the tangent Lie bialgebras to $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$ are linearizations of those specializations, see Proposition 7.1.

In this way we control tangent Lie bialgebras intrinsically and consequently we compute in Theorems 7.4 and 7.8 the tangent Lie bialgebras of the Poisson algebraic groups $M_{\mathfrak{q}}$, $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$ by means of a Manin pair. Since these algebraic groups are connected we describe them as Poisson algebraic groups in terms of Borel subgroups of complex semisimple algebraic groups of adjoint type. Also, $M_{\mathfrak{q}}^{\pm}$ are presented as Poisson homogeneous spaces.

Finally, we discuss in Section 8 the Poisson geometry of the Poisson algebraic groups $M_{\mathfrak{q}}$, $M_{\mathfrak{q}}^{\geqslant}$ and the Poisson homogeneous space $M_{\mathfrak{q}}^{+}$, and the applications to the irreducible representations of $U_{\mathfrak{q}}$, $U_{\mathfrak{q}}^{\geqslant}$ and $U_{\mathfrak{q}}^{+}$.

Besides, we discuss in Section 2 Poisson orders and their restrictions to central subalgebras, see Theorem 2.3; Section 3 is devoted to preliminaries on Hopf algebra theory while we present the main actors of this paper in Section 4.

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Notations. The base field is $\mathbb{C}$; all algebras, Hom's and tensor products are over $\mathbb{C}$. If $t \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $t < n$, then $\mathbb{I}_{t,n} := \{t, t+1, \dots, n\}$, $\mathbb{I}_n := \mathbb{I}_{1,n}$.

For each integer $N > 1$, let $\mathbb{G}_N$ be the group of N-th roots of unity in $\mathbb{C}$ and let $\mathbb{G}'_N$ be its subset of primitive roots (of order N). Also $\mathbb{G}_\infty = \bigcup_{N\in\mathbb{N}} \mathbb{G}_N$, $\mathbb{G}'_\infty = \mathbb{G}_\infty - \{1\}$.

2. Poisson orders and restrictions to central subalgebras

This section contains background on Poisson orders, their construction from specializations, and their relations to Hopf algebras. We prove a general result on restrictions of Poisson orders to central subalgebras, Theorem 2.3, which plays a key role later.

2.1. **Poisson orders.** Here we follow the exposition in [DP, Chapter 3, §11]. Consider
- a commutative $\mathbb{C}$-algebra A and $h \in A$ such that $A/h \simeq \mathbb{C}$,
- an A-algebra U such that h is not a zero divisor of U. The natural map $U \to U/(h)$ is denoted by $x \mapsto \overline{x}$.

For any $u \in U$ such that $\overline{u} \in \mathcal{Z}(U/(h))$ there is a linear map $D_u \in \operatorname{Hom} U/(h)$ given by

$$D_u(y) = \overline{\frac{[u,v]}{h}}, \qquad \text{if } y = \overline{v}. \tag{2.1}$$

Proposition 2.1. [DP, 11.7] *Let $u \in U$ such that $\overline{u} \in \mathcal{Z}(U/(h))$.*

(a) $D_u \in \operatorname{Der} U/(h)$.

(b) *Let $w \in U$. If $u' = u + hw$ so that $\overline{u} = \overline{u'}$, then $D_u - D_{u'} = \operatorname{ad} \overline{w}$ is an inner derivation. Conversely the inner derivation $\operatorname{ad} \overline{w}$ coincides with D_{hw}.*

(c) *Let $\varphi \in \operatorname{Aut}_{A-\mathrm{alg}}(U)$ and let $\overline{\varphi}$ be the induced automorphism of $U/(h)$. Then*
$$\overline{\varphi} \circ D_u \circ \overline{\varphi}^{-1} = D_{\varphi(u)}.$$

(d) *There is natural Poisson structure on $\mathcal{Z} := \mathcal{Z}(U/(h))$ given by*

$$\{x, y\} = D_u(y) = \overline{\frac{[u,v]}{h}}, \qquad \text{if } x = \overline{u}, y = \overline{v}. \tag{2.2}$$

(e) *The map $\varphi \mapsto \overline{\varphi}$ gives a group homomorphism $\operatorname{Aut}_{A-\mathrm{alg}}(U) \to \operatorname{Aut}_{\mathrm{Poisson}}(\mathcal{Z})$.*

(f) *$\mathcal{L} = \{D_v : v \in U, \overline{v} \in \mathcal{Z}\}$ is a Lie subalgebra of $\operatorname{Der} U/(h)$. Indeed*
$$[D_u, D_v] = D_{\frac{[u,v]}{h}}, \qquad v \in U,\ \overline{v} \in \mathcal{Z}.$$

(g) *The Poisson structure gives rise to a Lie subalgebra $\mathcal{L}'$ of $\operatorname{Der} \mathcal{Z}$ that fits into the complex*

$$0 \longrightarrow \operatorname{Innder}(U/(h)) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}' \longrightarrow 0. \tag{2.3}$$

The sequence (2.3) *is exact if and only if the Poisson center of $\mathcal{Z}$ is trivial (i.e., there are no* Casimir elements *except 0).* □

Brown and Gordon [BG] axiomatized the ingredients of the above setting as follows:

Definition 2.2. A pair of $\mathbb{C}$-algebras (R, Z) is called a *Poisson order* if Z is a central subalgebra of R, R is a Z-module of finite rank and the following two conditions hold:

(a) Z is equipped a structure of Poisson algebra $\{\cdot,\cdot\}$;

(b) There exists a linear map $D : Z \to \operatorname{Der}_{\mathbb{C}}(R)$ such that $D_z|_Z = \{z, -\}$ for all $z \in Z$.

Proposition 2.1 proves that the pair $(U/(h), \mathcal{Z}(U/(h)))$ has a canonical structure of Poisson order when $U/(h)$ is module finite over $\mathcal{Z}(U/(h))$. The Poisson bracket on $\mathcal{Z}(U/(h))$ is given by (2.2). The linear map D is the map induced from the one in (2.1) by taking a linear section of the canonical projection $U \to U/(h)$.

2.2. **Restrictions of Poisson orders from specializations.** In the setting of Proposition 2.1 the center $\mathcal{Z} = \mathcal{Z}(U/(h))$ can be singular and is more useful to work with suitable subalgebras $\mathcal{Z}'$. Next we prove a general fact for the construction of Poisson orders on pairs $(U/(h), \mathcal{Z}')$ for subalgebras $\mathcal{Z}'$ defined from algebra automorphisms and skew-derivations. For this purpose we fix:

- A-algebra endomorphisms $\varsigma_i : U \to U$, $i \in \mathbb{I}$. We denote by $\overline{\varsigma}_i$ the corresponding $\mathbb{C}$-algebra endomorphisms of $U/(h)$ induced by ς_i.
- A-linear $(\mathrm{id}, \varsigma_i)$-derivations $\partial_i : U \to U$, $i \in \mathbb{I}$. We denote by $\overline{\partial}_i$ the corresponding $\mathbb{C}$-linear $(\mathrm{id}, \overline{\varsigma}_i)$-derivations induced by ∂_i.

Theorem 2.3. *The Poisson order structure on $(U/(h), \mathcal{Z}(U/(h)))$ from Proposition 2.1 restricts to a Poisson order structure on $(U/(h), \mathcal{Z}')$, where*

$$\mathcal{Z}' := \mathcal{Z} \cap \left(\cap_{i\in\mathbb{I}} \ker \overline{\partial}_i\right) \cap \left(\cap_{i\in\mathbb{I}} \ker(\overline{\varsigma}_i - \mathrm{id})\right). \tag{2.4}$$

Proof. We have to check that $\{\mathcal{Z}', \mathcal{Z}'\} \subset \mathcal{Z}'$. Let $x_j \in \mathcal{Z}'$ and $u_j \in U$ such that $x_j = \overline{u_j}$, $j = 1, 2$. Fix $i \in \mathbb{I}$. As $\overline{\varsigma}_i(x_j) = x_j$ and $\partial_i(x_i) = 0$, there are $v_j, w_j \in U$ such that

$$\varsigma_i(u_j) = u_j + h\, v_j, \qquad \partial_i(u_j) = h\, w_j, \qquad j = 1, 2.$$

Now we compute

$$\begin{aligned}
\overline{\varsigma}_i\{x_1, x_2\} &= \overline{\varsigma}_i\left(\overline{\frac{[u_1, u_2]}{h}}\right) = \overline{\frac{[\varsigma_i(u_1), \varsigma_i(u_2)]}{h}} \\
&= \overline{\frac{[u_1, u_2]}{h}} + \overline{[u_1, v_2]} + \overline{[v_1, u_2]} = \{x_1, x_2\} + [x_1, \overline{v_2}] + [\overline{v_1}, x_2] = \{x_1, x_2\}, \\
\overline{\partial}_i\{x_1, x_2\} &= \overline{\partial}_i\left(\overline{\frac{[u_1, u_2]}{h}}\right) = \overline{\frac{\partial_i(u_1)\varsigma_i(u_2) + u_1\partial_i(u_2) - \partial_i(u_2)\varsigma_i(u_1) - u_2\partial_i(u_1)}{h}} \\
&= \overline{w_1(u_2 + h\, v_2) + u_1 w_2 - w_2(u_1 + h\, v_1) - u_2 w_1} = [x_1, \overline{w_2}] + [\overline{w_1}, x_2] = 0.
\end{aligned}$$

Hence $\{x_1, x_2\} \in \ker \overline{\partial}_i \cap \ker(\overline{\varsigma}_i - \mathrm{id})$ for all $i \in \mathbb{I}$ so $\{x, y\} \in \mathcal{Z}$. $\square$

2.3. **Poisson-Hopf algebras.** Assume that in the above setting U is a Hopf algebra over A. Then $U/(h)$ has a canonical structure of Hopf algebra over $\mathbb{C}$.

Let $u \in U$ such that $\overline{u} \in \mathcal{Z}(U/(h))$ and furthermore $\Delta(\overline{u}) \in \mathcal{Z}\left(U/(h) \otimes U/(h)\right)$. Then

$$D_{\Delta(u)}\Delta(y) = \Delta(D_u(y)), \qquad y \in U/(h). \tag{2.5}$$

Proposition 2.4. [DP, 11.7] *Let B be a central Hopf subalgebra of $U/(h)$. Then*

$$T := \text{ minimal subalgebra of } \mathcal{Z} \text{ containing } B \text{ and closed under the Poisson bracket}$$

is a central Hopf subalgebra of $U/(h)$, hence a Poisson-Hopf algebra.

We recall the elegant proof of [DP].

Proof. Apply (2.5) to $y \in \mathcal{Z}$ and $x = \overline{u}$ to get $\Delta(\{x, y\}) = \{\Delta(x), \Delta(y)\}$ for all $x, y \in \mathcal{Z}$. Hence $\widetilde{T} = \{t \in T : \Delta(t) \in T \otimes T\}$, which is a subalgebra containing B, is also closed under Poisson bracket; thus $\widetilde{T} = T$. $\square$

3. Hopf algebras

In this section we collect preliminaries on (braided) Hopf algebras (always with bijective antipode $\mathcal{S}$), bosonizations, braided vectors spaces of diagonal type, Nichols algebras, Weyl groupoids, distinguished pre-Nichols algebras and Lusztig algebras. We refer to [R, A] for more information on Hopf algebras, Nichols algebras, Nichols algebras of diagonal type, respectively.

3.1. **Cleft comodule algebras.** Let $\mathcal{H}$ be a Hopf algebra with a central Hopf subalgebra Z. Given $z \in G = \operatorname{Alg}(Z, \mathbb{C})$ (the pro-algebraic group defined by Z), let

$$\mathfrak{M}_z = \ker z, \qquad \mathfrak{I}_z = \mathcal{H}\mathfrak{M}_z, \qquad \mathcal{H}_z = \mathcal{H}/\mathfrak{I}_z;$$

thus $\mathcal{H}_z$ is an algebra (with multiplication m_z and unit u_z) and the natural projection $p_z : \mathcal{H} \to \mathcal{H}_z$ is an algebra map. Then $\mathcal{H}_\varepsilon$ is a quotient Hopf algebra of $\mathcal{H}$ and there is an exact sequence of Hopf algebras $Z \hookrightarrow \mathcal{H} \twoheadrightarrow \mathcal{H}_\varepsilon$. Also for any $z, z' \in G$ there are well-defined algebra morphisms $\Delta_{z,z'} : \mathcal{H}_{zz'} \to \mathcal{H}_z \otimes \mathcal{H}'_z$ and in particular the maps

$$\varrho_z := \Delta_{z,\varepsilon} : \mathcal{H}_z \to \mathcal{H}_z \otimes \mathcal{H}_\varepsilon, \qquad \lambda_z := \Delta_{\varepsilon,z} : \mathcal{H}_z \to \mathcal{H}_\varepsilon \otimes \mathcal{H}_z,$$

make $\mathcal{H}_z$ a $\mathcal{H}_\varepsilon$-bicomodule algebra for $z \in G$. Clearly

$$\varrho_z p_z = (p_z \otimes p_\varepsilon)\Delta_{\mathcal{H}}, \qquad \lambda_z p_z = (p_\varepsilon \otimes p_z)\Delta_{\mathcal{H}}. \tag{3.1}$$

Recall that a right K-comodule algebra A (over a Hopf algebra K) is cleft if there exists a convolution-invertible morphism of K-comodules $\chi : K \to A$.

Lemma 3.1. *If the $\mathcal{H}_\varepsilon$-comodule algebra $\mathcal{H}$ with coaction $\varrho = (\operatorname{id} \otimes p_\varepsilon)\Delta_{\mathcal{H}}$ is cleft, then so is $\mathcal{H}_z$ for any $z \in G$. In particular $\mathcal{H}_z$ is a Hopf-Galois object over $\mathcal{H}_e$.*

If $\mathcal{H}$ is a pointed Hopf algebra, then $\mathcal{H}_z$ is $\mathcal{H}_e$-cleft for all $z \in G$.

Proof. If $\chi : \mathcal{H}_\varepsilon \to \mathcal{H}$ is a morphism of $\mathcal{H}$-comodules, then so is $\chi_z := p_z\chi : \mathcal{H}_\varepsilon \to \mathcal{H}_z$:

$$\begin{aligned}(\chi_z \otimes \operatorname{id})\varrho_z &= (p_z \otimes \operatorname{id})(\chi \otimes \operatorname{id})\Delta_{\mathcal{H}_\varepsilon} \\ &= (p_z \otimes \operatorname{id})(\operatorname{id} \otimes p_\varepsilon)\Delta_{\mathcal{H}}\chi = \varrho_z p_z \chi = \varrho_z \chi_z.\end{aligned}$$

If χ is convolution-invertible, then so is χ_z since p_z is an algebra map.

For the last statement, $\mathcal{H}$ is $\mathcal{H}_\varepsilon$-cleft by [Sc, 4.3], and then we apply the first part. □

We refer to [S] for Hopf-Galois objects. In the setting of Cayley–Hamilton Hopf algebras, which is a refinement of the above setting for the pair $(\mathcal{H}, Z)$, a tensor product decomposition of the irreducible representations of $\mathcal{H}_z$ was obtained in [DPRR].

3.2. **Braided Hopf algebras and bosonization.** Recall that a braided vector space is a pair $(\mathcal{V}, c)$ where $\mathcal{V}$ is a vector space and $c \in GL(\mathcal{V} \otimes \mathcal{V})$ is a solution of the braid equation: $(c \otimes \operatorname{id})(\operatorname{id} \otimes c)(c \otimes \operatorname{id}) = (\operatorname{id} \otimes c)(c \otimes \operatorname{id})(\operatorname{id} \otimes c)$. There are natural notions of morphisms of braided vector spaces and braided Hopf algebras (braided vector spaces with compatible algebra and coalgebra structures), see [T] for details. To distinguish comultiplications of braided Hopf algebras from those of Hopf algebras, we use a variation of the Sweedler notation for the former: $\underline{\Delta}(r) = r^{(1)} \otimes r^{(2)}$.

Let H be a Hopf algebra. Then the category of (left) Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$ is a braided tensor category and there is a forgetful functor from ${}^H_H\mathcal{YD}$ to the category of braided vector spaces, namely $\mathcal{V} \in {}^H_H\mathcal{YD}$ goes to $(\mathcal{V}, c)$ where $c \in GL(\mathcal{V} \otimes \mathcal{V})$ is given by $c(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$ in Sweedler notation. This forgetful functor sends Hopf algebras in ${}^H_H\mathcal{YD}$ to braided Hopf algebras. In turn Hopf algebras in ${}^H_H\mathcal{YD}$ are noteworthy because of the Radford-Majid bosonization that provides a bijective correspondence between their collection and the collection of triples (A, π, ι) where $A \underset{\iota}{\overset{\pi}{\rightleftarrows}} H$ are morphisms of Hopf algebras with $\pi\iota = \mathrm{id}_H$. See [R] for an exposition. More precisely, the correspondence sends the Hopf algebra $R \in {}^H_H\mathcal{YD}$ to the bosonization $R\#H$ and the triple (A, π, ι) to the algebra of right coinvariants $R = A^{\mathrm{co}\,\pi}$.

Similar notions and results hold for the category of (right) Yetter-Drinfeld modules $\mathcal{YD}^H_H$ consisting of right H-modules and right H-comodules $\mathcal{V}$ satisfying the compatibility

$$(v \cdot h)_{(0)} \otimes (v \cdot h)_{(1)} = v_{(0)} \cdot h_{(2)} \otimes \mathcal{S}(h_{(1)}) v_{(1)} h_{(3)}, \qquad v \in \mathcal{V},\ h \in H.$$

For convenience of the reader we spell out the precise definitions. First, any $\mathcal{V} \in \mathcal{YD}^H_H$ becomes a braided vector space with $c \in GL(\mathcal{V} \otimes \mathcal{V})$ and its inverse given by

$$c(v \otimes w) = w_{(0)} \otimes v \cdot w_{(1)}, \qquad c^{-1}(v \otimes w) = w \cdot \mathcal{S}^{-1}(v_{(1)}) \otimes v_{(0)}, \qquad v, w \in V. \tag{3.2}$$

Let (A, π, ι) be a triple as before. Then the subalgebra of left coinvariants

$$S = {}^{\mathrm{co}\,\pi}A = \{s \in A : (\pi \otimes \mathrm{id})\Delta(s) = 1 \otimes s\}$$

becomes a Hopf algebra in $\mathcal{YD}^H_H$ with right action $\cdot$, right coaction ρ and comultiplication $\underline{\Delta}$ given by

$$s \cdot h = \mathcal{S}(h_{(1)}) s h_{(2)}, \quad \rho(s) = (\mathrm{id} \otimes \pi)\Delta(s), \quad \underline{\Delta}(s) = s_{(1)} \otimes \underline{\vartheta}(s_{(2)}), \quad s \in S, h \in H,$$

where $\underline{\vartheta} : A \to S$ is given by $\underline{\vartheta}(a) = \pi(\mathcal{S}(a_{(1)}))a_{(2)}$, $a \in A$. Conversely, the bosonization $H\#S$ of a Hopf algebra S in $\mathcal{YD}^H_H$ is the vector space $H \otimes S$ with the right smash product and coproduct. That is, given $s, \widetilde{s} \in S$ and $h, \widetilde{h} \in H$,

$$(h\#s)(\widetilde{h}\#\widetilde{s}) = h\widetilde{h}_{(1)}\#(s \cdot \widetilde{h}_{(2)})\widetilde{s}, \qquad \Delta(h\#s) = h_{(1)}\#(s^{(1)})_{(0)} \otimes h_{(2)}(s^{(1)})_{(1)}\#s^{(2)}.$$

3.3. **Nichols algebras.** Let $\mathcal{V} \in {}^H_H\mathcal{YD}$. Then the tensor algebra $T(\mathcal{V})$ is naturally a Hopf algebra in ${}^H_H\mathcal{YD}$. A *pre-Nichols algebra* of $\mathcal{V}$ is a factor of $T(\mathcal{V})$ by a graded Hopf ideal in ${}^H_H\mathcal{YD}$ supported in degrees $\geqslant 2$. The maximal Hopf ideal among those is denoted by $\mathcal{J}(\mathcal{V})$; the Nichols algebra of $\mathcal{V}$ is the quotient $\mathcal{B}(\mathcal{V}) = T(\mathcal{V})/\mathcal{J}(\mathcal{V})$.

The tensor algebra of a braided vector space $(\mathcal{V}, c)$ is also a braided Hopf algebra in the sense of [T]; a *pre-Nichols algebra* of $\mathcal{V}$ is a factor of $T(\mathcal{V})$ by a braided graded Hopf ideal supported in degrees $\geqslant 2$. The maximal Hopf ideal among those is denoted $\widetilde{\mathcal{J}}(\mathcal{V})$; the Nichols algebra of $\mathcal{V}$ is the quotient $\mathcal{B}(\mathcal{V}) = T(\mathcal{V})/\widetilde{\mathcal{J}}(\mathcal{V})$.

These two structures are compatible, i.e. if $\mathcal{V} \in {}^H_H\mathcal{YD}$ and $(\mathcal{V}, c)$ is the corresponding braided vector space, then $\mathcal{J}(\mathcal{V}) = \widetilde{\mathcal{J}}(\mathcal{V})$. But a pre-Nichols algebra of $(\mathcal{V}, c)$ does not necessarily come as the forgetful functor applied to a pre-Nichols algebra of $\mathcal{V} \in {}^H_H\mathcal{YD}$.

Remark 3.2. Let H be cosemisimple, $\mathcal{V} \in {}^H_H\mathcal{YD}$ and $G = \mathcal{B}(\mathcal{V})\#H = \oplus_{n \in \mathbb{N}_0} G^n$, where $G^n = \mathcal{B}^n(\mathcal{V})\#H$. By other characterizations of Nichols algebras, we know that

(a) $\mathcal{B}(\mathcal{V})$ is coradically graded and generated in degree 1;
(b) G is coradically graded and generated in degree 1.

Since the projection $\pi : G \to H$ is graded, the subalgebra of left coinvariants $S = {}^{\mathrm{co}\,\pi}G$ inherits the grading of G; by a standard argument it is also coradically graded and generated in degree 1. Thus S is a Nichols algebra in $\mathcal{YD}^H_H$.

3.4. **Hopf skew-pairings of bosonizations.** Let $\langle\cdot,\cdot\rangle : M\times V\to\mathbb{C}$ be a bilinear form between two vector spaces M and V. We denote by $\langle\cdot,\cdot\rangle : (M\otimes M)\times(V\otimes V)\to\mathbb{C}$ the bilinear form determined by

$$\langle m\otimes m', v\otimes v'\rangle = \langle m, v'\rangle\langle m', v\rangle, \qquad m, m'\in M,\ v, v'\in V. \tag{3.3}$$

Let H and K be two Hopf algebras. A bilinear form $\langle\cdot,\cdot\rangle : K\times H\to\mathbb{C}$ is a *Hopf skew-pairing* (or skew-pairing of Hopf algebras) if for all for $k, k'\in K$, $h, h'\in H$,

$$\begin{aligned} &\langle k, hh'\rangle = \langle\Delta^{\mathrm{op}}(k), h\otimes h'\rangle, &&\langle kk', h\rangle = \langle k\otimes k', \Delta(h)\rangle, \\ &\langle k, 1\rangle = \varepsilon(k), &&\langle 1, h\rangle = \varepsilon(h), &&\langle\mathcal{S}(k), h\rangle = \langle k, \mathcal{S}(h)\rangle. \end{aligned} \tag{3.4}$$

A skew-pairing of braided Hopf algebras is defined by (3.4) but with the convention

$$\underline{\Delta}^{\mathrm{op}} = c^{-1}\underline{\Delta}.$$

Let us fix a Hopf skew-pairing $\langle\cdot,\cdot\rangle : K\times H\to\mathbb{C}$. A *YD-pairing* between $\mathcal{M}\in\mathcal{YD}_K^K$ and $\mathcal{V}\in{}_H^H\mathcal{YD}$ is a bilinear form $\langle\cdot,\cdot\rangle : \mathcal{M}\times\mathcal{V}\to\mathbb{C}$ such that

$$\begin{aligned} \langle m\cdot k, v\rangle &= \langle k, v_{(-1)}\rangle\langle m, v_{(0)}\rangle, \\ \langle m, h\cdot v\rangle &= \langle m_{(1)}, h\rangle\langle m_{(0)}, v\rangle, \end{aligned} \qquad m\in\mathcal{M},\ k\in K,\ v\in\mathcal{V},\ h\in H. \tag{3.5}$$

We recall the following well-known result, whose proof is straightforward.

Lemma 3.3. *Let R be a Hopf algebra in ${}_H^H\mathcal{YD}$, S be a Hopf algebra in $\mathcal{YD}_K^K$ and $\langle\cdot,\cdot\rangle$: $(K\#S)\times(R\#H)\to\mathbb{C}$ be a bilinear form such that*

$$\langle ky, xh\rangle = \langle k, h\rangle\langle y, x\rangle, \qquad y\in S,\ k\in K,\ x\in R,\ h\in H. \tag{3.6}$$

Then the following are equivalent:

(a) *$\langle\cdot,\cdot\rangle$ is a Hopf skew-pairing.*
(b) *The restriction of $\langle\cdot,\cdot\rangle$ to $K\times H$ is a Hopf skew-pairing and the restriction of $\langle\cdot,\cdot\rangle$ to $S\times R$ is both a skew-pairing of braided Hopf algebras and a YD-pairing.* □

A YD-pairing between $\mathcal{M}\in\mathcal{YD}_K^K$ and $\mathcal{V}\in{}_H^H\mathcal{YD}$ extends canonically to a YD-pairing $\langle\cdot,\cdot\rangle : T(\mathcal{M})\times T(\mathcal{V})\to\mathbb{C}$. This extension is actually a braided Hopf skew-pairing, i.e., it satisfies (3.4) with respect to the braided comultiplications. The bilinear form

$$\langle\cdot,\cdot\rangle : (K\#T(\mathcal{M}))\times(T(\mathcal{V})\#H)\to\mathbb{C}, \qquad \langle k\#y, x\#h\rangle := \langle k, h\rangle\langle y, x\rangle,$$

$y\in T(\mathcal{M})$, $k\in K$, $x\in T(\mathcal{V})$, $h\in H$ is a Hopf skew-pairing by Lemma 3.3.

Assume that $\dim\mathcal{M}<\infty$. Then the radical $T(\mathcal{M}^*)^{\perp}$ with respect to $\langle\cdot,\cdot\rangle$ coincides with $\mathcal{J}(\mathcal{M})$. Hence, for any $\mathcal{V}$ YD-paired with $\mathcal{M}$ we have

$$T(\mathcal{V})^{\perp}\supseteq\mathcal{J}(\mathcal{M}).$$

Consequently, if $\dim\mathcal{M}<\infty$ and $\dim\mathcal{V}<\infty$, $\mathcal{B}$ is a pre-Nichols algebra of $\mathcal{M}$ in $\mathcal{YD}_K^K$ and $\mathcal{E}$ is a pre-Nichols algebra of $\mathcal{V}$ in ${}_H^H\mathcal{YD}$, then $\langle\cdot,\cdot\rangle$ descends to Hopf skew-pairings $\langle\cdot,\cdot\rangle : \mathcal{B}\times\mathcal{E}\to\mathbb{C}$ and $\langle\cdot,\cdot\rangle : (K\#\mathcal{B})\times(\mathcal{E}\#H)\to\mathbb{C}$.

3.5. **Nichols algebras of diagonal type.** We fix $\theta \in \mathbb{N}$ and set $\mathbb{I} = \mathbb{I}_\theta$. Let (V, c) be a (complex) braided vector space of diagonal type with braiding matrix

$$\mathfrak{q} = (q_{ij}) \in \left(\mathbb{C}^\times\right)^{\mathbb{I}\times\mathbb{I}} \tag{3.7}$$

with respect to a basis $(x_i)_{i\in\mathbb{I}}$, i.e. $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $i, j \in \mathbb{I}$. We assume that $\dim \mathcal{B}(V) < \infty$. These Nichols algebras are classified in [H2]. Throughout the paper we will also assume that the Dynkin diagram of $\mathfrak{q}$ is connected, for simplicity of the exposition.

The canonical basis of $\mathbb{Z}^{\mathbb{I}}$ is denoted $\alpha_1, \dots, \alpha_\theta$. The algebra $T(V)$ is $\mathbb{Z}^{\mathbb{I}}$-graded, with grading $\deg x_i = \alpha_i$, $i \in \mathbb{I}$. This grading naturally specializes to the standard $\mathbb{N}_0$-grading.

