PRIME FACTORS OF QUANTUM SCHUBERT CELL ALGEBRAS AND CLUSTERS FOR QUANTUM RICHARDSON VARIETIES

T. H. LENAGAN AND M. T. YAKIMOV

ABSTRACT. The understanding of the topology of the spectra of quantum Schubert cell algebras hinges on the description of their prime factors by ideals invariant under the maximal torus of the ambient Kac–Moody group. We give an explicit description of these prime quotients by expressing their Cauchon generators in terms of sequences of normal elements in chains of subalgebras. Based on this, we construct large families of quantum clusters for all of these algebras and the quantum Richardson varieties associated to arbitrary symmetrizable Kac–Moody algebras and all pairs of Weyl group elements. Along the way we develop a quantum version of the Fomin–Zelevinsky twist map for all quantum Richardson varieties. Furthermore, we establish an explicit relationship between the Goodearl–Letzter and Cauchon approaches to the descriptions of the spectra of symmetric CGL extensions.

1. INTRODUCTION

1.1. **Background.** The quantum Schubert cell algebras play an important role in representation theory (the Kashiwara–Lusztig theory of crystal/canonical bases [23, 25]), ring theory [22, 29, 31], Hopf algebras (coideal subalgebras [20]) and cluster algebras [13, 17]. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and w an element of its Weyl group. The corresponding quantum Schubert cell algebras $\mathcal{U}^{\pm}[w]$ are deformations of the universal enveloping algebras $\mathcal{U}(\mathfrak{n}_{\pm} \cap w(\mathfrak{n}_{\mp}))$ where $\mathfrak{n}_{\pm}$ are the nilradicals of the standard opposite Borel subalgebras of $\mathfrak{g}$. They were defined by De Concini–Kac–Procesi [7], Lusztig [26] and by Beck in the affine case [1].

In this paper we construct explicit models for the prime quotients of the quantum Schubert cell algebras by ideals invariant under the maximal torus of the ambient Kac–Moody group. These quotients play a key role in two problems that have attracted a lot of attention. One is of algebraic nature and is about the description of the topology of the spectra of quantum Schubert cell algebras. The other is of combinatorial nature – the construction of cluster algebra structures on quantum and classical Richardson varieties.

From the point of view of ring theory, the algebras $\mathcal{U}^{-}[w]$ are large families of deformations of universal enveloping algebras of nilpotent Lie algebras. It is a long-standing problem to carry out an analog of the orbit method [9] for these types of algebras. The canonical maximal torus $\mathcal{H}$ of the related Kac–Moody group G acts on $\mathcal{U}^{-}[w]$ by algebra automorphisms. The $\mathcal{H}$-invariant prime ideals of $\mathcal{U}^{-}[w]$ were classified in [29, 31], where

2010 *Mathematics Subject Classification.* Primary 16T20; Secondary 17B37, 14M15.

Key words and phrases. Quantum Schubert cell algebras, quantum Richardson varieties, prime factor rings, Cauchon diagrams, quantum cluster algebras.

The research of T.H.L. was partially supported by EPSRC grant EP/K035827/1 and a Leverhulme Trust Emeritus Fellowship. The research of M.T.Y. was partially supported by NSF grant DMS-1303038 and Louisiana Board of Regents grant Pfund-403.

it was shown that they are parametrized by $W^{\leq w} := \{u \in W \mid u \leq w\}$ – the corresponding ideals will be denoted by $I_w(u)$. By a general result of Goodearl and Letzter [15], $\mathrm{Spec}\,\mathcal{U}^-[w]$ is partitioned into

$$\mathrm{Spec}\,\mathcal{U}^-[w] = \bigsqcup_{u \in W^{\leq w}} \mathrm{Spec}_u\,\mathcal{U}^-[w]$$

where each stratum $\mathrm{Spec}_u\,\mathcal{U}^-[w]$ is homeomorphic to a torus and the ideals in it are obtained by extension and contraction from the center of $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$. The main ring-theoretic problem for $\mathcal{U}^-[w]$ is to describe the topology of their spectra, ideally by identifying it with the topological space of the symplectic foliation of the standard Poisson structure on the full flag variety of G (restricted to the Schubert cell corresponding to w). Understanding this topology amounts to solving the containment problem for the prime ideals of $\mathcal{U}^-[w]$ for which one needs an explicit model for the $\mathcal{H}$-prime quotients $\mathcal{U}^-[w]/I_w(u)$.

Recall that a quantum torus is an algebra of the form

$$\mathcal{T} := \frac{\mathbb{K}\langle Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}\rangle}{(Y_k Y_j - q_{kj} Y_j Y_k, k > j)}$$

for some $q_{kj} \in \mathbb{K}^*$. From each reduced expression

$$w = s_{i_1} \ldots s_{i_N} \tag{1.1}$$

one constructs Lusztig's root vectors $F_{\beta_1}, \ldots, F_{\beta_N}$ which form a generating set of $\mathcal{U}^-[w]$; here $\beta_1, \ldots, \beta_N$ are the roots of $n_+ \cap w(\mathfrak{n}_-)$. The algebra $\mathcal{U}^-[w]$ has two presentations as an iterated skew polynomial extension

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \ldots [F_{\beta_N}; \sigma_N, \delta_N] \tag{1.2}$$

$$= \mathbb{K}[F_{\beta_N}][F_{\beta_{N-1}}; \sigma^*_{N-1}, \delta^*_{N-1}] \ldots [F_{\beta_1}; \sigma^*_1, \delta^*_1]. \tag{1.3}$$

The Cauchon method of deleted derivations (applied to the first presentation) constructs in an iterative fashion a set of elements $Y_k \in \mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$, indexed by a subset $D(u) \subseteq [1, N]$ called the Cauchon diagram of $I_w(u)$, such that the elements $\{Y_k^{\pm 1} \mid k \in D(u)\}$ generate a copy of a quantum torus

$$\mathcal{T}(Y_k, k \in D(u)) \hookrightarrow \mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$$

satisfying

$$\mathcal{U}^-[w]/I_w(u) \hookrightarrow \mathcal{T}(Y_k, k \in D(u)).$$

This is the key construction that is currently used to understand the factors $\mathcal{U}^-[w]/I_w(u)$ and the topology of $\mathrm{Spec}\,\mathcal{U}^-[w]$. There are two main difficulties with it. Firstly, the elements Y_k are the result of an involved iterative construction and are not explicit in any way. Secondly, the elements Y_k do not lie in the algebra $\mathcal{U}^-[w]/I_w(u)$ in general but rather in its division ring of fractions. This leads to difficulties for the Goodearl–Letzter approach to $\mathrm{Spec}\,\mathcal{U}^-[w]$ because one needs to contract ideals of $\mathcal{T}(Y_k, k \in D(u))$ to $\mathcal{U}^-[w]/I_w(u)$.

The Cauchon method can be also applied to the reverse presentation (1.3) giving rise to another quantum torus in which $\mathcal{U}^-[w]/I_w(u)$ is embedded

$$\mathcal{U}^-[w]/I_w(u) \hookrightarrow \mathcal{T}(Y_{k,\mathrm{rev}}, k \in D_{\mathrm{rev}}(u)) \hookrightarrow \mathrm{Fract}(\mathcal{U}^-[w]/I_w(u)).$$

The are analogous difficulties in this situation.

1.2. **Results on the description of Cauchon generators.** We resolve the above problems and show that the Cauchon tori $\mathcal{T}(Y_k, k \in D(u))$ and $\mathcal{T}(Y_{k,\mathrm{rev}}, k \in D_{\mathrm{rev}}(u))$ have explicit generating sets that lie in $\mathcal{U}^-[w]/I_w(u)$. Our approach is based on the following general idea. Denote the canonical projection

$$p\colon \mathcal{U}^-[w] \to \mathcal{U}^-[w]/I_w(u).$$

Consider chains $\mathcal{C}$ of subalgebras

$$A_1 \subset A_2 \subset \ldots \subset A_N = \mathcal{U}^-[w]$$

such that

$$\operatorname{GK}\dim A_k - \operatorname{GK}\dim A_{k-1} = 1. \tag{1.4}$$

Here and below GK dim denotes the Gelfand–Kirillov dimension of an algebra. Given such a chain, we can project it

$$p(A_1) \subset p(A_2) \subset \ldots \subset p(A_N) = \mathcal{U}^-[w]/I_w(u)$$

and associate to it a subset of the form

$$\Sigma_{\mathcal{C}} := \{\text{a nonzero normal element } y_k \in p(A_k) \\ \text{for those } k \text{ such that } \operatorname{GK}\dim p(A_k) - \operatorname{GK}\dim p(A_{k-1}) = 1\}.$$

We will call the set of k's, the *jump set* of the chain $\mathcal{C}$. The main idea is to construct chains with the property that the projected subalgebras have sufficiently many nontrivial normal elements, and to show that sets $\Sigma_{\mathcal{C}}$ of the above form generate quantum tori and $\mathcal{U}^-[w]/I_w(u)$ embeds in them.

For each reduced expression (1.1), there are two special chains of subalgebras of $\mathcal{U}^-[w]$ obtained by adjoining the generators (1.2) of $\mathcal{U}^-[w]$ in the direct order and the reverse order (1.3). Denote them by $\mathcal{C}$ and $\mathcal{C}_{\mathrm{rev}}$. Our first result is as follows, see Theorems 5.1 and 8.1 for details.

Theorem A. *(i) The jump set of the chain $\mathcal{C}$ is the complement to the index set of the unique right positive subexpression of* (1.1) *with total product u in the sense of Deodhar, Marsh and Rietsch* [8, 27]*. A sequence of normal elements $\Sigma_{\mathcal{C}}$ is provided by the sequence of quantum minors*

$$p(\Delta_{u_{\leq k}\varpi_k, w_{\leq k}\varpi_k})$$

for the integers k in this index set. The set $\Sigma_{\mathcal{C}}$ is a set of independent generators of the Cauchon quantum torus $\mathcal{T}(Y_{k,\mathrm{rev}}, k \in D_{\mathrm{rev}}(u))$ and these generators belong to $\mathcal{U}^-[w]/I_w(u)$. Furthermore, the Cauchon generators $Y_{k,\mathrm{rev}}$ are Laurent monomials in the projected quantum minors whose exponents form a triangular matrix.

(ii) The jump set of the chain $\mathcal{C}_{\mathrm{rev}}$ is the complement to the index set of the unique left positive subexpression of (1.1) *with total product u. A sequence of normal elements $\Sigma_{\mathcal{C}_{\mathrm{rev}}}$ can be constructed using a similar sequence of quantum minors; $\Sigma_{\mathcal{C}_{\mathrm{rev}}} \subset \mathcal{U}^-[w]/I_w(u)$ is a set of independent generators of the Cauchon quantum torus $\mathcal{T}(Y_k, k \in D(u))$. The Cauchon generators Y_k are Laurent monomials in the elements of $\Sigma_{\mathcal{C}_{\mathrm{rev}}}$ with exponents forming a triangular matrix.*

In Theorems 4.2 and 7.1 we also describe explicitly all projected algebras $p(A_k)$ and $p(A_{k,\mathrm{rev}})$, in other words all contractions

$$I_w(u) \cap A_k \quad \text{and} \quad I_w(u) \cap A_{k,\mathrm{rev}}.$$

1.3. **Results on quantum twists maps for Richardson varieties in symmetrizable Kac–Moody groups.** The full flag variety of a symmetrizable Kac–Moody group G has the Schubert cell decompositions

$$G/B_+ = \bigsqcup_{w \in W} B_+ \cdot wB_+ = \bigsqcup_{u \in W} B_- \cdot uB_+$$

where $B_\pm$ is a pair of opposite Borel subgroups. The open Richardson varieties are defined by

$$R_{u,w} = B_+ \cdot wB_+ \cap B_- \cdot uB_+,$$

and $R_{u,w} \neq \varnothing$ if and only if $u \leq w$, [8, Corollary 1.2]. Since $\mathrm{Lie}\,(B_+) + \mathrm{Lie}\,(B-) = \mathrm{Lie}\,(G)$, they are smooth, irreducible varieties and $\dim R_{u,w} = \ell(w) - \ell(u)$. We have

$$G/B_+ = \bigsqcup_{u \leq w \in W} R_{u,w}.$$

The quantized coordinate ring of $R_{u,w}$ can be expressed as

$$R_q[R_{u,w}] := (\mathcal{U}^-[w]/I_w(u))[E_{u,w}^{-1}]$$

where $E_{u,w}$ is a multiplicative subset of $\mathcal{U}^-[w]/I_w(u)$ consisting of normal elements, see §2.5 for details.

Twist maps, defined by Fomin and Zelevinsky [10], are certain isomorphisms between double Bruhat cells. They play a major role in the study of the totally nonnegative part of G, canonical bases and cluster structures for double Bruhat cells. More recently, such were considered for open Richardson varieties in Grassmannians [28, 30], and were used to study cluster expansions and to prove local acyclicity for the related cluster algebras. We construct twist maps for all Richardson varieties and, in addition, do this in the quantum situation.

Theorem B. *There is an algebra antiisomorphism*

$$\Theta_w \colon \mathcal{U}^-[w^{-1}] \to \mathcal{U}^-[w]$$

given by (6.1)*. It satisfies* $\Theta_w(I_{u^{-1}}(w^{-1})) = I_u(w)$ *for all* $u \in W$*,* $u \leq w$ *and induces an antiisomorphism*

$$\Theta_w \colon R_q[R_{u^{-1},w^{-1}}] \to R_q[R_{u,w}].$$

The twist map Θ_w interchanges the statements in parts (i) and (ii) of Theorem A.

1.4. **Results on quantum clusters for Richardson varieties in symmetrizable Kac–Moody groups.** Recently, for each symmetric Kac–Moody algebra $\mathfrak{g}$, Leclerc [24] defined a cluster algebra structure inside the coordinate ring of each Richardson variety $R_{u,w}$ such that the two algebras have the same dimension. We apply Theorem A to obtain large families of toric frames for the algebras $\mathcal{U}^-[w]/I_w(u)$ and $R_q[R_{u,w}]$, with the ultimate goal of controlling the size of Leclerc's cluster algebra from below. Similarly to [17], consider the following subset of the symmetric group S_N:

$$\Xi_N := \{\pi \in S_N \mid \pi([1,k]) \text{ is an interval for all } k \in [2,N]\}.$$

The chain of subalgebras obtained by adjoining the Lusztig generators of $\mathcal{U}^-[w]$ in the order $F_{\beta_{\pi(1)}}, \dots, F_{\beta_{\pi(N)}}$ has the property (1.4). Denote this chain by $\mathcal{C}_\pi$. We recover the chains $\mathcal{C}$ and $\mathcal{C}_{\mathrm{rev}}$ for the identity and the longest element of S_N. Those are elements of the set Ξ_N which is very large.

Theorem C. *For all elements* $\pi \in \Xi_N$*, one can construct sets* $\Sigma_{\mathcal{C}_\pi}$ *consisting of projected quantum minors. Each of these sets produces toric frames for* $\mathcal{U}^-[w]/I_w(u)$ *and* $R_q[R_{u,w}]$

in the sense of Berenstein and Zelevinsky, [2]*.*

For the detailed formulation of this result we refer to Theorem 9.2.

1.5. **Unifying the Cauchon and Goodearl–Letzter approaches to the torus invariant prime ideals of CGL extensions.** The quantum Schubert cell algebras $\mathcal{U}^-[w]$ are members of the large axiomatic class of Cauchon–Goodearl–Letzter (CGL) extensions. These are iterated skew polynomial extensions with an action of a torus $\mathcal{H}$ that satisfy certain natural properties resembling the definition of (universal enveloping algebras of) nilpotent Lie algebras via derived series, see Definition 2.1. There are two approaches to describing the $\mathcal{H}$-invariant prime ideals of such algebras R. The Goodearl–Letzter [15] one describes these ideals via recursive contractions with the subalgebras of R and shows that at each step at most 2 ideals lead to the same contraction. The Cauchon approach first checks if a generator x of R belongs to an $\mathcal{H}$-prime ideal I and then maps the ideal to the leading coefficients of its elements written as polynomials in x, or to another contraction ideal. No connection between the two approaches was previously found.

In Theorem 3.2 we unify the two approaches for symmetric CGL extensions – extensions that satisfy the CGL axioms for the direct and reverse order of adjoining the generators of R. This relation interchanges the two approaches applied to the two opposite presentations.

The results of the paper have applications to the problems in §1.1 that will be described in forthcoming publications. Firstly, we will show that the toric frames in Theorem C are related by mutations and use this to control the size of Leclerc's cluster algebras [24] from below. Secondly, we use Theorem A to set up a torus equivariant map from the symplectic foliation of a Schubert cell to the primitive spectrum of the corresponding quantum Schubert cell algebra. This will be a conjectural candidate for the desired homeomorphism from §1.1 for all quantum Schubert cell algebras $\mathcal{U}^-[w]$. We believe that this will provide a framework in which one can attempt to settle the Brown–Goodearl conjecture [4, Conjecture 3.11] in the case of the algebras $\mathcal{U}^-[w]$; this is a general conjecture on the topology of spectra of quantum algebras that is only verified in very low GK dimension. Finally, we will also construct a direct relationship between the spectra of the quantum Schubert cell algebras $\mathcal{U}^-[w]$ and the totally nonnegative part of the corresponding Schubert cell; previously such was obtained for the algebras of quantum matrices [14].

Acknowledgements. We are thankful to Ken Goodearl, Ryan Kinser, Allen Knutson, Stéphane Launois, Bernard Leclerc and Jiang-Hua Lu for helpful discussions and for their comments on the first draft of the paper. We would also like to thank the anonymous referee for the very helpful suggestions.

2. Quantum Schubert cells, CGL extensions, and torus invariant primes

In this section we collect some facts on quantum groups and quantum Schubert cell algebras, as well as facts on their prime spectra, that will be used in the paper. For more details on quantum groups we refer the reader to [21, 22].

2.1. **Quantum algebras.** Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra of rank r with Weyl group W and set of simple roots α_i, $i \in [1, r]$. Let $\langle ., . \rangle$ be the invariant bilinear form on $\mathbb{R}\alpha_1 + \cdots + \mathbb{R}\alpha_r$ normalized by $\langle \alpha_i, \alpha_i \rangle = 2$ for short roots α_i. Denote by P^+ the set of dominant integral weights of $\mathfrak{g}$, and by P and Q the weight and root lattices of $\mathfrak{g}$. Let $\{\varpi_i\}$ and $\{\alpha_i^\vee\}$ be the fundamental weights and simple coroots of $\mathfrak{g}$. The corresponding simple reflections in W will be denoted by $\{s_i\}$.

