RIGIDITY OF QUANTUM TORI AND THE ANDRUSKIEWITSCH–DUMAS CONJECTURE

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Abstract. We prove the Andruskiewitsch–Dumas conjecture that the automorphism group of the positive part of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of an arbitrary finite dimensional simple Lie algebra $\mathfrak{g}$ is isomorphic to the semidirect product of the automorphism group of the Dynkin diagram of $\mathfrak{g}$ and a torus of rank equal to the rank of $\mathfrak{g}$.

The key step in our proof is a rigidity theorem for quantum tori. It has a broad range of applications. It allows one to control the (full) automorphism groups of large classes of associative algebras, for instance quantum cluster algebras.

1. Introduction

Automorphism groups of algebras are often difficult to describe and contain wild automorphisms. The latter fact was proved by Joseph [25] for $\operatorname{Aut}\mathcal{U}(\mathfrak{sl}_2)$, Alev [1] for $\operatorname{Aut}\mathcal{U}(\mathfrak{n})$ where $\mathfrak{n}$ is the nilradical of a Borel subalgebra of $\mathfrak{sl}_3$, and Shestakov and Umirbaev [38] for the Nagata automorphism of a polynomial algebra in three variables.

The Andruskiewitsch–Dumas conjecture [5] concerns the explicit structure of the automorphism groups of the quantum nilpotent algebras $\mathcal{U}_q^+(\mathfrak{g})$ for all simple Lie algebras $\mathfrak{g}$. It asserts that, in contrast to the above cases, the algebras $\mathcal{U}_q^+(\mathfrak{g})$ are rigid in the sense that they have small automorphism groups. Despite many attempts to prove the conjecture, it remained wide open for $\mathfrak{g} \neq \mathfrak{sl}_3, \mathfrak{sl}_4, \mathfrak{so}_5$. In this paper we prove the conjecture in full generality.

Let $\mathcal{U}_q(\mathfrak{g})$ be the quantized universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$, defined over an arbitrary base field $\mathbb{K}$ for a deformation parameter $q \in \mathbb{K}^*$ which is not a root of unity. It has Cartan generators E_α, F_α, and $K_\alpha^{\pm 1}$, where α runs over the set Π of simple roots of $\mathfrak{g}$. The algebra $\mathcal{U}_q^+(\mathfrak{g})$ is the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{E_\alpha \mid \alpha \in \Pi\}$. It is abstractly described as the $\mathbb{K}$-algebra with those generators subject to the quantum Serre relations, see (2.2). The torus $\mathbb{T}^r = (\mathbb{K}^*)^r$, where $r = |\Pi|$ is the rank of $\mathfrak{g}$, acts on $\mathcal{U}_q^+(\mathfrak{g})$ by algebra automorphisms by

$$t \cdot E_\alpha = t_\alpha E_\alpha, \;\; t = (t_{\alpha'})_{\alpha' \in \Pi}, \alpha \in \Pi.$$

The automorphism group of the Dynkin diagram Γ of $\mathfrak{g}$ has a natural embedding into $\operatorname{Aut}(\mathcal{U}_q^+(\mathfrak{g}))$. To $\theta \in \operatorname{Aut}(\Gamma)$ one associates the automorphism given by

$$E_\alpha \mapsto E_{\theta(\alpha)}, \;\; \alpha \in \Pi.$$

Andruskiewitsch and Dumas [5] have conjectured that the above generate the automorphism group $\operatorname{Aut}(\mathcal{U}_q^+(\mathfrak{g}))$.

Conjecture 1.1. (Andruskiewitsch–Dumas) For all simple Lie algebras $\mathfrak{g}$ of rank $r > 1$, base fields $\mathbb{K}$, and deformation parameters $q \in \mathbb{K}^*$ which are not roots of unity

$$\operatorname{Aut}(\mathcal{U}_q^+(\mathfrak{g})) \cong \mathbb{T}^r \ltimes \operatorname{Aut}(\Gamma).$$

Three cases of this conjecture were proved up to date: $\mathfrak{g} = \mathfrak{sl}_3$ by Alev–Dumas and Caldero [4, 12], $\mathfrak{g} = \mathfrak{so}_5$ by Launois [30] and Andruskiewitsch–Dumas [5], and $\mathfrak{g} = \mathfrak{sl}_4$ by Lopes–Launois [34]. They found important ways to study the automorphisms of $\mathcal{U}_q^+(\mathfrak{g})$ from the structure

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of the spectra Spec$\mathcal{U}_q^+(\mathfrak{g})$. Unfortunately this relation could not be used to obtain sufficient restrictions on the automorphisms of $\mathcal{U}_q^+(\mathfrak{g})$. As a result of this, the final steps of the proofs of the special cases relied on elaborate computations, which were specific to each case. Alev–Chamarie [3], Fleury [16], Launois–Lenagan [32, 33], and Rigal [37] studied the automorphism groups of quantum matrices, quantized universal enveloping algebras of Borel subalgebras, and quantum Weyl algebras. In their works, arguments with induced actions on prime spectra and relations to derivations of quantum tori lead to enough information for automorphisms only when there were few height one primes, lots of units, or when the algebras had low Gelfand–Kirillov dimension.