Let $\mathfrak{q} : \mathbb{Z}^{\mathbb{I}} \times \mathbb{Z}^{\mathbb{I}} \to \mathbb{C}^\times$ be the $\mathbb{Z}$-bilinear forms associated to the matrix $\mathfrak{q}$, i.e. $\mathfrak{q}(\alpha_j, \alpha_k) := q_{jk}$, $j, k \in \mathbb{I}$. If $\alpha, \beta \in \mathbb{Z}^{\mathbb{I}}$ and $i \in \mathbb{I}$, then we set

$$q_{\alpha\beta} = \mathfrak{q}(\alpha, \beta), \qquad q_{\alpha\alpha} = \mathfrak{q}(\alpha, \alpha), \qquad N_\alpha = \operatorname{ord} q_{\alpha\alpha}, \qquad N_i = \operatorname{ord} q_{\alpha_i\alpha_i} = N_{\alpha_i}. \tag{3.8}$$

Remark 3.4. Every $\mathbb{Z}^{\mathbb{I}}$-graded pre-Nichols algebra of V admits algebra automorphisms $\varsigma_i^{\mathfrak{q}}$ and $(\mathrm{id}, \varsigma_i^{\mathfrak{q}})$-derivations $\partial_i^{\mathfrak{q}}$ for each $i \in \mathbb{I}$; that is,

$$\partial_i^{\mathfrak{q}}(xy) = \partial_i^{\mathfrak{q}}(x)\varsigma_i^{\mathfrak{q}}(y) + x\partial_i^{\mathfrak{q}}(y), \qquad x, y \in T(V).$$

Indeed the algebra automorphism $\varsigma_i^{\mathfrak{q}} : T(V) \to T(V)$ is given by

$$\varsigma_i^{\mathfrak{q}}(x) = \mathfrak{q}(\alpha_i, \beta)x, \qquad x \in T(V) \text{ homogeneous of degree } \beta \in \mathbb{Z}^{\mathbb{I}}.$$

The linear endomorphisms $\partial_i^{\mathfrak{q}} : T(V) \to T(V)$ are defined as follows. Let $\Delta_{m,n}(x)$ be the homogeneous component of $\Delta(x) \in T(V) \otimes T(V)$ of degree $(m, n) \in \mathbb{N}_0^2$. Then

$$\Delta_{n-1,1}(x) = \sum_{i\in\mathbb{I}} \partial_i^{\mathfrak{q}}(x) \otimes x_i, \qquad x \in T^n(V).$$

It is easy to see that $\partial_i^{\mathfrak{q}}$ is a $(\mathrm{id}, \varsigma_i^{\mathfrak{q}})$-derivation. If $\mathcal{B}$ is a quotient of $T(V)$ by a $\mathbb{Z}^{\mathbb{I}}$-homogeneous ideal, then $\varsigma_i^{\mathfrak{q}}$ induces an algebra automorphism of $\mathcal{B}$, also denoted by $\varsigma_i^{\mathfrak{q}}$, and $\partial_i^{\mathfrak{q}}$ induces a $(\mathrm{id}, \varsigma_i^{\mathfrak{q}})$-derivation of $\mathcal{B}$, also denoted by $\partial_i^{\mathfrak{q}}$.

3.6. **Weyl groupoids.** The notions of Weyl groupoid and generalized root systems were introduced in [H1, HY1]. We recall the main features needed later. Let $(c_{ij}^{\mathfrak{q}})_{i,j\in\mathbb{I}} \in \mathbb{Z}^{\mathbb{I}\times\mathbb{I}}$ be the (generalized Cartan) matrix defined by $c_{ii}^{\mathfrak{q}} := 2$ and

$$c_{ij}^{\mathfrak{q}} := -\min\left\{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\right\}, \qquad i \neq j. \tag{3.9}$$

Let $i \in \mathbb{I}$. First, the reflection $s_i^{\mathfrak{q}} \in GL(\mathbb{Z}^{\mathbb{I}})$ is given by

$$s_i^{\mathfrak{q}}(\alpha_j) := \alpha_j - c_{ij}^{\mathfrak{q}}\alpha_i, \qquad j \in \mathbb{I}. \tag{3.10}$$

Second, the matrix $\rho_i(\mathfrak{q})$ is given by

$$(\rho_i(\mathfrak{q}))_{jk} := \mathfrak{q}(s_i^{\mathfrak{q}}(\alpha_j), s_i^{\mathfrak{q}}(\alpha_k)) = q_{jk} q_{ik}^{-c_{ij}^{\mathfrak{q}}} q_{ji}^{-c_{ik}^{\mathfrak{q}}} q_{ii}^{c_{ij}^{\mathfrak{q}} c_{ik}^{\mathfrak{q}}}, \qquad j, k \in \mathbb{I}. \tag{3.11}$$

Finally, the braided vector space $\rho_i(V)$ is of diagonal type with matrix $\rho_i(\mathfrak{q})$. Set

$$\mathcal{X} := \{\rho_{j_1} \dots \rho_{j_n}(\mathfrak{q}) : j_1, \dots, j_n \in \mathbb{I}, n \in \mathbb{N}\}.$$

The set $\mathcal{X}$ is called the Weyl-equivalence class of $\mathfrak{q}$. The set $\Delta_+^{\mathfrak{q}}$ of *positive roots* consists of the $\mathbb{Z}^{\mathbb{I}}$-degrees of the generators of a PBW-basis of $\mathcal{B}_{\mathfrak{q}}$, counted with multiplicities. Let $\Delta^{\mathfrak{q}} := \Delta_+^{\mathfrak{q}} \cup -\Delta_+^{\mathfrak{q}}$. Then the generalized root system of $\mathfrak{q}$ is the fibration $\Delta \to \mathcal{X}$, where the fiber of $\rho_{j_1} \dots \rho_{j_N}(\mathfrak{q})$ is $\Delta^{\rho_{j_1} \cdots \rho_{j_N}(\mathfrak{q})}$. The Weyl groupoid $\mathcal{W}_{\mathfrak{q}}$ of $\mathcal{B}_{\mathfrak{q}}$ acts on this fibration, generalizing the classical Weyl group. Here is another characterization of $\Delta_+^{\mathfrak{q}}$, valid because it is finite. Let $\omega_0^{\mathfrak{q}} \in \mathcal{W}_{\mathfrak{q}}$ be an element of maximal length and $\omega_0^{\mathfrak{q}} = \sigma_{i_1}^{\mathfrak{q}} \sigma_{i_2} \cdots \sigma_{i_\ell}$ be a reduced expression. Then

$$\beta_k := s_{i_1}^{\mathfrak{q}} \cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad k \in \mathbb{I}_\ell \tag{3.12}$$

are pairwise different vectors and $\Delta_+^{\mathfrak{q}} = \{\beta_k : k \in \mathbb{I}_\ell\}$ [CH, Prop. 2.12], so $|\Delta_+^{\mathfrak{q}}| = \ell$.

3.7. **Cartan roots** [An3]**.** This important notion is crucial for our purposes. First, $i \in \mathbb{I}$ is a *Cartan vertex* of $\mathfrak{q}$ if

$$q_{ij} q_{ji} = q_{ii}^{c_{ij}^{\mathfrak{q}}}, \qquad \text{for all } j \neq i. \tag{3.13}$$

Then the set of *Cartan roots* of $\mathfrak{q}$ is

$$\mathfrak{O}^{\mathfrak{q}} = \{s_{i_1}^{\mathfrak{q}} s_{i_2} \dots s_{i_k}(\alpha_i) \in \Delta^{\mathfrak{q}} : i \in \mathbb{I} \text{ is a Cartan vertex of } \rho_{i_k} \dots \rho_{i_2}\rho_{i_1}(\mathfrak{q})\}.$$

Set $\mathfrak{O}_+^{\mathfrak{q}} = \mathfrak{O}^{\mathfrak{q}} \cap \mathbb{N}_0^\theta$. Recall (3.8) and set $\widetilde{N}_\beta := N_\beta$, if $\beta \notin \mathfrak{O}^{\mathfrak{q}}$, or else ∞ if $\beta \in \mathfrak{O}^{\mathfrak{q}}$.

The set of Cartan roots gives rise to a root system up to a rescaling. Set

$$\underline{\mathfrak{O}}^{\mathfrak{q}} = \{N_\beta^{\mathfrak{q}} \beta : \beta \in \mathfrak{O}^{\mathfrak{q}}\}, \qquad \underline{\mathfrak{O}}_+^{\mathfrak{q}} = \underline{\mathfrak{O}}^{\mathfrak{q}} \cap \mathbb{N}_0^\theta, \qquad \underline{\beta} = N_\beta^{\mathfrak{q}} \beta,\ \beta \in \mathfrak{O}^{\mathfrak{q}}. \tag{3.14}$$

Theorem 3.5. [AAR3, Theorem 3.6] *The set $\underline{\mathfrak{O}}^{\mathfrak{q}}$ is either empty or a root system inside the real vector space generated by $\mathfrak{O}^{\mathfrak{q}}$. The set $\Pi^{\mathfrak{q}}$ of all indecomposable elements of $\underline{\mathfrak{O}}_+^{\mathfrak{q}}$ is a basis of this root system.*

Here $\underline{\gamma} \in \underline{\mathfrak{O}}_+^{\mathfrak{q}}$ is *indecomposable* if it can not be represented as a non-trivial positive linear combination of elements of $\underline{\mathfrak{O}}_+^{\mathfrak{q}}$. Let $\mathfrak{g}_{\mathfrak{q}}$ be either 0 or the semisimple Lie algebra with root system as in Theorem 3.5, accordingly. We fix a triangular decomposition

$$\mathfrak{g}_{\mathfrak{q}} = \mathfrak{n}_{\mathfrak{q}}^+ \oplus \mathfrak{h}_{\mathfrak{q}} \oplus \mathfrak{n}_{\mathfrak{q}}^- \tag{3.15}$$

and the Borel subalgebras $\mathfrak{b}_{\mathfrak{q}}^\pm = \mathfrak{h}_{\mathfrak{q}} \oplus \mathfrak{n}_{\mathfrak{q}}^\pm \subset \mathfrak{g}_{\mathfrak{q}}$; if $\mathfrak{g}_{\mathfrak{q}} = 0$, then $\mathfrak{n}_{\mathfrak{q}}^+ = \mathfrak{h}_{\mathfrak{q}} = \mathfrak{n}_{\mathfrak{q}}^- = 0$. We denote the root lattice by

$$\mathcal{Q}_{\mathfrak{q}} := \sum_{\gamma \in \underline{\mathfrak{O}}_+^{\mathfrak{q}}} \mathbb{Z}\gamma = \bigoplus_{\gamma \in \Pi^{\mathfrak{q}}} \mathbb{Z}\gamma. \tag{3.16}$$

3.8. **Distinguished pre-Nichols algebras.** The finite-dimensional Nichols algebras of diagonal type admit distinguished pre-Nichols algebras introduced in [An2, An3]. An ideal $\mathcal{I}(V)$ of $T(V)$ was introduced in [An3]; it is generated by all the defining relations of $\mathcal{B}_{\mathfrak{q}}$ in [An2, Theorem 3.1], but excluding the power root vectors $x_\alpha^{N_\alpha}$, $\alpha \in \underline{\mathfrak{O}}_{\mathfrak{q}}$, and adding some quantum Serre relations.

Definition 3.6. [An3] The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ of V is the quotient $\widetilde{\mathcal{B}}_{\mathfrak{q}} = T(V)/\mathcal{I}(V)$. Since $\mathcal{I}(V)$ is a Hopf ideal, $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ is a braided Hopf algebra.

By Remark 3.4, there are automorphisms $\varsigma_i^{\mathfrak{q}}$ and skew-derivations $\partial_i^{\mathfrak{q}}$ of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$, $i \in \mathbb{I}$.

3.9. **Lusztig algebras.** The *Lusztig algebra* $\mathcal{L}_{\mathfrak{q}}$ associated to $\mathfrak{q}$ is the graded dual of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ [AAR1]. Thus $\mathcal{L}_{\mathfrak{q}}$ is a braided Hopf algebra equipped with a bilinear form $(\,,\,)$: $\mathcal{L}_{\mathfrak{q}} \times \widetilde{\mathcal{B}}_{\mathfrak{q}} \to \mathbb{C}$, which satisfies

$$(y, xx') = (y^{(2)}, x)(y^{(1)}, x') \qquad \text{and} \qquad (yy', x) = (y, x^{(2)})(y', x^{(1)}) \tag{3.17}$$

for all $x, x' \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$, $y, y' \in \mathcal{L}_{\mathfrak{q}}$. Let $Z_{\mathfrak{q}} = {}^{\operatorname{co}\varpi}\widetilde{\mathcal{B}}_{\mathfrak{q}}$ be the subalgebra of coinvariants of the canonical projection

$$\varpi : \widetilde{\mathcal{B}}_{\mathfrak{q}} \to \mathcal{B}_{\mathfrak{q}}.$$

Then $Z_{\mathfrak{q}}$ is a *normal* Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ [An3, Theorems 4.10, 4.13] and we have an extension of braided Hopf algebras $Z_{\mathfrak{q}} \overset{\iota}{\hookrightarrow} \widetilde{\mathcal{B}}_{\mathfrak{q}} \overset{\varpi}{\twoheadrightarrow} \mathcal{B}_{\mathfrak{q}}$. Taking graded duals, we obtain a new extension of braided Hopf algebras, cf. [AAR2, Prop. 3.2]:

$$\mathcal{B}_{\mathfrak{q}^t} \overset{\varpi^*}{\hookrightarrow} \mathcal{L}_{\mathfrak{q}} \overset{\iota^*}{\twoheadrightarrow} \mathfrak{Z}_{\mathfrak{q}}, \tag{3.18}$$

Remark 3.7. Assume that (4.23) below holds. Then the braided Hopf algebra $\mathfrak{Z}_{\mathfrak{q}}$ is a Hopf algebra, isomorphic to the enveloping algebra of the Lie algebra $\mathcal{P}(\mathfrak{Z}_{\mathfrak{q}})$ [AAR2, 3.3]. Moreover $\mathcal{P}(\mathfrak{Z}_{\mathfrak{q}}) \simeq \mathfrak{n}_{\mathfrak{q}}^{-}$ as in (3.15) [AAR3].

4. Large quantum groups

In this section we describe the *large quantum groups* i.e. Drinfeld doubles of bozonizations of the distinguished pre-Nichols algebras belonging to a one-parameter family; these are the main focus of the paper. The large quantum Borel and unipotent subalgebras are also introduced here. Throughout the rest of the paper Γ^{+} and Γ^{-} denote free abelian groups of rank θ with bases denoted respectively $(K_i)_{i\in\mathbb{I}}$ and $(L_i)_{i\in\mathbb{I}}$. Let $\Gamma = \Gamma^{+} \times \Gamma^{-}$.

4.1. **Families of Nichols algebras.** From now on we assume that $\mathfrak{q}$ belongs to a one-parameter family (except when explicitly stated otherwise). This means that there exists an indecomposable matrix

$$\mathbf{q} = (\mathbf{q}_{ij}) \in \left(\mathbb{C}[\nu^{\pm 1}]^{\times}\right)^{\mathbb{I}\times\mathbb{I}} \tag{4.1}$$

such that:

- The Nichols algebra of the $\mathbb{C}(\nu)$-braided vector space of diagonal type $V_{\mathbb{C}(\nu)}$ with basis $(x_i)_{i\in\mathbb{I}}$ and braiding matrix (4.1) has finite root system thus listed in [H2].
- There exists an open subset $\emptyset \neq O \subseteq \mathbb{C}^{\times}$ such that for any $x \in O$, the matrix $\mathbf{q}(x)$ obtained by evaluation $\nu \mapsto x$ has the same finite root system as $\mathbf{q}$.
- There exists $\xi \in \mathbb{G}'_{\infty}$ such that $\mathfrak{q} = \mathbf{q}(\xi)$.

By inspection in [H2], all one-parameter families are listed in the Appendix A. We denote the Nichols algebras of V and $V_{\mathbb{C}(\nu)}$, with braidings given by $\mathfrak{q}$, respectively $\mathbf{q}$, by

$$\mathcal{B}_{\mathfrak{q}} := \mathcal{B}(V) \qquad \text{and} \qquad \mathcal{B}_{\mathbf{q}} := \mathcal{B}(V_{\mathbb{C}(\nu)}).$$

The defining relations and PBW-basis of $\mathcal{B}_{\mathfrak{q}}$ and $\mathcal{B}_{\mathbf{q}}$ are described in [AA] over an algebraically closed field of characteristic 0 but the same presentation and PBW-basis are valid over $\mathbb{C}(\nu)$. Indeed, apply to $\mathbb{F} = \mathbb{C}(\nu)$, $\mathbb{K} = \overline{\mathbb{C}(\nu)}$ the following remarks:

- Let $\mathbb{K}/\mathbb{F}$ be a field extension and (V, c) a braided $\mathbb{F}$-vector space. Then $(V \otimes_{\mathbb{F}} \mathbb{K}, c \otimes \mathrm{id})$ is a braided $\mathbb{K}$-vector space and $\mathcal{B}(V) \otimes_{\mathbb{F}} \mathbb{K} \simeq \mathcal{B}(V \otimes_{\mathbb{F}} \mathbb{K})$; use e.g. quantum symmetrizers.

- Let $\mathbb{K}/\mathbb{F}$ be a faithfully flat extension of commutative rings. Let U be a $\mathbb{F}$-algebra with generators $(y_j)_{j\in J}$ and $U_{\mathbb{K}} = U \otimes_{\mathbb{F}} \mathbb{K}$ which is also generated by $(y_j)_{j\in J}$. Let $(r_t)_{t\in T}$ be a set of elements in the tensor algebra over $\mathbb{F}$ of the free module $\mathbb{F}^{(J)}$. Then these are defining relations of U if and only if they are defining relations of $U_{\mathbb{K}}$.

The discussions in §3.5 and §3.6 apply to the matrix $\mathbf{q}$. Let $\mathbf{q} : \mathbb{Z}^{\mathbb{I}} \times \mathbb{Z}^{\mathbb{I}} \to (\mathbb{C}[\nu^{\pm 1}])^{\times}$ as in §3.5; we also have the notation $\mathbf{q}_{\alpha\beta}$ for $\alpha, \beta \in \mathbb{Z}^{\mathbb{I}}$ as in (3.8). We denote by $\mathcal{W}_{\mathbf{q}}$ the corresponding Weyl groupoid, by $\rho_i(\mathfrak{q})$ the related braiding matrices, etc. As in Remark 3.4, there are $\varsigma_i^{\mathbf{q}} \in \operatorname{Aut}_{\mathrm{alg}}(\mathcal{B}_{\mathbf{q}})$ and $(\mathrm{id}, \varsigma_i^{\mathbf{q}})$-derivations $\partial_i^{\mathbf{q}} : \mathcal{B}_{\mathbf{q}} \to \mathcal{B}_{\mathbf{q}}$, for every $i \in \mathbb{I}$.

Remark 4.1. Crucially, β is a Cartan root of $\mathfrak{q}$ if and only if $\operatorname{ord} \mathbf{q}_{\beta\beta} = \infty$.

4.2. **The quantum group $U_{\mathbf{q}}$.** Here we work over $\mathbb{C}(\nu)$. Let $W_{\mathbb{C}(\nu)}$ the $\mathbb{C}(\nu)$-vector space with basis $(y_i)_{i\in\mathbb{I}}$. The group Γ acts on $V_{\mathbb{C}(\nu)} \oplus W_{\mathbb{C}(\nu)}$ by

$$K_i \cdot x_j = \mathbf{q}_{ij} x_j, \qquad K_i \cdot y_j = \mathbf{q}_{ij}^{-1} y_j, \qquad L_i \cdot x_j = \mathbf{q}_{ji} x_j, \qquad L_i \cdot y_j = \mathbf{q}_{ji}^{-1} y_j, \tag{4.2}$$

$i, j \in \mathbb{I}$. The vector space $V_{\mathbb{C}(\nu)} \oplus W_{\mathbb{C}(\nu)}$ is Γ-graded by

$$\deg x_i = K_i, \qquad \deg y_i = L_i, \qquad i \in \mathbb{I}. \tag{4.3}$$

Thus $V_{\mathbb{C}(\nu)} \oplus W_{\mathbb{C}(\nu)} \in {}^{\mathbb{C}(\nu)\Gamma}_{\mathbb{C}(\nu)\Gamma}\mathcal{YD}$ with coaction given by the grading. In particular, $W_{\mathbb{C}(\nu)}$ is a braided vector space with braiding matrix $\mathbf{q}'$ where $\mathbf{q}'_{ij} = \mathbf{q}_{ji}^{-1}$, $i, j \in \mathbb{I}$.

We define $U_{\mathbf{q}}$ as the quotient Hopf algebra of the bosonization $T(V_{\mathbb{C}(\nu)} \oplus W_{\mathbb{C}(\nu)}) \# \mathbb{C}(\nu)\Gamma$ modulo the ideal generated by

$$\mathcal{J}(V_{\mathbb{C}(\nu)}), \qquad \mathcal{J}(W_{\mathbb{C}(\nu)}), \qquad x_i y_j - \mathbf{q}_{ij}^{-1} y_j x_i - \delta_{ij}(K_i L_i - 1), \qquad i, j \in \mathbb{I}.$$

The images of x_i, y_i, K_i and L_i in $U_{\mathbf{q}}$ will again be denoted by the same symbols. Let $E_i := x_i$, $F_i := y_i L_i^{-1}$ in $U_{\mathbf{q}}$, $i \in \mathbb{I}$. Then for all $i, j \in \mathbb{I}$ we have

$$K_i E_j = \mathbf{q}_{ij} E_j K_i, \qquad L_i E_j = \mathbf{q}_{ji} E_j L_i, \tag{4.4}$$

$$K_i F_j = \mathbf{q}_{ij}^{-1} F_j K_i, \qquad L_i F_j = \mathbf{q}_{ji}^{-1} F_j L_i, \tag{4.5}$$

$$E_i F_j - F_j E_i = \delta_{ij}(K_i - L_i^{-1}), \tag{4.6}$$

$$\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \qquad \Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i^{-1}. \tag{4.7}$$

We consider the following subalgebras of $U_{\mathbf{q}}$:

$$U_{\mathbf{q}}^{+0} = \mathbb{C}(\nu)[K_i^{\pm 1} : i \in \mathbb{I}], \qquad U_{\mathbf{q}}^{-0} = \mathbb{C}(\nu)[L_i^{\pm 1} : i \in \mathbb{I}], \quad U_{\mathbf{q}}^{0} = \mathbb{C}(\nu)[K_i^{\pm 1}, L_i^{\pm 1} : i \in \mathbb{I}],$$
$$U_{\mathbf{q}}^{+} = \mathbb{C}(\nu)\langle E_i : i \in \mathbb{I}\rangle, \qquad U_{\mathbf{q}}^{-} = \mathbb{C}(\nu)\langle F_i : i \in \mathbb{I}\rangle,$$
$$U_{\mathbf{q}}^{\geqslant} = \mathbb{C}(\nu)\langle E_i, K_i^{\pm 1} : i \in \mathbb{I}\rangle, \quad U_{\mathbf{q}}^{\leqslant} = \mathbb{C}(\nu)\langle F_i, L_i^{\pm 1} : i \in \mathbb{I}\rangle.$$

The multiplication map induces linear isomorphisms

$$U_{\mathbf{q}} \simeq U_{\mathbf{q}}^{+} \otimes_{\mathbb{C}(\nu)} U_{\mathbf{q}}^{0} \otimes_{\mathbb{C}(\nu)} U_{\mathbf{q}}^{-} \simeq U_{\mathbf{q}}^{\geqslant} \otimes_{\mathbb{C}(\nu)} U_{\mathbf{q}}^{\leqslant}. \tag{4.8}$$

We have canonical isomorphisms of Hopf algebras

$$U_{\mathbf{q}}^{+0} \simeq \mathbb{C}(\nu)\Gamma^{+}, \qquad U_{\mathbf{q}}^{-0} \simeq \mathbb{C}(\nu)\Gamma^{-}, \qquad U_{\mathbf{q}}^{0} \simeq \mathbb{C}(\nu)\Gamma.$$

The algebra $U_{\mathbf{q}}^{+}$ has a canonical structure of a Hopf algebra in ${}^{\mathbb{C}(\nu)\Gamma^{+}}_{\mathbb{C}(\nu)\Gamma^{+}}\mathcal{YD}$ and there are isomorphisms of (braided) Hopf algebras

$$U_{\mathbf{q}}^{+} \simeq \mathcal{B}_{\mathbf{q}}, \qquad U_{\mathbf{q}}^{\geqslant} \simeq U_{\mathbf{q}}^{+} \# U_{\mathbf{q}}^{+0},$$

see e.g. [ARS] for details.

Define the module $V^*_{\mathbb{C}(\nu)} \in {}^{\mathbb{C}(\nu)\Gamma^-}_{\mathbb{C}(\nu)\Gamma^-}\mathcal{YD}$ with basis $\{x_i^* : i \in \mathbb{I}\}$ by

$$x_j^* \cdot L_i = \mathbf{q}_{ji} x_j^*, \qquad \deg x_i^* = L_i^{-1}, \qquad i, j \in \mathbb{I}.$$

Let $\pi^- : U_{\mathbf{q}}^{\leqslant} \to U_{\mathbf{q}}^{-0}$ be the canonical Hopf algebra morphism; then ${}^{\operatorname{co}\pi^-}U_{\mathbf{q}}^{\leqslant} = U_{\mathbf{q}}^-$, cf. [ARS, Corollary 3.9 (2)]. Hence $U_{\mathbf{q}}^-$ has a canonical structure of a Hopf algebra in ${}^{\mathbb{C}(\nu)\Gamma^-}_{\mathbb{C}(\nu)\Gamma^-}\mathcal{YD}$. By Remark 3.2, we have isomorphisms of (braided) Hopf algebras

$$U_{\mathbf{q}}^- \simeq \mathcal{B}(V^*_{\mathbb{C}(\nu)}) \simeq \mathcal{B}_{\mathbf{q}^{(-1)}}, \qquad U_{\mathbf{q}}^{\leqslant} \simeq U_{\mathbf{q}}^{-0} \# U_{\mathbf{q}}^-.$$

Here $\mathbf{q}^{(-1)}$ means the matrix obtained by inverting every entry of $\mathbf{q}$.

Now there is a unique Hopf skew-pairing $\langle \cdot, \cdot \rangle : U_{\mathbf{q}}^{\leqslant} \times U_{\mathbf{q}}^{\geqslant} \to \mathbb{C}(\nu)$ determined by

$$\langle L_i, K_j \rangle = \mathbf{q}_{ji}^{-1}, \qquad \langle F_i, E_j \rangle = \delta_{ij}, \qquad \langle L_i, E_j \rangle = \langle F_i, K_j \rangle = 0, \qquad i, j \in \mathbb{I},$$

see [ARS, Theorem 3.7]. By [ARS, Theorem 3.11 (1)], we have

$$\langle x_- g_-, x_+ g_+ \rangle = \langle x_-, x_+ \rangle \langle g_-, g_+ \rangle, \qquad x_\pm \in U_{\mathbf{q}}^{\pm},\ g_\pm \in \Gamma^{\pm}.$$

The restriction $\langle \cdot, \cdot \rangle : U_{\mathbf{q}}^- \times U_{\mathbf{q}}^+ \to \mathbb{C}(\nu)$ is non-degenerate by [ARS, Theorem 3.11 (3)] and is a Hopf skew-pairing of braided Hopf algebras by Lemma 3.3.

4.3. **The large quantum group $U_{\mathfrak{q}}$.** Recall that $\mathfrak{q} \in \left(\mathbb{C}^\times\right)^{\mathbb{I} \times \mathbb{I}}$ belongs to a one parameter family given by a matrix $\mathbf{q}$, cf. §4.1.

Definition 4.2. The **large quantum group** $U_{\mathfrak{q}}$ is the Drinfeld double of the bosonization of the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_{\mathfrak{q}}$.