Let $\mathbb{K}$ be an arbitrary infinite base field and $q \in \mathbb{K}^*$ be a non-root of unity. Denote by $\mathcal{U}_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over the base field $\mathbb{K}$ with deformation parameter q. We will use the conventions of [21] for the (Hopf) algebra structure on $\mathcal{U}_q(\mathfrak{g})$, with the exception that the generators of $\mathcal{U}_q(\mathfrak{g})$ will be denoted by $E_i, F_i, K_i^{\pm 1}$, indexed by $i \in [1, r]$ rather than by the set of simple roots of $\mathfrak{g}$. Recall that the weight spaces of a $\mathcal{U}_q(\mathfrak{g})$-module V are defined by

$$V_\nu := \{v \in V \mid K_i v = q^{\langle \alpha_i, \nu \rangle} v\}, \quad \nu \in P.$$

For $\lambda \in P^+$ denote by $V(\lambda)$ the unique irreducible highest weight $\mathcal{U}_q(\mathfrak{g})$-module with highest weight λ. Let v_λ be a highest weight vector of $V(\lambda)$. We will use Lusztig's actions of the braid group of $\mathfrak{g}$ on $\mathcal{U}_q(\mathfrak{g})$ and $V(\lambda)$, $\lambda \in P^+$, in the conventions of [21].

Denote by $\mathcal{U}_q^\pm(\mathfrak{g})$ the unital subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{E_i\}$ and $\{F_i\}$, respectively. Given a Weyl group element w and a reduced expression

$$w = s_{i_1} \dots s_{i_N} \tag{2.1}$$

of w, consider the root vectors

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}).$$

De Concini, Kac, and Procesi [7], Lusztig [26, §40.2] and Beck [1] defined the quantum Schubert cell algebras $\mathcal{U}^\pm[w]$ as the unital subalgebras of $\mathcal{U}_q^\pm(\mathfrak{g})$ with generators

$$\{E_{\beta_j} := T_{i_1} \dots T_{i_{j-1}}(E_{\alpha_j}) \mid j \in [1, N]\} \quad \text{and} \quad \{F_{\beta_j} := T_{i_1} \dots T_{i_{j-1}}(F_{\alpha_j}) \mid j \in [1, N]\}, \tag{2.2}$$

respectively, and proved that these algebras do not depend on the choice of a reduced expression of w. Define the quantum R-matrix associated to w by

$$\mathcal{R}^w := \sum_{m_1, \dots, m_N \in \mathbb{N}} \left(\prod_{j=1}^N \frac{(q_{i_j}^{-1} - q_{i_j})^{m_j}}{{q_{i_j}}^{m_j(m_j-1)/2} [m_j]_{q_{i_j}}!} \right) E_{\beta_N}^{m_N} \dots E_{\beta_1}^{m_1} \otimes F_{\beta_N}^{m_N} \dots F_{\beta_1}^{m_1} \tag{2.3}$$

considered as an element of the completion of $\mathcal{U}^+ \otimes \mathcal{U}^-$ with respect to the descending filtration [26, §4.1.1]. As usual, q-integers and q-factorials are defined by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \; [n]_q! := [1]_q \dots [n]_q, \; n \in \mathbb{N}.$$

For $\lambda \in P^+$ and $w \in W$ set

$$v_{w\lambda} := T_{w^{-1}}^{-1} v_\lambda \in V(\lambda)_{w\lambda}.$$

It is well known that $v_{w\lambda}$ depends only on $w\lambda$ and not on the choice of w and λ. Since $\dim V(\lambda)_{w\lambda} = 1$, there is a unique dual vector

$$\xi_{w\lambda} \in (V(\lambda)^*)_{-w\lambda} \quad \text{such that} \quad \langle \xi_{w\lambda}, v_{w\lambda} \rangle = 1.$$

For a pair of Weyl group elements (u, w) one defines the quantum minor

$$c_{u\lambda, w\lambda} \in (\mathcal{U}_q(\mathfrak{g}))^* \quad \text{given by} \quad c_{u\lambda, w\lambda}(x) = \langle \xi_{u\lambda}, x v_{w\lambda} \rangle, \;\; \forall x \in \mathcal{U}_q(\mathfrak{g}).$$

This quantum minor does not depend on the choice of a highest weight vector of $V(\lambda)$. Given a reduced expression (2.1) and $k \in [1, N]$, set $w_{\leq k} := s_{i_1} \dots s_{i_k}$. The algebras

$\mathcal{U}^{\pm}[w_{\leq k}]$ coincide with the subalgebras of $\mathcal{U}^{\pm}[w]$ generated by the subsets of (2.2) with $j \in [1,k]$. For $u \in W$ and $k \in [1,N]$, consider the quantum minors

$$\Delta_{u\lambda, w_{\leq k}\lambda} = \langle c_{u\lambda, w_{\leq k}\lambda}\tau \otimes \mathrm{id}, \mathcal{R}^{w_{\leq k}} \rangle \in \mathcal{U}^{-}[w_{\leq k}] \subset \mathcal{U}^{-}[w] \tag{2.4}$$

where τ denotes the unique graded algebra antiautomorphism of $\mathcal{U}_q(\mathfrak{g})$ defined via

$$\tau(E_i) = E_i,\ \tau(F_i) = F_i,\ \tau(K_i) = K_i^{-1}, \quad \forall i \in [1,r], \tag{2.5}$$

see [21, Lemma 4.6(b)]. The quantum minor (2.4) is nonzero iff $u \leq w_{\leq k}$ in the Bruhat order. Using the form of $\mathcal{R}^w$ and the highest weight property of v_λ, one easily derives that

$$\Delta_{u\lambda, w_{\leq k}\lambda} = \langle c_{u\lambda, w_{\leq k}\lambda}\tau \otimes \mathrm{id}, \mathcal{R}^{w} \rangle. \tag{2.6}$$

2.2. **CGL extensions.** Consider an iterated skew polynomial extension of length N,

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N]. \tag{2.7}$$

For $k \in [0,N]$, denote the k-th algebra in the chain of extensions

$$R_k := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_k; \sigma_k, \delta_k].$$

In particular, $R_0 = \mathbb{K}$ and $R_N = R$.

Definition 2.1. An iterated skew polynomial extension R as in (2.7) is called a *Cauchon–Goodearl–Letzter* (*CGL*) *extension* if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(i) The elements $x_1, \ldots, x_N$ are $\mathcal{H}$-eigenvectors.
(ii) For every $k \in [2,N]$, δ_k is a locally nilpotent σ_k-derivation of R_{k-1}.
(iii) For every $k \in [1,N]$, there exists $h_k \in \mathcal{H}$ such that $\sigma_k = (h_k \cdot)$ and the eigenvalue of x_k, to be denoted by $\lambda_k \in \mathbb{K}^*$, is not a root of unity.

A CGL extension R possesses the following canonical chain of subalgebras which are CGL extensions:

$$R_1 \subset R_2 \subset \ldots \subset R_N = R, \tag{2.8}$$

where R_k are equipped with the restriction of the $\mathcal{H}$-action.

For $1 \leq j < k \leq N$ denote the eigenvalues $\lambda_{kj} \in \mathbb{K}$ given by

$$\sigma_k(x_j) = h_k \cdot x_j = \lambda_{kj} x_j.$$

Set $\lambda_{kk} = 1$ and $\lambda_{jk} = \lambda_{kj}^{-1}$ for $j > k$. For $j,k \in [1,N]$ denote by $R_{[j,k]}$ the unital subalgebra of R generated by $\{x_i \mid j \leq i \leq k\}$. In particular, $R_{[j,k]} = \mathbb{K}$ if $j \nleq k$.

Definition 2.2. A CGL extension R of length N as above is called *symmetric* if it can be presented as an iterated skew polynomial extension for the reverse order of its generators,

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \cdots [x_1; \sigma^*_1, \delta^*_1],$$

in such a way that conditions (ii)–(iii) in Definition 2.1 are satisfied for some choice of $h^*_N, \ldots, h^*_1 \in \mathcal{H}$.

The following proposition is easy to prove and is left to the reader.

Proposition 2.3. *A CGL extension R as above is symmetric if and only if it satisfies the Levendorskii–Soibelman type straightening law*

$$x_k x_j - \lambda_{kj} x_j x_k \in R_{[j,k]}, \quad \forall j < k$$

and there exist $h_j^* \in \mathcal{H}$, $\forall j \in [1, N]$, such that $h_j^* \cdot x_k = \lambda_{kj}^{-1} x_k$ for all $k > j$. In this case, the endomorphisms σ_j^* and δ_j^* of $R_{[j+1,N]}$ are given by $\sigma_k := (h_k^* \cdot)$ and

$$\delta_j^*(x_k) := x_j x_k - \lambda_{jk} x_k x_j = -\lambda_{jk} \delta_k(x_j), \ \ \forall k \in [j+1, N].$$

A symmetric CGL extension R possesses the following reverse chain of subalgebras which are CGL extensions:

$$R_{N,\mathrm{rev}} \subset R_{N-1,\mathrm{rev}} \subset \ldots \subset R_{1,\mathrm{rev}} = R, \tag{2.9}$$

where the intermediate subalgebras $R_{k,\mathrm{rev}}$ are given by

$$R_{k,\mathrm{rev}} = \mathbb{K}[x_N][x_{N-1}; \sigma_{N-1}^*, \delta_{N-1}^*] \cdots [x_k; \sigma_k^*, \delta_k^*]$$

and are equipped with the restriction of the $\mathcal{H}$-action.

2.3. **Cauchon's method of deleting derivations.** Consider a CGL extension R as above. The Cauchon map

$$\theta_{x_N} : R_{N-1} \to R_N[x_N^{-1}] \text{ is given by } \theta_{x_N}(b) = \sum_{m=0}^{\infty} \frac{(1-\lambda_N)^{-m}}{(m)_{\lambda_N}!} [\delta_N^m \sigma_N^{-m}(b)] x_N^{-m}$$

for $b \in R_{N-1}$. We set $(0)_q = 1$, $(m)_q = (1 - q^m)/(1 - q)$ for $m > 0$, and $(m)_q! = (0)_q \ldots (m)_q$ for $m \in \mathbb{N}$ and a non-root of unity $q \in \mathbb{K}^*$. This map is an injective $\mathcal{H}$-equivariant algebra homomorphism, [5]. Denote $R'_{N-1} := \theta_{x_N}(R_{N-1})$, and let R'_N be the subalgebra of $R[x_N^{-1}] = R_N[x_N^{-1}]$ generated by R'_{N-1} and x_N. The map σ_N extends to an automorphism of $R_N[x_N^{-1}]$ by setting $\sigma_N = (h_N \cdot)$ (in the notation of Definition 2.1) because x_N is an $\mathcal{H}$-eigenvector, and furthermore σ_N restricts to an automorphism of R'_{N-1}. Then, [5]

$$R'_N \cong R'_{N-1}[x_N; \sigma_N] \ \text{ and } \ R_N[x_N^{-1}] = R'_N[x_N^{-1}]. \tag{2.10}$$

For an element $a = b_n x_N^n + \cdots + b_m x_N^m \in R_N[x_N^{-1}]$ with $n \leq m$ and $b_m \neq 0$, denote its leading coefficient

$$\mathrm{lc}_{x_N}(a) := b_m$$

(called leading term in [12]). Set $\mathrm{lc}_{x_N}(0) := 0$. For a subset S of $R_N[x_N^{-1}]$ denote by $\mathrm{lc}_{x_N}(S)$ the set of leading coefficients of all elements of S.

Goodearl and Letzter proved [15, Proposition 4.2] all $\mathcal{H}$-prime ideals of a CGL extension are completely prime. Cauchon's method of deleting derivations associates to each $\mathcal{H}$-prime ideal I of a CGL extension R of length N as above, a subset $\mathcal{CD}(I) \subset [1, N]$, called the *Cauchon diagram* of I, and a sequence of nonzero elements

$$Y_k \in \mathrm{Fract}(R/I), \quad k \in [1, N] \backslash \mathcal{CD}(I), \tag{2.11}$$

which, together with their inverses, generate a copy of a quantum torus inside $\mathrm{Fract}(R/I)$ with commutation relations $Y_k Y_l = \lambda_{kl} Y_l Y_k$, $k, l \in [1, N] \backslash \mathcal{CD}(I)$. This quantum torus contains R/I, and is a localization of R/I.
We will call the elements (2.11) the *Cauchon generators* of R/I, and will denote the set of them by $\mathcal{CG}(R/I)$.

The sets $\mathcal{CD}(I)$ and $\mathcal{CG}(R/I)$ are defined recursively as follows:

Case 1, $x_N \in I$. In this case $I = I \cap R_{N-1} + R_N x_N$ and we have the isomorphism

$$\varphi \colon R_N/I \cong R_{N-1}/(I \cap R_{N-1}), \tag{2.12}$$

the inverse of which is induced by the embedding $R_{N-1} \hookrightarrow R_N$. One sets

$$\mathcal{CD}(I) := \mathcal{CD}(I \cap R_{N-1}) \sqcup \{N\} \text{ and } \mathcal{CG}(R_N/I) := \varphi^{-1}\mathcal{CG}(R_{N-1}/(I \cap R_{N-1}))$$

and continues recursively with the $\mathcal{H}$-prime ideal $I \cap R_{N-1}$ of R_{N-1}.

Case 2, $x_N \notin I$. In this case $I[x_N^{-1}] = \oplus_{m\in\mathbb{Z}} \theta_{x_N}(\mathrm{lc}_{x_N}(I)) x_N^m$, [12, Proposition 2.5(i)],

$$I' := I[x_N^{-1}] \cap R'_N = \theta_{x_N}(\mathrm{lc}_{x_N}(I))[x_N; \sigma_N] \text{ and } I[x_N^{-1}] = I'[x_N^{-1}]. \tag{2.13}$$

We have the isomorphisms

$$\begin{aligned} \varphi \colon (R_N/I)[x_N^{-1}] &\cong (R'_N[x_N^{-1}])/(I'[x_N^{-1}]) \cong \\ &\cong (R'_{N-1}/\theta_{x_N}(\mathrm{lc}_{x_N}(I)))[x_N^{\pm 1}; \sigma_N] \cong (R_{N-1}/\operatorname{lc}_{x_N}(I))[x_N^{\pm 1}; \sigma_N] \end{aligned} \tag{2.14}$$

where the last map is obtained by applying $\theta_{x_N}^{-1}$ and keeping x_N fixed. The first three maps are the canonical isomorphisms induced by (2.10) and (2.13). One sets

$$\mathcal{CD}(I) := \mathcal{CD}(\mathrm{lc}_{x_N}(I)) \text{ and } \mathcal{CG}(R_N/I) := \varphi^{-1}\mathcal{CG}(R_{N-1}/\operatorname{lc}_{x_N}(I)) \sqcup \{x_N\}$$

and continues recursively with the $\mathcal{H}$-prime ideal $\mathrm{lc}_{x_N}(I)$ of R_{N-1}.

2.4. **Torus invariant prime ideals of the quantum Schubert cell algebras.** The algebra $\mathcal{U}_q(\mathfrak{g})$ is Q-graded by setting $\deg E_i = \alpha_i$, $\deg F_i = -\alpha_i$, and $\deg K_i^{\pm 1} = 0$. The graded component of $\mathcal{U}_q(\mathfrak{g})$ of degree $\gamma \in Q$ will be denoted by $\mathcal{U}_q(\mathfrak{g})_\gamma$. The rational character lattice of the $\mathbb{K}$-torus

$$\mathcal{H} := (\mathbb{K}^*)^r$$

is identified with the weight lattice P of $\mathfrak{g}$ by mapping $\nu \in P$ to the character

$$(t_1, \dots, t_r) \mapsto (t_1, \dots, t_r)^\nu := \prod_{i=1}^r t_i^{\langle \nu, \alpha_i^\vee \rangle}, \quad \forall t_1, \dots, t_r \in \mathbb{K}^*.$$

The torus $\mathcal{H}$ acts rationally on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms by

$$h \cdot z = h^\gamma z \quad \text{for} \quad z \in \mathcal{U}_q(\mathfrak{g})_\gamma, \gamma \in Q.$$

This action preserves the subalgebras $\mathcal{U}^\pm[w]$. There is a unique algebra automorphism ω of $\mathcal{U}_q(\mathfrak{g})$ that satisfies

$$\omega(E_i) = F_i, \ \ \omega(F_i) = E_i, \ \ \omega(K_i) = K_i^{-1} \quad \forall i \in [1, r]. \tag{2.15}$$

This automorphism restricts to an isomorphism $\omega \colon \mathcal{U}^+[w] \cong \mathcal{U}^-[w]$, $\forall w \in W$.

The Levendorskii–Soibelman straightening law is the following commutation relation in $\mathcal{U}^-[w]$

$$\begin{aligned} F_{\beta_j} F_{\beta_k} - q^{-\langle \beta_k, \beta_j \rangle} F_{\beta_k} F_{\beta_j} \\ = \sum_{\mathbf{n}=(n_{k+1},\dots,n_{j-1}) \in \mathbb{N}^{\times(j-k-2)}} p_{\mathbf{n}} (F_{\beta_{j-1}})^{n_{j-1}} \dots (F_{\beta_{k+1}})^{n_{k+1}}, \ \ p_{\mathbf{n}} \in \mathbb{K}, \end{aligned} \tag{2.16}$$

for all $k < j$. It follows from (2.16) that each algebra $\mathcal{U}^-[w]$ is a symmetric CGL extension with an original presentation of the form

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \dots [F_{\beta_N}; \sigma_N, \delta_N] \tag{2.17}$$

and reverse presentation

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_N}][F_{\beta_{N-1}}; \sigma^*_{N-1}, \delta^*_{N-1}] \dots [F_{\beta_1}; \sigma^*_1, \delta^*_1]. \tag{2.18}$$

The automorphisms σ_k and σ_k^* are given by

$$\sigma_k = (h_k \cdot) \quad \text{and} \quad \sigma_k = (h_k^{-1} \cdot)$$

where h_k are the unique elements of $\mathcal{H}$ such that $h^\gamma = q^{\langle \gamma, \beta_k \rangle}$ for for all $\gamma \in P$. The skew derivations δ_k and δ_k^* are given by

$$\delta_k(z) = F_{\beta_k} z - (h_k \cdot z) F_{\beta_k} \quad \text{and} \quad \delta_k^*(z) = F_{\beta_k} z - (h_k^{-1} \cdot z) F_{\beta_k}.$$

The $\mathcal{H}$-primes of $\mathcal{U}^-[w]$ are classified by the following result [33, Theorem 3.1].

Theorem 2.4. *For all symmetrizable Kac–Moody algebras $\mathfrak{g}$, Weyl group elements w, base fields $\mathbb{K}$ and non-roots of unity $q \in \mathbb{K}^*$, the poset of $\mathcal{H}$-primes of $\mathcal{U}^-[w]$ is isomorphic to $W^{\leq w}$ equipped with the Bruhat order. The ideal corresponding to $u \in W$, $u \leq w$ is given by*

$$I_w(u) = \{ \langle c_{\xi, v_{w\lambda}} \tau \otimes \mathrm{id}, \mathcal{R}^w \rangle \mid \xi \in V(\lambda)^*, \xi \perp \mathcal{U}_q^-(\mathfrak{g}) v_{u\lambda}, \lambda \in P^+ \}.$$

In [33] this result was formulated for simple finite dimensional Lie algebras $\mathfrak{g}$. However, all proofs in [33] carry over word-by-word to all symmetrizable Kac–Moody algebras $\mathfrak{g}$. In all proofs one uses the quantized coordinate ring of the corresponding Kac–Moody group G instead of the quantized coordinate ring of the connected simply connected finite dimensional simple Lie group. The former is the subalgebra of the dual Hopf algebra $(\mathcal{U}_q(\mathfrak{g}))^*$ consisting of the matrix coefficients of all finitely generated integrable highest weight $\mathcal{U}_q(\mathfrak{g})$-modules.