The complex Hopf algebra $U_{\mathfrak{q}}$ was defined in [An3] for arbitrary $\mathfrak{q}$ with $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. Explicitly, let W the $\mathbb{C}$-vector space with basis $(y_i)_{i \in \mathbb{I}}$. The group Γ acts on $V \oplus W$ by

$$K_i \cdot x_j = q_{ij} x_j, \quad K_i \cdot y_j = q_{ij}^{-1} y_j, \quad L_i \cdot x_j = q_{ji} x_j, \quad L_i \cdot y_j = q_{ji}^{-1} y_j, \quad i, j \in \mathbb{I}.$$

Now $V \oplus W$ is Γ-graded by (4.3), so W is a braided vector space with braiding matrix $\mathfrak{q}'$ with entries $q'_{ij} = q_{ji}^{-1}$ for $i, j \in \mathbb{I}$. Recall the defining ideal $\mathcal{I}(V)$ of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Then $U_{\mathfrak{q}}$ is the bosonization $T(V \oplus W) \# \mathbb{C}\Gamma$ modulo the ideal generated by

$$\mathcal{I}(V), \qquad \mathcal{I}(W), \qquad x_i y_j - q_{ij}^{-1} y_j x_i - \delta_{ij}(K_i L_i - 1), \qquad i, j \in \mathbb{I}.$$

The images of x_i, y_i, K_i and L_i in $U_{\mathfrak{q}}$ will again be denoted by the same symbols. Let $e_i = x_i$, $f_i = y_i L_i^{-1}$ in $U_{\mathfrak{q}}$, $i \in \mathbb{I}$. Then for all $i, j \in \mathbb{I}$ we have

$$K_i e_j = q_{ij} e_j K_i, \qquad L_i e_j = q_{ji}^{-1} e_j L_i, \tag{4.9}$$

$$K_i f_j = q_{ij}^{-1} f_j K_i, \qquad L_i f_j = q_{ji} f_j L_i, \tag{4.10}$$

$$e_i f_j - f_j e_i = \delta_{ij}(K_i - L_i^{-1}), \tag{4.11}$$

$$\Delta(e_i) = K_i \otimes e_i + e_i \otimes 1, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes L_i^{-1}. \tag{4.12}$$

We consider the following subalgebras of $U_{\mathfrak{q}}$:

$$U_{\mathfrak{q}}^{+0} = \mathbb{C}[K_i^{\pm 1} : i \in \mathbb{I}], \qquad U_{\mathfrak{q}}^{-0} = \mathbb{C}[L_i^{\pm 1} : i \in \mathbb{I}], \qquad U_{\mathfrak{q}}^{0} = \mathbb{C}[K_i^{\pm 1}, L_i^{\pm 1} : i \in \mathbb{I}],$$
$$U_{\mathfrak{q}}^{+} = \mathbb{C}\langle e_i : i \in \mathbb{I} \rangle, \qquad U_{\mathfrak{q}}^{-} = \mathbb{C}\langle f_i : i \in \mathbb{I} \rangle,$$
$$U_{\mathfrak{q}}^{\geqslant} = \mathbb{C}\langle e_i, K_i^{\pm 1} : i \in \mathbb{I} \rangle, \qquad U_{\mathfrak{q}}^{\leqslant} = \mathbb{C}\langle f_i, L_i^{\pm 1} : i \in \mathbb{I} \rangle.$$

Definition 4.3. The algebras $U_{\mathfrak{q}}^{\geqslant}$ and $U_{\mathfrak{q}}^{\leqslant}$ will be called **large quantum Borel algebras** and the algebras $U_{\mathfrak{q}}^{\pm}$ **large quantum unipotent algebras**.

The multiplication map induces the linear isomorphisms

$$U_{\mathfrak{q}} \simeq U_{\mathfrak{q}}^{+} \otimes_{\mathbb{C}} U_{\mathfrak{q}}^{0} \otimes_{\mathbb{C}} U_{\mathfrak{q}}^{(-)} \simeq U_{\mathfrak{q}}^{\geqslant} \otimes_{\mathbb{C}} U_{\mathfrak{q}}^{\leqslant}. \tag{4.13}$$

We have canonical isomorphisms of Hopf algebras

$$U_{\mathfrak{q}}^{+0} \simeq \mathbb{C}\Gamma^{+}, \qquad U_{\mathfrak{q}}^{-0} \simeq \mathbb{C}\Gamma^{-}, \qquad U_{\mathfrak{q}}^{0} \simeq \mathbb{C}\Gamma.$$

The algebra $U_{\mathfrak{q}}^{+}$ has a canonical structure of a Hopf algebra in ${}^{\mathbb{C}\Gamma^{+}}_{\mathbb{C}\Gamma^{+}}\mathcal{YD}$. We have isomorphisms of (braided) Hopf algebras:

$$U_{\mathfrak{q}}^{+} \simeq \widetilde{\mathcal{B}}_{\mathfrak{q}}, \qquad U_{\mathfrak{q}}^{\geqslant} \simeq U_{\mathfrak{q}}^{+} \# U_{\mathfrak{q}}^{+0},$$

see [An3]. Define the module $V^{*} \in \mathcal{YD}^{\mathbb{C}\Gamma^{-}}_{\mathbb{C}\Gamma^{-}}$ with basis $\{x_i^{*} : i \in \mathbb{I}\}$ by

$$x_j^{*} \cdot L_i = \mathfrak{q}_{ji} x_j^{*}, \qquad \deg x_i^{*} = L_i^{-1}, \qquad i, j \in \mathbb{I}.$$

Let $\pi^{-} : U_{\mathfrak{q}}^{\leqslant} \to U_{\mathfrak{q}}^{-0}$ be the canonical Hopf algebra projection; then ${}^{\operatorname{co}\pi^{-}}U_{\mathfrak{q}}^{\leqslant} = U_{\mathfrak{q}}^{-}$ as in [ARS, Corollary 3.9 (2)]. Hence $U_{\mathfrak{q}}^{-}$ is a Hopf algebra in $\mathcal{YD}^{\mathbb{C}\Gamma^{-}}_{\mathbb{C}\Gamma^{-}}$ and because of the defining relations of $U_{\mathfrak{q}}^{-}$, it is isomorphic to the distinguished pre-Nichols algebra of $V^{*} \in \mathcal{YD}^{\mathbb{C}\Gamma^{-}}_{\mathbb{C}\Gamma^{-}}$. Combining the above, we get isomorphisms of (braided) Hopf algebras:

$$U_{\mathfrak{q}}^{-} \simeq \widetilde{\mathcal{B}}_{\mathfrak{q}^{(-1)}}, \qquad U_{\mathfrak{q}}^{\leqslant} \simeq U_{\mathfrak{q}}^{-0} \# U_{\mathfrak{q}}^{-}. \tag{4.14}$$

Here, again, $\mathfrak{q}^{(-1)}$ denotes the matrix obtained by inverting every entry of $\mathfrak{q}$.

4.4. **Lusztig isomorphisms and root vectors.** As in [H3, §3] we consider

$$\lambda_{ij}^{\mathbf{q}} = (\mathbf{q}_{ii}^{-c_{ij}^{\mathbf{q}}} \mathbf{q}_{ij}\mathbf{q}_{ji})^{c_{ij}^{\mathbf{q}}} (-c_{ij}^{\mathbf{q}})^{!}_{\mathbf{q}_{ii}} \prod_{0 \le s < -c_{ij}^{\mathbf{q}}} (\mathbf{q}_{ii}^{s}\mathbf{q}_{ij}\mathbf{q}_{ji} - 1) \in \mathbb{C}[\nu^{\pm 1}]^{\times}, \quad i \neq j \in \mathbb{I}. \tag{4.15}$$

By [H3, Proposition 6.8], there exist algebra isomorphisms $T_i^{\mathbf{q}} : U_{\rho_i(\mathbf{q})} \to U_{\mathbf{q}}$ such that

$$\begin{aligned} T_i^{\mathbf{q}}(\underline{K}_j) &= K_j K_i^{-c_{ij}^{\mathbf{q}}}; \quad T_i^{\mathbf{q}}(\underline{E}_i) = \begin{cases} F_i L_i, & j = i, \\ (\operatorname{ad}_c E_i)^{-c_{ij}^{\mathbf{q}}} E_j, & j \neq i, \end{cases} \\ T_i^{\mathbf{q}}(\underline{L}_j) &= L_j L_i^{-c_{ij}^{\mathbf{q}}}; \quad T_i^{\mathbf{q}}(\underline{F}_i) = \begin{cases} K_i^{-1} E_i, & j = i, \\ (\lambda_{ij}^{\mathbf{q}})^{-1} (\operatorname{ad}_c F_i)^{-c_{ij}^{\mathbf{q}}} F_j, & j \neq i, \end{cases} \end{aligned} \tag{4.16}$$

where the underlined letters denote the generators of $U_{\rho_i(\mathbf{q})}$.

Let $\omega_0^{\mathbf{q}}$ be the element of $\mathcal{W}_{\mathbf{q}}$ of maximal lenght ending at $\mathbf{q}$ and $\omega_0^{\mathbf{q}} = \sigma_{i_1}^{\mathbf{q}} \sigma_{i_2} \cdots \sigma_{i_\ell}$ be a reduced expression. By [H3, Theorem 6.20],

$$E_{\beta_k} := T_{i_1}^{\mathbf{q}} \dots T_{i_{k-1}}(E_{i_k}) \in U_{\mathbf{q}}^{+}, \qquad F_{\beta_k} := T_{i_1}^{\mathbf{q}} \dots T_{i_{k-1}}(F_{i_k}) \in U_{\mathbf{q}}^{-}, \qquad k \in \mathbb{I}_\ell. \tag{4.17}$$

By [HY2, Theorem 4.5] the sets

$$\left\{ E_{\beta_1}^{n_1} E_{\beta_2}^{n_2} \dots E_{\beta_\ell}^{n_\ell} : 0 \le n_j < \widetilde{N}_{\beta_j}, j \in \mathbb{I}_\ell \right\}, \left\{ F_{\beta_1}^{m_1} F_{\beta_2}^{m_2} \dots F_{\beta_\ell}^{m_\ell} : 0 \le m_j < \widetilde{N}_{\beta_j}, j \in \mathbb{I}_\ell \right\} \tag{4.18}$$

are bases of $U_{\mathbf{q}}^{+}$ and $U_{\mathbf{q}}^{-}$, respectively. Indeed, this follows from Property (c) in the Appendix A and Remark 4.1. Thus the following set is a basis of $U_{\mathbf{q}}$:

$$\{E_{\beta_1}^{m_1} \dots E_{\beta_\ell}^{m_\ell} K_1^{a_1} \dots K_\theta^{a_\theta} L_1^{b_1} \dots L_\theta^{b_\theta} F_{\beta_1}^{n_1} \dots F_{\beta_\ell}^{n_\ell} : 0 \le m_j,\, n_j < \widetilde{N}_\beta,\, a_i,\, b_i \in \mathbb{Z}\}. \tag{4.19}$$

We now turn to the algebras $U_{\mathfrak{q}}$. Let $\lambda_{ij}^{\mathfrak{q}}$ is defined as (4.15) with $\mathfrak{q}$ in place of $\mathbf{q}$. By [An3, Proposition 10], there exist algebra isomorphisms $T_i^{\mathfrak{q}} : U_{\rho_i(\mathfrak{q})} \to U_{\mathfrak{q}}$ such that

$$T_i^{\mathfrak{q}}(\underline{K}_j) = K_j K_i^{-c_{ij}^{\mathfrak{q}}}; \quad T_i^{\mathfrak{q}}(\underline{e}_i) = \begin{cases} f_i L_i, & j = i, \\ (\mathrm{ad}_c\, e_i)^{-c_{ij}^{\mathfrak{q}}} e_j, & j \neq i, \end{cases}$$

$$T_i^{\mathfrak{q}}(\underline{L}_j) = L_j L_i^{-c_{ij}^{\mathfrak{q}}}; \quad T_i^{\mathfrak{q}}(\underline{f}_i) = \begin{cases} K_i^{-1} e_i, & j = i, \\ (\lambda_{ij}^{\mathfrak{q}})^{-1}(\mathrm{ad}_c\, f_i)^{-c_{ij}^{\mathfrak{q}}} f_j, & j \neq i, \end{cases} \tag{4.20}$$

The underlined letters denote the generators of $U_{\rho_i(\mathfrak{q})}$.

Analogously, $e_{\beta_k} = T_{i_1}^{\mathfrak{q}} \dots T_{i_{k-1}}(e_{i_k})$ and $f_{\beta_k} = T_{i_1}^{\mathfrak{q}} \dots T_{i_{k-1}}(f_{i_k})$ belong to $U_{\mathfrak{q}}^+$ and $U_{\mathfrak{q}}^-$, respectively and by [An3, Theorem 11] the sets

$$\big\{e_{\beta_1}^{n_1} e_{\beta_2}^{n_2} \dots e_{\beta_\ell}^{n_\ell} : 0 \le n_i < \widetilde{N}_{\beta_i}\big\} \quad \text{and} \quad \big\{f_{\beta_1}^{m_1} f_{\beta_2}^{m_2} \dots f_{\beta_\ell}^{m_\ell} : 0 \le m_j < \widetilde{N}_{\beta_j}\big\} \tag{4.21}$$

are bases of $U_{\mathfrak{q}}^+$ and $U_{\mathfrak{q}}^-$, respectively. Thus the following set is a basis of $U_{\mathfrak{q}}$:

$$\big\{e_{\beta_1}^{m_1} \dots e_{\beta_\ell}^{m_\ell} K_1^{a_1} \dots K_\theta^{a_\theta} L_1^{b_1} \dots L_\theta^{b_\theta} f_{\beta_1}^{n_1} \dots f_{\beta_\ell}^{n_\ell} : 0 \le m_j,\, n_j < \widetilde{N}_{\beta_j},\, a_i,\, b_i \in \mathbb{Z}\big\}. \tag{4.22}$$

4.5. **The central subalgebras $Z_{\mathfrak{q}}$, $Z_{\mathfrak{q}}^{\pm}$, $Z_{\mathfrak{q}}^{\geqslant}$, $Z_{\mathfrak{q}}^{\leqslant}$.** In this subsection and the next $\mathfrak{q}$ does not need to be in a family, just $\dim \mathcal{B}_{\mathfrak{q}} < \infty$ is assumed. To start with, we consider the subalgebra $Z_{\mathfrak{q}}$ of $U_{\mathfrak{q}}$ introduced right after (3.17); as shown in [An3, p. 18], $Z_{\mathfrak{q}}$ is generated by

$$e_\beta^{N_\beta}, \qquad f_\beta^{N_\beta}, \qquad K_\beta^{\pm N_\beta}, \qquad L_\beta^{\pm N_\beta}, \qquad \beta \in \mathfrak{O}^{\mathfrak{q}};$$

this is a normal $\mathcal{Q}_{\mathfrak{q}}$-graded Hopf subalgebra of $U_{\mathfrak{q}}$ [An3, Proposition 21, Theorem 33]. For $Z_{\mathfrak{q}}$ to be central in $U_{\mathfrak{q}}$ we need the following condition that *we assume from now on:*

$$q_{\alpha\beta}^{N_\beta} = 1, \qquad \alpha \in \Delta^{\mathfrak{q}},\ \beta \in \mathfrak{O}^{\mathfrak{q}}. \tag{4.23}$$

Remark 4.4. (a) If (4.23) holds, then $q_{\beta\alpha}^{N_\beta} = 1$ [An3, Lemma 24].

(b) Condition (4.23) is equivalent to the following one:

$$q_{\alpha_i\beta}^{N_\beta} = 1, \qquad \text{for all } i \in \mathbb{I},\ \underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}. \tag{4.24}$$

The reduction to simple roots is clear. Since $q_{\alpha\beta}^{N_\beta} = q_{\alpha\underline{\beta}}$ and $\mathit{\Pi}^{\mathfrak{q}}$ is a basis of the root system $\mathfrak{O}^{\mathfrak{q}}$, the reduction from $\mathfrak{O}_+^{\mathfrak{q}}$ to $\mathit{\Pi}^{\mathfrak{q}}$ holds.

(c) Let $i \in \mathbb{I}$. Condition (4.23) holds for $\mathfrak{q}$ if and only if it holds for $\rho_i(\mathfrak{q})$.

Indeed, $\rho_i(\mathfrak{q})_{\alpha\beta} = \mathfrak{q}_{s_i^{\mathfrak{q}}(\alpha)s_i^{\mathfrak{q}}(\beta)}$ for all $\alpha, \beta \in \mathbb{Z}^\theta$ by (3.11), and by [AAR3, Lemma 2.3] we have $s_i^{\mathfrak{q}}(\mathfrak{O}^{\mathfrak{q}}) = \mathfrak{O}^{\rho_i(\mathfrak{q})}$, $N_{s_i^{\mathfrak{q}}(\beta)}^{\rho_i(\mathfrak{q})} = N_\beta^{\mathfrak{q}}$ for all β.

When $\mathfrak{q}$ is symmetric, we can quotient the large quantum by a central group subalgebra to remove the extra Cartan generators as in quantum groups. However the condition of $\mathfrak{q}$ being symmetric is not always compatible with (4.23) as we see next.

Example 4.5. Assume that $\mathfrak{q}$ has Dynkin diagram $\overset{-1}{\circ} \overset{\xi}{\text{———}} \overset{-1}{\circ}$, $\xi \in \mathbb{G}'_N$, $N > 2$: it is of super type $\mathbf{A}(1|0)$. In this case,

$$\Delta_+^{\mathfrak{q}} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}, \qquad \mathfrak{O}_+^{\mathfrak{q}} = \{\alpha_1 + \alpha_2\}.$$

Condition (4.24) becomes

$$1=(q_{11}q_{12})^N=(-q_{12})^N,\quad 1=(q_{21}q_{22})^N=(-q_{21})^N\qquad\Longleftrightarrow\ q_{12}^N=(-1)^N=q_{21}^N.$$

We have two possibilities: if N is even, then $q_{12}=\xi^k$ for some $k\in\mathbb{I}_N$, so $q_{21}=\xi^{1-k}$, and $\mathfrak{q}$ is not symmetric. If N is odd, then $q_{12}=-\xi^k$ for some $k\in\mathbb{I}_N$, so $q_{21}=-\xi^{1-k}$. In this case $\mathfrak{q}$ is symmetric only when $k=\frac{N+1}{2}$.

We consider also the Hopf subalgebras

$$\begin{aligned}
Z_{\mathfrak{q}}^{+0}&=\mathbb{C}\langle K_\beta^{\pm N_\beta}:\beta\in\mathfrak{O}_+^{\mathfrak{q}}\rangle, & Z_{\mathfrak{q}}^{-0}&=\mathbb{C}\langle L_\beta^{\pm N_\beta}:\beta\in\mathfrak{O}_+^{\mathfrak{q}}\rangle, & Z_{\mathfrak{q}}^{0}&=Z_{\mathfrak{q}}^{+0}Z_{\mathfrak{q}}^{-0},\\
Z_{\mathfrak{q}}^{+}&=\mathbb{C}\langle e_\beta^{N_\beta}:\beta\in\mathfrak{O}_+^{\mathfrak{q}}\rangle, & Z_{\mathfrak{q}}^{-}&=\mathbb{C}\langle f_\beta^{N_\beta}:\beta\in\mathfrak{O}_+^{\mathfrak{q}}\rangle, & Z_{\mathfrak{q}}^{\geqslant}&=Z_{\mathfrak{q}}^{+}Z_{\mathfrak{q}}^{+0},\ Z_{\mathfrak{q}}^{\leqslant}=Z_{\mathfrak{q}}^{-}Z_{\mathfrak{q}}^{-0}.
\end{aligned}$$

Remark 4.6. The following properties hold:

(a) [An3, Th. 23]. $Z_{\mathfrak{q}}^{\pm}$ is a polynomial ring in variables $e_\beta^{N_\beta}$, respectively $f_\beta^{N_\beta}$, $\beta\in\mathfrak{O}_+^{\mathfrak{q}}$.

(b) The multiplication gives linear isomorphisms $Z_{\mathfrak{q}}^{+}\otimes Z_{\mathfrak{q}}^{+0}\otimes Z_{\mathfrak{q}}^{-0}\otimes Z_{\mathfrak{q}}^{-}\simeq Z_{\mathfrak{q}}\simeq Z_{\mathfrak{q}}^{\geqslant}\otimes Z_{\mathfrak{q}}^{\leqslant}$.

(c) Recall the skew-derivations $\partial_i^{\mathfrak{q}}$, $\partial_i^{\mathfrak{q}^{(-1)}}$ of $U_{\mathfrak{q}}^{\pm}$, cf. (4.14). By [An3, Theorem 31],

$$Z_{\mathfrak{q}}^{+}=\bigcap_{i\in\mathbb{I}}\ker\partial_i^{\mathfrak{q}},\qquad\qquad Z_{\mathfrak{q}}^{-}=\bigcap_{i\in\mathbb{I}}\ker\partial_i^{\mathfrak{q}^{(-1)}}.\tag{4.25}$$

(d) The algebras $U_{\mathfrak{q}}$, $U_{\mathfrak{q}}^{\geqslant}$, $U_{\mathfrak{q}}^{\leqslant}$ and $U_{\mathfrak{q}}^{\pm}$ are module finite over their central subalgebras $Z_{\mathfrak{q}}$, $Z_{\mathfrak{q}}^{\geqslant}$, $Z_{\mathfrak{q}}^{\leqslant}$ and $Z_{\mathfrak{q}}^{\pm}$; just consider the PBW-bases in §4.4.

4.6. **Action of the Weyl groupoid on $Z_{\mathfrak{q}}$.** Next we prove invariance of the central Hopf subalgebras $Z_{\mathfrak{q}}$ under the Lusztig isomorphisms $T_i^{\mathfrak{q}}:U_{\rho_i(\mathfrak{q})}\to U_{\mathfrak{q}}$, cf. §4.4.

Theorem 4.7. *Let $i\in\mathbb{I}$. Then $T_i^{\mathfrak{q}}$ restricts to an algebra isomorphism $T_i^{\mathfrak{q}}:Z_{\rho_i(\mathfrak{q})}\to Z_{\mathfrak{q}}$.*

Proof. By (4.25), $Z_{\mathfrak{q}}$ does not depend on the expression of $\omega_0^{\mathfrak{q}}$; in particular we may choose $\omega_0^{\mathfrak{q}}=\sigma_{i_1}^{\mathfrak{q}}\dots\sigma_{i_\ell}$ such that $i_1=i$. For simplicity we set $\mathfrak{p}=\rho_i(\mathfrak{q})$. As $\sigma_{i_2}^{\mathfrak{p}}\dots\sigma_{i_\ell}$ is reduced, we may extend it to a reduced expression of $\omega^{\mathfrak{p}}$ [HY1, Corollary 3]:

$$\omega_0^{\mathfrak{p}}=\sigma_{i_2}^{\mathfrak{p}}\dots\sigma_{i_\ell}\sigma_j\qquad\qquad\text{for some } j\in\mathbb{I}.$$

We set $\beta'_k=\sigma_{i_1}(\beta_k)=\sigma_{i_2}^{\mathfrak{p}}\dots\sigma_{i_{k-1}}(\alpha_{i_k})$, $k\in\mathbb{I}_{2,\ell}$. Hence

$$\{\beta'_k:k\in\mathbb{I}_{2,\ell}\}=s_i^{\mathfrak{p}}\left(\varDelta_+^{\mathfrak{q}}-\{\alpha_i\}\right)=\varDelta_+^{\mathfrak{p}}-\{\alpha_i\}.$$

As $\sigma_{i_2}^{\mathfrak{p}}\dots\sigma_{i_\ell}(\alpha_j)\in\varDelta_+^{\mathfrak{p}}$, $\sigma_{i_2}^{\mathfrak{p}}\dots\sigma_{i_\ell}(\alpha_j)\neq\beta'_k$ for $k\in\mathbb{I}_{2,\ell}$, we have that $\sigma_{i_2}^{\mathfrak{p}}\dots\sigma_{i_\ell}(\alpha_j)=\alpha_i$.

Let $\beta\in\mathfrak{O}_{\rho_i(\mathfrak{q})}$. If $\beta=\beta'_k$ for some $k\in\mathbb{I}_{2,\ell}$, then $s_i^{\mathfrak{q}}(\beta'_k)=\beta_k$ and $N_{\beta'_k}=N_{\beta_k}$, hence

$$T_i^{\mathfrak{q}}\big(K_{\beta'_k}^{\pm N_{\beta'_k}}\big)=K_{s_i^{\mathfrak{q}}(\beta'_k)}^{\pm N_{\beta'_k}}=K_{\beta_k}^{\pm N_{\beta_k}}\in Z_{\mathfrak{q}},\quad T_i^{\mathfrak{q}}\big(e_{\beta'_k}^{N_{\beta'_k}}\big)=T_{i_1}^{\mathfrak{q}}T_{i_2}\dots T_{i_{k-1}}(e_{i_k}^{N_{\beta_k}})=e_{\beta_k}^{N_{\beta_k}}\in Z_{\mathfrak{q}}.$$

Otherwise $\beta=\alpha_i$, so i is a Cartan vertex and

$$T_i^{\mathfrak{q}}(K_\beta^{\pm N_\beta})=K_i^{\mp N_{\alpha_i}}\in Z_{\mathfrak{q}},\quad T_i^{\mathfrak{q}}(e_\beta^{N_\beta})=T_i^{\mathfrak{q}}(e_i^{N_{\alpha_i}})=(f_iL_i)^{N_{\alpha_i}}=q_{ii}^{\binom{N_{\alpha_i}}{2}}f_i^{N_{\alpha_i}}L_i^{N_{\alpha_i}}\in Z_{\mathfrak{q}}.$$

Analogously, $T_i^{\mathfrak{q}}(L_\beta^{\pm N_\beta}),T_i^{\mathfrak{q}}(f_\beta^{N_\beta})\in Z_{\mathfrak{q}}$ for all $\beta\in\mathfrak{O}_{\rho_i(\mathfrak{q})}$, so $T_i^{\mathfrak{q}}(Z_{\rho_i(\mathfrak{q})})\subseteq Z_{\mathfrak{q}}$. Applying $T_i^{\mathfrak{p}}$ we get the opposite inclusion. $\square$

5. The specialization setting for large quantum groups

In this section we construct the non-restricted integral form of $U_{\mathbf{q}}$ and prove that the large quantum group $U_{\mathfrak{q}}$ is a specialization of it. We also construct restricted integral forms of the subalgebras $U_{\mathbf{q}}^{\pm}$ and establish pairing results for the corresponding specializations. The latter integral forms will play a key role in our treatment of Poisson order structures on the large quantum groups $U_{\mathfrak{q}}$ and their Borel and unipotent subalgebras.

5.1. **Integral forms.** In order to implement the ideas of Section 2, we need to consider forms over suitable rings, generalizing [DP]. For simplicity, we set

$$\mathsf{A} := \mathbb{C}[\nu^{\pm 1}, (\mathbf{q}_{ii}^{s}\mathbf{q}_{ij}\mathbf{q}_{ji} - 1)^{-1} : i \neq j \in \mathbb{I}, 0 \leq s < -c_{ij}^{\mathbf{q}}] \subset \mathbb{C}(\nu). \tag{5.1}$$

We now define the *(non-restricted) integral forms* as the A-subalgebras

$$\begin{aligned}
U_{\mathbf{q},\mathsf{A}}^{+} &= \mathsf{A}\langle E_i : i \in \mathbb{I}\rangle \subset U_{\mathbf{q}}^{+}, & U_{\mathbf{q},\mathsf{A}}^{0} &= \mathsf{A}[K_i^{\pm 1}, L_i^{\pm 1} : i \in \mathbb{I}] \subset U_{\mathbf{q}}^{0},\\
U_{\mathbf{q},\mathsf{A}}^{-} &= \mathsf{A}\langle F_i : i \in \mathbb{I}\rangle \subset U_{\mathbf{q}}^{-}, & U_{\mathbf{q},\mathsf{A}} &= \mathsf{A}\langle K_i^{\pm 1}, L_i^{\pm 1}, E_i, F_i : i \in \mathbb{I}\rangle \subset U_{\mathbf{q}},\\
U_{\mathbf{q},\mathsf{A}}^{\geqslant} &= U_{\mathbf{q},\mathsf{A}}^{+} \otimes_{\mathsf{A}} \mathsf{A}[K_i^{\pm 1} : i \in \mathbb{I}], & U_{\mathbf{q},\mathsf{A}}^{\leqslant} &= U_{\mathbf{q},\mathsf{A}}^{-} \otimes_{\mathsf{A}} \mathsf{A}[L_i^{\pm 1} : i \in \mathbb{I}].
\end{aligned}$$

These are crucial for our purposes. We have again a triangular decomposition

$$U_{\mathbf{q},\mathsf{A}}^{+} \otimes_{\mathsf{A}} U_{\mathbf{q},\mathsf{A}}^{0} \otimes_{\mathsf{A}} U_{\mathbf{q},\mathsf{A}}^{-} \simeq U_{\mathbf{q},\mathsf{A}}. \tag{5.2}$$

The surjectivity of this multiplication map follows from the cross relations (4.4), (4.5) and (4.6), while the injectivity follows from (4.8). Recall (4.15) for the next result.

Lemma 5.1. *For all $i \neq j$, $(\lambda_{ij}^{\mathbf{q}})^{-1} \in \mathsf{A}$.*

Proof. If $\mathbf{q}_{ii}^{c_{ij}^{\mathbf{q}}}\mathbf{q}_{ij}\mathbf{q}_{ji} = 1$, then using that $\mathbf{q}_{ii} \in \mathbb{C}[\nu^{\pm 1}]^{\times}$ we have

$$(\lambda_{ij}^{\mathbf{q}})^{-1} = (-1)^{c_{ij}^{\mathbf{q}}} q_{ii}^{c_{ij}^{\mathbf{q}}(c_{ij}^{\mathbf{q}}-1)}(\mathbf{q}_{ii} - 1)^{-c_{ij}^{\mathbf{q}}} \prod_{0 \leq s < -c_{ij}^{\mathbf{q}}} (\mathbf{q}_{ii}^{s}\mathbf{q}_{ij}\mathbf{q}_{ji} - 1)^{-2} \in \mathsf{A}$$

Otherwise $\mathbf{q}_{ii}$ is a root of unity of order $1 - c_{ij}^{\mathbf{q}}$, so because $(-c_{ij}^{\mathbf{q}})_{\mathbf{q}_{ii}}^{!} \in \mathbb{C}^{\times}$, we have

$$(\lambda_{ij}^{\mathbf{q}})^{-1} = \frac{(\mathbf{q}_{ii}^{-1}\mathbf{q}_{ij}\mathbf{q}_{ji})^{-c_{ij}^{\mathbf{q}}}}{(-c_{ij}^{\mathbf{q}})_{\mathbf{q}_{ii}}^{!}} \prod_{0 \leq s < -c_{ij}^{\mathbf{q}}} (\mathbf{q}_{ii}^{s}\mathbf{q}_{ij}\mathbf{q}_{ji} - 1)^{-1} \in \mathsf{A}. \qquad \square$$

Example 5.2. Let $\mathbf{q}$ be of modular type $\mathtt{br}(2)$, respectively $\mathtt{wk}(4)$, see §A.3. Then $\mathsf{A} = \mathbb{C}[\nu^{\pm 1}, (\nu - 1)^{-1}, (\nu - \zeta)^{-1}]$, respectively $\mathsf{A} = \mathbb{C}[\nu^{\pm 1}, (\nu - 1)^{-1}, (\nu + 1)^{-1}]$.

We now define *restricted integral forms* that also play a central role in this paper. Recall the Hopf skew-pairing from §4.2. The A-submodules

$$U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,-} := \{y \in U_{\mathbf{q}}^{-} | \langle y, U_{\mathbf{q},\mathsf{A}}^{+}\rangle \subset \mathsf{A}\}, \qquad U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,+} := \{x \in U_{\mathbf{q}}^{+} | \langle U_{\mathbf{q},\mathsf{A}}^{-}, x\rangle \subset \mathsf{A}\}. \tag{5.3}$$

are A-subalgebras of $U_{\mathbf{q}}^{-}$ and $U_{\mathbf{q}}^{+}$, respectively. This follows from the fact that $U_{\mathbf{q},\mathsf{A}}^{\pm}$ are braided Hopf subalgebras of $U_{\mathbf{q}}^{\pm}$ over A and the properties of Hopf skew-pairings.

5.2. **PBW-bases of integral forms.** Recall the Lusztig isomorphisms $T_i^{\mathbf{q}}$ from §4.4.

Lemma 5.3. (a) *$T_i^{\mathbf{q}}$ restricts to an A-algebra isomorphism $T_i^{\mathbf{q}} : U_{\rho_i(\mathbf{q}),\mathsf{A}} \to U_{\mathbf{q},\mathsf{A}}$, $i \in \mathbb{I}$.*
(b) *Let $\beta \in \Delta_+$. Then $E_\beta, F_\beta \in U_{\mathbf{q},\mathsf{A}}$.*

Proof. (a) follows from (4.20) and Lemma 5.1, while (b) from (a) and (4.17). □

Proposition 5.4. *The sets* (4.18) *and* (4.19) *are A-bases of $U^{\pm}_{\mathbf{q},\mathsf{A}}$ and $U_{\mathbf{q},\mathsf{A}}$, respectively.*

Proof. We consider the case of $U^+_{\mathbf{q},\mathsf{A}}$, the other being analogous. Let Y be the set of PBW monomials of $U^+_{\mathbf{q}}$ from (4.18). By Lemma 5.3, $Y \subset U^+_{\mathbf{q},\mathsf{A}}$. The defining relations of $U^+_{\mathbf{q}}$ involve products of E_i with coefficients in A, hence we may prove recursively that, for $j > k$, $E_{\beta_j}E_{\beta_k} \in \mathsf{A}Y$, the A-module generated by Y, where each monomial in the expansion has letters E_{β_t}, $j > t > k$; see the proof of [HY2, Theorem 4.8]. Thus $\mathsf{A}Y$ is a left ideal containing 1, so $\mathsf{A}Y = U^+_{\mathbf{q},\mathsf{A}}$. This fact and the direct sum decomposition $U^+_{\mathbf{q}} = \oplus_{y\in Y}\mathbb{C}(\nu)y$ imply that $U^+_{\mathbf{q},\mathsf{A}} = \oplus_{y\in Y}\mathsf{A}y$. □

Recall the notation $\widetilde{N}_\beta$ in §3.7. Next we consider the quantum divided powers

$$F^{(n)}_{\beta_j} = \frac{F^n_{\beta_j}}{(n)^!_{\mathbf{q}_{\beta_j\beta_j}}}, \qquad E^{(n)}_{\beta_j} = \frac{E^n_{\beta_j}}{(n)^!_{\mathbf{q}_{\beta_j\beta_j}}}, \qquad 0 \le n < \widetilde{N}_{\beta_j}.$$

Proposition 5.5. *For $j \in \mathbb{I}_\ell$, let n_j, m_j be such that $0 \le n_j, m_j < \widetilde{N}_{\beta_j}$. Then*

$$\langle F^{(n_1)}_{\beta_1} \dots F^{(n_\ell)}_{\beta_\ell}, E^{m_1}_{\beta_1} \dots E^{m_\ell}_{\beta_\ell}\rangle = \delta_{n_1m_1} \dots \delta_{m_\ell n_\ell}.$$

Proof. Let $\eta_j = \langle F_{\beta_j}, E_{\beta_j}\rangle$, $j \in \mathbb{I}_\ell$. The same proof as [AnY, Proposition 4.6] shows that

$$\langle F^{(n_1)}_{\beta_1} \dots F^{(n_\ell)}_{\beta_\ell}, E^{m_1}_{\beta_1} \dots E^{m_\ell}_{\beta_\ell}\rangle = \delta_{n_1m_1} \dots \delta_{m_\ell n_\ell}\eta_1^{n_1} \cdots \eta_\ell^{n_\ell},$$

As in [AnY, 4.7], we see that $\eta_j = 1$: here $\langle F_i, E_i\rangle = 1$, there $\langle F_i, E_i\rangle = -1$ for $i \in \mathbb{I}$. □

Propositions 5.4 and 5.5 imply the following:

Corollary 5.6. *The following sets are A-basis of $U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}}$ and $U^{\mathrm{res}\,+}_{\mathbf{q},\mathsf{A}}$, respectively:*

$$\left\{F^{(n_1)}_{\beta_1} \dots F^{(n_\ell)}_{\beta_\ell} : 0 \le n_j < \widetilde{N}_{\beta_j}\right\} \quad \textit{and} \quad \left\{E^{(m_1)}_{\beta_1} \dots E^{(m_\ell)}_{\beta_\ell} : 0 \le m_j < \widetilde{N}_{\beta_j}\right\}. \tag{5.4}$$

5.3. **The specialization of $U_{\mathbf{q},\mathsf{A}}$.** We consider the setting in Section 2 assuming $R = \mathsf{A}$, $h = \nu - \xi$ and the R-algebra A being either $U_{\mathbf{q},\mathsf{A}}$ or its subalgebras $U^{\pm}_{\mathbf{q},\mathsf{A}}$. We claim that the map $\mathbb{C}[\nu^{\pm 1}] \to \mathbb{C}$, $q \mapsto \xi$ extends to an isomorphism $\mathsf{A}/(\nu - \xi) \simeq \mathbb{C}$. For, if

$$\mathbf{q}^s_{ii}\mathbf{q}_{ij}\mathbf{q}_{ji} - 1 \mapsto q^s_{ii}q_{ij}q_{ji} - 1 = 0 \qquad \text{for some } i \neq j,\ 0 \le s < -c^{\mathbf{q}}_{ij},$$

then $0 \le -c^{\mathfrak{q}}_{ij} \le s < -c^{\mathbf{q}}_{ij}$, which contradicts Property (c) in the Appendix A. Here and below we will use the bar notation $x \mapsto \overline{x}$ for specializations.

Theorem 5.7. *There are Hopf algebra (respectively, braided Hopf algebra) isomorphisms*

$$\Xi_{\mathfrak{q}} : U_{\mathfrak{q}} \to U_{\mathbf{q},\mathsf{A}}/(\nu - \xi) \qquad \textit{and} \qquad \Xi_{\mathfrak{q}}|_{U^{\pm}_{\mathfrak{q}}} : U^{\pm}_{\mathfrak{q}} \to U^{\pm}_{\mathbf{q},\mathsf{A}}/(\nu - \xi)$$

given by $e_i \mapsto \overline{E}_i, f_i \mapsto \overline{F}_i, K_i^{\pm 1} \mapsto \overline{K}_i^{\pm 1}, L_i^{\pm 1} \mapsto \overline{L}_i^{\pm 1}$ *for all* $i \in \mathbb{I}$. *For each* $i \in \mathbb{I}$, *the following diagram is commutative:*

$$\begin{tikzcd} U_{\rho_i(\mathfrak{q})} \arrow[r, "\Xi_{\rho_i(\mathfrak{q})}"] \arrow[d, "T_i^{\mathfrak{q}}"'] & U_{\rho_i(\mathfrak{q})f,\mathsf{A}}/(\nu-\xi) \arrow[d, "T_i^{\mathbf{q}}"] \\ U_{\mathfrak{q}} \arrow[r, "\Xi_{\mathfrak{q}}"] & U_{\mathbf{q},\mathsf{A}}/(\nu-\xi). \end{tikzcd} \tag{5.5}$$

Proof. The defining relations of $U_{\mathfrak{q}}$ hold in $U_{\mathbf{q},\mathsf{A}}/(\nu-\xi)$ by the definition of $U_{\mathfrak{q}}$ in [An3] and the presentation of $U_{\mathbf{q}}$ in [An2]. Therefore, the map $\Xi_{\mathfrak{q}}$ as above is well-defined. Moreover $\Xi_{\mathfrak{q}}$ is surjective, since $\overline{E}_i$, $\overline{F}_i$, $\overline{K}_i^{\pm 1}$, $\overline{L}_i^{\pm 1}$ generate $U_{\mathbf{q},\mathsf{A}}/(\nu-\xi)$ as $\mathbb{C}$-algebra.

Now we check that (5.5) is a commutative diagram. Indeed, since Property (c) in the Appendix A holds and $\mathbf{q} \mapsto \mathfrak{q}$ under the evaluation map, we have that

$$\Xi_{\mathfrak{q}} \circ T_i^{\mathfrak{q}}(e_j) = (\operatorname{ad}_c \overline{E}_i)^{-c_{ij}^{\mathfrak{q}}} \overline{E}_j = T_i^{\mathbf{q}} \circ \Xi_{\rho_i(\mathfrak{q})}(e_j),$$
$$\Xi_{\mathfrak{q}} \circ T_i^{\mathfrak{q}}(f_j) = (\operatorname{ad}_c \overline{F}_i)^{-c_{ij}^{\mathfrak{q}}} \overline{F}_j = T_i^{\mathbf{q}} \circ \Xi_{\rho_i(\mathfrak{q})}(f_j)$$

for $j \neq i$. Since $\Xi_{\mathfrak{q}} \circ T_i^{\mathfrak{q}}(X) = T_i^{\mathbf{q}} \circ \Xi_{\rho_i(\mathfrak{q})}(X)$ for $X \in \{e_i, f_i, K_j^{\pm 1}, L_j^{\pm 1}\}$, the claim follows. By (5.5), $\Xi_{\mathfrak{q}}(E_\beta) = \overline{E}_\beta$ and $\Xi_{\mathfrak{q}}(F_\beta) = \overline{F}_\beta$ for all $\beta \in \Delta_+$. Hence $\Xi_{\mathfrak{q}}$ sends the PBW basis of $U_{\mathfrak{q}}$ to that of $U_{\mathbf{q},\mathsf{A}}/(\nu-\xi)$, so $\Xi_{\mathfrak{q}}$, and its restrictions to $U_{\mathfrak{q}}^{\pm}$, are isomorphisms. Clearly $\Xi_{\mathfrak{q}}$ (and its restrictions) are isomorphisms of (braided) Hopf algebras. $\square$

5.4. **The specialization of $U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,\pm}$.** Recall the Lusztig algebra $\mathcal{L}_{\mathfrak{q}}$ §3.9 and the identification of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ with $U_{\mathfrak{q}}^+$ as in §4.3. For $\beta \in \mathfrak{O}^{\mathfrak{q}}$, $n \in \mathbb{N}_0$, define $\eta_\beta^{(n)} \in \mathcal{L}_{\mathfrak{q}}$ such that

$$(\eta_{\beta_j}^{(n)}, e_{\beta_1}^{m_1} \dots e_{\beta_\ell}^{m_\ell}) = \begin{cases} 1, & m_j = n,\ m_k = 0 \text{ for } k \neq j, \\ 0, & \text{otherwise,} \end{cases} \tag{5.6}$$

By [AAR1, Proposition 4.6], the set

$$\{\eta_{\beta_1}^{(n_1)} \cdots \eta_{\beta_\ell}^{(n_\ell)} : 0 \le n_j < \widetilde{N}_{\beta_j}\}$$

is a basis of $\mathcal{L}_{\mathfrak{q}}$ and the algebra $\mathcal{L}_{\mathfrak{q}}$ is generated by

$$\{\eta_{\alpha_i} : i \in \mathbb{I}\} \cup \{\eta_\beta^{(N_\beta)} : \underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}\}.$$

The Lie algebra $\mathfrak{n}_{\mathfrak{q}}^-$ from (3.15) has a $\mathbb{C}$-basis $\{\iota^*(\eta_\beta^{(N_\beta)}) : \beta \in \mathfrak{O}^{\mathfrak{q}}\}$ and set of simple root vectors $\{\iota^*(\eta_\beta^{(N_\beta)}) : \underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}\}$. Similar results hold for the Lusztig algebra $\mathcal{L}_{\mathfrak{q}^{(-1)}}$ associated to $\widetilde{\mathcal{B}}_{\mathfrak{q}^{(-)}} \simeq U_{\mathfrak{q}}^-$. The corresponding elements of $\mathcal{L}_{\mathfrak{q}^{(-1)}}$, defined as in (5.6) using f_β^m instead of e_β^m, will be denoted by $\theta_\beta^{(n)}$, where $\beta \in \mathfrak{O}^{\mathfrak{q}}$ and $n \in \mathbb{N}_0$.

Remark 5.8. The Lie algebras associated to $\mathcal{L}_{\mathfrak{q}}$ and $\mathcal{L}_{\mathfrak{q}^{(-1)}}$ as in Remark 3.7 are isomorphic to each other, see the list in the Appendix A. Hence we have a Lie algebra isomorphism

$$\mathfrak{n}_{\mathfrak{q}^{(-1)}}^- \simeq \mathfrak{n}_{\mathfrak{q}}^+ \tag{5.7}$$

where $\iota^*(\theta_\beta^{(N_\beta)}) \in \mathfrak{n}_{\mathfrak{q}^{(-1)}}^+$, $\underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}$ are mapped to the simple root vectors of $\mathfrak{n}_{\mathfrak{q}}^-$.

For a braided Hopf algebra B denote the braided opposite algebra $B^{\underline{\mathrm{op}}}$ with product $\mu^{\mathrm{op}} := \mu c^{-1}$ where $\mu : B \times B \to B$ is the product in B.

Proposition 5.9. *There are* $\mathbb{C}$*-algebra anti-isomorphisms*

$$\begin{aligned} \phi^- &: U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}}/(\nu-\xi) \to \mathcal{L}_{\mathfrak{q}}, & & \overline{F^{(n)}_\beta} \mapsto \eta^{(n)}_\beta, & \\ \phi^+ &: \left(U^{\mathrm{res}\,+}_{\mathbf{q},\mathsf{A}}/(\nu-\xi)\right)^{\mathrm{op}} \to \mathcal{L}_{\mathfrak{q}^{(-1)}}, & \text{given by } & \overline{E^{(n)}_\beta} \mapsto \theta^{(n)}_\beta, & \beta\in\mathfrak{O}^{\mathfrak{q}},\ n\in\mathbb{N}_0. \end{aligned}$$

Proof. We prove the statement in the minus case, the plus case is analogous. By Proposition 5.5 the Hopf skew-pairing $\langle\,,\,\rangle : U^-_{\mathbf{q}} \times U^+_{\mathbf{q}} \to \mathbb{C}(\nu)$ restricts to a perfect pairing

$$\langle\,,\,\rangle : U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}} \times U^+_{\mathbf{q},\mathsf{A}} \to \mathsf{A}.$$

Since $U^+_{\mathbf{q},\mathsf{A}}/(\nu-\xi) \simeq U^+_{\mathfrak{q}}$ as braided Hopf algebras, the latter pairing induces a non-degenerate pairing $\langle\,,\,\rangle : \left(U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}}/(\nu-\xi)\right) \times U^+_{\mathfrak{q}} \to \mathbb{C}$ such that

$$\langle yy', x\rangle = \langle y\otimes y', \Delta(x)\rangle, \qquad y,y'\in U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}}/(\nu-\xi), x\in U^+_{\mathfrak{q}} \tag{5.8}$$

and we have the commutative diagram

$$\begin{tikzcd} U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}} \times U^+_{\mathbf{q},\mathsf{A}} \arrow[r] \arrow[d] & \mathsf{A} \arrow[d] \\ \left(U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}}/(\nu-\xi)\right) \times U^+_{\mathfrak{q}} \arrow[r] & \mathbb{C} \end{tikzcd}$$

By the definition of $\mathcal{L}_{\mathfrak{q}}$, we have a canonical vector space isomorphism

$$\phi^- : U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}}/(\nu-\xi) \to \mathcal{L}_{\mathfrak{q}} \qquad \text{such that} \qquad \langle Y, x\rangle = (\phi^-(Y), x)$$

for all $Y\in U^{\mathrm{res}\,-}_{\mathbf{q},\mathsf{A}}/(\nu-\xi), x\in U^+_{\mathfrak{q}}$. Comparing (3.17) and (5.8), we see that ϕ^- is an algebra anti-isomorphism. Using again Proposition 5.5 and the definition (5.6) of $\eta^{(n)}_\beta$, we get that ϕ^- is given by $\overline{F^{(n)}_\beta} \mapsto \eta^{(n)}_\beta$ for $\beta\in\mathfrak{O}^{\mathfrak{q}}$, $n\in\mathbb{N}_0$. □

6. Poisson orders on large quantum groups

By Theorem 5.7, the large quantum group $U_{\mathfrak{q}}$ fits in the context of Section 2 and consequently the pair $(U_{\mathfrak{q}}, \mathcal{Z}(U_{\mathfrak{q}}))$ inherits a structure of Poisson order from deformation theory. However the Poisson algebra $\mathcal{Z}(U_{\mathfrak{q}})$ is often singular. We prove that the central Hopf subalgebra $Z_{\mathfrak{q}}$ introduced in §4.5 (which is of course regular) is a Poisson subalgebra of $\mathcal{Z}(U_{\mathfrak{q}})$ of the same dimension. Thus $(U_{\mathfrak{q}}, Z_{\mathfrak{q}})$ has a structure of Poisson order that restricts to the corresponding large quantum Borel and unipotent algebras.

6.1. **Poisson structure on** $Z_{\mathfrak{q}}$**.** We show that $Z_{\mathfrak{q}}$, $Z^{\geqslant}_{\mathfrak{q}}$, $Z^{\leqslant}_{\mathfrak{q}}$, $Z^+_{\mathfrak{q}}$ and $Z^-_{\mathfrak{q}}$ are Poisson subalgebras of $\mathcal{Z}(U_{\mathfrak{q}})$, respectively $\mathcal{Z}(U^{\geqslant}_{\mathfrak{q}})$, $\mathcal{Z}(U^{\leqslant}_{\mathfrak{q}})$, $\mathcal{Z}(U^+_{\mathfrak{q}})$ and $\mathcal{Z}(U^-_{\mathfrak{q}})$.

We need first to introduce the matrix $\mathscr{P}^{\mathbf{q}} \in \mathbb{C}^{\Pi^{\mathfrak{q}}\times\Pi^{\mathfrak{q}}}$. Let $\beta,\gamma\in\mathfrak{O}^{\mathfrak{q}}_+$. As $q_{\beta\gamma} = \mathbf{q}_{\beta\gamma}(\xi)$, (4.23) implies that there exists $\wp^{\mathbf{q}}_{\beta\gamma}(\nu)\in\mathsf{A}$ such that

$$1 - \mathbf{q}^{N_\beta N_\gamma}_{\beta\gamma} = (\nu-\xi)\wp^{\mathbf{q}}_{\beta\gamma}(\nu). \tag{6.1}$$

Recall the notation $\underline{\beta}$ from (3.14) and the set $\underline{\mathfrak{O}}^{\mathfrak{q}}$ from Theorem 3.5. Then we define

$$\mathscr{P}^{\mathbf{q}} = (\wp^{\mathbf{q}}_{\beta\gamma}(\xi))_{\underline{\beta},\underline{\gamma}\in\Pi^{\mathfrak{q}}}. \tag{6.2}$$

Lemma 6.1. *Let* $i\in\mathbb{I}$*. Then* $\mathscr{P}^{\rho_i(\mathbf{q})} = \mathscr{P}^{\mathbf{q}}$.

Proof. First, $\Pi^{\rho_i(\mathfrak{q})} = s^{\mathfrak{q}}_i(\Pi^{\mathfrak{q}})$. Thus $\wp^{\rho_i(\mathbf{q})}_{s^{\mathfrak{q}}_i(\beta)s^{\mathfrak{q}}_i(\gamma)}(\nu) = \wp^{\mathbf{q}}_{\beta\gamma}(\nu)$ for $\underline{\beta},\underline{\gamma}\in\Pi^{\mathfrak{q}}$ by (3.11). □

The next theorem is the main result of this section.

Theorem 6.2. *There are structures of Poisson order on the pairs*

$$(U_{\mathfrak{q}}, Z_{\mathfrak{q}}), \qquad (U_{\mathfrak{q}}^{\geqslant}, Z_{\mathfrak{q}}^{\geqslant}), \qquad (U_{\mathfrak{q}}^{\leqslant}, Z_{\mathfrak{q}}^{\leqslant}), \qquad (U_{\mathfrak{q}}^{+}, Z_{\mathfrak{q}}^{+}) \qquad \text{and} \qquad (U_{\mathfrak{q}}^{-}, Z_{\mathfrak{q}}^{-}) \tag{6.3}$$

arising by restriction from the Poisson order on the corresponding algebra and its center with Poisson bracket (2.2)*. The central algebras $Z_{\mathfrak{q}}$, $Z_{\mathfrak{q}}^{\geqslant}$ and $Z_{\mathfrak{q}}^{\leqslant}$ are Poisson-Hopf while $Z_{\mathfrak{q}}^{\pm}$ are coideal Poisson subalgebras over the former.*

Because of Theorem 5.7 and Proposition 2.4 we are reduced to prove:

Proposition 6.3. *The subalgebras $Z_{\mathfrak{q}}^{\pm}$, $Z_{\mathfrak{q}}^{\geqslant}$, $Z_{\mathfrak{q}}^{\leqslant}$ and $Z_{\mathfrak{q}}$ are Poisson subalgebras of $\mathcal{Z}(U_{\mathfrak{q}}^{\pm})$, $\mathcal{Z}(U_{\mathfrak{q}}^{\geqslant})$, $\mathcal{Z}(U_{\mathfrak{q}}^{\leqslant})$ and $\mathcal{Z}(U_{\mathfrak{q}})$, respectively, under the Poisson bracket* (2.2)*.*

Observe that $Z_{\mathfrak{q}}^{\pm}$, $Z_{\mathfrak{q}}^{\geqslant}$ and $Z_{\mathfrak{q}}^{\leqslant}$ are Poisson subalgebras of $Z_{\mathfrak{q}}$.

Proof. We apply Theorem 2.3 to the algebra $U_{\mathbf{q},\mathsf{A}}^{+}$, the automorphisms $\varsigma_i^{\mathbf{q}}$ and the $(\mathrm{id}, \varsigma_i^{\mathbf{q}})$-derivations $\partial_i^{\mathbf{q}}$, $i \in \mathbb{I}$ to conclude that $\mathcal{Z}'$ defined as in (2.4) is a Poisson subalgebra of $\mathcal{Z}(U_{\mathfrak{q}}^{+})$. Now we have that

$$\overline{\varsigma}_i^{\mathbf{q}} \overset{\star}{=} \varsigma_i^{\mathfrak{q}}, \qquad\qquad \overline{\partial}_i^{\mathbf{q}} \overset{*}{=} \partial_i^{\mathfrak{q}}, \qquad\qquad Z_{\mathfrak{q}}^{+} \subset \cap_{i\in\mathbb{I}} \ker(\varsigma_i^{\mathfrak{q}} - \mathrm{id}).$$

The equality $\star$ holds since $\mathfrak{q} = \mathbf{q}(\xi)$, while $*$ follows by a direct computation on the generators of $U_{\mathfrak{q}}^{+}$. The inclusion holds since $Z_{\mathfrak{q}}^{+} \subset \mathcal{Z}(U_{\mathfrak{q}})$: indeed $\varsigma_i^{\mathfrak{q}}(x) = K_i x K_i^{-1} = x$ for all $x \in Z_{\mathfrak{q}}^{+}$. From this inclusion and (4.25), $\mathcal{Z}' = Z_{\mathfrak{q}}^{+}$. The proof for $Z_{\mathfrak{q}}^{-}$ is analogous. The restriction of the Poisson structure to $Z_{\mathfrak{q}}^{0\pm}$ vanishes by the definition (2.2).