2.5. **Quantum Richardson varieties.** For $u \in W$, $u \leq w$, denote the canonical projection

$$p \colon \mathcal{U}^-[w] \to \mathcal{U}^-[w]/I_w(u).$$

The elements $\{ p(\Delta_{u\lambda, w\lambda}) \mid \lambda \in P^+ \}$ are nonzero normal elements of $\mathcal{U}^-[w]/I_w(u)$ and

$$p(\Delta_{u\lambda, w\lambda}) z = q^{-\langle (w+u)\lambda, \gamma \rangle} z p(\Delta_{u\lambda, w\lambda}), \quad \forall z \in (\mathcal{U}^-[w]/I_w(u))_\gamma, \gamma \in Q, \tag{2.19}$$

see [33, Theorem 3.1 (b), Eq. (3.1)]. It follows from [31, Theorem 2.6] that they satisfy

$$\Delta_{u\lambda_1, w\lambda_1} \Delta_{u\lambda_2, w\lambda_2} = q^{\langle (w-u)\lambda_1, u\lambda_2 \rangle} \Delta_{u(\lambda_1+\lambda_2), w(\lambda_1+\lambda_2)}, \quad \forall \lambda_1, \lambda_2 \in P^+. \tag{2.20}$$

Given $u \leq w$, one defines the open Richardson variety

$$R_{u,w} := B_+ \cdot w B_+ \cap B_- \cdot u B_+ \subset G/B_+.$$

Its quantized coordinate ring is defined by

$$R_q[R_{u,w}] := \mathcal{U}^-[w]/I_w(u)[p(\Delta_{u\lambda, w\lambda})^{-1}, \lambda \in P^+].$$

This algebra has a canonical rational form over $\mathbb{Q}[q^{\pm 1}]$ whose specialization is isomorphic to the coordinate ring of $R_{u,w}$ in the case when $\mathbb{K} = \mathbb{C}$, see e.g. [32, Sect. 4]. In the finite dimensional case, this is proved in [32, Sect. 4]; the general case is analogous.

Given an integral weight $\lambda \in P$, let $\lambda_1, \lambda_2 \in P^+$ be such that $\lambda = \lambda_1 - \lambda_2$. Denote the localized quantum minors

$$\Delta_{u\lambda, w\lambda} := q^{-\langle (w-u)\lambda_1, u\lambda_2 \rangle} \Delta_{u\lambda_1, w\lambda_1} \Delta^{-1}_{u\lambda_2, w\lambda_2} \in \mathrm{Fract}(\mathcal{U}^-[w]). \tag{2.21}$$

It follows from (2.20) that this definition does not depend on the choice of $\lambda_1, \lambda_2 \in P^+$.

3. Contraction of $\mathcal{H}$-primes in symmetric CGL extensions

In this section we prove a very general contraction formula for the $\mathcal{H}$–prime ideals of a CGL extension R with the subalgebras in the chain (2.8). This formula is given in terms of the Cauchon diagrams with respect to the reverse presentation of R which has to do with the chain (2.9).

3.1. **Statement of main result.** Let

$$R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \dots [x_N; \sigma_N, \delta_N] \tag{3.1}$$

be a symmetric CGL extension as in Definition 2.2 with a rational action of the $\mathbb{K}$-torus $\mathcal{H}$. Consider the reverse CGL extension presentation of R

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \dots [x_1; \sigma^*_1, \delta^*_1] \tag{3.2}$$

and the reverse CGL extension presentation of R_{N-1}

$$R_{N-1} = \mathbb{K}[x_{N-1}][x_{N-2}; \sigma^*_{N-2}, \delta^*_{N-2}] \dots [x_1; \sigma^*_1, \delta^*_1]. \tag{3.3}$$

The maps σ^*_k, δ^*_k, $k \in [1, N-2]$ in (3.3) are restrictions of the corresponding maps in (3.2). We use the same symbols for simplicity of the notation.

Definition 3.1. For every $\mathcal{H}$-prime ideal I of a symmetric CGL extension R as in (3.1), we will denote by $\mathcal{CD}_{\text{rev}}(I)$ the Cauchon diagram of I with respect to the reverse presentation (3.2), where the indices in $\mathcal{CD}_{\text{rev}}(R)$ are recorded in the same way as those of the generators x (without any change of the enumeration).

Given a prime ideal J of the symmetric CGL extension R_{N-1}, denote its Cauchon diagram with respect to the presentation (3.3) by $\mathcal{CD}_{\text{rev}}(J)$ with the same convention as the one for the ideals of R.

To clarify the convention in Definition 3.1, we give an example. If Rx_1 is a prime ideal of R, then $\mathcal{CD}_{\text{rev}}(Rx_1) = \{1\}$ rather than $\{N\}$.

The next theorem provides a very general contraction statement for $\mathcal{H}$-primes of symmetric CGL extensions. It provides a bridge between the two approaches of Goodearl–Letzter (contractions) [15] and Cauchon (deleting derivations) [5] to the $\mathcal{H}$-prime ideals of symmetric CGL extensions.

Theorem 3.2. *Let R be a symmetric CGL extension of length N. Assume that I and J are two $\mathcal{H}$-prime ideals of R and R_{N-1}, respectively, such that*

$$\mathcal{CD}_{\text{rev}}(J) = \mathcal{CD}_{\text{rev}}(I) \cap [1, N-1].$$

Then

$$J = I \cap R_{N-1}.$$

3.2. **Proof of Theorem 3.2.** We argue by induction on N. First, consider the case $1 \in \mathcal{CD}_{\text{rev}}(J)$. The assumption $\mathcal{CD}_{\text{rev}}(I) \cap [1, N-1] = \mathcal{CD}_{\text{rev}}(J)$ implies $1 \in \mathcal{CD}_{\text{rev}}(I)$. Therefore

$$I = x_1 R + I \cap R_{[2,N]}, \quad J = x_1 R_{N-1} + J \cap R_{[2,N-1]}.$$

and

$$\begin{aligned} I \cap R_{N-1} &= \big(x_1 R + I \cap R_{[2,N]}\big) \cap R_{N-1} \\ &= x_1 R_{N-1} + I \cap R_{[2,N]} \cap R_{N-1}. \end{aligned}$$

The second equality follows from the fact that $\{x_1^{m_1} \dots x_N^{m_N} \mid m_1, \dots, m_N \in \mathbb{N}\}$ is a basis of R.

Consider the CGL extension presentations of $R_{[2,N]}$ and $R_{[2,N-1]}$ obtained from (3.2) and (3.3) by removing the last step of the extensions associated to adjoining x_1. The Cauchon diagrams of the $\mathcal{H}$-prime ideals $I \cap R_{[2,N]}$ and $J \cap R_{[2,N-1]}$ of $R_{[2,N]}$ and $R_{[2,N-1]}$ with respect to these presentations are

$$\mathcal{CD}_{\mathrm{rev}}(I)\backslash\{1\} \quad \text{and} \quad \mathcal{CD}_{\mathrm{rev}}(J)\backslash\{1\}.$$

Note that in the second case the generators of $R_{[2,N]}$ are indexed by $[2,N]$ in the definition of the diagram. The inductive assumption implies that

$$I \cap R_{[2,N]} \cap R_{N-1} = (I \cap R_{[2,N]}) \cap R_{[2,N-1]} = J \cap R_{[2,N-1]}.$$

Thus,

$$I \cap R_{N-1} = x_1 R_{N-1} + J \cap R_{[2,N-1]} = J.$$

Next, we consider the case $1 \notin \mathcal{CD}_{\mathrm{rev}}(J)$, which implies $1 \notin \mathcal{CD}_{\mathrm{rev}}(I)$. Hence,

$$I = \theta_{x_1}\left(\mathrm{lc}_{x_1}(I)\right)[x_1^{\pm 1}; \tau_1] \cap R \quad \text{and} \quad J = \theta_{x_1}\left(\mathrm{lc}_{x_1}(J)\right)[x_1^{\pm 1}; \tau_1] \cap R_{N-1},$$

see Section 2.3. We have

$$\begin{aligned} I \cap R_{N-1} &= \theta_{x_1}\left(\mathrm{lc}_{x_1}(I)\right)[x_1^{\pm 1}; \tau_1] \cap R_{N-1} \\ &= \theta_{x_1}\left(\mathrm{lc}_{x_1}(I) \cap R_{[2,N-1]}\right)[x_1^{\pm 1}; \tau_1] \cap R_{N-1}. \end{aligned}$$

The second equality is proved by recursively applying the property

$$\theta_{x_1}(a) - a \in \oplus_{m=1}^{\infty} x_1^{-m} R_{[2,k]}, \forall a \in R_{[2,k]}, k \in [2,N].$$

This property follows from the definition of the Cauchon map θ_{x_1} and Proposition 2.3. In this case $\mathcal{CD}_{\mathrm{rev}}(I)$ and $\mathcal{CD}_{\mathrm{rev}}(J)$ coincide with the Cauchon diagrams of the $\mathcal{H}$-primes $\mathrm{lc}_{x_1}(I)$ and $\mathrm{lc}_{x_1}(J)$ of $R_{[2,N]}$ and $R_{[2,N-1]}$ obtained from (3.2) and (3.3) by removing the last step of the extensions. The inductive assumption implies

$$\mathrm{lc}_{x_1}(I) \cap R_{[2,N-1]} = \mathrm{lc}_{x_1}(J)$$

and thus

$$I \cap R_{N-1} = \theta_{x_1}\left(\mathrm{lc}_{x_1}(J)\right)[x_1^{\pm 1}; \tau_1] \cap R_{N-1} = J.$$

This completes the proof of the theorem. □

4. Contractions of $\mathcal{H}$-primes of $\mathcal{U}^-[w]$ and sequences of normal elements

In this section we prove an explicit formula for the contractions of all $\mathcal{H}$-prime ideals of the quantum Schubert cell algebras $\mathcal{U}^-[w]$ with the intermediate subalgebras corresponding to the presentation (2.17). For each such ideal, the projection of the chain to the prime factor gives a chain of subalgebras of the prime factor. We define an explicit sequence of normal elements for each such chain.

4.1. **Contractions.** Throughout the section we fix a Weyl group element $w \in W$ of length N and a reduced expression of w as in (2.1). The subexpressions of the latter are parametrized by the subsets $D \subseteq [1,N]$. For such a subset D, denote

$$s_k^D := \begin{cases} s_{i_k}, & \text{if } k \in D, \\ 1, & \text{if } k \notin D \end{cases}$$

and

$$w_{\leq k}^D := s_1^D \dots s_k^D, \quad w_{\geq k}^D := s_k^D \dots s_N^D, \quad w_{[j,k]}^D := s_j^D \dots, s_k^D, \quad w^D := w_{\leq N}^D = w_{\geq 1}^D.$$

A subexpression is called right positive (respectively left positive) if its index set D satisfies $w^D_{\leq k} \leq w^D_{\leq k} s^D_{i_{k+1}}$ $\forall k \in [1, N-1]$, (respectively $w^D_{\geq k} \leq s^D_{i_{k-1}} w^D_{\geq k}$, $\forall k \in [2, N]$). Deodhar and Marsh–Rietsch [8, 27] proved that for each $u \in W$ such that $u \leq w$ there exists a unique right positive subexpression of (2.1) with total product equal to u, i.e., $w^D = u$. Its index set will be denoted by $\mathcal{RP}_w(u)$. When one passes from subexpressions of w to those of w^{-1}, the sets of left and right positive subexpressions are interchanged. Hence, for each $u \in W$ such that $u \leq w$ there exists a unique left positive subexpression of (2.1) with total product equal to u. Its index set will be denoted by $\mathcal{LP}_w(u)$.

We will use the convention of Definition 3.1 for diagrams with respect to reverse presentations.

Theorem 4.1. *Let w be a Weyl group element with reduced expression* (2.1)*. For all Weyl group elements $u \in W$, $u \leq w$, the following hold:*

(a) the Cauchon diagram $\mathcal{CD}(I_w(u))$ of the $\mathcal{H}$-prime ideal $I_w(u)$ of $\mathcal{U}^-[w]$ with respect to the presentation (2.17) *equals the index set $\mathcal{LP}_w(u)$ of the left positive subexpression of* (2.1) *with total product u, and*

(b) the Cauchon diagram $\mathcal{CD}_{\mathrm{rev}}(I_w(u))$ of $I_w(u)$ of $\mathcal{U}^-[w]$ with respect to the presentation (2.18) *equals the index set $\mathcal{RP}_w(u)$ of the right positive subexpression of* (2.1) *with total product u.*

The first part of the theorem is [12, Theorem 1.1]. The proof of the second part is completely analogous.

The intermediate subalgebras for the direct CGL extension presentation (2.17) of $\mathcal{U}^-[w]$ are given by

$$\mathcal{U}^-[w]_k = \mathcal{U}^-[w_{\leq k}], \quad k \in [0, N].$$

For a Weyl group element $u \in W$ such that $u \leq w$, set for brevity

$$\vec{u}_{\leq k} := w^{\mathcal{RP}_w(u)}_{\leq k}.$$

The vector notation is suggestive of the definition of right positive subexpression. (Right positive subexpressions of reduced expressions of Weyl group elements are picking up indices to the far right of the reduced expression.) A reverse vector notation will be used in relation to left positive subexpressions in Section 7.

Theorems 3.2 and 4.1(b) imply at once the following result describing all contractions of the $\mathcal{H}$-prime ideals of $\mathcal{U}^-[w]$ with the intermediate subalgebras for the direct presentation of $\mathcal{U}^-[w]$:

Theorem 4.2. *Let w be a Weyl group element with reduced expression* (2.1)*. For all Weyl group elements $u \in W$ such that $u \leq w$ and $k \in [1, N]$, the contractions of the ideal $I_w(u)$ with the subalgebras $\mathcal{U}[w]_k$ are given by*

$$I_w(u) \cap \mathcal{U}^-[w_{\leq k}] = I_{w_{\leq k}}(\vec{u}_{\leq k}).$$

4.2. **Sequences of normal elements.**

Definition 4.3. For an algebra B, by *a sequence of nonzero normal elements*

$$\Delta_1, \ldots, \Delta_l \tag{4.1}$$

we will mean a sequence of nonzero elements for which there exists a chain of subalgebras

$$B_1 \subset \ldots \subset B_l = B \tag{4.2}$$

such that for all k, Δ_k is a normal element of B_k. We will also say that (4.1) is a normal sequence for the chain (4.2) when it is necessary to emphasize the chain of subalgebras in the background.

Note that in general a sequence of normal elements for an algebra B does not consist of normal elements of B. We will construct quantum clusters for an algebra B from sequences of normal elements by removing intermediate terms Δ_k that are algebraically dependent on the previous terms.

As in Section 2.5, for a pair of Weyl group elements $u \leq w$, we will denote by

$$p \colon \mathcal{U}^-[w] \to \mathcal{U}^-[w]/I_w(u)$$

the canonical projection. Consider the chain of subalgebras of the prime factor $\mathcal{U}^-[w]/I_w(u)$ obtained by projecting the intermediate subalgebras for the extension presentation (2.18)

$$p(\mathcal{U}^-[w_{\leq 1}]) \subseteq p(\mathcal{U}^-[w_{\leq 2}]) \subseteq \ldots \subseteq p(\mathcal{U}^-[w_{\leq N}]) \cong \mathcal{U}^-[w]/I_w(u). \tag{4.3}$$

Theorem 4.2 implies that

$$\operatorname{Ker} p|_{\mathcal{U}^-[w_{\leq k}]} = I_{w_{\leq k}}(\vec{u}_{\leq k}) \quad \text{and} \quad p(\mathcal{U}^-[w_{\leq k}]) \cong \mathcal{U}^-[w_{\leq k}]/I_{w_{\leq k}}(\vec{u}_{\leq k}). \tag{4.4}$$

The next theorem constructs a sequence of normal elements for the chain (4.3).

Theorem 4.4. *Let w be a Weyl group element with reduced expression* (2.1) *and u be a Weyl group element such that $u \leq w$. Then for all $k \in [1, N]$ and $\lambda \in P^+$, $p(\Delta_{\vec{u}_{\leq k}\lambda, w_{\leq k}\lambda})$ is a nonzero normal element of the k-algebra $p(\mathcal{U}^-[w_{\leq k}])$ in the chain* (4.3) *and more precisely*

$$p(\Delta_{\vec{u}_{\leq k}\lambda, w_{\leq k}\lambda}) x = q^{-\langle (w_{\leq k} + \vec{u}_{\leq k})\lambda, \gamma\rangle} x p(\Delta_{\vec{u}_{\leq k}\lambda, w_{\leq k}\lambda}), \quad \forall \gamma \in Q, x \in p(\mathcal{U}^-[w_{\leq k}])_\gamma. \tag{4.5}$$

In particular,

$$p(\Delta_{\vec{u}_{\leq 1}\varpi_{i_1}, w_{\leq 1}\varpi_{i_1}}), p(\Delta_{\vec{u}_{\leq 2}\varpi_{i_2}, w_{\leq 2}\varpi_{i_2}}), \ldots, p(\Delta_{\vec{u}_{\leq N}\varpi_{i_N}, w_{\leq N}\varpi_{i_N}}) \in \mathcal{U}^-[w]/I_w(u) \tag{4.6}$$

is a sequence with the property that its k-th element is a nonzero normal element of the k-algebra $p(\mathcal{U}^-[w_{\leq k}])$ in the chain (4.3) *and*

$$p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) z = q^{-\langle (w_{\leq k} + \vec{u}_{\leq k})\varpi_{i_k}, \gamma\rangle} z p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}),$$

for all $k \in [1, N], \gamma \in Q, z \in p(\mathcal{U}^-[w_{\leq k}])_\gamma$.