Next we prove the statement for $Z_{\mathfrak{q}}^{\geqslant}$. Let $\beta, \gamma \in \mathfrak{O}_{+}^{\mathfrak{q}}$. We compute

$$\{e_\beta^{N_\beta}, K_\gamma^{N_\gamma}\} = \overline{\frac{[E_\beta^{N_\beta}, K_\gamma^{N_\gamma}]}{\nu - \xi}} = \overline{\frac{1 - \mathbf{q}_{\beta\gamma}^{N_\beta N_\gamma}}{\nu - \xi} E_\beta^{N_\beta} K_\gamma^{N_\gamma}} \overset{(6.1)}{=} \wp_{\beta\gamma}^{\mathbf{q}}(\xi) e_\beta^{N_\beta} K_\gamma^{N_\gamma} \in Z_{\mathfrak{q}}^{\geqslant}.$$

This proves the claim since it suffices to check the bracket between generators. Similarly,

$$\{e_\beta^{N_\beta}, L_\gamma^{N_\gamma}\} \in \mathsf{A} e_\beta^{N_\beta} L_\gamma^{N_\gamma}, \qquad \{f_\beta^{N_\beta}, K_\gamma^{N_\gamma}\} \in \mathsf{A} e_\beta^{N_\beta} K_\gamma^{N_\gamma}, \qquad \{f_\beta^{N_\beta}, L_\gamma^{N_\gamma}\} \in Z_{\mathfrak{q}}^{\leqslant}.$$

This finishes the proof for $Z_{\mathfrak{q}}^{\leqslant}$ and reduces that of $Z_{\mathfrak{q}}$ to prove that $\{e_\beta^{N_\beta}, f_\gamma^{N_\gamma}\} \in Z_{\mathfrak{q}}$. For this we use the enumeration of the positive roots using the longest element of the Weyl groupoid. First we assume that $\beta = \beta_j$, $\gamma = \beta_k$ for $1 \le j < k \le \ell$. Let $\mathfrak{p} = \rho_{i_j} \dots \rho_{i_1}(\mathfrak{q})$, $\gamma' = s_{i_j}^{\mathfrak{p}} \dots s_{i_1}(\gamma)$, so $N_{\gamma'} = N_\gamma$. We have that

$$\begin{aligned}
\{e_\beta^{N_\beta}, f_\gamma^{N_\gamma}\} &= \overline{\frac{[E_\beta^{N_\beta}, F_\gamma^{N_\gamma}]}{\nu - \xi}} = \overline{\frac{T_{i_1}^{\mathbf{q}} \dots T_{i_j}([K_{i_j}^{-N_{i_j}} F_{i_j}^{N_{i_j}}, F_{\gamma'}^{N_{\gamma'}}])}{\nu - \xi}} \\
&= T_{i_1}^{\mathfrak{q}} \dots T_{i_j} \left(\overline{\frac{[K_{i_j}^{-N_{i_j}} F_{i_j}^{N_{i_j}}, F_{\gamma'}^{N_{\gamma'}}]}{\nu - \xi}} \right) = T_{i_1}^{\mathfrak{q}} \dots T_{i_j} \left(\left\{ K_{i_j}^{-N_{i_j}} f_{i_j}^{N_{i_j}}, f_{\gamma'}^{N_{\gamma'}} \right\} \right).
\end{aligned}$$

By the statements already proved, $\left\{ K_{i_j}^{-N_{i_j}} f_{i_j}^{N_{i_j}}, f_{\gamma'}^{N_{\gamma'}} \right\} \in Z_{\mathfrak{p}}$. Hence

$$\{e_\beta^{N_\beta}, f_\gamma^{N_\gamma}\} \in T_{i_1}^{\mathfrak{q}} \dots T_{i_j}(Z_{\mathfrak{p}}) \overset{\text{Theorem 4.7}}{=} Z_{\mathfrak{q}}.$$

The case $j > k$ is proved analogously. Now assume that $\beta = \gamma$. We start with the case $\beta = \alpha_i$ for some $i \in \mathbb{I}$ (a simple Cartan root). Using (4.6) we prove recursively that

$$[E_i^N, F_i^N] = \sum_{t=1}^{N} (t)^!_{\mathbf{q}_{ii}} \binom{N}{t}^2_{\mathbf{q}_{ii}} F_i^{N-t} \prod_{s=0}^{t-1} \left(K_i \mathbf{q}_{ii}^{2t-2N-s} - L_i^{-1} \right) E_i^{N-t}, \qquad N \in \mathbb{N}. \tag{6.4}$$

Let $t \in \mathbb{I}_{N_i-1}$. As q_{ii} is a primitive N_i-th root of unity and $\overline{\mathbf{q}_{ii}} = q_{ii}$,

$$\wp^{\mathbf{q}}_{\alpha_i\alpha_i}(\xi) = \overline{\frac{1-\mathbf{q}_{ii}^{N_i^2}}{\nu - \xi}} = \overline{\frac{(1-\mathbf{q}_{ii}^{N_i})(1+\mathbf{q}_{ii}^{N_i} + \cdots + \mathbf{q}_{ii}^{N_i(N_i-1)})}{\nu-\xi}} = N_i \overline{\frac{1-\mathbf{q}_{ii}^{N_i}}{\nu-\xi}}.$$

Hence,

$$\overline{\frac{(N_i)^!_{\mathbf{q}_{ii}}}{\nu-\xi}} = \overline{\frac{1-\mathbf{q}_{ii}^{N_i}}{\nu-\xi}} \cdot \frac{(1-q_{ii})\dots(1-q_{ii}^{N_i-1})}{(1-q_{ii})^{N_i}} = \frac{\wp^{\mathbf{q}}_{\alpha_i\alpha_i}(\xi)}{(1-q_{ii})^{N_i}}. \tag{6.5}$$

From this we obtain,

$$\{e_i^{N_i}, f_i^{N_i}\} = \overline{\frac{[E_i^{N_i}, F_i^{N_i}]}{\nu-\xi}} = \overline{\frac{(N_i)^!_{\mathbf{q}_{ii}}}{\nu-\xi}} \prod_{s=0}^{N_i-1} (K_i q_{ii}^{-s} - L_i) = \frac{-\wp^{\mathbf{q}}_{\alpha_i\alpha_i}(\xi)}{(q_{ii}-1)^{N_i}} (K_i^{N_i} - L_i^{-N_i}) \in Z_{\mathfrak{q}}.$$

Next, if β is not simple, say $\beta = \beta_j$ for some $j \in \mathbb{I}_\ell$, then using Theorem 4.7 again

$$\{e_\beta^{N_\beta}, f_\beta^{N_\beta}\} = T^{\mathfrak{q}}_{i_1} \dots T_{i_{j-1}} \Big(\overline{\frac{[E_{i_j}^{N_{i_j}}, F_{i_j}^{N_{i_j}}]}{\nu-\xi}} \Big) = \frac{-\wp^{\mathbf{q}}_{\beta\beta}(\xi)}{(q_{\beta\beta}-1)^{N_\beta}} (K_\beta^{N_\beta} - L_\beta^{N_\beta}) \in Z_{\mathfrak{q}}. \quad \square \tag{6.6}$$

7. The associated Poisson algebraic groups

In this section we describe the Poisson algebraic groups that correspond to the Poisson-Hopf algebras $Z_{\mathfrak{q}}$, $Z_{\mathfrak{q}}^{\geqslant}$ and $Z_{\mathfrak{q}}^{\leqslant}$. We prove that, as algebraic groups, they are isomorphic to Borel subgroups of connected semisimple algebraic groups but of adjoint type (and not of simply connected type as in previous works) and direct products of such Borel subgroups. The dual Lie bialgebras of the three tangent Lie algebras are proved to constitute a Manin triple, the ample Lie algebra in which is reductive. It is shown that the resulting Lie bialgebra structures are the ones from the Belavin–Drinfeld classification [BD] for the standard BD-triple (containing the empty subsets of the Dynkin graph) and arbitrary choice of the continuous parameters The results completely determine the Poisson structures on the three kinds of algebraic groups in question.

7.1. **The positive and negative parts of the dual tangent Lie bialgebra of $M_{\mathfrak{q}}$.** Let $M_{\mathfrak{q}}$, $M_{\mathfrak{q}}^{\pm}$, $M_{\mathfrak{q}}^{\pm 0}$, $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$ be the complex algebraic groups which are equal to the maximal spectra of the commutative Hopf algebras $Z_{\mathfrak{q}}$, $Z_{\mathfrak{q}}^{\pm}$, $Z_{\mathfrak{q}}^{\pm 0}$, $Z_{\mathfrak{q}}^{\geqslant}$ and $Z_{\mathfrak{q}}^{\leqslant}$, respectively. Here the Hopf algebra structures on $Z_{\mathfrak{q}}^{\pm}$ are the restrictions of the braided Hopf algebra structures on $U_{\mathfrak{q}}^{\pm}$ to $Z_{\mathfrak{q}}^{\pm}$ [An3].

Since $Z_{\mathfrak{q}}$ is a finitely generated Poisson-Hopf algebra which is an integral domain, $M_{\mathfrak{q}}$ is a connected Poisson algebraic group (see §B.2 for background). Analogously, $M_{\mathfrak{q}}^{\geqslant}$, $M_{\mathfrak{q}}^{\leqslant}$ and $M_{\mathfrak{q}}^{\pm 0}$ are connected Poisson algebraic groups, and $M_{\mathfrak{q}}^{\pm}$ are connected unipotent algebraic groups. The latter are not Poisson algebraic groups; they are isomorphic to certain Poisson homogeneous spaces for $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$ (see §8.3). The tensor product decompositions $Z_{\mathfrak{q}} \simeq Z_{\mathfrak{q}}^{\geqslant} \otimes Z_{\mathfrak{q}}^{\leqslant}$ from §4.5 give rise to the decomposition of algebraic groups

$$M_{\mathfrak{q}} \simeq M_{\mathfrak{q}}^{\geqslant} \times M_{\mathfrak{q}}^{\leqslant}. \tag{7.1}$$

This is not a direct product decomposition of Poisson algebraic groups (because $Z_{\mathfrak{q}} \simeq Z_{\mathfrak{q}}^{\geqslant} \otimes Z_{\mathfrak{q}}^{\leqslant}$ is a tensor product decomposition of commutative but not Poisson algebras). However, the canonical projections $M_{\mathfrak{q}} \twoheadrightarrow M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}} \twoheadrightarrow M_{\mathfrak{q}}^{\leqslant}$ are homomorphisms of Poisson algebraic groups because $Z_{\mathfrak{q}}^{\geqslant}$ and $Z_{\mathfrak{q}}^{\leqslant}$ are Poisson-Hopf subalgebras of $Z_{\mathfrak{q}}$.

Denote by $\mathfrak{m}_{\mathfrak{q}}$, $\mathfrak{m}_{\mathfrak{q}}^{\geqslant}$ and $\mathfrak{m}_{\mathfrak{q}}^{\leqslant}$ the tangent Lie bialgebras of $M_{\mathfrak{q}}$, $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$ (see §B.1 for background and notations). Eq. (7.1) gives rise to the direct sum decomposition of Lie algebras

$$\mathfrak{m}_{\mathfrak{q}} \simeq \mathfrak{m}_{\mathfrak{q}}^{\leqslant} \oplus \mathfrak{m}_{\mathfrak{q}}^{\geqslant}.$$

The Lie coalgebra structure on $\mathfrak{m}_{\mathfrak{q}}$, fully described below, has cross terms. The dual of the tangent Lie bialgebra $\mathfrak{m}_{\mathfrak{q}}^* = T_1^* M_{\mathfrak{q}}$ is computed as the linearization at the identity element 1 of $M_{\mathfrak{q}}$ of its Poisson structure by using (B.1). The maximal ideal $\mathfrak{M}_1$ of $\mathbb{C}[M_{\mathfrak{q}}] \simeq Z_{\mathfrak{q}}$ of functions vanishing at 1 coincides with the augmentation ideal of $Z_{\mathfrak{q}}$. In the proofs below we will use the identification $T_1^* M \simeq \mathfrak{M}_1/\mathfrak{M}_1^2$ where the differential $d_1(g)$ of a function $g \in \mathbb{C}[M_{\mathfrak{q}}]$ at $1 \in M_{\mathfrak{q}}$ is sent to the class of $g - g(1)$ in $\mathfrak{M}_1/\mathfrak{M}_1^2$ for $g \in \mathbb{C}[M_{\mathfrak{q}}]$. The Lie algebra $\mathfrak{m}_{\mathfrak{q}}^*$ has the $\mathbb{C}$-basis:

$$\big\{ d_1(e_\beta^{N_\beta}), d_1(f_\beta^{N_\beta}), d_1(K_\gamma^{N_\gamma}), d_1(L_\gamma^{N_\gamma}) : \beta \in \mathfrak{O}_+^{\mathfrak{q}}, \underline{\gamma} \in \Pi^{\mathfrak{q}} \big\}. \tag{7.2}$$

By Proposition 6.3, the subspaces

$$(\mathfrak{m}_{\mathfrak{q}}^{+})^* := \oplus_{\beta \in \mathfrak{O}_+^{\mathfrak{q}}} \mathbb{C}\, d_1(e_\beta^{N_\beta}) \quad \text{and} \quad (\mathfrak{m}_{\mathfrak{q}}^{-})^* := \oplus_{\beta \in \mathfrak{O}_+^{\mathfrak{q}}} \mathbb{C}\, d_1(f_\beta^{N_\beta})$$

are Lie subalgebras of $\mathfrak{m}_{\mathfrak{q}}^*$. The dual Lie bialgebras $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^* := (\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^* := (\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ are canonically identified with the Lie sub-bialgebras of $\mathfrak{m}_{\mathfrak{q}}^*$

$$(\mathfrak{m}_{\mathfrak{q}}^{+})^* \oplus \big(\oplus_{\underline{\gamma} \in \Pi^{\mathfrak{q}}} d_1(K_\gamma^{N_\gamma}) \big) \quad \text{and} \quad (\mathfrak{m}_{\mathfrak{q}}^{-})^* \oplus \big(\oplus_{\underline{\gamma} \in \Pi^{\mathfrak{q}}} d_1(L_\gamma^{N_\gamma}) \big).$$

Recall the notation from §5.4.

It follows from the triangular decomposition (3.15) of the semisimple Lie algebra $\mathfrak{g}_{\mathfrak{q}}$ associated to the large quantum group $U_{\mathfrak{q}}$ that the set of simple root of $\mathfrak{g}_{\mathfrak{q}}$ can be identified with $\Pi^{\mathfrak{q}}$. Denote the entries of the Cartan matrix of $\mathfrak{g}_{\mathfrak{q}}$ by

$$c_{\underline{\beta}\,\underline{\gamma}}, \qquad\qquad \underline{\beta}, \underline{\gamma} \in \Pi^{\mathfrak{q}}.$$

Throughout the section we will assume the identification $\mathfrak{n}_{\mathfrak{q}^{(-1)}}^{-} \simeq \mathfrak{n}_{\mathfrak{q}}^{+}$ from (5.7), so $\mathfrak{g}_{\mathfrak{q}} = \mathfrak{n}_{\mathfrak{q}}^{+} \oplus \mathfrak{h}_{\mathfrak{q}} \oplus \mathfrak{n}_{\mathfrak{q}}^{-}$ will be identified with $\mathfrak{n}_{\mathfrak{q}^{(-1)}}^{-} \oplus \mathfrak{h}_{\mathfrak{q}} \oplus \mathfrak{n}_{\mathfrak{q}}^{-}$. By the definitions of $\mathfrak{n}_{\mathfrak{q}^{(-1)}}^{-}$ and $\mathfrak{n}_{\mathfrak{q}}^{-}$, $\mathfrak{g}_{\mathfrak{q}}$ has a set of Chevalley generators

$$\{x_\beta, y_\beta, h_\beta : \underline{\beta} \in \Pi^{\mathfrak{q}}\}$$

such that $x_\beta \in \mathbb{C}^\times \iota^*(\theta_\beta)$ and $y_\beta \in \mathbb{C}^\times \iota^*(\eta_\beta)$, respectively. In this way the root lattice of $\mathfrak{g}_{\mathfrak{q}}$ is identified with $\mathcal{Q}_{\mathfrak{q}}$ by setting $\deg x_\beta = -\deg y_\beta = N_\beta \beta$, $\deg h_\beta = 0$ for $\underline{\beta} \in \Pi^{\mathfrak{q}}$.

Proposition 7.1. *We have a $\mathcal{Q}_{\mathfrak{q}}$-graded Lie algebra isomorphism $(\mathfrak{m}_{\mathfrak{q}}^{\pm})^* \simeq \mathfrak{n}_{\mathfrak{q}}^{\pm}$ given by*

$$d_1(e_\beta^{N_\beta}) \mapsto s_\beta \iota^*(\theta_\beta^{(N_\beta)}), \qquad \textit{respectively} \qquad d_1(f_\beta^{N_\beta}) \mapsto -s_\beta \iota^*(\eta_\beta^{(N_\beta)})$$

for all $\beta \in \mathfrak{O}_+^{\mathfrak{q}}$, where $s_\beta := \dfrac{\wp_{\beta\beta}^{\mathfrak{q}}(\xi)}{(1-q_{\beta\beta})^{N_\beta}}$. In the plus case we use the identification (5.7).

Proof. First we prove the minus case. Let $\beta, \gamma \in \mathfrak{O}_+^{\mathfrak{q}}$. Since $\overline{F_\beta^{N_\beta}} = f_\beta^{N_\beta}, \overline{F_\gamma^{N_\gamma}} = f_\gamma^{N_\gamma} \in Z_{\mathfrak{q}}$ and the subalgebra $Z_{\mathfrak{q}}$ is closed under the Poisson bracket $\{\cdot,\cdot\}$ by Proposition 6.3, using Proposition 5.4 we obtain

$$[F_\beta^{N_\beta}, F_\gamma^{N_\gamma}] \equiv \sum_{\delta \in \mathfrak{O}_+^{\mathfrak{q}}} (\nu - \xi) a_{\beta\gamma}^{\delta}(\nu) F_\delta^{N_\delta} + (\nu-\xi) g_{\beta\gamma} \mod (\nu-\xi)^2 U_{\mathbf{q},\mathsf{A}}^{-}, \tag{7.3}$$

where $a_{\beta\gamma}^{\delta}(\nu) \in \mathsf{A}$ and $g_{\beta\gamma}$ is a non-commutative polynomial in $\{F_\delta^{N_\delta} : \delta \in \mathfrak{O}_+^{\mathfrak{q}}\}$ involving monomials of degree ≥ 2. Since $U_{\mathbf{q}}$ is $\mathbb{Z}^{\mathbb{I}}$-graded, the sum in the right-hand side has at most one non-zero term, when $N_\beta \beta + N_\gamma \gamma = N_\delta \delta$ for some $\delta \in \mathfrak{O}_+^{\mathfrak{q}}$. Therefore

$$\begin{aligned}[d_1(f_\beta^{N_\beta}), d_1(f_\gamma^{N_\gamma})] &= d_1\big(\{f_\beta^{N_\beta}, f_\gamma^{N_\gamma}\}\big) = d_1\left(\overline{\frac{[F_\beta^{N_\beta}, F_\gamma^{N_\gamma}]}{\nu-\xi}}\right) \\ &= \sum_{\delta \in \mathfrak{O}_+^{\mathfrak{q}}} a_{\beta\gamma}^{\delta}(\xi) d_1(f_\delta^{N_\delta}) + d_1(g_{\beta\gamma}) = \sum_{\delta \in \mathfrak{O}_+^{\mathfrak{q}}} a_{\beta\gamma}^{\delta}(\xi) d_1(f_\delta^{N_\delta}),\end{aligned} \tag{7.4}$$

because $g_{\beta\gamma} \in \mathfrak{M}_1{}^2$. From (7.3) and since $U_{\mathbf{q},\mathsf{A}}^{-}$ is $\mathbb{N}_0$-graded connected, we see that

$$[F_\beta^{(N_\beta)}, F_\gamma^{(N_\gamma)}] \equiv \sum_{\delta \in \mathfrak{O}_+^{\mathfrak{q}}} a_{\beta\gamma}^{\delta}(\nu) \frac{(\nu-\xi)(N_\delta)^!_{q_{\delta\delta}}}{(N_\beta)^!_{q_{\beta\beta}} (N_\gamma)^!_{q_{\gamma\gamma}}} F_\delta^{(N_\delta)} \mod (\nu-\xi) U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,-}.$$

It follows from (6.5) that

$$\frac{\overline{(N_\beta)^!_{q_{\beta\beta}}}}{\nu - \xi} = \frac{\wp_{\beta\beta}^{\mathbf{q}}(\xi)}{(1-q_{\beta\beta})^{N_\beta}} = s_\beta.$$

Hence in $U_{\mathbf{q},\mathsf{A}}^{\mathrm{res}\,-}/(\nu-\xi)$ we have

$$[s_\beta \overline{F_\beta^{(N_\beta)}}, s_\gamma \overline{F_\gamma^{(N_\gamma)}}] = \sum_{\delta \in \mathfrak{O}_+^{\mathfrak{q}}} a_{\beta\gamma}^{\delta}(\xi) s_\delta \overline{F_\delta^{(N_\delta)}}. \tag{7.5}$$

The statement of the lemma follows from this identity, (7.4) and Proposition 5.9. The plus case is proved analogously, using Remark 5.8 and that $\mathfrak{q}_{\beta\delta} = 1$ for all $\beta, \gamma \in \underline{\mathfrak{O}}_+^{\mathfrak{q}}$. □

The last part of the proof gives the following fact about the structure of Lusztig algebras which is of independent interest:

Corollary 7.2. *The braided Hopf algebra projection $\iota^* : \mathcal{L}_{\mathfrak{q}} \twoheadrightarrow U(\mathfrak{n}_{\mathfrak{q}}^-)$ (recall (3.18)) has an algebra section $U(\mathfrak{n}_{\mathfrak{q}}^-) \to \mathcal{L}_{\mathfrak{q}}$ given by*

$$\iota^*(\eta_\beta^{(N_\beta)}) \mapsto \eta_\beta^{(N_\beta)}, \qquad \beta \in \mathfrak{O}_+^{\mathfrak{q}}.$$

7.2. **The dual tangent Lie bialgebra of $M_{\mathfrak{q}}$.**

Lemma 7.3. *The following hold in the Lie algebra $\mathfrak{m}_{\mathfrak{q}}^*$:*

$$[d_1(e_\beta^{N_\beta}), d_1(f_\gamma^{N_\gamma})] = -\delta_{\beta\gamma} \frac{\wp_{\beta\beta}^{\mathbf{q}}(\xi)}{(q_{\beta\beta}-1)^{N_\beta}} (d_1(K_\beta^{N_\beta}) + d_1(L_\beta^{N_\beta})), \qquad \underline{\beta}, \underline{\gamma} \in \Pi^{\mathfrak{q}},$$

and

$$\begin{aligned}[d_1(K_\beta^{N_\beta}), d_1(e_\gamma^{N_\gamma})] &= -\wp_{\beta\gamma}^{\mathbf{q}}(\xi) d_1(e_\gamma^{N_\gamma}), \qquad [d_1(K_\beta^{N_\beta}), d_1(f_\gamma^{N_\gamma})] = \wp_{\beta\gamma}^{\mathbf{q}}(\xi) d_1(f_\gamma^{N_\gamma}), \\ [d_1(L_\beta^{N_\beta}), d_1(e_\gamma^{N_\gamma})] &= -\wp_{\gamma\beta}^{\mathbf{q}}(\xi) d_1(e_\gamma^{N_\gamma}), \qquad [d_1(L_\beta^{N_\beta}), d_1(f_\gamma^{N_\gamma})] = \wp_{\gamma\beta}^{\mathbf{q}}(\xi) d_1(f_\gamma^{N_\gamma})\end{aligned}$$

for all $\beta, \gamma \in \mathfrak{O}_+^{\mathfrak{q}}$.

Proof. The case of $\underline{\beta} \neq \underline{\gamma} \in \varPi^{\mathfrak{q}}$ of the first identity follows from the fact that (7.2) is a basis of the Lie algebra $\mathfrak{m}_{\mathfrak{q}}^*$ and that the latter is $\mathcal{Q}_{\mathfrak{q}}$-graded. The case $\underline{\beta} = \underline{\gamma} \in \varPi^{\mathfrak{q}}$ is a consequence of (6.6) since $d_1(L_\beta^{N_\beta}) = -d_1(L_\beta^{-N_\beta})$, which in turn follows since the value of $L_\beta^{N_\beta}$ at the identity of $M_{\mathfrak{q}}$ equals 1. The other four identities follow from (4.4)–(4.5). $\square$

Since the polynomials $\nu^n - a$ are separable over $\mathbb{C}$ for $a \neq 0$, we infer from (6.1) that

$$\wp^{\mathfrak{q}}_{\beta\beta}(\xi) \neq 0 \qquad \text{for all } \beta \in \mathfrak{O}_+^{\mathfrak{q}}.$$

Theorem 7.4. (a) *The Cartan matrix of the semisimple Lie algebra* $\mathfrak{g}_{\mathfrak{q}}$ *is given by*

$$c_{\underline{\beta}\,\underline{\gamma}} = \frac{\wp^{\mathfrak{q}}_{\beta\gamma}(\xi) + \wp^{\mathfrak{q}}_{\gamma\beta}(\xi)}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)}, \qquad \underline{\beta}, \underline{\gamma} \in \varPi^{\mathfrak{q}}.$$

(b) *There is a ($\mathcal{Q}_{\mathfrak{q}}$-graded) Lie algebra isomorphism* $\mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}} \simeq \mathfrak{m}_{\mathfrak{q}}^*$ *such that*

$$x_\beta \mapsto d_1(f_\beta^{N_\beta}), \;\; y_\beta \mapsto \frac{(q_\beta - 1)^{N_\beta}}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)^2} d_1(e_\beta^{N_\beta}), \;\; h_\beta \mapsto \frac{1}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)}(d_1(K_\beta^{N_\beta}) + d_1(L_\beta^{N_\beta}))$$

for $\underline{\beta} \in \varPi^{\mathfrak{q}}$ *and* $\mathfrak{h}_{\mathfrak{q}}$ *maps to the subspace*

$$\Big\{ \sum_{\underline{\beta} \in \varPi^{\mathfrak{q}}} a_\beta d_1(K_\beta^{N_\beta}) + b_\beta d_1(L_\beta^{N_\beta}) : \sum_{\underline{\beta} \in \varPi^{\mathfrak{q}}} \wp^{\mathfrak{q}}_{\beta\gamma}(\xi) a_\beta + \wp^{\mathfrak{q}}_{\gamma\beta}(\xi) b_\beta = 0, \forall \underline{\gamma} \in \varPi^{\mathfrak{q}} \Big\}$$

of the abelian Lie algebra $\oplus_{\underline{\beta} \in \varPi^{\mathfrak{q}}} (\mathbb{C} d_1(K_\beta^{N_\beta}) + \mathbb{C} d_1(L_\beta^{N_\beta}))$.

Proof. (a) For $\underline{\beta} \in \varPi^{\mathfrak{q}}$, define the following elements of $\mathfrak{m}_{\mathfrak{q}}^*$:

$$\widetilde{x}_\beta := d_1(f_\gamma^{N_\gamma}), \;\; \widetilde{y}_\beta := \frac{(q_\beta - 1)^{N_\beta}}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)^2} d_1(e_\beta^{N_\beta}), \;\; \widetilde{h}_\beta := \frac{1}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)}(d_1(K_\beta^{N_\beta}) + d_1(L_\beta^{N_\beta}))$$

and the Lie subalgebra $\mathfrak{g}_{\mathfrak{q}}(\beta) := \mathbb{C}\widetilde{x}_\beta \oplus \mathbb{C}\widetilde{h}_\beta \oplus \mathbb{C}\widetilde{y}_\beta$. Lemma 7.3 implies that $[\widetilde{h}_\beta, \widetilde{x}_\beta] = 2\widetilde{x}_\beta$, $[\widetilde{h}_\beta, \widetilde{y}_\beta] = -2\widetilde{y}_\beta$, $[\widetilde{x}_\beta, \widetilde{y}_\beta] = \widetilde{h}_\beta$, so $\mathfrak{g}_{\mathfrak{q}}(\beta) \simeq \mathfrak{sl}_2$.

Now take $\underline{\beta} \neq \underline{\gamma} \in \varPi^{\mathfrak{q}}$ and consider $\mathfrak{g}_{\mathfrak{q}}$ as a $\mathfrak{g}_{\mathfrak{q}}(\beta)$-module under the adjoint action. It follows from Lemma 7.3 that

$$[\widetilde{x}_\beta, \widetilde{y}_\gamma] = 0 \quad \text{and} \quad [\widetilde{h}_\beta, \widetilde{y}_\gamma] = -\frac{\wp^{\mathfrak{q}}_{\beta\gamma}(\xi) + \wp^{\mathfrak{q}}_{\gamma\beta}(\xi)}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)} \widetilde{x}_\gamma,$$

so $\widetilde{y}_\gamma$ is a highest weight vector for $\mathfrak{g}_{\mathfrak{q}}(\beta) \simeq \mathfrak{sl}_2$ of weight $-\frac{\wp^{\mathfrak{q}}_{\beta\gamma}(\xi) + \wp^{\mathfrak{q}}_{\gamma\beta}(\xi)}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)} \omega$ where ω denotes the fundamental weight of $\mathfrak{sl}_2$. The isomorphism of Proposition 7.1 and the Serre relations in $\mathfrak{n}_{\mathfrak{q}}^-$ imply that

$$\operatorname{ad}_{\widetilde{y}_\beta}^{-c_{\beta\gamma}+1}(\widetilde{y}_\gamma) = 0 \quad \text{and} \quad \operatorname{ad}_{\widetilde{y}_\beta}^{j}(\widetilde{y}_\gamma) \neq 0 \;\text{ for }\; j \leqslant -c_{\beta\gamma}.$$

Hence, $\operatorname{ad}_{\widetilde{y}_\beta}^{-c_{\beta\gamma}}(\widetilde{y}_\gamma)$ is the lowest weight vector of the (irreducible) $\mathfrak{g}_{\mathfrak{q}}(\beta)$-module generated by $\widetilde{y}_\gamma$, which forces

$$c_{\underline{\beta}\,\underline{\gamma}} = \frac{\wp^{\mathfrak{q}}_{\beta\gamma}(\xi) + \wp^{\mathfrak{q}}_{\gamma\beta}(\xi)}{\wp^{\mathfrak{q}}_{\beta\beta}(\xi)}.$$

This proves part (a). It also proves that the assignment $x_\beta \mapsto \widetilde{x}_\beta$, $y_\beta \mapsto \widetilde{y}_\beta$, $h_\beta \mapsto \widetilde{h}_\beta$ for $\underline{\beta} \in \varPi^{\mathfrak{q}}$ defines a $\mathcal{Q}_{\mathfrak{q}}$-graded Lie algebra homomorphism $\eta : \mathfrak{g}_{\mathfrak{q}} \to \mathfrak{m}_{\mathfrak{q}}^*$ which is an embedding by Proposition 7.1 and the linear independence of $\{d_1(K_\beta^{N_\beta}), d_1(L_\beta^{N_\beta}) : \beta \in \mathfrak{O}_+^{\mathfrak{q}}\}$. Here we use the canonical isomorphism $\mathfrak{n}_{\mathfrak{q}}^{\pm} \to \mathfrak{n}_{\mathfrak{q}}^{\mp}$ obtained by restricting the Chevalley involution of $\mathfrak{g}_{\mathfrak{q}}^{\pm}$. Denote

$$(\mathfrak{m}_{\mathfrak{q}}^0)^* := \oplus_{\underline{\beta} \in \varPi^{\mathfrak{q}}} \Big(\mathbb{C} d_1(K_\beta^{N_\beta}) \oplus \mathbb{C} d_1(L_\beta^{N_\beta}) \Big).$$

Let $(\mathfrak{m}_{\mathfrak{q}}^0)'$ be the intersection of the kernels of the functionals $\{l_\beta : \underline{\beta} \in \varPi^{\mathfrak{q}}\}$ on $(\mathfrak{m}_{\mathfrak{q}}^0)^*$ given by

$$l_\beta(d_1(K_\gamma^{N_\gamma})) := \wp^{\mathbf{q}}_{\gamma\beta}(\xi), \qquad\qquad l_\beta(d_1(L_\gamma^{N_\gamma})) := \wp^{\mathbf{q}}_{\beta\gamma}(\xi).$$

Proposition 7.1 and Lemma 7.3 imply that $[(\mathfrak{m}_{\mathfrak{q}}^0)', \operatorname{Im} \eta] = 0$. Since the number of the above functionals equals $|\mathfrak{O}_+^{\mathfrak{q}}| = \dim \mathfrak{h}_{\mathfrak{q}}$, we have $\dim(\mathfrak{m}_{\mathfrak{q}}^0)' \geqslant \dim \mathfrak{h}_{\mathfrak{q}}$. It follows from part (a) that $\dim(\mathfrak{m}_{\mathfrak{q}}^0)' \leqslant \dim \mathfrak{h}_{\mathfrak{q}}$, Hence, $\dim \mathfrak{h}_{\mathfrak{q}} = \dim(\mathfrak{m}_{\mathfrak{q}}^0)'$ and taking a linear isomorphism $\mathfrak{h}_{\mathfrak{q}} \simeq (\mathfrak{m}_{\mathfrak{q}}^0)'$ extends η to the needed Lie algebra isomorphism for part (b). □

Let $(\cdot, \cdot)$ be the invariant symmetric bilinear form on $\mathfrak{g}_{\mathfrak{q}}$ for which the induced form on the dual of the Cartan subalgebra of $\mathfrak{g}_{\mathfrak{q}}$ satisfies $(\underline{\beta}, \underline{\beta}) = 2$ for short roots $\underline{\beta}$. As it is common, we will identify $\mathfrak{h}_{\mathfrak{q}}$ with $\mathfrak{h}_{\mathfrak{q}}^*$ via this form. The scalar

$$\kappa_\beta := 2 \wp^{\mathbf{q}}_{\beta\beta}(\xi) (\underline{\beta}, \underline{\beta})^{-1}$$

only depends on the simple factor of $\mathfrak{g}_{\mathfrak{q}}$ of which $\underline{\beta}$ is a root, because by Theorem 7.4(a),

$$c_{\beta\gamma} = \frac{\wp^{\mathbf{q}}_{\beta\gamma}(\xi) + \wp^{\mathbf{q}}_{\gamma\beta}(\xi)}{\wp^{\mathbf{q}}_{\beta\beta}(\xi)} = \frac{2(\underline{\beta}, \underline{\gamma})}{(\underline{\beta}, \underline{\beta})}. \tag{7.6}$$

Proposition A.3 (i) tells us that each large quantum group $U_{\mathfrak{q}}$ is realized as a specilization of an integral form of a one-parameter quantum group $U_{\mathbf{q}}$ in infinitely many different ways parametrized by integers $t_{ij} \in \mathbb{Z}$ for $i < j \in \mathbb{I}$. Furthermore, by part (ii) of that proposition, for a generic choice of the parameters $t_{ij} \in \mathbb{Z}$, $i < j \in \mathbb{I}$, the matrix with entries $\wp^{\mathbf{q}}_{\beta\gamma}(\xi)$ for $\underline{\beta}, \underline{\gamma} \in \varPi^{\mathfrak{q}}$ is non-degenerate. In the remaining part of the paper we will assume the following:

Non-degeneracy Assumption 7.5. The specialization parameters $t_{ij} \in \mathbb{Z}$, $i < j \in \mathbb{I}$ in Proposition A.3 are chosen in such a way that the matrix $\mathscr{P}^{\mathbf{q}}$ in (6.2) is non-degenerate.

Remark 7.6. In what follows we will identify the Lie algebras

$$\mathfrak{m}_{\mathfrak{q}}^* \simeq \mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}} \tag{7.7}$$

via the isomorphism from Theorem 7.4. In particular, $x_\beta, y_\beta, h_\beta$ for $\underline{\beta} \in \varPi^{\mathfrak{q}}$ will be viewed as elements of $\mathfrak{m}_{\mathfrak{q}}^*$. We also fix the identification of abelian Lie algebras

$$\Big\{ \sum_{\underline{\beta} \in \varPi^{\mathfrak{q}}} a_\beta d_1(K_\beta^{N_\beta}) + b_\beta d_1(L_\beta^{N_\beta}) : \sum_{\underline{\beta} \in \varPi^{\mathfrak{q}}} \wp^{\mathbf{q}}_{\beta\gamma}(\xi) a_\beta + \wp^{\mathbf{q}}_{\gamma\beta}(\xi) b_\beta = 0, \forall \underline{\gamma} \in \varPi^{\mathfrak{q}} \Big\} \simeq \mathfrak{h}_{\mathfrak{q}} \tag{7.8}$$

for Theorem 7.4(b) by sending $\sum_{\underline{\beta} \in \varPi^{\mathfrak{q}}} a_\beta d_1(K_\beta^{N_\beta}) + b_\beta d_1(L_\beta^{N_\beta}) \mapsto \sum_{\underline{\beta} \in \varPi^{\mathfrak{q}}} b_\beta \kappa_\beta \underline{\beta}$, using the identification of $\mathfrak{h}_{\mathfrak{q}}$ with $\mathfrak{h}_{\mathfrak{q}}^*$ via the form $(.,.)$. Since both Lie algebras in (7.8) have the same dimensions, we only need to show that this map is injective. An element in the kernel has $b_\beta = 0$ for $\beta \in \varPi^{\mathfrak{q}}$ and thus, $\sum \wp^{\mathbf{q}}_{\beta\gamma}(\xi) a_\beta = 0$. The Non-degeneracy Assumption 7.5 implies that $a_\beta = 0$ for $\beta \in \varPi^{\mathfrak{q}}$.

Let $\mathfrak{b}^{\pm}_{\mathfrak{q}}$ be the Borel subalgebras of $\mathfrak{g}_{\mathfrak{q}}$ with respect to these Chevalley generators. Then

$$(\mathfrak{m}^{\geqslant}_{\mathfrak{q}})^* \subset \mathfrak{b}^{-}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}} \quad \text{and} \quad (\mathfrak{m}^{\leqslant}_{\mathfrak{q}})^* \subset \mathfrak{b}^{+}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}. \tag{7.9}$$

Using the Non-degeneracy Assumption 7.5 one more time, we obtain that the projection into the first component $\mathfrak{m}^*_{\mathfrak{q}} \simeq \mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}} \to \mathfrak{g}_{\mathfrak{q}}$ restricts to the Lie algebra isomorphisms

$$(\mathfrak{m}^{\geqslant}_{\mathfrak{q}})^* \simeq \mathfrak{b}^{-}_{\mathfrak{q}} \quad \text{and} \quad (\mathfrak{m}^{\leqslant}_{\mathfrak{q}})^* \simeq \mathfrak{b}^{+}_{\mathfrak{q}}. \tag{7.10}$$

We next describe the embeddings (7.9). Denote the linear maps $\mathscr{P}, \mathscr{P}^{\mathrm{T}} \in \operatorname{End}(\mathfrak{h}_{\mathfrak{q}})$:

$$\mathscr{P}(\underline{\beta}) = \sum_{\underline{\gamma}\in\Pi^{\mathfrak{q}}} \wp^{\mathbf{q}}_{\beta\gamma}(\xi)\underline{\gamma}, \qquad \mathscr{P}^{\mathrm{T}}(\underline{\beta}) = \sum_{\underline{\gamma}\in\Pi^{\mathfrak{q}}} \wp^{\mathbf{q}}_{\gamma\beta}(\xi)\underline{\gamma}. \tag{7.11}$$

Because of the Non-degeneracy Assumption 7.5 both endomorphisms are invertible.

Denote by $(\!(\cdot,\cdot)\!)$ the invariant symmetric bilinear form on $\mathfrak{g}_{\mathfrak{q}}$ which is a rescaling of $(\cdot,\cdot)$ on each simple factor of $\mathfrak{g}_{\mathfrak{q}}$ by κ_β^{-1}. It satisfies

$$\begin{aligned}
&(\!(d_1(K_\beta^{N_\beta}) + d_1(L_\beta^{N_\beta}), d_1(K_\gamma^{N_\gamma}) + d_1(L_\gamma^{N_\gamma}))\!) = \wp^{\mathbf{q}}_{\beta\beta}(\xi)\wp^{\mathbf{q}}_{\gamma\gamma}(\xi)(\!(h_\beta, h_\gamma)\!) \\
&= \wp^{\mathbf{q}}_{\beta\beta}(\xi)\wp^{\mathbf{q}}_{\gamma\gamma}(\xi)\kappa_\gamma^{-1}\frac{2c_{\beta\gamma}}{(\underline{\gamma},\underline{\gamma})} = \wp^{\mathbf{q}}_{\beta\gamma}(\xi) + \wp^{\mathbf{q}}_{\gamma\beta}(\xi).\cdot
\end{aligned}$$

This implies that the form $(\!(\cdot,\cdot)\!)$ has a unique extension to an invariant symmetric bilinear form on $\mathfrak{m}^*_{\mathfrak{q}}$ such that

$$(\!(d_1(K_\beta^{N_\beta}), d_1(L_\gamma^{N_\gamma}))\!) = \wp^{\mathbf{q}}_{\beta\gamma}(\xi), \tag{7.12}$$

$$(\!(d_1(K_\beta^{N_\beta}), d_1(L_\beta^{N_\beta}))\!) = (\!(d_1(K_\gamma^{N_\gamma}), d_1(L_\gamma^{N_\gamma}))\!) = 0 \tag{7.13}$$

for $\underline{\beta}, \underline{\gamma} \in \Pi^{\mathfrak{q}}$. The Non-degeneracy Assumption 7.5 implies that the bilinear form $(\!(\cdot,\cdot)\!)$ on $\mathfrak{m}^*_{\mathfrak{q}}$ is non-degenerate.

One easily verifies that the orthogonal complement in $\mathfrak{m}^*_{\mathfrak{q}}$ of $\mathfrak{g}_{\mathfrak{q}}$ equals $\mathfrak{h}_{\mathfrak{q}}$.

Proposition 7.7. *For all large quantum groups $U_{\mathfrak{q}}$ satisfying the Non-degeneracy Assumption 7.5, the subalgebras $(\mathfrak{m}^{\geqslant}_{\mathfrak{q}})^* \subset \mathfrak{b}^{-}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$ and $(\mathfrak{m}^{\leqslant}_{\mathfrak{q}})^* \subset \mathfrak{b}^{+}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$ are given by*

$$(\mathfrak{m}^{\geqslant}_{\mathfrak{q}})^* = \{(y+h, -h) : y \in \mathfrak{n}^{-}_{\mathfrak{q}}, h \in \mathfrak{h}_{\mathfrak{q}}\}, \quad (\mathfrak{m}^{\leqslant}_{\mathfrak{q}})^* = \{(x+h, \mathscr{P}^{-1}\mathscr{P}^{\mathrm{T}}(h)) : x \in \mathfrak{n}^{+}_{\mathfrak{q}}, h \in \mathfrak{h}_{\mathfrak{q}}\}.$$

Proof. Denote the first (abelian) Lie algebra in (7.8) by $\mathfrak{h}^{(2)}_{\mathfrak{q}}$. Fix

$$h := \sum_{\underline{\beta}\in\Pi^{\mathfrak{q}}} c_\beta(d_1(K_\beta^{N_\beta}) + d_1(L_\beta^{N_\beta})), \quad h_1 := \sum_{\underline{\beta}\in\Pi^{\mathfrak{q}}} a_\beta d_1(K_\beta^{N_\beta}), \qquad h_2 := \sum_{\underline{\beta}\in\Pi^{\mathfrak{q}}} b_\beta d_1(L_\beta^{N_\beta}).$$

If $h_1 + h_2 \in \mathfrak{h}^{(2)}_{\mathfrak{q}}$, then $l_\gamma(h_1) = -l_\gamma(h_2)$ for all $\underline{\gamma} \in \Pi^{\mathfrak{q}}$, which is equivalent to

$$\mathscr{P}\Big(\sum_{\underline{\beta}\in\Pi^{\mathfrak{q}}} a_\beta\underline{\beta}\Big) = -\mathscr{P}^{\mathrm{T}}\Big(\sum_{\underline{\beta}\in\Pi^{\mathfrak{q}}} b_\beta\underline{\beta}\Big). \tag{7.14}$$

By Theorem 7.4(b), in the identification (7.7), $d_1(K_\beta^{N_\beta}) + d_1(L_\beta^{N_\beta})$ corresponds to $\kappa_\beta\underline{\beta}$. Hence, the first statement of the proposition is equivalent to proving that for all h, h_1, h_2 as above, if $h_1 + h_2 \in \mathfrak{h}^{(2)}_{\mathfrak{q}}$ and $h + h_1 + h_2 \in (\mathfrak{m}^{\geqslant}_{\mathfrak{q}})^*$, then $c_\beta = -b_\beta$ for $\underline{\beta} \in \Pi^{\mathfrak{q}}$. From the condition $h_1 + h_2 \in \mathfrak{h}^{(2)}_{\mathfrak{q}}$ we obtain $(\!(d_1(L_\gamma^{N_\gamma}), h + h_2)\!) = 0$ for all $\underline{\gamma} \in \Pi^{\mathfrak{q}}$. Thus

$$\sum_{\underline{\beta}\in\Pi^{\mathfrak{q}}} \wp^{\mathbf{q}}_{\beta\gamma}(\xi)(c_\beta + b_\beta) = 0, \qquad \forall \underline{\gamma} \in \Pi^{\mathfrak{q}}.$$

Now the first statement follows from the Assumption 7.5. The second follows from the first by interchanging the roles of $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ and applying (7.14). $\square$

We next describe the Lie coalgebra structure on $\mathfrak{m}_{\mathfrak{q}}^*$ and the corresponding Manin triple; see §B.1 for background.

Theorem 7.8. *For every choice of the specialization parameters $t_{ij} \in \mathbb{Z}$ satisfying the Non-degeneracy Assumption 7.5 the following hold:*

(a) *The Lie coalgebra structure of the Lie bialgebra $\mathfrak{m}_{\mathfrak{q}}^*$ is given by*

$$\delta(x_\beta) = d_1(L_\beta^{N_\beta}) \wedge x_\beta, \quad \delta(y_\beta) = d_1(K_\beta^{N_\beta}) \wedge y_\beta, \quad \delta(d_1(L_\beta^{N_\beta})) = \delta(d_1(L_\beta^{N_\beta})) = 0$$

for all $\underline{\beta} \in \Pi^{\mathfrak{q}}$.

(b) *With respect to the bilinear form $(\!(\cdot,\cdot)\!)$, $(\mathfrak{m}_{\mathfrak{q}}^*, (\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*, (\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*)$ is a Manin triple.*

(c) *The Lie coalgebra structures of $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ satisfy*

$$(\!(\delta(y), x_1 \otimes x_2)\!) = -(\!(y, [x_1, x_2])\!), \qquad (\!(\delta(x), y_1 \otimes y_2)\!) = (\!(x, [y_1, y_2])\!) \tag{7.15}$$

for all $x, x_1, x_2 \in (\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ and $y, y_1, y_2 \in (\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$.

Remark 7.9. (a) Part (a) of the theorem uniquely determines the Lie coalgebra structures of $\mathfrak{m}_{\mathfrak{q}}^*$, $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$, since the set $\{x_\beta, y_\beta, d_1(L_\beta^{N_\beta}), d_1(L_\beta^{N_\beta}) : \underline{\beta} \in \Pi^{\mathfrak{q}}\}$ and its appropriate subsets generate $\mathfrak{m}_{\mathfrak{q}}^*$, $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$.

(b) By part (c) of the theorem, the Lie coalgebra structures of $\mathfrak{m}_{\mathfrak{q}}^*$, $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$, are precisely the ones that are associated to a Manin triple as in Remark B.1(c). In particular, we have the isomorphism of Lie bialgebras

$$\mathfrak{m}_{\mathfrak{q}}^* \simeq D((\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*), \qquad (\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^* \simeq (((\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*)^*)^{\mathrm{op}} \simeq (\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^{\mathrm{op}}. \tag{7.16}$$

(c) The Lie bialgebra structures on the reductive Lie algebras $\mathfrak{m}_{\mathfrak{q}}^* \simeq \mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$ from part (a) of the theorem correspond to standard Belavin–Drinfeld triples (containing the empty subsets of the Dynkin graph) and arbitrary choice of the continuous parameters in their classification [BD].

Proof of Theorem 7.8. Part (a) follows from Lemma B.2 and the identities

$$\Delta(e_\beta^{N_\beta}) = K_\beta^{N_\beta} \otimes e_\beta^{N_\beta} + e_\beta^{N_\beta} \otimes 1, \qquad \Delta(f_\beta^{N_\beta}) = 1 \otimes f_\beta^{N_\beta} + f_\beta^{N_\beta} \otimes L_\beta^{-N_\beta}. \tag{7.17}$$

for $\underline{\beta} \in \Pi^{\mathfrak{q}}$ and the fact that $K_\beta^{N_\beta}$ and $L_\beta^{N_\beta}$ are group-like elements.

(b) The subalgebras $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ are orthogonal to their nilradicals because of the embeddings (7.9). This, combined with (7.13), implies that they are isotropic subalgebras of $\mathfrak{m}_{\mathfrak{q}}^*$ with respect to the form $(\!(\cdot,\cdot)\!)$. The direct sum decomposition $\mathfrak{m}_{\mathfrak{q}} \simeq \mathfrak{m}_{\mathfrak{q}}^{\leqslant} \oplus \mathfrak{m}_{\mathfrak{q}}^{\geqslant}$ yields the desired result.

(c) Part (a) of the theorem and the isomorphism in Theorem 7.4(b) imply at once the validity of the identities (7.15) for $y = d_1(e_\beta^{N_\beta})$, $y = d_1(K_\beta^{N_\beta})$, $x = d_1(f_\beta^{N_\beta})$, $x = d_1(L_\beta^{N_\beta})$, $\underline{\beta} \in \Pi^{\mathfrak{q}}$ and all possible choices of x_1, x_2, y_2, y_2. The general case follows by induction on root height when x, y are chosen to be root vectors by using the invariance of the bilinear form $(\!(\cdot,\cdot)\!)$. $\square$

7.3. **The Poisson algebraic groups $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$.** Combining the isomorphisms (7.10) and (7.16), we get the Lie algebra isomorphisms

$$\mathfrak{m}_{\mathfrak{q}}^{\geqslant} \simeq ((\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*)_{\mathrm{op}} \simeq (\mathfrak{b}_{\mathfrak{q}}^{+})_{\mathrm{op}} \simeq \mathfrak{b}_{\mathfrak{q}}^{+} \quad \text{and} \quad \mathfrak{m}_{\mathfrak{q}}^{\leqslant} \simeq (\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^* \simeq \mathfrak{b}_{\mathfrak{q}}^{-}, \tag{7.18}$$

where $(.)_{\mathrm{op}}$ stands for the opposite Lie algebra structure and $(\mathfrak{b}_{\mathfrak{q}}^{+})_{\mathrm{op}} \simeq \mathfrak{b}_{\mathfrak{q}}^{+}$ is the standard Lie algebra isomorphism $x \mapsto -x$. The proof of Proposition 7.7 shows that the corresponding pull back maps on the level of duals send

$$\underline{\beta} \mapsto -d_1(K_\beta^{N_\beta}), \qquad\qquad \underline{\beta} \mapsto d_1(L_\beta^{N_\beta}), \qquad\qquad \forall \underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}. \tag{7.19}$$

The scalars κ_β do not appear here because the form $(\!(\cdot,\cdot)\!)$ is a rescaling of the form $(\cdot,\cdot)$ on each simple factor of $\mathfrak{g}_{\mathfrak{q}}$ by κ_β^{-1}.

Denote by $G_{\mathfrak{q}}$ the adjoint semisimple algebraic group with Lie algebra $\mathfrak{g}_{\mathfrak{q}}$ and by $B_{\mathfrak{q}}^{\pm}$ its Borel subgroups corresponding to $\mathfrak{b}_{\mathfrak{q}}^{\pm}$. Let $T_{\mathfrak{q}} := B_{\mathfrak{q}}^{+} \cap B_{\mathfrak{q}}^{-}$ be the corresponding maximal torus of $G_{\mathfrak{q}}$. Denote by $N_{\mathfrak{q}}^{\pm}$ the unipotent radicals of $B_{\mathfrak{q}}^{\pm}$.

The groups of group-like elements of $Z_{\mathfrak{q}}^{\geqslant}$ and $Z_{\mathfrak{q}}^{\leqslant}$ are the free abelian groups on $K_\beta^{\pm N_\beta}$, $\underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}$ and $L_\beta^{\pm N_\beta}$, $\underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}$, respectively.

Theorem 7.10. *For every choice of the specialization parameters $t_{ij} \in \mathbb{Z}$ satisfying the Non-degeneracy Assumption 7.5, the Lie algebra isomorphisms* (7.18) *integrate to isomorphisms of algebraic groups.*

$$\tau_+ : M_{\mathfrak{q}}^{\geqslant} \xrightarrow{\simeq} B_{\mathfrak{q}}^{+} \quad \textit{and} \quad \tau_- : M_{\mathfrak{q}}^{\leqslant} \xrightarrow{\simeq} B_{\mathfrak{q}}^{-}.$$

Theorem 7.10 describes explicitly the algebraic groups $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$. As an algebraic group, $M_{\mathfrak{q}} \simeq B_{\mathfrak{q}}^{+} \times B_{\mathfrak{q}}^{-}$. The Poisson structures on $M_{\mathfrak{q}}^{\geqslant}$, $M_{\mathfrak{q}}^{\leqslant}$ and $M_{\mathfrak{q}}$ are the unique Poisson algebraic group structures that integrate the Lie bialgebras $\mathfrak{m}_{\mathfrak{q}}^{\geqslant}$, $\mathfrak{m}_{\mathfrak{q}}^{\leqslant}$ and $\mathfrak{m}_{\mathfrak{q}}$, whose dual Lie bialgebras are described in Theorem 7.8.

Proof. We prove the first statement, the second being analogous. Since $G_{\mathfrak{q}}$ is of adjoint type, the Borel subgroup $B_{\mathfrak{q}}^{+}$ is canonically identified with the identity component of $\mathrm{Aut}(\mathfrak{b}_{\mathfrak{q}}^{+})$. The adjoint action of $M_{\mathfrak{q}}^{\geqslant}$ on $\mathfrak{m}_{\mathfrak{q}}^{\geqslant} \simeq \mathfrak{b}_{\mathfrak{q}}^{+}$ induces a surjective homomorphism $\tau_+ : M_{\mathfrak{q}}^{\geqslant} \twoheadrightarrow B_{\mathfrak{q}}^{+}$. The latter restricts to an isomorphism $\tau_+ : N(M_{\mathfrak{q}}^{\geqslant}) \xrightarrow{\simeq} N_{\mathfrak{q}}^{+}$, where $N(M_{\mathfrak{q}}^{\geqslant})$ is the unipotent radical of $M_{\mathfrak{q}}^{\geqslant}$. The homomorphism τ_+ also restricts to a surjective homomorphism

$$\tau_+ : T(M_{\mathfrak{q}}^{\geqslant}) \twoheadrightarrow T_{\mathfrak{q}}, \tag{7.20}$$

where $T(M_{\mathfrak{q}}^{\geqslant})$ is a maximal torus of $M_{\mathfrak{q}}^{\geqslant}$. The tori $T(M_{\mathfrak{q}}^{\geqslant})$ and $T_{\mathfrak{q}}$ are connected because $M_{\mathfrak{q}}^{\geqslant}$ and $B_{\mathfrak{q}}^{+}$ are connected algebraic groups. Invoking the Levi decompositions of $M_{\mathfrak{q}}^{\geqslant}$ and $B_{\mathfrak{q}}^{+}$, to show that τ_+ is an isomorphism, it is sufficient to show that the restriction (7.20) is an isomorphism. However,

$$\mathbb{C}[T(M_{\mathfrak{q}}^{\geqslant})] \simeq \mathbb{C}[M_{\mathfrak{q}}^{\geqslant}/N(M_{\mathfrak{q}}^{\geqslant})] \simeq \mathbb{C}[G(\mathbb{C}[M_{\mathfrak{q}}^{\geqslant}])], \quad \mathbb{C}[T_{\mathfrak{q}}] \simeq \mathbb{C}[B_{\mathfrak{q}}^{+}/N_{\mathfrak{q}}^{+}] \simeq \mathbb{C}[G(\mathbb{C}[B_{\mathfrak{q}}^{+}])],$$

where $G(H)$ denotes the group of group-like elements of a Hopf algebra H.

The group of group-like elements of $\mathbb{C}[M_{\mathfrak{q}}^{\geqslant}] \simeq Z_{\mathfrak{q}}^{\geqslant}$ is the free abelian group with generators $K_\beta^{N_\beta}$, $\underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}$ and the group of group-like elements of $B_{\mathfrak{q}}^{+}$ is canonically identified with the roots lattice $\mathbb{Z}\mathit{\Pi}^{\mathfrak{q}}$ of $G_{\mathfrak{q}}$. The differentials at the identity element of the two generating sets are respectively $d_1(K_\beta^{N_\beta})$ and $\underline{\beta}$, where $\underline{\beta} \in \mathit{\Pi}^{\mathfrak{q}}$. Eq. (7.19) implies that $\tau_+^* : G(\mathbb{C}[B_{\mathfrak{q}}^+]) \to G(\mathbb{C}[M_{\mathfrak{q}}^{\geqslant}])$ is an isomorphism. Hence, $\tau_+^* : \mathbb{C}[T_{\mathfrak{q}}] \to \mathbb{C}[T(M_{\mathfrak{q}}^{\geqslant}))]$ is an isomorphism and same holds for (7.20). This completes the proof of the theorem. $\square$

Example 7.11. Let $\mathfrak{q}$ be of type $\mathtt{wk}(4)$ and fix $N = \operatorname{ord} q$, $M = \operatorname{ord}(-q)$, see §A.3. Let $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$. Then $N_{\alpha_1} = N_{\alpha_2} = N$, $N_{\alpha_4} = N_\gamma = M$,

$$\underline{\mathfrak{O}}_+^{\mathfrak{q}} = \{\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_1 + \underline{\alpha}_2, \underline{\alpha}_4, \underline{\gamma}, \underline{\alpha}_4 + \underline{\gamma}\}.$$

As shown previously, $\Delta(e_\beta^{N_\beta}) = e_\beta^{N_\beta} \otimes 1 + K_\beta^{N_\beta} \otimes e_\beta^{N_\beta}$ for $\underline{\beta} \in \varPi^{\mathfrak{q}} = \{\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_4, \underline{\gamma}\}$. We can check that $e_{\alpha_1+\alpha_2} = [e_1, e_2]_c$, $e_{\alpha_4+\gamma} = [e_\gamma, e_4]_c$ and

$$\Delta(e^N_{\alpha_1+\alpha_2}) = e^N_{\alpha_1+\alpha_2} \otimes 1 + (q-1)^N e^N_{\alpha_1} K^N_{\alpha_2} \otimes e^N_{\alpha_2} + K^N_{\alpha_1} K^N_{\alpha_2} \otimes e^N_{\alpha_1+\alpha_2},$$
$$\Delta(e^M_{\alpha_4+\gamma}) = e^M_{\alpha_4+\gamma} \otimes 1 + (q+1)^M e^M_\gamma K^M_{\alpha_4} \otimes e^M_{\alpha_4} + K^M_{\alpha_4} K^M_\gamma \otimes e^M_{\alpha_4+\gamma}.$$

We now construct an explicit isomorphism between $Z_{\mathfrak{q}}^{\geqslant}$ and the algebra of functions over the Borel subgroup of $\mathrm{PSL}_3(\mathbb{C}) \times \mathrm{PSL}_3(\mathbb{C})$. Consider the Levi decomposition $\widetilde{B}_3 \simeq N_3 \rtimes \widetilde{T}_3$ of the Borel subgroup of $\mathrm{SL}_3(\mathbb{C})$, where

$$\widetilde{T}_3 = \{\operatorname{diag}(a_1, a_2, a_3) : a_i \in \mathbb{C}^\times, a_1a_2a_3 = 1\}, \quad N_3 = \left\{ \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{12} \\ 0 & 0 & 1 \end{pmatrix} : t_{ij} \in \mathbb{C} \right\}.$$

The coproducts of these coordinate functions are given by $\Delta(a_i) = a_i \otimes a_i$ and

$$\Delta(x_{12}) = x_{12} \otimes 1 + a_1 a_2^{-1} \otimes x_{12}, \qquad \Delta(x_{23}) = x_{23} \otimes 1 + a_2 a_3^{-1} \otimes x_{23},$$
$$\Delta(x_{13}) = x_{13} \otimes 1 + x_{12} a_2 a_3^{-1} \otimes x_{23} + a_1 a_3^{-1} \otimes x_{13}.$$

Denote $\mathbb{Z}_3 = \langle(\zeta, \zeta, \zeta)\rangle$, where ζ is a primitive 3rd root of unity. The Borel subgroup B_3 of $\mathrm{SL}_3(\mathbb{C})$ has Levi decomposition $B_3 \simeq N_3 \rtimes T_3$ where $T_3 = \widetilde{T}/\mathbb{Z}_3$, so

$$\mathbb{C}[B_3] = \mathbb{C}[N_3] \otimes \mathbb{C}[\widetilde{T}_3]^{\mathbb{Z}_3} = \mathbb{C}[x_{12}, x_{23}, x_{13}, a_{12}^{\pm 1}, a_{23}^{\pm 1}],$$

where $a_{12} := a_1 a_2^{-1}$ and $a_{23} = a_2 a_3^{-1}$. The coproducts of the coordinate functions on B_3 are given by $\Delta(a_{ii+1}) = a_{ii+1} \otimes a_{ii+1}$ and

$$\Delta(x_{12}) = x_{12} \otimes 1 + a_{12} \otimes x_{12}, \qquad \Delta(x_{23}) = x_{23} \otimes 1 + a_{23} \otimes x_{23},$$
$$\Delta(x_{13}) = x_{13} \otimes 1 + x_{12} a_{23} \otimes x_{23} + a_{12} a_{23} \otimes x_{13}.$$

The Borel subgroup of $\mathrm{PSL}_3(\mathbb{C}) \times \mathrm{PSL}_3(\mathbb{C})$ is isomorphic to $B_3 \times B_3$. We denote the coordinate functions a_{ii+1} and x_{ij} on the first and second copy of B_3 by superscripts 1 and 2. Now, clearly the map $\tau_+ : Z_{\mathfrak{q}}^{\geqslant} \to \mathbb{C}[B_3 \times B_3]$ given by

$$K^N_{\alpha_1} \mapsto a^1_{12}, \qquad K^N_{\alpha_2} \mapsto a^1_{23}, \qquad K^N_{\alpha_3} \mapsto a^2_{12}, \qquad K^M_\gamma \mapsto a^2_{23}$$

and

$$e^N_{\alpha_1} \mapsto x^1_{12}, \qquad e^N_{\alpha_2} \mapsto x^1_{23}, \qquad \frac{e^N_{\alpha_1+\alpha_2}}{(q-1)^N} \mapsto x^1_{13},$$
$$e^M_{\alpha_4} \mapsto x^2_{12}, \qquad e^M_\gamma \mapsto x^2_{23}, \qquad \frac{e^M_{\alpha_4+\gamma}}{(q+1)^M} \mapsto x^2_{13}$$

is a Hopf algebra isomorphism.