Proof. Theorem 4.2 (or equivalently (4.4)) implies that to prove (4.5) it is sufficient to establish it for $k = N$, in which case it follows from (2.19). $\square$

Taking into account that

$$p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) \in p(\mathcal{U}^-[w_{\leq k}])_{(w_{\leq k} - \vec{u}_{\leq k})\varpi_{i_k}}$$

for $k \in [1, N]$, we obtain the following:

Corollary 4.5. *For all pairs of Weyl group elements $u, w \in W$ with $u \leq w$ and reduced expressions of w, the sequence of elements* (4.6) *of the prime factor $\mathcal{U}^-[w]/I_w(u)$ is quasi-commuting, more precisely,*

$$\begin{aligned} p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) p(\Delta_{\vec{u}_{\leq j}\varpi_{i_j}, w_{\leq j}\varpi_{i_j}}) = \\ q^{-\langle (w_{\leq k} + \vec{u}_{\leq k})\varpi_{i_k}, (w_{\leq j} - \vec{u}_{\leq j})\varpi_{i_j}\rangle} p(\Delta_{\vec{u}_{\leq j}\varpi_{i_j}, w_{\leq j}\varpi_{i_j}}) p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) \end{aligned} \tag{4.7}$$

for all $1 \leq j < k \leq N$.

5. Sequences of normal elements vs. reverse Cauchon generators for prime factors of $\mathcal{U}^-[w]$

In this section we derive an explicit formula expressing the Cauchon generators of the $\mathcal{H}$-prime factors of $\mathcal{U}^-[w]$ with respect to the reverse presentation (2.18) in terms of the sequences of normal elements associated to the direct presentation of $\mathcal{U}^-[w]$ from Theorem 4.4. This formula is of monomial nature and the exponents have triangular form. As a consequence, the formula yields explicit quantum minor generators of the Cauchon quantum tori for all $\mathcal{H}$-prime factors of $\mathcal{U}^-[w]$ (for the reverse presentation of $\mathcal{U}^-[w]$). In particular, this constructs quantum clusters of the $\mathcal{H}$-prime factors of $\mathcal{U}^-[w]$ in terms of quantum minors.

5.1. **Statement of the main result.** As in the previous section we fix a Weyl group element $w \in W$ and a reduced expression (2.1) of w. Let u be another Weyl group element such that $u \leq w$. Recall from Section 4.1 that $\mathcal{RP}_w(u)$ denotes the index set of the right positive subexpression of (2.1) whose product is u. By Theorem 4.1(b), the Cauchon diagram $\mathcal{CD}_{\mathrm{rev}}(I_w(u))$ of the $\mathcal{H}$-prime ideal $I_w(u)$ of $\mathcal{U}^-[w]$ for the reverse presentation (2.18) is $\mathcal{RP}_w(u)$. Thus, the Cauchon deleting derivation method applied to the reverse presentation (2.18) of $\mathcal{U}^-[w]$ defines a sequence of nonzero elements

$$Y_{k,\mathrm{rev}} \in \mathrm{Fract}(\mathcal{U}^-[w]/I_w(u)), \quad k \in [1,N]\backslash\mathcal{RP}_w(u)$$

which, together with their inverses, generate a copy of a quantum torus inside $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$. This quantum torus contains $\mathcal{U}^-[w]/I_w(u)$, and is a localization of $\mathcal{U}^-[w]/I_w(u)$. Following the convention of Definition 3.1, the indices of the Y-elements match those of the x-elements even though the x-generators are adjoined in the reverse order.

The expression (2.1) gives rise to a partial order on $[1,N]$ given by

$$j \prec k \text{ if } i_j = i_k \text{ and } j < k. \tag{5.1}$$

Set $j \preceq k$ if $j \prec k$ or $j = k$. Extending the notation $\vec{u}_{\leq k}$ from the previous section, set

$$\vec{u}_{[j,k]} := w_{[j,k]}^{\mathcal{RP}_w(u)}.$$

Define the following integer matrix of size $(N - |\mathcal{RP}_w(u)|) \times N$ whose rows are indexed by the set $[1,N]\backslash\mathcal{RP}_w(u)$:

$$a_{jk} = \begin{cases} 0, & \text{if } j > k \\ \langle \alpha_{i_j}^\vee, \vec{u}_{[j+1,k]}(\varpi_{i_k})\rangle = \delta_{j\preceq k} - \sum_{j<l\preceq k, l\in\mathcal{RP}_w(u)} \langle \alpha_{i_j}^\vee, \vec{u}_{[j+1,l-1]}(\alpha_{i_l})\rangle, & \text{if } j \leq k, \end{cases}$$

where $\delta_{j\prec k} := 1$ if $j \prec k$ and $\delta_{j\prec k} := 0$ otherwise. Recall the definition of the quantum minors (2.4). The equality in the second case follows from Eq. (5.5) in Proposition 5.5.

Theorem 5.1. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and w be a Weyl group element with reduced expression* (2.1)*. Let $u \in W$, $u \leq w$. For all base fields $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$, in* $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$,

$$p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) = \prod_{j\in[1,k]\backslash\mathcal{RP}_w(u)} \frac{(q_{i_j}^{-1} - q_{i_j})^{a_{jk}}}{q_{i_j}^{a_{jk}(a_{jk}-1)/2}} Y_{j,\mathrm{rev}}^{a_{jk}}, \quad \forall k \in [1,N]$$

where $p\colon \mathcal{U}^-[w] \to \mathcal{U}^-[w]/I_w(u)$ is the canonical projection. The product in the right hand side is taken in decreasing order from left to right.

The special case of the theorem when $\mathcal{U}^-[w]$ equals the algebra of quantum matrices is due to Cauchon [6], the case $u = 1$ (all w and $\mathfrak{g}$) was obtained in [16].

Remark 5.2. (1) The principal minor of the matrix (a_{jk}) of size $N - \mathcal{RP}_w(u)$ (whose rows and columns are indexed by $[1, N]\backslash\mathcal{RP}_w(u)$) is triangular with ones on the diagonal. Thus it is invertible and its inverse

$$(b_{jk})_{j,k\in[1,N]\backslash\mathcal{RP}_w(u)}$$

is an integral matrix with the same properties. Theorem 5.1 implies that

$$Y_{k,\mathrm{rev}} = \zeta_k \prod_{j\in[1,N]\backslash\mathcal{RP}_w(u)} p(\Delta_{\vec{u}_{\leq j}\varpi_{i_j}, w_{\leq j}\varpi_{i_j}})^{b_{jk}}, \quad \forall k \in [1, N]\backslash\mathcal{RP}_w(u)$$

where $\zeta_k \in \mathbb{K}^*$ can be computed explicitly using the q-commutation relations between the elements $Y_{k,\mathrm{rev}}$. The product in the right hand side can be taken in any order, but since the terms q-commute (Corollary 4.5), the scalars ζ_k depend on the choice of order.

(2) Theorem 5.1 and the first part of the remark imply that

$$p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) = \theta_k \prod_{j\in[1,k]\backslash\mathcal{RP}_w(u)} p(\Delta_{\vec{u}_{\leq j}\varpi_{i_j}, w_{\leq j}\varpi_{i_j}})^{a'_{jk}}, \quad \forall k \in \mathcal{RP}_w(u)$$

for some integers a'_{jk} and scalars $\theta_k \in \mathbb{K}^*$ which can be computed explicitly. The integers a'_{jk} have the property that $a'_{jk} = 0$ for $j > k$.

(3) The matrix (a_{jk}) has stronger properties than plain triangularity, for example,

$$a_{jk} = \delta_{j\preceq k}, \quad \forall j \in [\max\{l \preceq k\} \cap \mathcal{RP}_w(u), k], k \in [1, N].$$

Corollary 5.3. *For all symmetrizable Kac–Moody algebras $\mathfrak{g}$, pairs of Weyl group elements $u \leq w$, base fields $\mathbb{K}$ and non-roots of unity $q \in \mathbb{K}^*$, the nonzero elements*

$$p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) \in \mathcal{U}^-[w]/I_w(u), \quad k \in [1, N]\backslash\mathcal{RP}_w(u)$$

and their inverses generate a copy of a quantum torus inside $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ *with commutation relations* (4.7). *This quantum torus contains $\mathcal{U}^-[w]/I_w(u)$, and is a localization of $\mathcal{U}^-[w]/I_w(u)$*

By the Cauchon procedure (Section 2.3) the elements $\{Y_{k,\mathrm{rev}}^{\pm 1} \mid k \in [1, N]\backslash\mathcal{RP}_w(u)\}$ generate a quantum torus inside $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ and this quantum torus contains $\mathcal{U}^-[w]/I_w(u)$. The corollary follows from the fact that, by Theorem 5.1, the elements $\{p(\Delta_{\vec{u}_{\leq k\varpi_{i_k}}, w_{\leq k}\varpi_{i_k}}) \mid k \in [1, N]\backslash\mathcal{RP}_w(u)\}$ generate the same quantum torus.

Recall that a toric frame (with index set $\mathcal{I} \subseteq \mathbb{Z}$, $|\mathcal{I}| < \infty|$) for an algebra R is a map

$$M \colon \mathbb{Z}^{\mathcal{I}} \to \mathrm{Fract}(R)$$

which satisfies the following conditions:

- For some multiplicatively skewsymmetric group bicharacter $\Lambda \colon \mathbb{Z}^{\mathcal{I}} \times \mathbb{Z}^{\mathcal{I}} \to \mathbb{K}^*$,

  $$M(f_1)M(f_2) = \Lambda(f_1, f_2)M(f_1 + f_2), \quad \forall f_1, f_2 \in \mathbb{Z}^{\mathcal{I}}$$

  (in particular, for the standard basis $\{e_k \mid k \in \mathcal{I}\}$ of $\mathbb{Z}^{\mathcal{I}}$, $M(e_k)^{\pm 1}$ generate a quantum torus inside $\mathrm{Fract}(R)$ with commutation relations $M(e_k)M(e_j) = \Lambda(e_k, e_j)^2 M(e_j)M(e_k)$),
- $M(\mathbb{N}^{\mathcal{I}}) \subset R$, and
- the quantum torus generated by $M(e_k)^{\pm 1}$, $k \in \mathcal{I}$, contains R.

A quantum seed for R is a pair consisting of a toric frame and an integral $\mathcal{I} \times \mathcal{I}'$ matrix whose principal part is skewsymmetrizable and which is compatible with the cocycle Λ in the sense of [2, Definition 3.1] and [17, §2.3]. (Here $\mathcal{I}' \subseteq \mathcal{I}$ is a set of exchangeable indices.) We refer to Berenstein–Zelevinsky [2] where these notions were introduced. The case of algebras over $\mathbb{Q}(q)$ was considered in [2] and the general case of arbitrary base fields in [17].

We note that [2] defines toric frames for division algebras without requiring the third condition above. However, if one has a quantum cluster algebra structure on a given algebra R, then the third condition is a consequence of the quantum Laurent phenomenon. It was shown in [17, Sect. 7] that the presence of the third property for a family of frames can be used in an essential way for the construction of a quantum cluster algebra structure on R. This is the reason for making it part of the definition.

Corollary 5.4. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, $u \leq w$ a pair of Weyl group elements, $\mathbb{K}$ a base field, and $q \in \mathbb{K}^*$ a non-root of unity such that $\sqrt{q} \in \mathbb{K}$. The prime factor $\mathcal{U}^-[w]/I_w(u)$ admits a toric frame $M \colon \mathbb{Z}^{[1,N] \backslash \mathcal{RP}_w(u)} \to \operatorname{Fract}(\mathcal{U}^-[w]/I_w(u))$ defined by*

$$M(e_k) := p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}), \quad \forall k \in [1,N] \backslash \mathcal{RP}_w(u)$$

with respect to the multiplicatively skewsymetric bicharacter given by

$$\Lambda(e_k, e_j) := \sqrt{q}^{\,-\langle (w_{\leq k} + \vec{u}_{\leq k})\varpi_{i_k}, (w_{\leq j} - \vec{u}_{\leq j})\varpi_{i_j} \rangle}, \quad \forall k > j \in \mathbb{Z}^{[1,N] \backslash \mathcal{RP}_w(u)}.$$

The toric frame can be extended to a quantum seed of $\mathcal{U}^-[w]/I_w(u)$ using Leclerc's matrices [24, Theorem 4.5 and Corollary 4.4].

The compatibility in the last part of the corollary was established by Leclerc in [24, Sect. 6].

The proof of Theorem 5.1 is given in Sections 5.3 and 5.4; Section 5.2 contains some auxiliary degree results that are needed for the proof. Proposition 5.8 in Section 5.5 is a generalization of the theorem that is needed for the proof of Theorem 8.1. For simplicity of the exposition we provide full details of the proof of Theorem 5.1 and leave the (analogous) proof of Proposition 5.8 to the reader.

5.2. **Degree considerations.** Given a Weyl group element w, consider an expression $w = s_{i_1} \dots s_{i_N}$ which is not necessarily reduced. Let

$$\mathcal{S} := \{d_1 < \dots < d_m\} \subseteq [1,N] \quad \text{and} \quad u = s_{i_{d_1}} \dots s_{i_{d_m}}. \tag{5.2}$$

We will use the generalization to this setting of the notation β_k, $w_{\leq k}$, $w_{[j,k]}$, and the partial order $\prec$ on $[1,N]$. For simplicity of the notation set $u_{\leq k} := w^{\mathcal{S}}_{\leq k}$, $u_{[j,k]} := w^{\mathcal{S}}_{[j,k]}$, and $u_\varnothing := 1$. For $j \in [1,k]$ and $\lambda \in P$, denote

$$a_{jk}(\lambda) := \langle \alpha_{i_j}^\vee, u_{[j+1,k]}(\lambda) \rangle. \tag{5.3}$$

The matrix in Section 5.1 is a special case of this notation: for $\mathcal{S} = \mathcal{RP}_w(u)$ and $j \leq k$, $a_{jk} = a_{jk}(\varpi_{i_k})$.

Proposition 5.5. *Let $w = s_{i_1} \dots s_{i_N} \in W$, and $\mathcal{S} \subseteq [1,N]$ and $u \in W$ be given by* (5.2)*. For all $k \in [1,N]$ and $\lambda \in P$,*

$$(w_{\leq k} - u_{\leq k})\lambda = - \sum_{j \leq k, j \notin \mathcal{S}} a_{jk}(\lambda)\beta_j. \tag{5.4}$$

Furthermore, for all $i \in [1, r]$ (where r is the rank of $\mathfrak{g}$) and $j \leq k$,

$$u_{[j+1,k]}(\varpi_i) = \omega_i - \sum_{j<l\leq k, i_l=i, l\in\mathcal{S}} \langle \alpha_{i_j}^\vee, u_{[j+1,l-1]}(\alpha_i)\rangle \beta_j.$$

In particular,

$$a_{jk}(\varpi_i) = \delta_{i_j,i} - \sum_{j<l\leq k, i_l=i, l\in\mathcal{S}} \langle \alpha_{i_j}^\vee, u_{[j+1,l-1]}(\alpha_i)\rangle. \tag{5.5}$$

Proof. For all $j \in [1, k]$, $j \notin \mathcal{S}$,

$$\begin{aligned} w_{\leq j}\big(u_{[j+1,k]}(\lambda)\big) &= w_{\leq j-1} s_{i_j}\big(u_{[j+1,k]}(\lambda)\big) \\ &= w_{\leq j-1}\big(u_{[j+1,k]}(\lambda)\big) - \langle \alpha_{i_j}^\vee, u_{[j+1,k]}(\lambda)\rangle \beta_j \\ &= w_{\leq j-1}\big(u_{[j,k]}(\lambda)\big) - a_{jk}(\lambda)\beta_j. \end{aligned}$$

Adding these identities, proves (5.4).

If the set $\{n \in [j+1, k] \mid i_n = i, n \in \mathcal{S}\}$ is empty, then the second identity in the proposition is trivial. Otherwise, denote by l the maximal element of the set and compute

$$u_{[j+1,k]}\varpi_i = u_{[j+1,l]}\varpi_i = u_{[j+1,l-1]}(\varpi_i - \alpha_i).$$

Then iterate this with $u_{[j+1,k]}$ replaced by $u_{[j+1,l-1]}$. □

Corollary 5.6. *In the setting of Proposition 5.5, if $l \notin \mathcal{S}$, then*

$$\sum_{j<l, j\notin\mathcal{S}} \langle \beta_j, \beta_l\rangle a_{jk}(\lambda) - \sum_{l<j\leq k, j\notin\mathcal{S}} \langle \beta_j, \beta_l\rangle a_{jk}(\lambda) = \langle (w_{\leq k} + u_{\leq k})\lambda, \beta_l\rangle. \tag{5.6}$$

Proof. Note that the integers $a_{jk}(\lambda)$ only depend on the expression $s_{i_j} \ldots s_{i_k}$ and not on the rest of the expression defining w. Let $m \in [1, N]$. Applying (5.4) for the expression $s_{i_m} \ldots s_{i_N}$ of $w_{[m,N]}$ gives

$$(w_{[m,k]} - u_{[m,k]})\lambda = - \sum_{m\leq j\leq k, j\notin\mathcal{S}} a_{jk}(\lambda) w_{\leq m-1}^{-1}(\beta_j)$$

and thus

$$(w_{\leq k} - w_{\leq m-1} u_{[m,k]})\lambda = - \sum_{m\leq j\leq k, j\notin\mathcal{S}} a_{jk}(\lambda)\beta_j.$$

Using this identity for $m = l$ and $l + 1$ and once again (5.4), we obtain that the left hand side of (5.6) equals

$$\begin{aligned} &\sum_{j\leq k, j\notin\mathcal{S}} \langle \beta_j, \beta_l\rangle a_{jk}(\lambda) - \sum_{l\leq j\leq k, j\notin\mathcal{S}} \langle \beta_j, \beta_l\rangle a_{jk}(\lambda) - \sum_{l<j\leq k, j\notin\mathcal{S}} \langle \beta_j, \beta_l\rangle a_{jk}(\lambda) \\ &= \langle (w_{\leq k} + u_{\leq k})\lambda, \beta_l\rangle - \langle (w_{\leq l} u_{[l+1,k]} + w_{\leq l-1} u_{[l,k]})\lambda, \beta_l\rangle. \end{aligned}$$

The corollary follows from the fact that the last term vanishes,

$$\begin{aligned} \langle (w_{\leq l} u_{[l+1,k]} + w_{\leq l-1} u_{[l,k]})\lambda, \beta_l\rangle &= \langle w_{\leq l-1}(s_{i_l} + 1) u_{[l,k]}\lambda, w_{\leq l-1}\alpha_{i_l}\rangle = \\ &= \langle u_{[l,k]}\lambda, (s_{i_l} + 1)\alpha_{i_l}\rangle = 0, \end{aligned}$$

where we used the fact that $l \notin \mathcal{S}$, thus $u_{[l+1,k]} = u_{[l,k]}$. □

5.3. **Proof of Theorem 5.1, part I.** Here we prove that

$$p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}) = \eta_k \prod_{j\in[1,k]\backslash\mathcal{RP}_w(u)} Y_{j,\mathrm{rev}}^{a_{jk}}, \quad \forall k \in [1,N] \tag{5.7}$$

for some $\eta_k \in \mathbb{K}^*$. In the next subsection we compute the scalars η_k explicitly.