8. Poisson geometry and representations

In this section we describe the symplectic foliations and the torus orbits of symplectic leaves of the Poisson algebraic groups $M_{\mathfrak{q}}$, $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$, and the Poisson homogeneous spaces $M_{\mathfrak{q}}^{+}$ and $M_{\mathfrak{q}}^{-}$. Previous work in this direction dealt with the so called standard Poisson structures on simple algebraic groups (and their Borel subgroups) [HL], the dual Poisson algebraic groups [DKP] and the related flag varieties [GY]. The Poisson structures in Remark 7.9 are not of standard type in general and the results in this section can not be deduced from [HL, DKP, GY]. For $z \in M_{\mathfrak{q}}$, respectively $M_{\mathfrak{q}}^{\geqslant}$, $M_{\mathfrak{q}}^{\leqslant}$, $M_{\mathfrak{q}}^{+}$, $M_{\mathfrak{q}}^{-}$, let $\mathcal{H}_z$, respectively $\mathcal{H}_z^{\geqslant}$, $\mathcal{H}_z^{\leqslant}$, $\mathcal{H}_z^{+}$, $\mathcal{H}_z^{-}$ be the algebra defined in Theorem A (c), respectively (1.2). The Poisson geometric results described above provide information on the irreducible representations of the large quantum groups $U_{\mathfrak{q}}$ by reduction to the sheaf of algebras $\mathcal{H}_z$, $z \in M_{\mathfrak{q}}$. Analogous results hold for $U_{\mathfrak{q}}^{\geqslant}$, $U_{\mathfrak{q}}^{\leqslant}$, and $U_{\mathfrak{q}}^{\pm}$.

8.1. **Representations of the large quantum groups and symplectic foliations.** The Manin triple described in Theorem 7.8 and the identification $\mathfrak{m}_{\mathfrak{q}}^* \simeq \mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$ equip $\mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$ with a quasitriangular Lie bialgebra structure, which turns $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$ into a Poisson algebraic group. The Poisson structure on $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$ equals $L_g(r) - R_q(r)$ for $g \in G_{\mathfrak{q}} \times T_{\mathfrak{q}}$, where $r \in \wedge^2(\mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}})$ is the r-matrix for the Lie bialgebra structure on $\mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$, and $L_g(-)$ and $R_g(-)$ refer to the left and right-invariant bivector fields on $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$.

Let $\widetilde{M}_{\mathfrak{q}}^{\geqslant}$ and $\widetilde{M}_{\mathfrak{q}}^{\leqslant}$ be the connected Lie subgroups of $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$ with Lie algebras $(\mathfrak{m}_{\mathfrak{q}}^{\leqslant})^*$ and $(\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^*$. Proposition 7.7 implies that $\widetilde{M}_{\mathfrak{q}}^{\leqslant}$ is an algebraic subgroup, while $\widetilde{M}_{\mathfrak{q}}^{\geqslant}$ is not necessarily a closed Lie subgroup. The projection onto the first component $\pi : G_{\mathfrak{q}} \times T_{\mathfrak{q}} \twoheadrightarrow G_{\mathfrak{q}}$ gives the surjective Lie group homomorphisms

$$\pi_+ : \widetilde{M}_{\mathfrak{q}}^{\geqslant} \twoheadrightarrow B_{\mathfrak{q}}^{+}, \qquad \pi_- : \widetilde{M}_{\mathfrak{q}}^{\leqslant} \twoheadrightarrow B_{\mathfrak{q}}^{-}.$$

Since $G_{\mathfrak{q}}$ is of adjoint type, the kernel of the exponential map $\exp : \mathfrak{h}_{\mathfrak{q}} \twoheadrightarrow T_{\mathfrak{q}}$ equals $2\pi i P_{\mathfrak{q}}^{\vee}$, where $P_{\mathfrak{q}}^{\vee}$ denotes the coweight lattice of $\mathfrak{g}_{\mathfrak{q}}$. Denote the subgroup

$$C_{\mathfrak{q}} := \exp\big(2\pi i \mathscr{P}^{-1}\mathscr{P}^{\mathrm{T}}(P_{\mathfrak{q}}^{\vee})\big) \subset T_{\mathfrak{q}}, \tag{8.1}$$

cf. (7.11). Proposition 7.7 and the solvability of $\widetilde{M}_{\mathfrak{q}}^{\geqslant}$ and $\widetilde{M}_{\mathfrak{q}}^{\leqslant}$ give that

$$\begin{aligned} \widetilde{M}_{\mathfrak{q}}^{\geqslant} &= (N_{\mathfrak{q}}^{+} \times \{1\}) \times \{\exp(h, \mathscr{P}^{-1}\mathscr{P}^{\mathrm{T}}(h)) : h \in \mathfrak{h}_{\mathfrak{q}}\}, \\ \widetilde{M}_{\mathfrak{q}}^{\leqslant} &= (N_{\mathfrak{q}}^{-} \times \{1\}) \times \{(t, t^{-1}) : t \in T_{\mathfrak{q}}\}, \end{aligned}$$

from which one obtains that

$$\operatorname{Ker} \pi_+ = \{1\}, \qquad \operatorname{Ker} \pi_- = \{1\} \times C_{\mathfrak{q}}.$$

Composing $\pi_{\pm}$ with the isomorphisms from Theorem 7.10 leads to the isomorphisms

$$\tau_+^{-1}\pi_+ : \widetilde{M}_{\mathfrak{q}}^{\geqslant} \xrightarrow{\simeq} M_{\mathfrak{q}}^{\geqslant}, \qquad \tau_-^{-1}\pi_- : \widetilde{M}_{\mathfrak{q}}^{\leqslant}/\operatorname{Ker} \pi_- \xrightarrow{\simeq} M_{\mathfrak{q}}^{\leqslant}.$$

Their inverses give the canonical embeddings

$$j_+ : M_{\mathfrak{q}}^{\geqslant} \hookrightarrow G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big), \qquad j_- : M_{\mathfrak{q}}^{\leqslant} \hookrightarrow G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big). \tag{8.2}$$

Here we use that $G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big) \simeq \big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big)/\operatorname{Ker} \pi_-$ and $\widetilde{M}_{\mathfrak{q}}^{\leqslant} \cap \operatorname{Ker} \pi_- = \{1\}$.

Remark 8.1. If the matrix $\mathbf{q}$ is symmetric, then so is the matrix $\mathscr{P}^{\mathbf{q}}$. This implies that $\mathscr{P} = \mathscr{P}^{\mathrm{T}}$ and that the group $C_{\mathbf{q}}$ is trivial. Then the continuous parameter accompanying the BD triple is trivial and the Poisson structure is the standard one.

Theorem 8.2. *Let $U_{\mathfrak{q}}$ be a large quantum group. For every choice of the specialization parameters $t_{ij} \in \mathbb{Z}$ satisfying the Non-degeneracy Assumption 7.5 the following hold:*

(a) *The symplectic leaves of the Poisson algebraic group $M_{\mathfrak{q}} \simeq M_{\mathfrak{q}}^{\geqslant} \times M_{\mathfrak{q}}^{\leqslant}$ are the inverse images $j^{-1}(\mathcal{O} \times t)$ under the map*

$$j : M_{\mathfrak{q}} \to G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big), \quad j(m_+, m_-) := j_+(m_+)^{-1} j_-(m_-), \qquad m_+ \in M_{\mathfrak{q}}^{\geqslant}, m_- \in M_{\mathfrak{q}}^{\leqslant},$$

where $\mathcal{O}$ is a conjugacy class of $G_{\mathfrak{q}}$ and $t \in T_{\mathfrak{q}}/C_{\mathfrak{q}}$. The dimension of the symplectic leaf $j^{-1}(\mathcal{O} \times \{t\})$ equals $\dim \mathcal{O}$.

(b) *If $j(z)$ and $j(z')$ are in the same conjugacy class of $G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big)$, then there is an algebra isomorphism*

$$\mathcal{H}_z \simeq \mathcal{H}_{z'}.$$

Note that, since $T_{\mathfrak{q}}/C_{\mathfrak{q}}$ is abelian, each conjugacy class of $G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big)$ has the form $\mathcal{O} \times \{t\}$, where $\mathcal{O}$ is a conjugacy class of $G_{\mathfrak{q}}$ and $t \in \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big)$.

Proof. (a) By [RSTS], since the Poisson algebraic group $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$ is quasitriangular, its double Poisson algebraic group is canonically isomorphic to

$$D(G_{\mathfrak{q}} \times T_{\mathfrak{q}}) \simeq \big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big) \times \big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big).$$

Theorem 7.8(b) implies that the dual Poisson Lie group of $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$ is

$$\widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}} \hookrightarrow \big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big) \times \big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big)$$

with the opposite Poisson structure to the restriction of the one of the double. Both $\widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}}$ and $M_{\mathfrak{q}} \simeq M_{\mathfrak{q}}^{\geqslant} \times M_{\mathfrak{q}}^{\leqslant}$ have the same tangent Lie bialgebra, hence the map

$$\tau := (\tau_+^{-1}\pi_+, \tau_-^{-1}\pi_-) : \widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}} \twoheadrightarrow M_{\mathfrak{q}}^{\geqslant} \times M_{\mathfrak{q}}^{\leqslant} \simeq M_{\mathfrak{q}}$$

is a Poisson covering map. By the Semenov-Tian-Shansky dressing method [STS], we get that the symplectic leaves of $\widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}}$ are the connected components of the intersections

$$\widetilde{M_{\mathfrak{q}}^{\leqslant}} \cap \big(\operatorname{diag}(G_{\mathfrak{q}} \times T_{\mathfrak{q}}) \cdot g \cdot \operatorname{diag}(G_{\mathfrak{q}} \times T_{\mathfrak{q}})\big),$$

where $\operatorname{diag}\big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big)$ denotes the diagonal of $(G_{\mathfrak{q}} \times T_{\mathfrak{q}})^{\times 2}$ and $g \in (G_{\mathfrak{q}} \times T_{\mathfrak{q}})^{\times 2}$. Now we apply [Y, Theorem 1.10] to obtain that each such intersection is a dense, open and connected subset of $\operatorname{diag}\big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big) \cdot g \cdot \operatorname{diag}\big(G_{\mathfrak{q}} \times T_{\mathfrak{q}}\big)$. Consider the map

$$\widetilde{j} : \widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}} \to G_{\mathfrak{q}} \times T_{\mathfrak{q}}, \quad \widetilde{j}(m_+, m_-) := m_+^{-1} m_-, \qquad m_+ \in \widetilde{M_{\mathfrak{q}}^{\geqslant}}, m_- \in \widetilde{M_{\mathfrak{q}}^{\leqslant}}.$$

By a direct argument we conclude that each symplectic leaf of $\widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}}$ is of the form

$$\mathcal{S}_{\mathcal{O}'} := \big(\widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}}\big) \cap \mathcal{O}',$$

where $\mathcal{O}'$ is a conjugacy class of $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$, and that

$$\dim \mathcal{S}_{\mathcal{O}'} = \dim \mathcal{O}'.$$

Since $\tau : \widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}} \twoheadrightarrow M_{\mathfrak{q}}$ is a covering of Poisson Lie groups, each symplectic leaf of $M_{\mathfrak{q}}$ is of the form $\tau(\mathcal{S}_{\mathcal{O}'})$. One easily verifies that the diagram

$$\begin{tikzcd}
\widetilde{M_{\mathfrak{q}}^{\geqslant}} \times \widetilde{M_{\mathfrak{q}}^{\leqslant}} \arrow[r, "\widetilde{j}"] \arrow[d, "\tau"'] & G_{\mathfrak{q}} \times T_{\mathfrak{q}} \arrow[d, "\psi"] \\
M_{\mathfrak{q}} \arrow[r, "j"] & G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big)
\end{tikzcd}$$

commutes, where $\psi : G_{\mathfrak{q}} \times T_{\mathfrak{q}} \twoheadrightarrow G_{\mathfrak{q}} \times \big(T_{\mathfrak{q}}/C_{\mathfrak{q}}\big)$ is the canonical projection. Clearly, $\psi(\mathcal{O}') = \mathcal{O} \times \{t\}$, where $\mathcal{O}$ is a conjugacy class of $G_{\mathfrak{q}}$ and $t \in T_{\mathfrak{q}}/C_{\mathfrak{q}}$. Therefore all symplectic leaves of $M_{\mathfrak{q}}$ are of the form $\tau(\mathcal{S}_{\mathcal{O}'}) = j^{-1}\psi(\mathcal{O}') = j^{-1}(\mathcal{O} \times \{t\})$ and $\dim j^{-1}(\mathcal{O} \times \{t\}) = \dim \mathcal{O}' = \dim \mathcal{O}$.

Part (b) follows from part (a), Theorem 6.2, and the theorem for isomorphisms of central quotients across symplectic leaves [BG, Theorem 4.1]. □

In regard to the irreducible representations of $U_{\mathfrak{q}}$ we wonder whether the De Concini–Kac–Procesi conjecture could be extended to the setting of Theorem 8.2, see [DKP].

Question 8.3. *Let $\mathcal{O}$ be a conjugacy class of $G_{\mathfrak{q}}$, $t \in T_{\mathfrak{q}}/C_{\mathfrak{q}}$ and $z \in j^{-1}(\mathcal{O} \times \{t\})$. Does $\ell^{\dim \mathcal{O}/2}$ divide the dimension of any irreducible representation of $\mathcal{H}_z$?*

8.2. **The torus orbits of symplectic leaves and the representations of the large quantum Borel algebras.** The algebras $U_{\mathfrak{q}}$, $U_{\mathfrak{q}}^{\geqslant}$, $U_{\mathfrak{q}}^{\leqslant}$ and $U_{\mathfrak{q}}^{\pm}$ are $\mathbb{Z}^{\mathbb{I}}$-graded with grading $\deg e_i = -\deg f_i = \alpha_i$, $\deg K_i = \deg L_i = 0$ for $i \in \mathbb{I}$. This leads to a canonical action of the torus $(\mathbb{C}^{\times})^{\mathbb{I}}$ on these algebras by algebra automorphisms, which preserves the central subalgebras $Z_{\mathfrak{q}}$, $Z_{\mathfrak{q}}^{\geqslant}$, $Z_{\mathfrak{q}}^{\leqslant}$ and $Z_{\mathfrak{q}}^{\pm}$.

By a direct comparison, one obtains that the $(\mathbb{C}^{\times})^{\mathbb{I}}$-action on $Z_{\mathfrak{q}}^{\geqslant}$ corresponds to the left action of $\tau_{+}^{-1}(T_{\mathfrak{q}})$ on $M_{\mathfrak{q}}^{\geqslant}$ in the sense that every automorphism from the first one corresponds to an automorphism from the second and vice versa. Similarly, the $(\mathbb{C}^{\times})^{\mathbb{I}}$-action on $Z_{\mathfrak{q}}^{\leqslant}$ corresponds to the left action of $\tau_{-}^{-1}(T_{\mathfrak{q}})$ on $M_{\mathfrak{q}}^{\leqslant}$. Theorem 8.2(a) implies that the induced action of $(\mathbb{C}^{\times})^{\mathbb{I}}$ on $M_{\mathfrak{q}}$ preserves the symplectic leaves of $M_{\mathfrak{q}}$. So, in regard to irreps of $U_{\mathfrak{q}}$, the $(\mathbb{C}^{\times})^{\mathbb{I}}$-automorphisms of $U_{\mathfrak{q}}$ do not provide any additional information to that in Theorem 8.2(a).

But for $U_{\mathfrak{q}}^{\geqslant}$ and $U_{\mathfrak{q}}^{\leqslant}$, we do obtain additional representation theoretic information from the $(\mathbb{C}^{\times})^{\mathbb{I}}$-action, as stated in next theorem. Let $W_{\mathfrak{q}}$ be the Weyl group of $G_{\mathfrak{q}}$.

Theorem 8.4. *For every choice of the specialization parameters $t_{ij} \in \mathbb{Z}$ satisfying the Non-degeneracy Assumption 7.5 the following hold:*

(a) *The Poisson structure on $M_{\mathfrak{q}}^{\geqslant}$ is invariant under the left and right actions of $\tau_{+}^{-1}(T_{\mathfrak{q}})$. The $\tau_{+}^{-1}(T_{\mathfrak{q}})$-orbits of symplectic leaves of $M_{\mathfrak{q}}^{\geqslant}$ are the double Bruhat cells*
$$\tau_{+}^{-1}(B_{\mathfrak{q}}^{+} \cap B_{\mathfrak{q}}^{-} w B_{\mathfrak{q}}^{-}), \qquad\qquad w \in W_{\mathfrak{q}}.$$
(b) *If $\tau_{+}(z)$ and $\tau_{+}(z')$ are in the same double Bruhat cell, then there is an algebra isomorphism*
$$\mathcal{H}_{z}^{\geqslant} \simeq \mathcal{H}_{z'}^{\geqslant}.$$

Proof. (a) For a Lie subalgebra of $\mathfrak{g}_{\mathfrak{q}} \oplus \mathfrak{h}_{\mathfrak{q}}$, denote by $N(-)$ its normalizer in $G_{\mathfrak{q}} \times T_{\mathfrak{q}}$. By [LY, Lemma 2.12], the left and right actions of $\widetilde{M}_{\geqslant} \cap N((\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^{*})$ on the Lie group $\widetilde{M}_{\geqslant}$ preserve its Poisson structure. By the definition of τ_{+}, these actions correspond to the left and right actions of $\tau_{+}^{-1}(T_{\mathfrak{q}})$ on $M_{\mathfrak{q}}^{\geqslant}$, so the latter preserve the Poisson structure on $M_{\mathfrak{q}}^{\geqslant}$, because $\widetilde{M}_{\mathfrak{q}}^{\geqslant} \twoheadrightarrow M_{\mathfrak{q}}^{\geqslant}$ is a Poisson map.

Applying [LY, Theorem 2.7 and Proposition 2.15] and the Bruhat decomposition of $G_{\mathfrak{q}}$, we obtain that the $\widetilde{M}_{\geqslant} \cap N((\mathfrak{m}_{\mathfrak{q}}^{\geqslant})^{*})$-orbits of symplectic leaves of $\widetilde{M}_{\geqslant}$ (with respect to either action) are the intersections
$$\widetilde{M}_{\geqslant} \cap \big((G_{\mathfrak{q}} \times T_{\mathfrak{q}}) w (G_{\mathfrak{q}} \times T_{\mathfrak{q}})\big)$$

for $w \in W_{\mathfrak{q}}$. Since $\widetilde{M_{\mathfrak{q}}^{\geqslant}} \twoheadrightarrow M_{\mathfrak{q}}^{\geqslant}$ is a Poisson covering map and $\tau_+ : M_{\mathfrak{q}}^{\geqslant} \xrightarrow{\simeq} B_{\mathfrak{q}}^+$ is an isomorphism (Theorem 7.10), the $\tau_+^{-1}(T_{\mathfrak{q}})$-orbits of symplectic leaves of $M_{\mathfrak{q}}^{\geqslant}$ (with respect to either action) are the double Bruhat cells $\tau_+^{-1}(B_{\mathfrak{q}}^+ \cap B_{\mathfrak{q}}^- w B_{\mathfrak{q}}^-)$ for $w \in W_{\mathfrak{q}}$.

Part (b) follows from part (a), the constructed Poisson orders in Theorem 6.2, [BG, Theorem 4.1] and the fact that the left action of $\tau_+^{-1}(T_{\mathfrak{q}})$ on $M_{\mathfrak{q}}^{\geqslant}$ comes from the $(\mathbb{C}^\times)^{\mathbb{I}}$-action on $U_{\mathfrak{q}}^{\geqslant}$ by algebra automorphisms. $\square$

Example 8.5. Let $\mathfrak{q}$ be of type $\mathtt{wk}(4)$. By Example 7.11, the corresponding algebraic group $G_{\mathfrak{q}}$ is isomorphic to $\mathrm{PSL}_3(\mathbb{C})\times\mathrm{PSL}_3(\mathbb{C})$ whose Weyl groups is $S_3\times S_3$. Theorem 8.4 implies that among the quotients $U_{\mathfrak{q}}^{\geqslant}/\mathfrak{M}_z^{\leqslant} U_{\mathfrak{q}}^{\geqslant}$ for z in the maximal spectrum of $Z_{\mathfrak{q}}^{\geqslant}$, there are at most $|S_3 \times S_3| = (3!)^2 = 36$ isomorphism classes of finite dimensional algebras.

Analogously to Theorem 8.4 one proves the following:

Proposition 8.6. *For every choice of the specialization parameters $t_{ij} \in \mathbb{Z}$ satisfying the Non-degeneracy Assumption 7.5 the following hold:*

(a) *The Poisson structure on $M_{\mathfrak{q}}^{\leqslant}$ is invariant under the left and right actions of $\tau_-^{-1}(T_{\mathfrak{q}})$. The $\tau_-^{-1}(T_{\mathfrak{q}})$-orbits of symplectic leaves of $M_{\mathfrak{q}}^{\leqslant}$ are the double Bruhat cells*
$$\tau_-^{-1}(B_{\mathfrak{q}}^- \cap B_{\mathfrak{q}}^+ w B_{\mathfrak{q}}^+), \qquad\qquad w \in W_{\mathfrak{q}}.$$
(b) *If $\tau_-(z)$ and $\tau_-(z')$ are in the same double Bruhat cell, then $\mathcal{H}_z^{\leqslant} \simeq \mathcal{H}_{z'}^{\leqslant}$ as algebras.*

8.3. **Poisson homogeneous spaces and irreps of large quantum unipotent algebras.** Since $Z_{\mathfrak{q}}^+$ is the algebra of coinvariants for the coaction of $Z_{\mathfrak{q}}^{0+}$ on $Z_{\mathfrak{q}}^{\geqslant}$ obtained by restricting the coaction of $U_{\mathfrak{q}}^{0+}$ on $Z_{\mathfrak{q}}^{\geqslant}$, and analogously for the negative part, we have isomorphisms of Poisson algebras

$$Z_{\mathfrak{q}}^+ \simeq \mathbb{C}[M_{\mathfrak{q}}^{\geqslant}/\tau_+^{-1}(T_{\mathfrak{q}})], \qquad\qquad Z_{\mathfrak{q}}^- \simeq \mathbb{C}[M_{\mathfrak{q}}^{\leqslant}/\tau_-^{-1}(T_{\mathfrak{q}})]. \tag{8.3}$$

As shown in the previous subsection, the left and right actions of $\tau_+^{-1}(T_{\mathfrak{q}})$ and $\tau_-^{-1}(T_{\mathfrak{q}})$ on the Poisson algebraic groups $M_{\mathfrak{q}}^{\geqslant}$ and $M_{\mathfrak{q}}^{\leqslant}$ preserve their Poisson structures. The right hand sides of the isomorphisms (8.3) involve the coordinate rings of the resulting Poisson homogeneous spaces $M_{\mathfrak{q}}^{\geqslant}/\tau_+^{-1}(T_{\mathfrak{q}})$ and $M_{\mathfrak{q}}^{\leqslant}/\tau_-^{-1}(T_{\mathfrak{q}})$ obtained by taking quotients with respect to the right actions. The Poisson structures on $M_{\mathfrak{q}}^{\geqslant}/\tau_+^{-1}(T_{\mathfrak{q}})$ and $M_{\mathfrak{q}}^{\leqslant}/\tau_-^{-1}(T_{\mathfrak{q}})$ are invariant under the induced left actions of $\tau_+^{-1}(T_{\mathfrak{q}})$ and $\tau_-^{-1}(T_{\mathfrak{q}})$. By Theorem 7.10, τ_+ restricts to the isomorphism of homogeneous spaces $\tau_+ : M_{\mathfrak{q}}^{\geqslant}/j_+^{-1}(T_{\mathfrak{q}}) \xrightarrow{\simeq} B_{\mathfrak{q}}^+/T_{\mathfrak{q}}$. Denote the canonical isomorphism
$$\upsilon : B_{\mathfrak{q}}^+/T_{\mathfrak{q}} \xrightarrow{\simeq} B_{\mathfrak{q}}^+ B_{\mathfrak{q}}^-/B_{\mathfrak{q}}^- \subset G_{\mathfrak{q}}/B_{\mathfrak{q}}^-.$$

Theorem 8.7. *For every choice of the specialization parameters $t_{ij} \in \mathbb{Z}$ satisfying the Non-degeneracy Assumption 7.5 the following hold:*

(a) *The $\tau_+^{-1}(T_{\mathfrak{q}})$-orbits of symplectic leaves of $M_{\mathfrak{q}}^{\geqslant}/\tau_+^{-1}(T_{\mathfrak{q}})$ are the open Richardson varieties*
$$\tau_+^{-1}\upsilon^{-1}\big((B_{\mathfrak{q}}^+ B_{\mathfrak{q}}^- \cap B_{\mathfrak{q}}^- w B_{\mathfrak{q}}^-)/B_{\mathfrak{q}}^-\big), \qquad\qquad w \in W_{\mathfrak{q}}.$$
(b) *If $\upsilon\tau_+(z)$ and $\upsilon\tau_+(z')$ are in the same open Richardson variety, then there is an isomorphism of algebras*
$$\mathcal{H}_z^+ \simeq \mathcal{H}_{z'}^+.$$

Proof. Part (a) is proved arguing as in the proof of Theorem 8.4(a). Then (b) is a consequence of (a), Theorem 6.2 and [BG, Theorem 4.1]. $\square$

An analogous result holds for the large quantum unipotent algebra $U^-_{\mathfrak{q}}$ and the torus orbits of symplectic leaves of the Poisson homogeneous space $M^{\leqslant}_{\mathfrak{q}}/\tau_-^{-1}(T_{\mathfrak{q}})$.

Appendix A. Families of finite-dimensional Nichols algebras

Let $\theta \in \mathbb{N}$, $\mathbb{I} = \mathbb{I}_\theta$. We fix a matrix $\mathfrak{q} = (q_{ij}) \in \mathbb{C}^{\mathbb{I}\times\mathbb{I}}$ such that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. To insure centrality of $Z_{\mathfrak{q}}$ we require

(a). *The matrix* $\mathfrak{q}$ *satisfies* (4.24), *i.e.* $q_{\alpha_i\beta}^{N_\beta} = 1$, *for all* $i \in \mathbb{I}$, $\underline{\beta} \in \varPi^{\mathfrak{q}}$.

Remark A.1. If the Dynkin diagram of $\mathfrak{q}'$ is as in Tables 1, 2 and 3, then there is $\mathfrak{q}$ with the same Dynkin diagram that satisfies (4.24); the proof is straightforward.