Denote for simplicity

$$\overline{\Delta}_k := p(\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_k}) \in p(\mathcal{U}^-[w_{\leq k}]) \subseteq p(\mathcal{U}^-[w]).$$

The last part of Theorem 4.4 implies that

$$\overline{\Delta}_k z = q^{-\langle (w_{\leq k}+\vec{u}_{\leq k})\varpi_{i_k}, \gamma\rangle} z\overline{\Delta}_k, \quad \forall z \in \mathrm{Fract}(p(\mathcal{U}^-[w_{\leq k}]))_\gamma, \gamma \in Q.$$

It follows from the description of the Cauchon procedure in Section 2.3 that

$$Y_{l,\mathrm{rev}} \in \mathrm{Fract}(p(\mathcal{U}^-[w_{\leq k}])) \subset \mathrm{Fract}(p(\mathcal{U}^-[w]))$$

for all $l \leq k$, $l \notin \mathcal{RP}_w(u)$, and that $Y_{l,\mathrm{rev}}$ has the same degree as F_{β_l}. Therefore,

$$\overline{\Delta}_k Y_{l,\mathrm{rev}} = q^{\langle (w_{\leq k}+\vec{u}_{\leq k})\varpi_{i_k}, \beta_l\rangle} Y_{l,\mathrm{rev}}\overline{\Delta}_k, \quad \forall l \leq k, l \notin \mathcal{RP}_w(u).$$

Since $\{Y_{l,\mathrm{rev}}^{\pm 1} \mid l \in [1,k], l \notin \mathcal{RP}_w(u)\}$ generate a quantum torus inside $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ with commutation relations $Y_{j,\mathrm{rev}}Y_{l,\mathrm{rev}} = q^{-\langle\beta_j,\beta_l\rangle}Y_{l,\mathrm{rev}}Y_{j,\mathrm{rev}}$, for $j > l$, we have

$$\Bigg(\prod_{j\in[1,k]\backslash\mathcal{RP}_w(u)} Y_{j,\mathrm{rev}}^{a_{jk}}\Bigg) Y_{l,\mathrm{rev}} = q^m Y_{l,\mathrm{rev}} \Bigg(\prod_{j\in[1,k]\backslash\mathcal{RP}_w(u)} Y_{j,\mathrm{rev}}^{a_{jk}}\Bigg)$$

where

$$m = \sum_{j<l, j\notin\mathcal{RP}_w(u)} \langle\beta_j,\beta_l\rangle a_{jk} - \sum_{l<j\leq k, j\notin\mathcal{RP}_w(u)} \langle\beta_j,\beta_l\rangle a_{jk}.$$

By Corollary 5.6, applied to $\lambda = \varpi_{i_k}$, we have $m = \langle (w_{\leq k} + \vec{u}_{\leq k})\varpi_{i_k}, \beta_l\rangle$. Since the division algebra $\mathrm{Fract}(p(\mathcal{U}^-[w_{\leq k}]))$ is generated by the set $\{Y_{l,\mathrm{rev}} \mid l \leq N, l \notin \mathcal{RP}_w(u)\}$,

$$\Bigg(\prod_{j\in[1,k]\backslash\mathcal{RP}_w(u)} Y_{j,\mathrm{rev}}^{a_{jk}}\Bigg) \overline{\Delta}_k^{-1} \in Z(\mathrm{Fract}(p(\mathcal{U}^-[w_{\leq k}])).$$

Applying Eq. (5.4) in Proposition 5.5 for $\lambda = \varpi_{i_k}$, we get that the element is in $Z(\mathrm{Fract}(p(\mathcal{U}^-[w_{\leq k}]))^{\mathcal{H}}$.

The Goodearl strong $\mathcal{H}$-rationality result [3, Theorem II.6.4] for the $\mathcal{H}$-prime ideals of CGL extensions implies that

$$Z(\mathrm{Fract}(p(\mathcal{U}^-[w_{\leq k}])))^{\mathcal{H}} \cong Z(\mathrm{Fract}(\mathcal{U}^-[w_{\leq k}]/I_{w_{\leq k}}(\vec{u}_{\leq k})))^{\mathcal{H}} = \mathbb{K}.$$

This proves that (5.7) is satisfied for some $\eta_k \in \mathbb{K}^*$.

5.4. **Proof of Theorem 5.1, part II.** Here we obtain an explicit formula for the scalars η_k in (5.7) and complete the proof of Theorem 5.1. The definition of right positive subexpression implies that for all $l \in [1,N]$,

($*$) $\mathcal{RP}_w(u) \cap [l,N]$ *is the index set of the right positive subexpression of* $w_{i_l} \dots w_{i_N}$ *with total product* $\vec{u}_{[l,N]}$.

Denote the scalar in the right hand side of (5.7) for the triple $(w_{[l,N]}, \vec{u}_{[l,N]}, k)$ by $\eta_{w_{[l,N]},\vec{u}_{[l,N]},k}$. Here $k \geq l$. Theorem 5.1, now follows by iterating the next lemma and using ($*$) at each step.

Lemma 5.7. *For all Weyl group elements $u \leq w$ and $1 \leq k \leq N = \ell(w)$,*

$$\eta_{w,u,k} = \begin{cases} \eta_{w_{[2,N]},\vec{u}_{[2,N]},k}, & \text{if } 1 \in \mathcal{RP}_w(u) \\ \frac{(q_{i_1}^{-1} - q_{i_1})^{a_{1k}}}{q_{i_1}^{a_{1k}(a_{1k}-1)/2}} \eta_{w_{[2,N]},\vec{u}_{[2,N]},k}, & \text{if } 1 \notin \mathcal{RP}_w(u) \end{cases}$$

where we set $\eta_{w_{[2,N]},\vec{u}_{[2,N]},1} := 1$.

Proof. The last extension in the reverse presentation (2.18) of $\mathcal{U}^-[w]$ is

$$\mathcal{U}^-[w] = T_{s_{i_1}}(\mathcal{U}^-[w_{[2,N]}])[F_{\beta_1}; \sigma_1^*, \delta_1^*]. \tag{5.8}$$

To analyze the nature of the Cauchon procedure applied to this step coupled with the effects of the representation theory of $\mathcal{U}_q(\mathfrak{g})$, we need to consider three cases: (1) $1 \in \mathcal{RP}_w(u)$, (2) $1 \notin \mathcal{RP}_w(u)$ and $\vec{u}_{\leq k}(\alpha_{i_1}) \in Q^+$, (3) $1 \notin \mathcal{RP}_w(u)$ and $\vec{u}_{\leq k}(\alpha_{i_1}) \in -Q^+$.

Case (1) $1 \in \mathcal{RP}_w(u)$. Then $\vec{u}_{\leq k}^{-1}(\alpha_{i_1}) \in -Q^+$ and

$$\langle \xi_{\vec{u}_{\leq k-1}\varpi_{i_k}}, E_{i_1}^m v \rangle = \langle S^{-1}(E_{i_1})^m \xi_{\vec{u}_{\leq k-1}\varpi_{i_k}}, v \rangle = 0, \quad \forall m > 0, v \in V(\varpi_{i_k}).$$

It follows from (2.6) that

$$\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_k} = T_{i_1}(\Delta_{\vec{u}_{[2,k]}\varpi_{i_k}, w_{[2,k]}\varpi_k}).$$

Theorem 4.1 (b) implies that we are in the situation of Case 1 in Section 2.3. So,

$$F_{\beta_1} \in I_w(u) \quad \text{and} \quad I_w(u) \cap T_{i_1}(\mathcal{U}^-[w_{[2,N]}]) = T_{i_1}(I_{w_{[2,N]}}(u_{[2,N]})).$$

We have the isomorphism

$$\mathcal{U}^-[w]/I_w(u) \cong T_{i_1}(\mathcal{U}^-[w_{[2,N]}])/T_{i_1}(I_{w_{[2,N]}}(u_{[2,N]}))$$

the reverse of which is induced by the embedding $T_{i_1}(\mathcal{U}^-[w_{[2,N]}]) \hookrightarrow \mathcal{U}^-[w]$. Under this isomorphism,

$$\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_k} + I_w(u) \mapsto T_{i_1}(\Delta_{\vec{u}_{[2,k]}\varpi_{i_k}, w_{[2,k]}\varpi_k}) + T_{i_1}(I_{w_{[2,N]}}(u_{[2,N]}))$$

which proves the lemma in this case.

Case (2) $1 \notin \mathcal{RP}_w(u)$ and $\vec{u}_{\leq k}(\alpha_{i_1}) \in Q^+$. Now $\vec{u}_{\leq k} = \vec{u}_{[2,k]}$, $a_{1k} = \langle \alpha_{i_1}^\vee, \vec{u}_{[2,k]}\varpi_{i_k} \rangle \geq 0$ and $v_{\vec{u}_{\leq k}\varpi_{i_1}}$ is a highest weight vector for the $\mathcal{U}_q(\mathfrak{sl}_2)$-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ corresponding to the i_1-th root with highest weight $a_{1k}\varpi_{i_1}$.

We will need the following properties of $\mathcal{U}_q(\mathfrak{sl}_2)$-modules and the Lusztig braid group action on them:

$$T_1 v_{n\varpi_1} = \frac{(-q)^n}{[n]_q!} F_1^n v_{n\varpi_1}, \quad T_1^{-1} v_{n\varpi_1} = \frac{1}{[n]_q!} F_1^n v_{n\varpi_1} \quad \text{and} \tag{5.9}$$

$$E_1^n F_1^n v_{n\varpi_1} = ([n]_q!)^2 v_{n\varpi_1}, \quad \forall n > 0, \tag{5.10}$$

see [21, Eq. 8.6 (3-7) and Lemma 1.7]. From them we obtain

$$E_{i_1}^{a_{1k}} T_{i_1}^{-1} v_{\vec{u}_{\leq k}\varpi_{i_k}} = \frac{1}{[a_{1k}]_{q_{i_1}}!} E_{i_1}^{a_{1k}} F_{i_1}^{a_{1k}} v_{\vec{u}_{\leq k}\varpi_{i_k}} = ([a_{1k}]_{q_{i_1}}!) v_{\vec{u}_{\leq k}\varpi_{i_k}}.$$

Thus,

$$\langle \xi_{\vec{u}_{\leq k}\varpi_{i_k}}, E_{i_1}^{a_{1k}} T_{i_1}^{-1} v \rangle = ([a_{1k}]_{q_{i_1}}!) \langle \xi_{\vec{u}_{\leq k}\varpi_{i_k}}, v \rangle$$

and

$$\langle \xi_{\vec{u}_{\leq k}\varpi_{i_k}}, E_{i_1}^{a_{1k}+m} T_{i_1}^{-1} v \rangle = 0$$

for all $v \in V(\varpi_{i_k})$, $m > 0$. This and the definition of quantum minors (2.3)–(2.6) imply that in the extension (5.8)

$$\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} = \frac{(q_{i_1}^{-1} - q_{i_1})^{a_{1k}}}{q_{i_1}^{a_{1k}(a_{1k}-1)/2}} T_{i_1}(\Delta_{\vec{u}_{[2,k]}\varpi_{i_k}, w_{[2,k]}\varpi_{i_k}}) F_{\beta_1}^{a_{1k}} + \text{lower order terms in } F_{\beta_1}.$$

Applying the result in Section 5.3, the fact that the leading term of the Cauchon map (from Section 2.3) is the identity and the equality $Y_{1,\mathrm{rev}} = F_{\beta_1}$, proves the lemma in this case.

Case (3) $1 \notin \mathcal{RP}_w(u)$ and $\vec{u}_{\leq k}(\alpha_{i_1}) \in -Q^+$. Now

$$\vec{u}_{\leq k} = s_{i_1} u' \quad \text{for some} \quad u' \in W, \ell(u') = \ell(\vec{u}_{\leq k}) - 1.$$

The above Weyl group elements satisfy

$$u' < \vec{u}_{\leq k} = \vec{u}_{[2,k]} \leq w_{[2,k]}$$

with respect to the Bruhat order. Moreover, $a_{1k} = \langle \alpha_{i_1}^\vee, \vec{u}_{[2,k]}\varpi_{i_k} \rangle \leq 0$. For brevity, set

$$a := |a_{1k}|.$$

Then $a_{1k} = -a$ and $v_{\vec{u}_{\leq k}\varpi_{i_1}}$ is a lowest weight vector for the $\mathcal{U}_q(\mathfrak{sl}_2)$-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ corresponding to the i_1-th root with lowest weight $-a\varpi_{i_1}$. The highest weight vector of this module is $v_{u'\varpi_{i_1}}$. This implies that

$$S^{-1}(E_{i_1})^m \xi_{u'\varpi_{i_1}} \in (\mathcal{U}_q^-(\mathfrak{g})\varpi_{\vec{u}_{\leq k}\varpi_{i_1}})^\perp, \quad \forall m \in [0, a-1].$$

So,

$$\langle c_{S^{-1}(E_{i_1})^m, v_{w_{\leq k}\varpi_k}} \tau \otimes \mathrm{id}, \mathcal{R}^w \rangle \in I_{w_{\leq k}}(\vec{u}_{\leq k}), \quad \forall m \in [0, a-1]. \tag{5.11}$$

Given a linear operator L on a vector space V, denote its adjoint by $L^*\colon V^* \to V^*$, satisfying $\langle L^*\xi, v \rangle = \langle \xi, Lv \rangle$, $\forall v \in V, \xi \in V^*$. For $m \in [0, a]$, denote

$$D_m = T_{i_1}\Big(\langle c_{(T_{i_1}^{-1})^* S^{-1}(E_i)^m \xi_{u'\varpi_{i_k}}, v_{w_{[2,k]}\varpi_{i_k}}} \tau \otimes \mathrm{id} \rangle \mathcal{R}^{w_{[2,k]}} \Big) \in T_{i_1}(\mathcal{U}^-[w_{[2,k]}]).$$

The properties (5.9)–(5.10) imply

$$T_{i_1}^{-1} v_{\vec{u}_{\leq k}\varpi_{i_k}} = T_{i_1}^{-1} T_{i_1}^{-1} v_{u'\varpi_{i_1}} = (-q_{i_1})^{-a} v_{u'\varpi_{i_k}}$$

and

$$E_{i_1}^a v_{\vec{u}_{\leq k}\varpi_{i_k}} = E_{i_1}^a T_{i_1}^{-1} v_{u'\varpi_{i_1}} = ([a]_{q_{i_1}}!) v_{u'\varpi_{i_1}}. \tag{5.12}$$

It follows from the first equality that $(T_{i_1}^{-1})^* \xi_{\vec{u}_{\leq k}\varpi_{i_k}} = \xi_{u'\varpi_k}$. Combining this with the definition of the quantum minors (2.4), leads to

$$D_0 = (-q_{i_1})^{-a} T_{s_1}(\Delta_{\vec{u}_{[2,k]}\varpi_{i_k}, w_{[2,k]}\varpi_{i_k}}). \tag{5.13}$$

By (5.12), $(T_{i_1}^{-1})^* S^{-1}(E_i)^a \xi_{u'\varpi_{i_k}} = ([a]_{q_{i_1}}!) \xi_{u'\varpi_{i_1}}$. This and the fact that $S^{-1}(E_{i_1}) \xi_{\vec{u}_{\leq k}\varpi_{i_1}} = 0$ imply

$$D_a = ([a]_{q_{i_1}}!) T_{i_1}(\Delta_{u'\varpi_{i_k}, w_{[2,k]}\varpi_{i_1}}) = ([a]_{q_{i_1}}!) \Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}. \tag{5.14}$$

The R-matrices (2.3) satisfy $\mathcal{R}^{w_{\leq k}} = (T_{s_{i_1}} \mathcal{R}^{w_{[2,k]}}) \mathcal{R}^{s_1}$. Using this and the above mentioned highest weight property of $v_{u'\varpi_{i_1}}$ with respect to the $\mathcal{U}_q(\mathfrak{sl}_2)$-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ associated to the i_1-th root, after some computations, we obtain

$$\langle c_{S^{-1}(E_{i_1})^{a-m}, v_{w_{\leq k}\varpi_k}} \tau \otimes \mathrm{id}, \mathcal{R}^w \rangle = \sum_{n=0}^{m} \frac{(q_{i_1}^{-1} - q_{i_1})^n}{q_{i_1}^{n(n-1)/2} [n]_{q_{i_1}}!} D_{a-m+n} F_{\beta_1}^n, \quad \forall m \in [0, a].$$

The left hand side belongs to the ideal $I_w(u)$ for $m \in [1,a]$. By induction on $n = 0, \ldots, a$, applying the q-binomial formula

$$\sum_{n=0}^{m} \frac{(-1)^{m-n} q^{(m-n)(m-n-1)/2}}{q^{n(n-1)/2}[m-n]_q![n]_q!} = \frac{(-1)^m q^{m^2/2}}{[m]_q!} \sum_{n=0}^{m} (-1)^n q^{(m-1)n} \begin{bmatrix} a-m \\ n \end{bmatrix}_q =$$
$$= \prod_{n=0}^{m-1} (1 - q^{2(n+1-m)}),$$

we obtain

$$D_{a-n} = \frac{(-1)^n q_{i_1}^{n(n-1)/2} (q_{i_1}^{-1} - q_{i_1})^n}{[n]_{q_{i_1}}!} D_a F_{\beta_1}^n \mod I_w(u).$$

Combining this with (5.13) and (5.14), and taking into account that $-a = a_{1k}$, gives

$$\Delta_{\vec{u}_{\leq k}\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} F_{\beta_1}^{-a_{1k}} = \frac{(q_{i_1}^{-1} - q_{i_1})^{a_{1k}}}{q^{a_{1k}(a_{1k}-1)/2}} T_{s_1}(\Delta_{\vec{u}_{[2,k]}\varpi_{i_k}, w_{[2,k]}\varpi_{i_k}}) \mod I_w(u).$$

The lemma now follows from the fact that the leading term of the Cauchon map is the identity, the result in Section 5.3 and the fact that $Y_{1,\mathrm{rev}} = F_{\beta_1}$. □

5.5. **A generalization of Theorem 5.1.** For an integral weight $\lambda \in P$, define the $([1,N] \backslash \mathcal{RP}_w(u)) \times [1,N]$ matrix

$$a_{jk}(\lambda) = \begin{cases} 0, & \text{if } j > k \\ \langle \alpha_{i_j}^\vee, \vec{u}_{[j+1,k]}(\lambda) \rangle, & \text{if } j \leq k, \end{cases}$$

which is a specialization of the notation in (5.3) to the case $\mathcal{S} = \mathcal{RP}_w(u)$. The following result generalizes Theorem 5.1. Its proof is analogous and is left to the reader.