If $\mathfrak{q}$ satisfies (4.24), then any matrix in its Weyl-equivalence class also does. Let $\mathbb{C}[\nu^{\pm 1}]$ be the algebra of Laurent polynomials; its group of units is $\mathbb{C}[\nu^{\pm 1}]^\times = \mathbb{C}^\times \nu^{\mathbb{Z}}$. Let

$$\mathbf{q} = (\mathbf{q}_{ij}) \in \left(\mathbb{C}[\nu^{\pm 1}]^\times\right)^{\mathbb{I}\times\mathbb{I}} \tag{A.1}$$

For $x \in \mathbb{C}^\times$, we denote by $\mathbf{q}(x)$ the matrix obtained by the evaluation $\operatorname{ev} : \mathbb{C}[\nu^{\pm 1}] \to \mathbb{C}$, $\operatorname{ev}(\nu) = x$. We seek for matrices (A.1) with the following properties (b) and (d).

(b). *The Nichols algebra of the* $\mathbb{C}(\nu)$*-braided vector space of diagonal type with braiding matrix* (A.1) *has the same arithmetic root system as* $\mathfrak{q}$.

By inspection of the list in [H2]–see also the exposition in [AA]–we conclude that the only possible matrices (A.1) are those Weyl-equivalent to the ones with Dynkin diagrams as in Tables 1, 2 and 3 and that the following property holds.

(c). *There exists an open subset* $\emptyset \neq O \subseteq \mathbb{C}^\times$ *such that for any* $x \in O$, *the root systems and Weyl groupoids associated to* $\mathbf{q}$ *and* $\mathbf{q}(x)$ *are isomorphic. Also there exists* $\xi \in \mathbb{G}'_\infty \cap O$ *with* $N := \operatorname{ord}\xi \in [2, \infty)$ *such that* $\mathfrak{q} = \mathbf{q}(\xi)$.

Remark A.2. (i). The Dynkin diagrams of the matrices $\mathbf{q}$ and $\mathfrak{q}$ locally have the form $\overset{\mathbf{q}_{ii}}{\circ}\xrightarrow{\ \widetilde{\mathbf{q}}_{ij}\ }\overset{\mathbf{q}_{jj}}{\circ}$, respectively $\overset{q_{ii}}{\circ}\xrightarrow{\ \widetilde{q}_{ij}\ }\overset{q_{jj}}{\circ}$, where $\widetilde{\mathbf{q}}_{ij} = \mathbf{q}_{ij}\mathbf{q}_{ji}$, $\widetilde{q}_{ij} = q_{ij}q_{ji}$; i.e. the Dynkin diagram does not determine completely the braiding matrix. We deal with this as follows. Let $\mathfrak{p} = (p_{ij}) \in \mathbb{C}^{\mathbb{I}\times\mathbb{I}}$ with the same Dynkin diagram as $\mathfrak{q}$. Then there exists $\mathbf{p} \in \left(\mathbb{C}[\nu^{\pm 1}]^\times\right)^{\mathbb{I}\times\mathbb{I}}$ with the same Dynkin diagram as $\mathbf{q}$ such that $\mathfrak{p} = \mathbf{p}(\xi)$. For, take $\mathbf{p}_{ii} = \mathbf{q}_{ii}$ and $\mathbf{p}_{ij} \in \mathbb{C}[\nu^{\pm 1}]^\times$ such that $p_{ij} = \mathbf{p}_{ij}(\xi)$ for $i < j$; then $\mathbf{p}_{ji} = \widetilde{\mathbf{q}}_{ij}\mathbf{p}_{ij}^{-1}$.

(ii). Assume that $\mathbf{q}$ satisfies (b). Let $\mathbf{p}$ be another matrix with the same Dynkin diagram as (A.1). Then $\mathbf{q}_{ij} = \mathbf{p}_{ij} q^{h_{ij}N}$, $i < j$ for a unique family $(h_{ij})_{i<j\in\mathbb{I}}$ with $h_{ij} \in \mathbb{Z}$.

(d). $\mathscr{P}^{\mathbf{q}}$ *defined in* (6.2) *is invertible.*

Let $\mathcal{N}$ be the diagonal matrix with entries N_β, $\underline{\beta} \in \varPi^{\mathfrak{q}}$. The matrix $\mathscr{P}^{\mathbf{q}}$ is invertible if and only if the auxiliary matrix $\mathcal{T}^{\mathbf{q}}$ is so, where

$$\mathscr{P}^{\mathbf{q}} = -\xi^{-1}\mathcal{N}\mathcal{T}^{\mathbf{q}}\mathcal{N}.$$

Proposition A.3. *There exist matrices* $C = (c_{ij}) \in \mathbb{Z}^{\mathbb{I}\times\mathbb{I}}$ *and* $(\mathrm{p}_{ij}) \in (\mathbb{C}^\times)^{\mathbb{I}\times\mathbb{I}}$ *such that* C *is symmetric and:*

(i) *There are infinitely many matrices* $T = (t_{ij}) \in \mathbb{Z}^{\mathbb{I}\times\mathbb{I}}$ *fulfilling*

$$t_{ii} = c_{ii}, \qquad t_{ij} + t_{ji} = c_{ij} \qquad \textit{for all } i \neq j \in \mathbb{I} \tag{A.2}$$

such that the matrix $\mathbf{q} = (\mathbf{q}_{ij})$ *defined by*

$$\mathbf{q}_{ij} = \mathtt{p}_{ij}\nu^{t_{ij}}, \qquad \textit{for all } i, j \in \mathbb{I} \tag{A.3}$$

satisfies (b).

(ii) *Among those* T *in* (i)*, there infinitely many such that* $\mathbf{q}$ *satisfies (d).*

Proof. It suffices to fix one matrix for each Weyl-equivalence class, see Lemma 6.1. We check below (i) by case-by-case considerations computing also $\mathcal{T}^{\mathbf{q}}$ and proving that it is invertible for infinitely many T.

A.1. **Cartan type.** Let $\mathfrak{q}$ be in this class; then there is a Cartan matrix $A = (a_{ij})_{i,j\in\mathbb{I}}$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$. We fix $d_i \in \mathbb{I}_3$ such that $d_ia_{ij} = d_ja_{ji}$ for all $i, j \in \mathbb{I}$. The Lie algebra $\mathfrak{g}_{\mathfrak{q}}$ has the same type except when N is even and A is of type B_θ or C_θ, when they are interchanged. In this case $\mathit{\Pi}^{\mathfrak{q}} = \{N_i\alpha_i : i \in \mathbb{I}\}$, so (4.24) becomes:

$$q_{ij}^{N_j} = 1, \qquad \text{for all } i, j \in \mathbb{I}. \tag{A.4}$$

The matrix $\mathbf{q}$ we are looking for should also satisfy $\mathbf{q}_{ij}\mathbf{q}_{ji} = \mathbf{q}_{ii}^{a_{ij}}$ for all $i \neq j$. In all cases we take $\xi = q_{11}$ except for B_θ, where $\xi = q_{\theta\theta}$; see Table 1. Set $t_{ii} = d_i$ and $\mathbf{q}_{ii} = \nu^{t_{ii}}$. Thus $\mathbf{q}_{ii}(\xi) = q_{ii}$ for all $i \in \mathbb{I}$. Recall that

$$(\nu - \xi)\wp^{\mathbf{q}}_{\alpha_i\alpha_j}(\nu) = 1 - \mathbf{q}_{ij}^{N_iN_j}.$$

For instance $\wp^{\mathbf{q}}_{\alpha_i\alpha_i}(\nu) = \frac{1-\nu^{d_iN_i^2}}{\nu-\xi}$ hence

$$\wp^{\mathbf{q}}_{\alpha_i\alpha_i}(\xi) = -\xi^{-1}d_iN_i^2 = -\xi^{-1}t_{ii}N_i^2. \tag{A.5}$$

Let $i < j$. We see that there exists $d_j \in \mathbb{I}_3$ such that $N_j = N/d_j$. By (A.4), q_{ij} is a power of ξ^{d_j}; choose $t_{ij} \in d_j\mathbb{Z}$ such that $\mathbf{q}_{ij} = \nu^{t_{ij}}$ satisfies $\mathbf{q}_{ij}(\xi) = \xi^{t_{ij}} = q_{ij}$. Set $t_{ji} = d_ia_{ij} - t_{ij}$ and $\mathbf{q}_{ji} = \nu^{t_{ji}}$. We have defined T satisfying (A.2) and $\mathbf{q}$ turns out to be given by (A.3) with $\mathtt{p}_{ij} = 1$ for all i, j, i.e. (i) holds. Also for all $i \neq j$, $(\nu - \xi)\wp^{\mathbf{q}}_{\alpha_i\alpha_j}(\nu) = 1 - \nu^{t_{ij}N_iN_j}$ and

$$\wp^{\mathbf{q}}_{\alpha_i\alpha_j}(\xi) = -\xi^{-1}t_{ij}N_iN_j. \tag{A.6}$$

Therefore $\mathcal{T}^{\mathbf{q}} = T$. Observe that if $t_{ij} = 0$ for $i < j$, then $\det\mathcal{T}^{\mathbf{q}} \neq 0$. By a standard argument, (ii) holds.

A.2. **Super type.** Assume that the braiding matrix $\mathfrak{q}$ is of super type; see [AA] for details (see [AA] for details and below for $\mathbf{D}(2, 1; \alpha)$). Going over the list, we see that there exist

- $\xi \in \mathbb{C}^\times$, a root of 1 of order $N > 1$;
- a symmetric matrix $B = (b_{ij})_{i,j\in\mathbb{I}} \in \mathbb{Z}^{\mathbb{I}\times\mathbb{I}}$ with $b_{ij} = 1$ for at least one pair (i, j);
- a parity vector $\mathtt{p} = (\mathtt{p}_1, \dots, \mathtt{p}_\theta) \in \{\pm1\}^{\mathbb{I}}$ with $\mathtt{p}_i = -1$ when $b_{ii} = 0$; such that

$$q_{ij}q_{ji} = \xi^{b_{ij}}, \qquad i \neq j; \qquad q_{ii} = \mathtt{p}_i\xi^{b_{ii}}, \qquad i \in \mathbb{I}.$$

We describe in Table 2 matrices $\mathbf{q}$ of super type, one for each Weyl-equivalence class (here $\alpha_{(ij)} := \alpha_i + \dots + \alpha_j$ for $i < j$). Since the matrix $\mathfrak{q}$ has an analogous shape, we may assume that

- there exists $k \in \mathbb{I}$ such that $\{i \in \mathbb{I} : \mathtt{p}_i = -1\} = \{k\}$;
- there exists $h \in \mathbb{I}$, $h \neq k$, such that $\xi = q_{hh}$.

Therefore we have:

TABLE 1. Cartan type

Type	$\mathbf{q}$	N
A_θ	$\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{——}}\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{——}}\overset{\nu}{\circ}\cdots\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{——}}\overset{\nu}{\circ}$	
$B_\theta,\ \theta \geq 2$	$\overset{\nu^2}{\circ}\overset{\nu^{-2}}{\text{——}}\overset{\nu^2}{\circ}\cdots\overset{\nu^2}{\circ}\overset{\nu^{-2}}{\text{——}}\overset{\nu}{\circ}$	> 2
$C_\theta,\ \theta \geq 3$	$\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{——}}\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{——}}\overset{\nu}{\circ}\cdots\overset{\nu}{\circ}\overset{\nu^{-2}}{\text{——}}\overset{\nu^2}{\circ}$	> 2
$D_\theta,\ \theta \geq 4$	$\begin{array}{ccccccccc} &&&&&& \overset{\nu}{\circ} && \\ &&&&&& \vert\,\nu^{-1} && \\ \overset{\nu}{\circ} & \overset{\nu^{-1}}{\text{——}} & \overset{\nu}{\circ} & \cdots & \overset{\nu}{\circ} & \overset{\nu^{-1}}{\text{——}} & \overset{\nu}{\circ} & \overset{\nu^{-1}}{\text{——}} & \overset{\nu}{\circ} \end{array}$	
$B_\theta,\ \theta \in \mathbb{I}_{6,8}$	$\begin{array}{ccccccccc} &&&& \overset{\nu}{\circ} &&&& \\ &&&& \vert\,\nu^{-1} &&&& \\ \overset{\nu}{\circ} & \overset{\nu^{-1}}{\text{——}} & \overset{\nu}{\circ} & \cdots & \overset{\nu}{\circ} & \overset{\nu^{-1}}{\text{——}} & \overset{\nu}{\circ} & \overset{\nu^{-1}}{\text{——}} & \overset{\nu}{\circ} \end{array}$	
F_4	$\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{——}}\overset{\nu}{\circ}\overset{\nu^{-2}}{\text{——}}\overset{\nu^2}{\circ}\overset{\nu^{-2}}{\text{——}}\overset{\nu^2}{\circ}$	> 2
G_2	$\overset{\nu}{\circ}\overset{\nu^{-3}}{\text{——}}\overset{\nu^3}{\circ}$	> 3

- either $\Pi^{\mathfrak{q}} = \{N_i\alpha_i : i \in \mathbb{I}, i \neq k\}$ for type $\mathbf{A}(k-1|\theta-k)$ or else there exists a unique positive non-simple root β such that $\Pi^{\mathfrak{q}} = \{N_i\alpha_i : i \in \mathbb{I}, i \neq k\} \cup \{N_\beta\beta\}$;
- for $i \in \mathbb{I}$, $i \neq k$, we may (and do) choose $b_{ii} \in \{\pm1, \pm2, \pm3\}$. Then $N_i = LCD(b_{ii}, N)$; set $d_i = N/N_i$.

We start defining the matrix $\mathbf{q}$. First we take $t_{ii} = b_{ii}$ and $\mathbf{q}_{ii} = \mathtt{p}_i q^{b_{ii}}$ for all $i \in \mathbb{I}$.

Condition (4.24) says that $q_{ij}^{N_j} = 1$, for all $i \in \mathbb{I}$, $j \in \mathbb{I}\backslash\{k\}$. Let $i < j$ with $j \neq k$; choose $t_{ij} \in d_j\mathbb{Z}$ and set $t_{ji} = b_{ji} - t_{ij}$. Then $\mathbf{q}_{ij} = \nu^{t_{ij}}$ and $\mathbf{q}_{ji} = \nu^{t_{ji}}$ satisfy $\mathbf{q}_{ij}(\xi) = \xi^{t_{ij}} = q_{ij}$ and $\mathbf{q}_{ij}\mathbf{q}_{ji} = \nu^{b_{ij}}$.

Similarly, $q_{ki}^{N_i} = 1$ for $k > i$, so choose $t_{ik} \in d_i\mathbb{Z}$ and set $t_{ki} = b_{ki} - t_{ik}$, $\mathbf{q}_{ki} = \nu^{b_{ki}-t_{ik}}$ and $\mathbf{q}_{ik} = \nu^{t_{ik}}$ so that $\mathbf{q}_{ki}(\xi) = \xi^{t_{ki}} = q_{ki}$. We have defined T satisfying (A.2) and $\mathbf{q}$ turns out to be given by (A.3) with $\mathtt{p}_{ii} = \mathtt{p}_i$ and $\mathtt{p}_{ij} = 1$ for all $i \neq j$, i.e. (i) holds.

It remains to compute the matrix $\mathcal{T}^{\mathbf{q}}$. Arguing as in the Cartan case we see that

$$\wp^{\mathbf{q}}_{\alpha_i\alpha_i}(\xi) = -\xi^{-1}\mathtt{p}_i b_{ii} N_i^2, \qquad \wp^{\mathbf{q}}_{\alpha_i\alpha_j}(\xi) = -\xi^{-1} t_{ij} N_i N_j, \qquad i, j \in \mathbb{I}\backslash\{k\}.$$

Assume that there exists $\gamma \in \Pi^{\mathfrak{q}}\backslash\mathbb{I}$ (a non-simple Cartan root). Then there exist $\mathtt{p}_\gamma \in \{\pm1\}$ and $b_{\gamma\gamma}, b_{i\gamma} \in \mathbb{Z}$ such that

$$\mathbf{q}_{\gamma\gamma} = \mathtt{p}_\gamma q^{b_{\gamma\gamma}}, \qquad \mathbf{q}_{i\gamma}\mathbf{q}_{\gamma i} = \nu^{b_{i\gamma}}.$$

Extend (t_{ij}) to a bilinear form $t : \mathbb{Z}^{\mathbb{I}} \times \mathbb{Z}^{\mathbb{I}} \to \mathbb{Z}$. Then for $k \neq i \in \mathbb{I}$,

$$\wp^{\mathbf{q}}_{\gamma\gamma}(\xi) = -\xi^{-1}\mathtt{p}_\gamma b_{\gamma\gamma} N_\gamma^2, \quad \wp^{\mathbf{q}}_{\alpha_i\gamma}(\xi) = -\xi^{-1} t_{i\gamma} N_i N_\gamma, \quad (\nu - \xi)\wp^{\mathbf{q}}_{\gamma\alpha_i}(\nu) = -\xi^{-1} t_{\gamma i} N_i N_\gamma.$$

All in all, $\mathcal{T}^{\mathfrak{q}}$ is of the form $(t_{\alpha\beta}) \in \mathbb{Z}^{\mathit{\Pi}^{\mathfrak{q}} \times \mathit{\Pi}^{\mathfrak{q}}}$, where $t_{\alpha\alpha} = \mathtt{p}_\alpha b_{\alpha\alpha}$, $t_{r\alpha\beta} + t_{\beta\alpha} = b_{\alpha\beta}$ for $\alpha \neq \beta$. Arguing as in the Cartan case, we conclude that (ii) holds.

Table 2. Super type

Type	$\mathbf{q}$	N	$\mathit{\Pi}^{\mathfrak{q}}$	$\mathfrak{g}_{\mathfrak{q}}$
$\mathbf{A}(k-1\vert\theta-k)$, $k \in \mathbb{I}_{\lfloor\frac{\theta+1}{2}\rfloor}$	$\overset{\nu^{-1}}{\circ}\overset{\nu}{\text{—}}\overset{\nu^{-1}}{\circ}\cdots\underset{k}{\overset{-1}{\circ}}\cdots\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{—}}\overset{\nu}{\circ}$	>2	$\{N\alpha_j \,\vert j \neq k\}$	$A_{k-1} \times A_{\theta-k}$
$\mathbf{B}(k\vert\theta-k)$, $k \in \mathbb{I}_{\theta-1}$	$\overset{\nu^{-2}}{\circ}\overset{\nu^{2}}{\text{—}}\overset{\nu^{-2}}{\circ}\cdots\underset{k}{\overset{-1}{\circ}}\cdots\overset{\nu^{2}}{\circ}\overset{\nu^{-2}}{\text{—}}\overset{\nu}{\circ}$	$\neq 2,4$	(A.7)	$C_k \times B_{\theta-k}$ $C_k \times C_{\theta-k}$
$\mathbf{D}(k\vert\theta-k)$, $k < \frac{\theta}{2}$	$\overset{\nu^{-1}}{\circ}\overset{\nu}{\text{—}}\overset{\nu^{-1}}{\circ}\cdots\underset{k}{\overset{-1}{\circ}}\cdots\overset{\nu}{\circ}\overset{\nu^{-2}}{\text{—}}\overset{\nu^{2}}{\circ}$	>2	(A.8)	$D_k \times C_{\theta-k}$ $D_k \times B_{\theta-k}$
$\mathbf{D}(2,1;\alpha)$, $d_1, d_3 \in \mathbb{N}$	$\overset{\nu^{d_1}}{\circ}\overset{\nu^{-d_1}}{\text{—}}\overset{-1}{\circ}\overset{\nu^{-d_3}}{\text{—}}\overset{\nu^{d_3}}{\circ}$		$\{1,3,12^23\}$	$A_1 \times A_1 \times A_1$
$\mathbf{F}(4)$	$\overset{\nu^{2}}{\circ}\overset{\nu^{-2}}{\text{—}}\overset{\nu^{2}}{\circ}\overset{\nu^{-2}}{\text{—}}\overset{\nu}{\circ}\overset{\nu^{-1}}{\text{—}}\overset{-1}{\circ}$	>2	(A.9)	$A_1 \times B_3$
$\mathbf{G}(3)$	$\overset{-1}{\circ}\overset{\nu^{-1}}{\text{—}}\overset{\nu}{\circ}\overset{\nu^{-3}}{\text{—}}\overset{\nu^{3}}{\circ}$,	$N > 3$	(A.10)	$A_1 \times G_2$

(A.7) $\{N_j\alpha_j \,|j \neq k\} \cup \{N_{\alpha_{(k\theta)}}\alpha_{(k\theta)}\}$,

(A.8) $\{N_j\alpha_j \,|j \neq k\} \cup \{N_{\alpha_{(k-1\,\theta)}+\alpha_{(k\,\theta-1)}}(\alpha_{(k-1\,\theta)} + \alpha_{(k\,\theta-1)})\}$,

(A.9) $\{N_1\alpha_1, N_2\alpha_2, N_3\alpha_3, N_{\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)\}$,

(A.10) $\{N_{\alpha_1+2\alpha_2+\alpha_3}(\alpha_1 + 2\alpha_2 + \alpha_3), N_2\alpha_2, N_3\alpha_3\}$.

Type $\mathbf{D}(2,1;\alpha)$. The diagrams of this type are Weyl equivalent to the following one $\overset{r}{\circ}\overset{r^{-1}}{\text{—}}\overset{-1}{\circ}\overset{s^{-1}}{\text{—}}\overset{s}{\circ}$, with $r, s, rs \neq 1$. The corresponding Nichols algebra has finite dimension if and only if $r, s \in \mathbb{G}'_\infty$, $rs \neq 1$. Let $\mathfrak{q}$ be a braiding matrix with this diagram satisfying (4.24). Fix a generator ξ of the subgroup of $\mathbb{G}_\infty$ generated by r, s; we choose $d_1, d_3 \in \mathbb{N}$ minimal such that $r = \xi^{d_1}$, $s = \xi^{d_3}$. Then there exists a braiding matrix $\mathbf{q}$ as in Table 2 such that $\mathfrak{q} = \mathbf{q}(\xi)$.

A.3. **Modular type.** The Nichols algebras in this family could be thought of as quantizations in char 0 of the 34-dimensional Lie algebras in char 2 from [KaW], respectively the 10-dimensional Lie algebras in char 3 introduced in [Br]. The information on this type is given in Table 2. The matrices T and $\mathcal{T}^{\mathfrak{q}}$ are worked out as in the super case. $\square$

Appendix B. Lie bialgebras and Poisson algebraic groups

We gather minimal background material on Lie bialgebras and Poisson algebraic groups for Sections 7 and 8. We refer to [ES, Section 2-7] for a full treatment.

Table 3. Modular type

Type	$\mathbf{q}$	N	$\mathit{\Pi}^{\mathfrak{q}}$	$\mathfrak{g}_{\mathfrak{q}}$
$\mathtt{wk}(4)$	$\overset{q}{\circ}\overset{q^{-1}}{\text{---}}\overset{q}{\circ}\overset{q^{-1}}{\text{---}}\overset{-1}{\circ}\overset{-q}{\text{---}}\overset{-q^{-1}}{\circ}$	>2	(A.11)	$A_2 \times A_2$
$\mathtt{br}(2)$	$\overset{\zeta}{\circ}\overset{\xi^{-1}}{\text{---}}\overset{\xi}{\circ}$, $\zeta \in \mathbb{G}'_3$	$\neq 3$	$\{2M\alpha_1 + M\alpha_2, N\alpha_2\}$	$A_1 \times A_1$

$$\{N\alpha_1, N\alpha_2, M\alpha_4, M\alpha_1 + 2M\alpha_2 + 3M\alpha_3 + M\alpha_4\}. \tag{A.11}$$

B.1. **Lie bialgebras.** Recall that a Lie bialgebra is a Lie algebra $\mathfrak{g}$ equipped with a linear map $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ such that

(i) the dual of the map δ defines a Lie algebra structure of $\mathfrak{g}^*$ and
(ii) δ is a 1-cocycle, i.e., $\delta([a, b]) = \mathrm{ad}_a(\delta(b)) - \mathrm{ad}_b(\delta(a))$ for all $a, b \in \mathfrak{g}$.

The Lie bialgebras with opposite cobracket (same bracket) and opposite bracket (same cobracket) will be denoted by $\mathfrak{g}_{\mathrm{op}}$ and $\mathfrak{g}^{\mathrm{op}}$, respectively. The dual Lie bialgebra $\mathfrak{g}^*$ of $\mathfrak{g}$ is the Lie bialgebra with Lie bracket and cobracket given by

$$\langle [f, g], a\rangle = \langle f \otimes g, \delta(a)\rangle, \qquad \langle \delta(f), a \otimes b\rangle = \langle f, [a, b]\rangle, \qquad \forall a, b \in \mathfrak{g}, f, g \in \mathfrak{g}^*.$$

The Drinfeld double $D(\mathfrak{g})$ of the Lie bialgebra $\mathfrak{g}$ is a Lie bialgebra which is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space and is uniquely defined by the conditions:

(a) The canonical embeddings $\iota : \mathfrak{g} \hookrightarrow D(\mathfrak{g})$ and $\iota^* : (\mathfrak{g}^*)^{\mathrm{op}} \hookrightarrow D(\mathfrak{g})$ are embeddings of Lie bialgebras;
(b) For $a \in \mathfrak{g} \subset D(\mathfrak{g})$, $f \in \mathfrak{g}^* \subset D(\mathfrak{g})$, $[x, f] = \mathrm{ad}^*_x(f) - \mathrm{ad}^*_f(x)$ in terms of the coadjoint actions of $\mathfrak{g}$ and $\mathfrak{g}^*$.

A quadratic Lie algebra is a Lie algebra $\mathfrak{g}$ equipped with an non-degenerate invariant symmetric bilinear form $(.,.)$. A Manin triple is a triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ consisting of a quadratic Lie algebra $(\mathfrak{g}, (.,.))$ and a pair of isotropic Lie subalgebras $\mathfrak{g}_\pm \subset \mathfrak{g}$.

Remark B.1. The notions of Drinfeld double and Manin triple are equivalent in the case of finite dimensional Lie algebras:

(a) Each Drinfeld double $D(\mathfrak{g})$ is a quadratic Lie algebra with symmetric bilinear form
$$(a + f, b + g) = \langle f, b\rangle + \langle g, a\rangle, \qquad a, b \in \mathfrak{g}, f, g \in \mathfrak{g}^*.$$
With respect to this form, $(D(\mathfrak{g}), \mathfrak{g}, \mathfrak{g}^*)$ is a Manin triple.
(b) For a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, $\mathfrak{g}_\pm$ have canonical Lie bialgebra structures given by
$$(\delta(a), f \otimes g) = (a, [f, g]), \qquad (\delta(f), a \otimes b) = -(f, [a, b]), \qquad \forall a, b \in \mathfrak{g}_+, f, g \in \mathfrak{g}_-.$$
Then $\mathfrak{g}$, equipped with the Lie cobracket $\delta_{\mathfrak{g}_+} + \delta_{\mathfrak{g}_-}$, is isomorphic to the Drinfeld double of $\mathfrak{g}_+$, and $\mathfrak{g}_- \simeq (\mathfrak{g}^*_+)^{\mathrm{op}}$.

B.2. **Poisson algebraic groups.** A (complex) Poisson algebraic group is an algebraic group G equipped with a bivector field π such that the product map

$$(G, \pi) \times (G, \pi) \to (G, \pi)$$

is Poisson. The coordinate ring $\mathbb{C}[G]$ has a canonical structure of commutative Poisson-Hopf algebra with Poisson bracket given by

$$\{f, g\} := \langle df \otimes dg, \pi\rangle, \qquad f, g \in \mathbb{C}[G],$$

where df denotes the differential of f. Conversely, every finitely generated commutative Poisson-Hopf algebra H gives rise to the Poisson algebraic group $\operatorname{MaxSpec} H$.

The tangent Lie algebra $\mathfrak{g} = T_1 G$ of every Poisson algebraic group G has a canonical Lie bialgebra structure. The Poisson structure π automatically vanishes at the identity element 1 of G. The Lie cobracket on $\mathfrak{g}$, or equivalently the Lie bracket on $\mathfrak{g}^* \simeq T_1^* G$, is defined as the linearization of π at 1:

$$[d_1(f), d_1(g)] := d_1\big(\{f, g\}\big), \qquad f, g \in \mathbb{C}[G]. \tag{B.1}$$

In Hopf algebra situations it is advantageous to describe the tangent Lie algebra $\mathfrak{g}$ of an algebraic group G by describing the corresponding Lie cobracket on $\mathfrak{g}^* = T_1^* G$.

Lemma B.2. *Let G be a complex algebraic group; as usual $\Delta(f) = f_{(1)} \otimes f_{(2)}$ for $f \in \mathbb{C}[G]$. Then the canonical Lie coalgebra structure on $T_1^* G \simeq \mathfrak{g}^*$ is given by*

$$\delta(d_1 f) = d_1 f_{(1)} \wedge d_1 f_{(2)}, \qquad f \in \mathbb{C}[G].$$

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