Proposition 5.8. *In the setting of Theorem 5.1, for all $\lambda \in P$, the localized quantum minors* (2.21) *satisfy*

$$p(\Delta_{\vec{u}_{\leq k}\lambda, w_{\leq k}\lambda}) = \prod_{j \in [1,k] \backslash \mathcal{RP}_w(u)} \frac{(q_{i_j}^{-1} - q_{i_j})^{a_{jk}(\lambda)}}{q_{i_j}^{a_{jk}(\lambda)(a_{jk}(\lambda)-1)/2}} Y_{j,\mathrm{rev}}^{a_{jk}(\lambda)}, \quad \forall k \in [1,N]$$

in $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$*. The product in the right hand side is taken in decreasing order from left to right.*

6. Quantum twist maps for quantum Schubert cell algebras and Richardson varieties

In this section we define a quantum twist map $\Theta_w : \mathcal{U}^-[w^{-1}] \to \mathcal{U}^-[w]$ that interchanges the direct and reverse presentation of the two algebras. It is a quantum version of the Fomin–Zelevinsky twist map [10]. This map is an algebra antiisomorphism. We furthermore prove that it restricts to antiisomorphisms

$$\Theta_w : \mathcal{U}^-[w^{-1}]/I_{w^{-1}}(u^{-1}) \to \mathcal{U}^-[w]/I_w(u) \quad \text{and} \quad \Theta_w : R_q[R_{u^{-1},w^{-1}}] \to R_q[R_{u,w}].$$

6.1. **The quantum twist maps.** For $w \in W$ consider the algebra antiautomorphism

$$\Theta_w := T_w \tau S \tau \omega \colon \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g}) \tag{6.1}$$

where S is the antipode of $\mathcal{U}_q(\mathfrak{g})$, τ is the antiautomorphism of $\mathcal{U}_q(\mathfrak{g})$ defined in (2.5), and ω is the automorphism of $\mathcal{U}_q(\mathfrak{g})$ defined in (2.15). The repetitive use of the map τ is needed because of the presence of this map in Theorem 2.4. One checks that $\omega\tau = \tau\omega$ and $\omega S = S\omega$, so Θ_w is also given by

$$\Theta_w = T_w \omega \tau S \tau.$$

Proposition 6.1. *For a reduced expression $w = s_{i_1} \dots s_{i_N}$ consider the reduced expression $w^{-1} = s_{i_N} \dots s_{i_1}$. For all $k \in [1, N]$, the antiisomorphism Θ_w satisifies*

$$\Theta_w(T_{i_N} \dots T_{i_{k+1}}(F_{i_k})) = \zeta_{w,k} T_{i_1} \dots T_{i_{k-1}}(F_{i_k}) \tag{6.2}$$

for some $\zeta_{w,k} \in \mathbb{K}^*$. In particular, Θ_w restricts to an algebra antiisomorphism

$$\Theta_w \colon \mathcal{U}^-[w^{-1}] \to \mathcal{U}^-[w].$$

Proof. For $\gamma = \sum_i n_i \alpha_i \in Q$, denote $K_\gamma := \prod_i K_i^{n_i} \in \mathcal{U}_q(\mathfrak{g})$. The antipode satisfies

$$S(x) = \zeta_\gamma \tau(x) K_\gamma^{\mp 1}, \quad \forall x \in \mathcal{U}_q(\mathfrak{b}_\pm)_\gamma, \gamma \in Q \tag{6.3}$$

for some $\zeta_\gamma \in \mathbb{K}^*$ ([19, Lemma 2.2]). This property and the following compatibility property of τ and the braid group action [21, Eq. 8.18(6)]

$$\tau(T_w x) = T_{w^{-1}}^{-1}(\tau(x)), \quad \forall x \in \mathcal{U}_q(\mathfrak{g}),\ w \in W,$$

imply

$$\begin{aligned}
\tau S \tau\big(T_{i_N} \dots T_{i_{k+1}}(F_{i_k})\big) &= \zeta_1 \tau\big(T_{i_N} \dots T_{i_{k+1}}(F_{i_k})\big) K_{\pm s_{i_N} \dots s_{i_{k+1}}(\alpha_{i_k})} \\
&= \zeta_1 T_{i_N}^{-1} \dots T_{i_{k+1}}^{-1}(F_{i_k}) K_{\pm s_{i_N} \dots s_{i_{k+1}}(\alpha_{i_k})}
\end{aligned}$$

for some $\zeta_1 \in \mathbb{K}^*$. Hence,

$$\begin{aligned}
\Theta_w(T_{i_N} \dots T_{i_{k+1}}(F_{i_k})) &= \zeta_1 (T_{i_1} \dots T_{i_N} \omega)\big(T_{i_N}^{-1} \dots T_{i_{k+1}}^{-1}(F_{i_k}) K_{\pm s_{i_N} \dots s_{i_{k+1}}(\alpha_{i_k})}\big) \\
&= \zeta_2 T_{i_1} \dots T_{i_N} \big(T_{i_N}^{-1} \dots T_{i_{k+1}}^{-1}(E_{i_k}) K_{\mp s_{i_N} \dots s_{i_{k+1}}(\alpha_{i_k})}\big) \\
&= \zeta_2 T_{i_1} \dots T_{i_k} \big(E_{i_k} K_{i_k}^{\mp 1}\big) = \zeta_3 T_{i_1} \dots T_{i_{k-1}}(F_{i_k})
\end{aligned}$$

for some $\zeta_2, \zeta_3 \in \mathbb{K}^*$ where in the second equality we used the commutation property [21, Eq. 8.18(5)]

$$\omega T_w(x) = \zeta T_w \omega(x), \forall x \in \mathcal{U}_q(\mathfrak{g})_\gamma, \gamma \in Q$$

for some $\zeta \in \mathbb{K}^*$ depending on γ. □

6.2. **Properties of the quantum twist maps.** For $u \leq w$, denote the canonical projection

$$p' \colon \mathcal{U}^-[w^{-1}] \to \mathcal{U}^-[w^{-1}]/I_{w^{-1}}(u^{-1}). \tag{6.4}$$

By [11, Theorem 3.3(i)], $\{\Delta_{u\lambda, w\lambda}, \lambda \in P^+\}$ is an Ore subset of $\mathcal{U}^-[w]$. We extend p and p' to projections

$$\mathcal{U}^-[w][\Delta_{u\lambda,w\lambda}, \lambda \in P^+]^{-1} \to R_q[R_{u,w}],\ \mathcal{U}^-[w^{-1}][\Delta_{u^{-1}\lambda,w^{-1}\lambda}, \lambda \in P^+]^{-1} \to R_q[R_{u^{-1},w^{-1}}].$$

Theorem 6.2. *The following hold for an arbitrary symmetrizable Kac–Moody algebra* $\mathfrak{g}$*, Weyl group element* w*, base field* $\mathbb{K}$*, and a non-root of unity* $q \in \mathbb{K}^*$*:*

(a) The quantum twist map Θ_w *restricts to an algebra antiisomorphism* $\Theta_w \colon \mathcal{U}^-[w^{-1}] \to \mathcal{U}^-[w]$ *which interchanges the direct and reverse CGL extension presentations* (2.17)–(2.18) *of the two algebras.*

(b) For all $u \in W$*,* $u \leq w$*,*

$$\Theta_w(I_{u^{-1}}(w^{-1})) = I_u(w).$$

(c) The algebra antiisomorphism

$$\Theta_w \colon \mathcal{U}^-[w^{-1}]/I_{w^{-1}}(u^{-1}) \to \mathcal{U}^-[w]/I_w(u) \tag{6.5}$$

induces an antiisomorphism

$$\Theta_w \colon R_q[R_{u^{-1},w^{-1}}] \to R_q[R_{u,w}] \tag{6.6}$$

and

$$\Theta_w(p'(\Delta_{u^{-1}\lambda,w^{-1}\lambda})) = \zeta_\lambda p(\Delta_{u(u^{-1}\lambda),w(u^{-1}\lambda)}), \quad \forall \lambda \in P \tag{6.7}$$

for some $\zeta_\lambda \in \mathbb{K}^*$*. In the last equality the notation for localized quantum minors* (2.21) *is used for* λ *and* $u^{-1}\lambda \in P$*, respectively.*

Proof. Part (a) of the theorem follows at once from Proposition 6.1. Part (b) follows from Theorem 4.1 and the first part.

(c) The antiisomorphism (6.5) induces an antiisomorphism

$$\Theta_w \colon \operatorname{Fract}(\mathcal{U}^-[w^{-1}]/I_{w^{-1}}(u^{-1})) \to \operatorname{Fract}(\mathcal{U}^-[w]/I_w(u)).$$

We will prove that (6.7) holds in $\operatorname{Fract}(\mathcal{U}^-[w]/I_w(u))$. This implies (6.6) and establishes part (c).

The definition of the quantum twist map Θ_w gives that

$$\Theta_w(z) \in (\mathcal{U}^-[w]/I_w(u))_{-w\gamma}, \quad \forall z \in (\mathcal{U}^-[w^{-1}]/I_{w^{-1}}(u^{-1}))_\gamma, \gamma \in Q. \tag{6.8}$$

Eq. (2.19) holds for all $\lambda \in P$. This equation, the fact that Θ_w is an antiisomorphism and the identity

$$w(w^{-1} \pm u^{-1})\lambda = (u \pm w)u^{-1}\lambda$$

imply that $\Theta_w(p(\Delta_{u^{-1}\lambda,w^{-1}\lambda}))p(\Delta_{u(u^{-1}\lambda),w(u^{-1}\lambda)}))^{-1}$ is in the center of the division ring of factions of $\mathcal{U}^-[w]/I_w(u)$. Furthermore, this identity and (6.8) imply

$$\Theta_w(p'(\Delta_{u^{-1}\lambda,w^{-1}\lambda}))p(\Delta_{u(u^{-1}\lambda),w(u^{-1}\lambda)})^{-1} \in Z(\operatorname{Fract}(\mathcal{U}^-[w]/I_w(u)))^{\mathcal{H}}.$$

The Goodearl strong rationality result [3, Theorem II.6.4] for the torus invariant prime ideals of CGL extensions gives $Z(\operatorname{Fract}(\mathcal{U}^-[w]/I_w(u)))^{\mathcal{H}} = \mathbb{K}$ which completes the proof of (6.7) and the theorem. □

The quantum twist map Θ_w will be used in an essential way in the proof of the Berenstein–Zelevinsky conjecture [2] in [18].

The theorem has the following corollary for elements of the form $p(\Delta_{u\lambda,w\lambda}) \in R_q[R_{u,w}]$ for $\lambda \in P$ that belong to the subalgebra $\mathcal{U}^-[w]/I_w(u)$.

Corollary 6.3. *We have*

$$p(\Delta_{u\lambda,w\lambda}) \in \mathcal{U}^-[w]/I_w(u) \quad \textit{for } \lambda \in P^+ \cup u^{-1}(P^+),$$

7. Reverse contractions of $\mathcal{H}$-primes of $\mathcal{U}^-[w]$ and sequences of reverse normal elements

In this section we describe the contractions of the $\mathcal{H}$-prime ideals of each of the quantum Schubert cell algebras $\mathcal{U}^-[w]$ with the intermediate subalgebras associated to the reverse presentation (2.18). Using the quantum twist map, we also construct an explicit sequence of normal elements for each of these chains which we call a sequence of reverse normal elements.

7.1. **Reverse contractions.** As before, $w \in W$ denotes a fixed Weyl group element and we work with a fixed reduced expression (2.1) of it. Denote $w_{<k} := w_{\leq k-1}$, ($w_{<1} := 1$) and $w_{\geq k} := w_{[k,N]}$.

The intermediate subalgebras for the reverse presentation (2.18) of $\mathcal{U}^-[w]$ are given by

$$\mathcal{U}^-[w]_{k,\mathrm{rev}} = \mathbb{K}\langle F_{\beta_k}, \ldots, F_{\beta_N}\rangle = T_{w_{<k}}\mathcal{U}^-[w_{\geq k}], \quad k \in [1, N]. \tag{7.1}$$

For a Weyl group element $u \in W$, $u \leq w$, set

$$\overleftarrow{u}_{[j,k]} := w_{[j,k]}^{\mathcal{LP}_w(u)} \quad \text{and} \quad \overleftarrow{u}_{\geq k} = \overleftarrow{u}_{[k,N]}.$$

The reverse vector notation is suggestive of the definition of left positive subexpression; the point being that left positive subexpressions of reduced expressions are picking up indices to the far left of the reduced expression.

The following result describes the contractions of all $\mathcal{H}$-prime ideals of $\mathcal{U}^-[w]$ with the intermediate subalgebras (7.1) for the reverse presentation of $\mathcal{U}^-[w]$. It follows from Theorems 3.2 and 4.1(a). (One can also use Theorem 6.2 (a)-(b), but this is not really needed at this point.)

Theorem 7.1. *For all pairs of Weyl group elements $u \leq w$ and reduced expressions (2.1) of w, the contractions of the ideal $I_w(u)$ with the subalgebras $\mathcal{U}^-[w]_{k,\mathrm{rev}}$ are given by*

$$I_w(u) \cap T_{w_{<k}}\mathcal{U}^-[w_{\geq k}] = T_{w_{<k}}\Big(I_{w_{\geq k}}(\overleftarrow{u}_{\geq k})\Big), \quad \forall k \in [1, N].$$

7.2. **Sequences of reverse normal elements.** Consider the canonical projection

$$p \colon \mathcal{U}^-[w] \to \mathcal{U}^-[w]/I_w(u).$$

The chain of subalgebras

$$\mathcal{U}^-[w]_{N,\mathrm{rev}} \subset \mathcal{U}^-[w]_{N-1,\mathrm{rev}} \subset \ldots \subset \mathcal{U}^-[w]_{1,\mathrm{rev}} = \mathcal{U}^-[w]$$

gives rise to the chain of subalgebras of the prime quotient

$$p(T_{w_{<N}}\mathcal{U}^-[w_{\geq N}]) \subseteq \ldots \subseteq p(T_{w_{<2}}\mathcal{U}^-[w_{\geq 2}]) \subseteq p(\mathcal{U}^-[w]) \cong \mathcal{U}^-[w]/I_w(u). \tag{7.2}$$

By Theorem 7.1, the k-th term in this chain is given by

$$\begin{aligned} p(T_{w_{<k}}\mathcal{U}^-[w_{\geq k}]) &\cong T_{w_{<k}}\mathcal{U}^-[w_{\geq k}]/(T_{w_{<k}}\mathcal{U}^-[w_{\geq k}] \cap I_w(u)) \\ &\cong T_{w_{<k}}\mathcal{U}^-[w_{\geq k}]/T_{w_{<k}}(I_{w \geq k}(\overleftarrow{u}_{\geq k})) \cong \mathcal{U}^-[w_{\geq k}]/I_{w \geq k}(\overleftarrow{u}_{\geq k}). \end{aligned} \tag{7.3}$$

For simplicity of the notation we will write

$$w_{\geq k}^{-1} := (w_{\geq k})^{-1}, \quad \overleftarrow{u}_{\geq k}^{-1} := (\overleftarrow{u}_{\geq k})^{-1}.$$

Theorem 7.2. *Assume the setting of Theorem 7.1. For all* $k \in [1, N]$ *and* $\lambda \in P^+$, $p\big(T_{w<k}(\Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k})\lambda, w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k})\lambda})\big)$ *is a nonzero normal element of* $p(\mathcal{U}^-[w]_{k,\mathrm{rev}})$, *and more precisely*

$$p\Big(T_{w>k}\Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\lambda), w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\lambda)}\Big)z = q^{-\langle w_{<k}(w_{\geq k}\overleftarrow{u}^{-1}_{\geq k}+1)\lambda, \gamma\rangle} zp\Big(T_{w>k}\Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\lambda), w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\lambda)}\Big) \tag{7.4}$$

for all $z \in p(\mathcal{U}^-[w]_{k,\mathrm{rev}})_\gamma$, $\gamma \in Q$.

The sequence

$$\widetilde{\Delta}_k := p\big(T_{w_{<N}}\Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k}), w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k})}\big), k = N, \dots, 1 \tag{7.5}$$

has the property that its k-th element is a nonzero normal element of the k-th algebra $p(\mathcal{U}^-[w]_{k,\mathrm{rev}})$ *in the chain* (7.2). *In particular, the elements in the sequence quasi-commute,*

$$\widetilde{\Delta}_l\widetilde{\Delta}_k = q^{\langle (w^{-1}_{\geq k}+\overleftarrow{u}^{-1}_{\geq k})\varpi_{i_k}, (w^{-1}_{\geq l}-\overleftarrow{u}^{-1}_{\geq l})\varpi_{i_l}\rangle}\widetilde{\Delta}_k\widetilde{\Delta}_l$$

for all $1 \leq k < l \leq N$.

The theorem follows by applying the quantum twist map to the sequence of normal elements from Theorem 4.4 for the algebra $\mathcal{U}^-[w^{-1}]$, using Theorem 6.2 and the identity

$$\Theta_{w_1w_2} = T_{w_1}\Theta_{w_2} \quad \text{for} \;\; w_1, w_2 \in W \;\; \text{such that} \;\; \ell(w_1w_2) = \ell(w_1) + \ell(w_2). \tag{7.6}$$

The inverses of Weyl group elements arise from the application of Theorem 6.2 (c). There are simpler sequences of normal elements but they do not have the property proved in the next section characterizing the Cauchon generators for $\mathcal{U}^-[w]/I_w(u)$.

We will call the sequence (7.5), *a sequence of reverse normal elements* for $\mathcal{U}^-[w]/I_w(u)$. It is a sequence of normal elements in the sense of Definition 4.3 for the chain of subalgebras (7.2).

8. Sequences of reverse normal elements vs. Cauchon generators for prime factors of $\mathcal{U}^-[w]$

In this section we use the quantum twist maps to obtain explicit expressions for the Cauchon generators of the $\mathcal{H}$-prime factors of $\mathcal{U}^-[w]$ with respect to the direct presentation (2.17) of $\mathcal{U}^-[w]$ in terms of the sequences of reverse normal elements from the previous section. The latter are associated to the reverse presentation (2.18) of $\mathcal{U}^-[w]$. This produces another quantum cluster for each $\mathcal{H}$-prime factor of the algebra $\mathcal{U}^-[w]$.

In the next section we show that a recursive combination of the results of this section and Section 5 applied in a recursive fashion to subalgebras of $\mathcal{U}^-[w]$ can be used to construct whole families of toric frames for the $\mathcal{H}$-prime factors of $\mathcal{U}^-[w]$.

8.1. **Statement of the main result.** As before, w denotes a Weyl group element with a fixed reduced expression (2.1), and u is a Weyl group element with $u \leq w$. Theorem 4.1(a) gives that the Cauchon diagram $\mathcal{CD}(I_w(u))$ of the $\mathcal{H}$-prime ideal $I_w(u)$ of $\mathcal{U}^-[w]$ for the direct presentation (2.17) equals the index set $\mathcal{LP}_w(u)$ of the left positive subexpression of (2.1) with product u:

$$\mathcal{CD}(I_w(u)) = \mathcal{LP}_w(u).$$

So, the Cauchon deleting derivation method applied to the direct presentation (2.17) of $\mathcal{U}^-[w]$ defines a sequence of nonzero elements

$$Y_k \in \mathrm{Fract}(\mathcal{U}^-[w]/I_w(u)), \quad k \in [1,N] \backslash \mathcal{LP}_w(u).$$

The elements $\{Y_k^{\pm 1} \mid k \in [1,N] \backslash \mathcal{LP}_w(u)\}$ generate a copy of a quantum torus inside $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ and this quantum torus contains $\mathcal{U}^-[w]/I_w(u)$.

Recall the partial order $\prec$ on $[1,N]$ from (5.1). Consider the following integer matrix of size $(N - |\mathcal{LP}_w(u)|) \times N$ whose rows are indexed by the set $[1,N] \backslash \mathcal{LP}_w(u)$:

$$b_{lk} = \begin{cases} 0, & \text{if } l < k \\ 1, & \text{if } l = k \\ \langle \alpha_{i_l}^\vee, \overleftarrow{u}^{-1}_{[k,l-1]}(\varpi_{i_k}) \rangle = \delta_{k \prec l} - \sum_{k \preceq j < l, j \in \mathcal{LP}_w(u)} \langle \alpha_{i_l}^\vee, \overleftarrow{u}^{-1}_{[j+1,l-1]}(\alpha_{i_j}) \rangle, & \text{if } l > k. \end{cases}$$

The equality in the third case follows from Eq. (5.5) in Proposition 5.5. As in the previous section, we write

$$\overleftarrow{u}^{-1}_{[k,l]} := (\overleftarrow{u}_{[k,l]})^{-1}.$$

Theorem 8.1. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and w be a Weyl group element with reduced expression* (2.1)*. Let $u \in W$, $u \leq w$. For all base fields $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$, in* $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$,

$$p\big(T_{w_{<k}} \Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k}), w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k})}\big) = \prod_{l \in [k,N] \backslash \mathcal{LP}_w(u)} \frac{(q_{i_l}^{-1} - q_{i_l})^{b_{lk}}}{q_{i_l}^{b_{lk}(b_{lk}-1)/2}} Y_l^{b_{lk}}, \quad \forall k \in [1,N]$$

where $p \colon \mathcal{U}^-[w] \to \mathcal{U}^-[w]/I_w(u)$ is the canonical projection. The product in the right hand side is taken in a decreasing order from left to right.

The case of the theorem when $\mathcal{U}^-[w]$ equals the algebra of quantum matrices was proved by Cauchon in [6], the case $u = 1$ (all w and $\mathfrak{g}$) was obtained in [12].

Remark 8.2. Up to a reordering of rows and columns, the matrix in Theorem 8.1 equals the one in Theorem 5.1 for the Weyl group elements u^{-1} and w^{-1} (with the reversed to (2.1) reduced expression). Because of this the matrix in Theorem 8.1 has the triangular properties in Remark 5.2 (after reordering of rows and columns).

The formulas in Theorem 8.1 prove that the quantum torus inside $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ generated by

$$\{Y_k^{\pm 1} \mid k \in [1,N] \backslash \mathcal{LP}_w(u)\}$$

also has generators

$$\big\{p\big(T_{w_{<k}} \Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k}), w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k})}\big) \mid k \in [1,N] \backslash \mathcal{LP}_w(u)\big\}. \tag{8.1}$$

This quantum torus contains $\mathcal{U}^-[w]/I_w(u)$, and is a localizations of the prime factor. The elements of the second set are monomials in the elements of the first set with exponents given by a triangular integral matrix, and vice versa the elements of the first set are monomials in the elements of the second set. Finally, Theorem 8.1 also implies that the elements

$$\big\{p\big(T_{w_{<k}} \Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k}), w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k}\varpi_{i_k})}\big) \mid k \in \mathcal{LP}_w(u)\big\}$$

are mononomials in the elements of the set (8.1). One can easily derive explicit formulas for this; we leave the details to the reader.

Corollary 8.3. *For every symmetrizable Kac–Moody algebra $\mathfrak{g}$, pair of Weyl group elements $u \leq w$, base field $\mathbb{K}$, and a non-root of unity $q \in \mathbb{K}^*$ such that $\sqrt{q} \in \mathbb{K}^*$, the prime factor $\mathcal{U}^-[w]/I_w(u)$ has a toric frame $M \colon \mathbb{Z}^{[1,N]\setminus \mathcal{LP}_w(u)} \to \mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ given by*

$$M(e_k) := p\big(T_{w_{<k}} \Delta_{\overleftarrow{u}_{\geq k}(\overleftarrow{u}^{-1}_{\geq k} \varpi_{i_k}), w_{\geq k}(\overleftarrow{u}^{-1}_{\geq k} \varpi_{i_k})}\big).$$

The corresponding multiplicatively skewsymmetric bicharacter is given by

$$\Lambda(e_l, e_k) := \sqrt{q}^{\langle (w^{-1}_{\geq k} + \overleftarrow{u}^{-1}_{\geq k}) \varpi_{i_k}, (w^{-1}_{\geq l} - \overleftarrow{u}^{-1}_{\geq l}) \varpi_{i_l} \rangle}, \quad \forall l > k \in \mathbb{Z}^{[1,N]\setminus \mathcal{LP}_w(u)}.$$

The toric frame can be augmented to a quantum seed of $\mathcal{U}^-[w]/I_w(u)$ using Leclerc's matrices [24, Theorem 4.5 and Corollary 4.4], *cf. Remark 8.2.*

Theorem 8.1 is proved in the next subsections. For the purposes of an induction argument, we establish a stronger result. For $\lambda \in P$ and $l \in [1, N]$, denote

$$b_l(\lambda) = \langle \alpha^\vee_{i_l}, \overleftarrow{u}_{\geq l}(\lambda) \rangle.$$

Then $b_{lk} = b_l\big(\overleftarrow{u}^{-1}_{\geq k} \varpi_{i_k}\big)$.

Proposition 8.4. *In the setting of Theorem 8.1, for all $\lambda \in P$, we have*

$$p(T_{w_{<k}} \Delta_{\overleftarrow{u}_{\geq k}\lambda, w_{\geq k}\lambda}) = \prod_{l \in [k,N]\setminus \mathcal{LP}_w(u)} \frac{(q^{-1}_{i_l} - q_{i_l})^{b_l(\lambda)}}{q_{i_l}^{b_l(\lambda)(b_l(\lambda)-1)/2}} Y_l^{b_l(\lambda)}, \quad \forall k \in [1, N]$$

in $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$, recall the notation (2.21) *for localized quantum minors. The product in the right hand side is taken in a decreasing order from left to right.*

8.2. **Proof of Proposition 8.4.** For $\lambda \in P$, set

$$\overline{\Delta}_{\lambda,k,\mathrm{rev}} := p(T_{w_{<k}} \Delta_{\overleftarrow{u}_{\geq k}\lambda, w_{\geq k}\lambda}).$$

The identity (7.6) for the quantum twist maps and Theorem 6.2 imply that

$$\overline{\Delta}_{\lambda,k,\mathrm{rev}} = \zeta_{k,w} \Theta_w p'\big(\Delta_{\overleftarrow{u}^{-1}_{\geq k}(\overleftarrow{u}_{\geq k}\lambda), w^{-1}_{\geq k}(\overleftarrow{u}_{\geq k}\lambda)}\big) \quad \text{for some } \ \zeta_{w,k} \in \mathbb{K}^*,$$

recall (6.4). It follows from Theorem 5.1 (applied to the Weyl group elements u^{-1} and w^{-1}) and Theorem 6.2 (a) that

$$\overline{\Delta}_{\lambda,k,\mathrm{rev}} = \zeta_k \prod_{l \in [k,N]\setminus \mathcal{LP}_w(u)} Y_l^{b_l(\lambda)}, \quad \forall k \in [1, N] \tag{8.2}$$

for some $\zeta_k \in \mathbb{K}^*$.

We obtain an explicit formula for the scalars in (8.2) by induction. The arguments for the inductive statement are different from those in Section 5.4. It follows from the definition of left positive subexpressions that for all $j \in [1, N]$,

($**$) *$\mathcal{LP}_w(u) \cap [1, j]$ is the index set of the left positive subexpression of $w_{i_1} \ldots w_{i_j}$ with total product $\vec{u}_{\leq j}$.*

For $k \leq j$, let $\zeta_{\lambda, w_{\leq j}, \vec{u}_{\leq j}, k}$ be the scalar in the right hand side of (8.2) for the quadruple $(\lambda, w_{\leq j}, \vec{u}_{\leq j}, k)$ and the above choices of reduced expressions of $w_{\leq j}$. Proposition 8.4 follows by induction from the next lemma and ($**$).

Lemma 8.5. *For all Weyl group elements $u \leq w$ and $1 \leq k \leq N = \ell(w)$,*

$$\zeta_{\lambda,w,u,k} = \begin{cases} \zeta_{s_{i_N}\lambda, w_{\leq N-1}, \overleftarrow{u}_{\leq N-1}, k}, & \text{if } N \in \mathcal{LP}_w(u) \\ \frac{(q^{-1}_{i_N} - q_{i_N})^{b_N(\lambda)}}{q_{i_N}^{b_N(\lambda)(b_N(\lambda)-1)/2}} \zeta_{\lambda, w_{\leq N-1}, \overleftarrow{u}_{\leq N-1}, k}, & \text{if } N \notin \mathcal{LP}_w(u) \end{cases}$$

where $\zeta_{\lambda, w_{\leq N-1}, \vec{u}_{\leq N-1}, N} := 1$.

Before we proceed with the proof of Lemma 8.5, we establish a general fact on Weyl group invariance of localized quantum minors (2.21) which is of independent interest. This fact will also play a role in the next section in connection to toric frames for the quantum Richardson varieties.

Proposition 8.6. *Let* $\lambda \in P$, w *be a Weyl group element with reduced expression* (2.1) *and* $u \leq w$ *be such that* $N \in \mathcal{RP}_w(u)$. *Then*

$$p(\Delta_{u(\lambda), w(\lambda)}) = p(\Delta_{\vec{u}_{\leq N-1}(s_{i_N}\lambda), w_{\leq N-1}(s_{i_N}\lambda)}) \tag{8.3}$$

where the localized quantum minors use the notation from (2.21) *with* λ *and* $s_i(\lambda) \in P$, *respectively.*

Proof. By Theorem 4.2

$$I_w(u) \cap \mathcal{U}^-[w_{\leq N-1}] = I_{w_{\leq N-1}}(\vec{u}_{\leq N-1}).$$

The embedding

$$\mathcal{U}^-[w_{\leq N-1}] \hookrightarrow \mathcal{U}^-[w]$$

induces the embedding

$$\varphi\colon \mathcal{U}^-[w_{\leq N-1}]/I_{w_{\leq N-1}}(\vec{u}_{\leq N-1}) \hookrightarrow \mathcal{U}^-[w]/I_w(u).$$

We will denote by the same letter the extension of this embedding to the corresponding division rings of fractions. It is easy to see that the Cauchon generators of the prime factor on the left with respect to the reverse presentation

$$\mathcal{U}^-[w_{\leq N-1}] = \mathbb{K}[F_{\beta_{N-1}}][F_{\beta_{N-2}}; \sigma^*_{N-1}, \delta^*_{N-1}] \ldots [F_{\beta_1}; \sigma^*_1, \delta^*_1]$$

are precisely

$$\{\varphi^{-1}(Y_{k,\mathrm{rev}}) \mid k \in [1, N-1] \backslash \mathcal{RP}_w(u)\}.$$

The equality (8.3) now follows from the fact that the two sides have the same expressions in the sets $\{Y_{k,\mathrm{rev}} \mid k \in [1, N-1] \backslash \mathcal{RP}_w(u)\}$ and $\{\varphi^{-1}(Y_{k,\mathrm{rev}}) \mid k \in [1, N-1] \backslash \mathcal{RP}_w(u)\}$ given by Proposition 8.4. (It is straightforward to see that the exponents in the two expressions are the same.) □

Proof of Lemma 8.5. We consider two cases: (1) $N \in \mathcal{LP}_w(u)$ and (2) $N \notin \mathcal{LP}_w(u)$.

Case (1) $N \in \mathcal{LP}_w(u)$. This implies $us_{i_N} < u$. Hence, $N \in \mathcal{RP}_w(u)$ and $\vec{u}_{\leq N-1} = \overleftarrow{u}_{\leq N-1}$. The lemma now follows from Proposition 8.6.

Case (2) $N \notin \mathcal{LP}_w(u)$. We prove the statement of the lemma for $\lambda \in P^+$. The general case follows from the commutation relations between the elements Y_l and the definition of the localized quantum minors (2.21).

The end of the direct presentation (2.17) of $\mathcal{U}^-[w]$ is

$$\mathcal{U}^-[w] = \mathcal{U}^-[w_{\leq N-1}][F_{\beta_N}; \sigma_N, \delta_N]. \tag{8.4}$$

In this case we are in the situation of case 2 in Section 2.3 and $\overleftarrow{u}_{[k,N-1]} = \overleftarrow{u}_{\geq k}$. Using the fact that $v_{w_{\geq k}\lambda}$ is a lowest weight vector for the $\mathcal{U}_q(\mathfrak{sl}_2)$-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ spanned by $T_{w_{[k,N-1]}}\{E_{i_N}, F_{i_N}, K_{i_N}^{\pm 1}\}$ with lowest weight $-b_N(\lambda)\varpi_{i_N} = -\langle \lambda, \alpha_{i_N}^\vee \rangle \varpi_{i_N}$, one easily obtains that, with respect to the presentation (8.4), the leading term of

$$T_{w_{<k}} \Delta_{\overleftarrow{u}_{\geq k}\lambda, w_{\geq k}\lambda} \quad \text{is} \quad \frac{(q_{i_N}^{-1} - q_{i_N})^{b_N(\lambda)}}{q_{i_N}^{b_N(\lambda)(b_N(\lambda)-1)/2}} F_{\beta_N}^{b_N(\lambda)} \big(T_{w_{<k}} \Delta_{\overleftarrow{u}_{[k,N-1]}\lambda, w_{[k,N-1]}\lambda}\big).$$

Now the lemma follows from this, (8.2) and the fact that the leading term of the Cauchon map from Section 2.3 is the identity. □

8.3. **A second Weyl group invariance of localized quantum minors.** Analogously to the proof of Proposition 8.6 one derives the following mirror version of it using Proposition 8.4. This fact will be needed in the next section for the construction of toric frames for the quantum Richardson varieties.

Proposition 8.7. *Let $\lambda \in P$, w be a Weyl group element with reduced expression* (2.1) *and $u \leq w$ be such that $1 \in \mathcal{RP}_w(u)$. Then*

$$p(\Delta_{u\lambda, w\lambda}) = p(T_{s_1} \Delta_{\vec{u}_{\geq 1}\lambda, w_{\geq 1}\lambda})$$

in the notation from (2.21).

9. Families of toric frames for $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$

In this section we construct families of toric frames for the $\mathcal{H}$-prime factors of $\mathcal{U}^-[w]$ and the quantum Richardson algebras $R_q[R_{u,w}]$. This is done by a recursive application of the results of Sections 5 and 8 to different chains of subalgebras of $\mathcal{U}^-[w]/I_w(u)$.

9.1. **Families of chains of subalgebras of $\mathcal{U}^-[w]$ and contractions of prime ideals.** Let Ξ_N be the subset of the symmetric group S_N which consists of all permutations $\pi \in S_N$ such that

$$\pi(k) = \max \pi([1, k-1]) + 1 \ \text{ or } \ \pi(k) = \min \pi([1, k-1]) - 1, \ \ \forall k \in [2, N].$$

The subset Ξ_N can be equivalently described as the set of all $\pi \in S_N$ such that $\pi([1, k])$ is an interval for all $k \in [2, N]$.

Consider a symmetric CGL extension R, recall Definition 2.2. Each $\pi \in \Xi_N$ gives rise to a CGL extension presentation [16, Remark 6.5] of R,

$$R = \mathbb{K}[x_{\pi(1)}][x_{\pi(2)}; \sigma'_{\pi(2)}, \delta'_{\pi(2)}] \cdots [x_{\pi(N)}; \sigma'_{\pi(N)}, \delta'_{\pi(N)}], \tag{9.1}$$

where

$$\sigma'_{\pi(k)} := \sigma_{\pi(k)}, \ \ h'_{\pi(k)} := h_{\pi(k)} \ \text{ and } \ \delta'_{\pi(k)} := \delta_{\pi(k)}, \quad \text{if } \ \pi(k) = \max \pi([1, k-1]) + 1$$

and

$$\sigma'_{\pi(k)} := \sigma^*_{\pi(k)}, \ \ h'_{\pi(k)} := h^*_{\pi(k)} \ \text{ and } \ \delta'_{\pi(k)} := \delta^*_{\pi(k)}, \quad \text{if } \ \pi(k) = \min \pi([1, k-1]) - 1.$$

The direct presentation (3.1) of R corresponds to the identity element $\pi = 1$ and the reverse presentation (3.2) to π being equal to the longest element of S_N. It was proved in [17, Theorem 8.2] that, under very mild assumptions, each $\pi \in \Xi_N$ gives rise to a quantum seed of R and that those seeds are related by mutations. This was used to develop a general theory of quantum cluster algebra structures on symmetric CGL extensions in [17]. In what follows we use the results of Sections 5 and 8 to construct families of toric frames for all $\mathcal{H}$-prime factors of the algebras $\mathcal{U}^-[w]$ indexed by the elements of Ξ_N. The prime quotients of an algebra usually behave in much more complicated fashion than the algebra itself and cluster structures for such are more difficult to construct.

Fix a Weyl group element w and a reduced expression (2.1) of it. Let $\pi \in \Xi_N$ where $N := \ell(w)$. For $k \in [1, N]$, define $c(k) \leq d(k) \in [1, N]$ by

$$[c(k), d(k)] := \pi([1, k]).$$

By the definition of Ξ_N,

$$\pi(k) = c(k) \ \text{ or } \ d(k). \tag{9.2}$$

The k-th intermediate subalgebra of R with respect to the presentation (9.1) is $R_{[c(k),d(k)]}$. For the quantum Schubert cell algebra $\mathcal{U}^-[w]$,

$$\mathcal{U}^-[w]_{[c(k),d(k)]} = T_{w_{<c(k)}} \mathcal{U}^-[w_{[c(k),d(k)]}]. \tag{9.3}$$

We have the direct CGL extension presentation of the algebra $\mathcal{U}^-[w]_{[c(k),d(k)]}$

$$\mathcal{U}^-[w]_{[c(k),d(k)]} = \mathbb{K}[F_{\beta_{c(k)}}][F_{\beta_{c(k)+1}}; \sigma_{c(k)+1}, \delta_{c(k)+1}] \dots [F_{\beta_{d(k)}}; \sigma_{d(k)}, \delta_{d(k)}]$$

and the reverse CGL extension presentation of it

$$\mathcal{U}^-[w]_{[c(k),d(k)]} = \mathbb{K}[F_{\beta_{d(k)}}][F_{\beta_{d(k)-1}}; \sigma^*_{d(k)-1}, \delta^*_{d(k)-1}] \dots [F_{\beta_{c(k)}}; \sigma^*_{c(k)}, \delta^*_{c(k)}].$$

The automorphisms σ_j, σ_j^* and skew derivations δ_j, δ_j^* are the ones from (2.17) and (2.18), restricted to the appropriate subalgebras. Another way to look at these presentations is to take the two CGL extension presentations (2.17)–(2.18) of $\mathcal{U}^-[w_{[c(k),d(k)]}]$ associated to the reduced expression

$$w_{[c(k),d(k)]} = s_{i_{c(k)}} \dots s_{i_{d(k)}}$$

and to apply the automorphism $T_{w_{<c(k)}}$ to the corresponding Lusztig root vectors, taking into account (9.3). The index sets of the left and right positive subexpressions of this expression will be computed as subsets of $[c(k), d(k)]$ (not of $[1, d(k) - c(k) + 1]$).

Let u be a Weyl group element such that $u \leq w$. Next, we describe the projections of the chain of subalgebras

$$\mathcal{U}^-[w]_{[c(1),d(1)]} \subset \mathcal{U}^-[w]_{[c(2),d(2)]} \subset \dots \subset \mathcal{U}^-[w]_{[c(N),d(N)]} = \mathcal{U}^-[w] \tag{9.4}$$

into each prime factor $\mathcal{U}^-[w]/I_w(u)$. Define recursively a sequence of Weyl group elements

$$u(N) := u, u(N-1), \dots, u(1) \in W$$

as follows. Recall (9.2).

Case (1) $\pi(k+1) = d(k+1)$. Set

$$u(k) = \min(u(k+1) s_{i_{d(k+1)}}, u(k+1))$$

with respect to the Bruhat order.

Case (2) $\pi(k+1) = c(k+1)$. Set

$$u(k) = \min(s_{i_{c(k+1)}} u(k+1), u(k+1)).$$

Note that the sequence depends on π. This dependence will not be shown explicitly for simplicity of the notation. The sequence can be equivalently defined by setting

$$u(k) := \begin{cases} u(k+1) s_{i_{d(k+1)}}, & \text{if } d(k+1) \in \mathcal{RP}_{w_{[c(k+1),d(k+1)]}}(u(k+1)) \\ u(k+1), & \text{if } d(k+1) \notin \mathcal{RP}_{w_{[c(k+1),d(k+1)]}}(u(k+1)). \end{cases}$$

in the first case and

$$u(k) := \begin{cases} s_{i_{c(k+1)}} u(k+1), & \text{if } c(k+1) \in \mathcal{LP}_{w_{[c(k+1),d(k+1)]}}(u(k+1)) \\ u(k+1), & \text{if } c(k+1) \notin \mathcal{LP}_{w_{[c(k+1),d(k+1)]}}(u(k+1)). \end{cases}$$

in the second.

Since $\pi(k+1) = c(k+1)$ or $d(k+1)$, and $[c(k), d(k)] = [c(k+1), d(k+1)] \backslash \{\pi(k+1)\}$, each of the extensions

$$\mathcal{U}^-[w]_{[c(k),d(k)]} \subset \mathcal{U}^-[w]_{[c(k+1),d(k+1)]}$$

falls within the framework of subalgebras of quantum Schubert cell algebras treated in Sections 4 or 7, i.e., subalgebras obtained by removing the first or last of the Lusztig root vectors. Recursively applying Theorems 4.2 and 7.1 to the chain of subalgebras (9.4), we obtain the following,

Corollary 9.1. *For all pairs of Weyl group elements $u \leq w$, reduced expressions* (2.1) *of w, and elements $\pi \in \Xi_N$, the contractions of the ideal $I_w(u)$ with the subalgebras $\mathcal{U}^-[w]_{[c(k),d(k)]}$ are given by*

$$I_w(u) \cap \mathcal{U}^-[w]_{[c(k),d(k)]} = T_{w_{<c(k)}}\Big(I_{w_{[c(k),d(k)]}}(u(k))\Big), \quad \forall k \in [1,N].$$

Therefore, in the framework of Corollary 9.1, the images of the subalgebras (9.4) under the projection $p\colon \mathcal{U}^-[w] \to \mathcal{U}^-[w]/I_w(u)$ are given by

$$\begin{aligned} p(\mathcal{U}^-[w]_{[c(k),d(k)]}) &\cong \mathcal{U}^-[w]_{[c(k),d(k)]}/T_{w_{<c(k)}}\Big(I_{w_{[c(k),d(k)]}}(u(k))\Big) \\ &\cong \mathcal{U}^-[w_{[c(k),d(k)]}]/I_{w_{[c(k),d(k)]}}(u(k)). \end{aligned}$$

For an arbitrary $\pi \in \Xi_N$, each of the extensions

$$p(\mathcal{U}^-[w]_{[c(k),d(k)]}) \subset p(\mathcal{U}^-[w]_{[c(k+1),d(k+1)]})$$

falls within the framework of those treated in Theorems 5.1 and 8.1. We will use those results to construct sequences of normal elements inside the prime factors $\mathcal{U}^-[w]/I_w(u)$.

9.2. **Families of toric frames for the algebras $\mathcal{U}^-[w]/I_w(u)$ and $R_q[R_{u,w}]$.** Define the following subset of $[1,N]$,

$$\begin{aligned} D(\pi) :=& \{d(k) \mid k \in [1,N], \pi(k) = d(k), d(k) \in \mathcal{RP}_{w_{[c(k),d(k)]}}(u(k))\} \\ & \cup \{c(k) \mid k \in [1,N], \pi(k) = c(k), c(k) \in \mathcal{LP}_{w_{[c(k),d(k)]}}(u(k))\}. \end{aligned}$$

(The dependence of the set $D(\pi)$ on u is not explicitly shown for simplicity of the notation.) The second definition of $u(k)$ implies that

$$u(k) = w_{[c(k),d(k)]}^{D(\pi)\cap[c(k),d(k)]}, \quad \forall k \in [1,N]$$

in the notation of Section 4.1. For $k \in [1,N]$, define the weights

$$\lambda^{\pm}_{\pi,k} := \begin{cases} w_{<c(k)}(w_{[c(k),d(k)]} \pm u(k))\varpi_{i_{d(k)}}, & \text{if } \pi(k) = d(k) \\ w_{<c(k)}(w_{[c(k),d(k)]} \pm u(k))u(k)^{-1}\varpi_{i_{c(k)}}, & \text{if } \pi(k) = c(k) \end{cases}$$

and the sequence of elements

$$\Delta_{\pi,k} := \begin{cases} T_{w_{<c(k)}}\Delta_{u(k)\varpi_{i_{d(k)}}, w_{[c(k),d(k)]}\varpi_{i_{d(k)}}}, & \text{if } \pi(k) = d(k) \\ T_{w_{<c(k)}}\Delta_{u(k)(u(k)^{-1}\varpi_{i_{c(k)}}), w_{[c(k),d(k)]}(u(k)^{-1}\varpi_{i_{c(k)}})}, & \text{if } \pi(k) = c(k) \end{cases}$$

in the notation (2.21) for localized minors. It follows from Corollary 6.3 that

$$p(\Delta_{\pi,k}) \in (\mathcal{U}^-[w]/I_w(u))_{\lambda^-_{\pi,k}}, \quad \forall k \in [1,N].$$

By Theorems 4.4 and 7.2, $p(\Delta_{\pi,k})$ are nonzero normal elements of $p(\mathcal{U}^-[w]_{[c(k),d(k)]})$ for all $\lambda \in P$, and more precisely,

$$p(\Delta_{\pi,k})z = q^{-\langle \lambda^+_{\pi,k}, \gamma\rangle} z p(\Delta_{\pi,k})$$

for all $z \in p(\mathcal{U}^-[w]_{[c(k),d(k)]})_\gamma, \gamma \in Q$. In particular,

$$p(\Delta_{\pi,1}), \dots, p(\Delta_{\pi,N})$$

is a sequence of normal elements for the chain of subalgebras of $\mathcal{U}^-[w]/I_w(u)$ consisting of the images of the intermediate subalgebras (9.4) of $\mathcal{U}^-[w]$ with respect to the presentation (2.5). Furthermore,

$$p(\Delta_{\pi,k})p(\Delta_{\pi,j}) = q^{-\langle \lambda^+_{\pi,k}, \lambda^-_{\pi,j}\rangle} p(\Delta_{\pi,j})p(\Delta_{\pi,k}), \quad \forall k > j \in [1,N].$$

Theorem 9.2. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, $u \leq w$ a pair of Weyl group elements, $\mathbb{K}$ a base field, and $q \in \mathbb{K}^*$ a non-root of unity such that $\sqrt{q} \in \mathbb{K}^*$. For all $\pi \in \Xi_N$, the algebras $\mathcal{U}^-[w]/I_w(u)$ and $R_q[R_{u,w}]$ have a toric frame $M \colon \mathbb{Z}^{[1,N]\backslash D(\pi)} \to \operatorname{Fract}(\mathcal{U}^-[w]/I_w(u))$ given by*

$$M(e_k) = p(\Delta_{\pi,k}) \quad \forall k \in [1,N]\backslash D(\pi).$$

The corresponding multiplicatively skewsymmetric bicharacter is given by

$$\Lambda(e_k, e_j) := \sqrt{q}^{\,-\langle \lambda^+_{\pi,k}, \lambda^-_{\pi,j}\rangle}, \quad \forall k > j \in [1,N]\backslash D(\pi).$$

We have $D(1) = \mathcal{RP}_w(u)$ and $D(w_\circ) = \mathcal{LP}_w(u)$ where $w_\circ$ denotes the longest element of Ξ_N. In the special cases of $\pi = 1$ and $\pi = w_\circ$, the toric frames in Theorem 9.2 recover the ones in Corollaries 5.4 and 8.3.

9.3. **Proof of Theorem 9.2.** Before we proceed with the proof of the theorem we establish two lemmas.

Lemma 9.3. *For all symmetrizable Kac–Moody algebras $\mathfrak{g}$, Weyl group elements $u \leq w$ and $\pi \in \Sigma_N$ (where $N = \ell(w)$), the elements $p(\Delta_{u\lambda,w\lambda})$ are Laurent monomials in the quantum torus generators $\{M(e_k) \mid k \in [1,N]\backslash D(\pi)\}$ for all integral weights λ.*

The elements in the lemma are precisely the normal elements of $\mathcal{U}^-[w]/I_w(u)$ for the localization defining quantum Richardson varieties in Section 2.5. The lemma follows from Propositions 8.6 and 8.7 and the identity

$$\Delta_{u\varpi_i, w\varpi_i} = \Delta_{\vec{u}_{\leq N-1}\varpi_i, w_{\leq N-1}\varpi_i}, \quad \forall i \in [1,N], i \neq i_N,$$

the latter in terms of the reduced expression (2.1).

Lemma 9.4. *In the setting of the previous lemma and the reduced expression* (2.1)

$$\begin{aligned} p(\Delta_{\vec{u}_{\leq N-1}\varpi_{i_N}, w\varpi_{i_N}}) &= (q_{i_N}^{-1} - q_{i_N}) p(F_{\beta_N}) p(\Delta_{\vec{u}_{\leq N-1}\varpi_{i_N}, w_{\leq N-1}\varpi_{i_N}}) - p(x) \\ &= \begin{cases} 0, & \text{if } N \in \mathcal{RP}_w(u) \\ p(\Delta_{u\varpi_{i_N}, w\varpi_{i_N}}), & \text{if } N \notin \mathcal{RP}_w(u) \end{cases} \end{aligned}$$

for some $x \in \mathcal{U}^-[w_{\leq N-1}]$.

Proof. The first equality follows from [12, Proposition 4.7]. The second case of the second equality is straightforward. In the first case, $N \in \mathcal{RP}_w(u)$ implies

$$(\vec{u}_{\leq N-1} - u)\varpi_{i_N} = \vec{u}_{\leq N-1}\alpha_{i_N} \in Q^+\backslash\{0\}.$$

Thus, $\xi_{\vec{u}_{\leq N-1}\varpi_{i_N}} \perp \mathcal{U}_q^-(\mathfrak{g})v_{u\varpi_{i_N}}$ and $\Delta_{\vec{u}_{\leq N-1}\varpi_{i_N}, w\varpi_{i_N}} \in I_w(u)$. □

Proof of Theorem 9.2. The only part of the theorem that has not been proved yet is that $\mathcal{U}^-[w]/I_w(u)$ and $R_q[R_{u,w}]$ are subalgebras of the quantum torus generated by $\{M(e_k)^{\pm 1} \mid k \in [1,N]\backslash D(\pi)\}$. The second statement follows from the first and Lemma 9.3. To establish the first statement, we prove by induction on $l = 1, \ldots, N$, the stronger fact that

$p(\mathcal{U}^-[w]_{[c(l),d(l)]})$ is a subalgebra of the quantum subtorus of $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ *generated by* $\{M(e_k)^{\pm 1} \mid k \in [1,l]\backslash D(\pi)\}$.

Denote by $\mathcal{T}_l$ the quantum subtorus of $\mathrm{Fract}(\mathcal{U}^-[w]/I_w(u))$ generated by $\{M(e_k)^{\pm 1} \mid k \in [1,l]\backslash D(\pi)\}$. Assume the validity of the statement for $l-1$. Consider the case $\pi(l) = d(l)$. Applying Lemmas 9.3 and 9.4, we obtain that

$$p(F_{\beta_l}) \in \big(\mathcal{T}_l - p(\mathcal{U}^-[w]_{[c(l-1),d(l-1)]})\big) p\big(\Delta_{u(l-1)\varpi_{i_l}, w_{[c(l-1),d(l-1)]}\varpi_{i_l}}\big)^{-1}$$

The space on the right is a subspace of $\mathcal{T}_l$ because of the inductive assumption and the fact that the last term is a Laurent monomial in the generators of $\mathcal{T}_l$ by Lemma 9.3. The case $\pi(l) = c(l)$ is handled similarly by applying the quantum twist map $\Theta_{w_{[c(l),d(l)]}}$ to the equalities in Lemma 9.4. □

References

[1] J. Beck, *Convex bases of PBW type for quantum affine algebras*, Comm. Math. Phys. **165** (1994), 193–199.

[2] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Adv. Math. **195** (2005), 405–455.

[3] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2002.

[4] K. A. Brown and K. R. Goodearl, *Zariski topologies on stratified spectra of quantum algebras*, In: Commutative Algebra and Noncommutative Algebraic Geometry II: Research Articles, Eds: D. Eisenbud et al., pp 63-91, MSRI Publ. Vol 68. Cambridge Univ Press, 2015.

[5] G. Cauchon, *Effacement des dérivations et spectres premiers d'algébres quantiques,* J. Algebra **260** (2003), 476–518.

[6] G. Cauchon, *Spectre premier de $O_q(M_n(k))$: image canonique et séparation normale*, J. Algebra **260** (2003), 519–569.

[7] C. De Concini, V. Kac and C. Procesi, *Some quantum analogues of solvable Lie groups,* In: Geometry and analysis (Bombay, 1992), pp. 41–65, Tata Inst. Fund. Res., Bombay, 1995.

[8] V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Invent. Math. **79** (1985) 499–511.

[9] J. Dixmier, *Enveloping algebras*, Revised reprint of the 1977 translation. Grad. Stud. in Math., **11**, Amer. Math. Soc., Providence, RI, 1996.

[10] S. Fomin and A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), 335–380.

[11] S. Fryer and M. Yakimov, *Separating Ore sets for prime ideals of quantum algebras*, preprint arXiv:1602.05052.

[12] J. Geiger and M. Yakimov, *Quantum Schubert cells via representation theory and ring theory*, Michigan Math. J. **63**, (2014), no. 1, 125–157.

[13] C. Geiß, B. Leclerc and J. Schröer, *Cluster structures on quantum coordinate rings*, Selecta Math. (N.S.) **19** (2013), 337–397.

[14] K. R. Goodearl, S. Launois and T. H. Lenagan, *Torus-invariant prime ideals in quantum matrices, totally nonnegative cells and symplectic leaves*, Math. Z. **269** (2011), 29–45.

[15] K. R. Goodearl and E. S. Letzter, *The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras,* Trans. Amer. Math. Soc. **352** (2000), 1381–1403.

[16] K. R. Goodearl and M. T. Yakimov, *From quantum Ore extensions to quantum tori via noncommutative UFDs*, preprint arXiv:1208.6267, to appear in Adv. Math.

[17] K. R. Goodearl and M. T. Yakimov, *Quantum cluster algebra structures on quantum nilpotent algebras*, preprint arXiv:1309.7869, to appear in Memoirs Amer. Math. Soc.

[18] K. R. Goodearl and M. T. Yakimov, *The Berenstein–Zelevinsky quantum cluster algebra conjecture*, preprint arXiv:1602.00498.

[19] I. Heckenberger and S. Kolb, *Homogeneous right coideal subalgebras of quantized enveloping algebras*, Bull. Lond. Math. Soc. **44** (2012), 837–848.

[20] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid*, Israel J. Math. **197** (2013), 139–187.

[21] J. C. Jantzen, *Lectures on Quantum Groups,* Grad. Studies in Math., 6. Amer. Math. Soc., Providence, RI, 1996.
[22] A. Joseph, *Quantum Groups and Their Primitive Ideals,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1995.
[23] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Commun. Math. Phys. **133** (1990) 249–260.
[24] B. Leclerc, *Cluster structures on strata of flag varieties*, preprint arXiv:1402.4435, to appear in Adv. Math.
[25] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.
[26] G. Lusztig, *Introduction to Quantum Groups*, Progr. Math. 110, Birkhäuser, 1993.
[27] R. J. Marsh and K. Rietsch, *Parametrizations of flag varieties*, Represent. Theory **8** (2004), 212–242.
[28] R. J. Marsh and J. Scott, *Twists of Plücker coordinates as dimer partition functions*, Comm. Math. Phys. **341** (2016), 821–884.
[29] A. Mériaux and G. Cauchon, *Admissible diagrams in $U_q^w(\mathfrak{g})$ and combinatoric properties of Weyl groups*, Represent. Theory **14** (2010), 645–687.
[30] G. Muller and D. E. Speyer, *Cluster algebras of Grassmannians are locally acyclic*, preprint arXiv:1401.5137.
[31] M. Yakimov, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Proc. London Math. Soc. (3) **101** (2010), 454–476.
[32] M. Yakimov, *Strata of prime ideals of De Concini–Kac–Procesi algebras and Poisson geometry*, In: New trends in noncommutative algebra, eds: P. Ara et al., Contem. Math. 562, pp. 265–278, Amer. Math. Soc. 2012.
[33] M. Yakimov, *Spectra and catenarity of multiparameter quantum Schubert cells*, Glasgow Math. J. **55A** (2013), 169–194.

Maxwell Institute for Mathematical Sciences, School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, King's Buildings, Peter Guthrie Tait Road, Edinburgh EH9 3FD, Scotland, United Kingdom
E-mail address: T.Lenagan@ed.ac.uk

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.
E-mail address: yakimov@math.lsu.edu