We give a proof of Conjecture 1.1 and exhibit a general classification method for automorphism groups of related algebras. The key components of this new classification scheme are a relationship between $\mathrm{Aut}(\mathcal{U}_q^+(\mathfrak{g}))$ and the group of certain continuous bi-finite automorphisms of completed quantum tori, and a rigidity result for the latter. In order to state those, we need to introduce some more terminology and notation. Denote by $M_N(\mathbb{K}^*)$ the set of $N\times N$ matrices with entries in $\mathbb{K}^*$. An $N\times N$ matrix $\mathbf{q}=(q_{kl})_{k,l=1}^N \in M_N(\mathbb{K}^*)$ is called multiplicatively skew-symmetric if $q_{kl}q_{lk}=1$ for $k\neq l$ and $q_{ll}=1$. Such gives rise to the rank N quantum torus

$$\mathcal{T}_{\mathbf{q}} = \frac{\mathbb{K}\langle X_1^{\pm 1},\dots,X_N^{\pm 1}\rangle}{(X_kX_k^{-1}-1, X_k^{-1}X_k-1, X_kX_l-q_{kl}X_lX_k, 1\le k<l\le N)}. \tag{1.1}$$

Denote the multiplicative kernel of the matrix $\mathbf{q}$

$$\mathrm{Ker}(\mathbf{q}) = \Big\{(j_1,\dots,j_N)\in\mathbb{Z}^N \mid \prod_{l=1}^N q_{kl}^{j_l}=1,\ \forall 1\le k\le N\Big\}. \tag{1.2}$$

Let $\mathbb{Z}_+ := \{1,2,\dots\}$. We say the quantum torus $\mathcal{T}_{\mathbf{q}}$ is *saturated* if

$$f\in\mathbb{Z}^N, n\in\mathbb{Z}_+,\ \ nf\in\mathrm{Ker}(\mathbf{q}) \Rightarrow f\in\mathrm{Ker}(\mathbf{q}). \tag{1.3}$$

For example, $\mathcal{T}_{\mathbf{q}}$ is saturated if the subgroup of $\mathbb{K}^*$ generated by q_{kl}, $1\le k<l\le N$ is torsion-free. The condition (1.3) has several other equivalent formulations, see §3.1. It is equivalent to the condition that for $u\in\mathcal{T}_{\mathbf{q}}$, $n\in\mathbb{Z}_+$, $u^n\in Z(\mathcal{T}_{\mathbf{q}})$ implies $u\in Z(\mathcal{T}_{\mathbf{q}})$. Here and below for an algebra B, $Z(B)$ denotes its center. We call an N-tuple $\mathbf{d}=(d_1,\dots,d_N)\in\mathbb{Z}_+^N$ a degree vector and use it to define a $\mathbb{Z}$-grading on $\mathcal{T}_{\mathbf{q}}$ by assigning $\deg X_l = d_l$. Consider the completion

$$\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}} := \{u_m+u_{m+1}+\dots \mid m\in\mathbb{Z}, u_j\in\mathcal{T}_{\mathbf{q}}, \deg u_j = j\}.$$

We will call a continuous automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ *unipotent* if

$$\phi(X_l)-X_l\in(\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}})^{\ge d_l+1},\ \ \forall 1\le l\le N,$$

where $(\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}})^{\ge m} := \{u_m+u_{m+1}+\dots \mid u_j\in\mathcal{T}_{\mathbf{q}}, \deg u_j = j\}$ for $m\in\mathbb{Z}$. The N-tuple $(\phi(X_1),\dots,\phi(X_N))$ consists of units of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and uniquely determines the continuous automorphism ϕ. A unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ will be called *bi-finite* if

$$\phi(X_l) \text{ and } \phi^{-1}(X_l)\in\mathcal{T}_{\mathbf{q}},\ \ \forall 1\le l\le N.$$

We refer the reader to §3.1 for properties of the above types of automorphisms. To this end we note that in general bi-finite unipotent automorphisms do not satisfy $\phi(\mathcal{T}_{\mathbf{q}})\subseteq\mathcal{T}_{\mathbf{q}}$ since $\phi(X_l^{-1})=\phi(X_l)^{-1}$ belongs to $\mathcal{T}_{\mathbf{q}}$ only in very special cases. In Section 3 we prove the following result:

Theorem 1.2. *Let $\mathcal{T}_{\mathbf{q}}$ be a saturated quantum torus of rank N over an arbitrary base field $\mathbb{K}$. Let $\mathbf{d}\in\mathbb{Z}_+^N$ be a degree vector. For every bi-finite unipotent automorphism ϕ of the completed quantum torus $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, there exist N elements*

$$u_1,u_2,\dots,u_N\in Z(\mathcal{T}_{\mathbf{q}})^{\ge 1}$$

such that $\phi(X_l)=(1+u_l)X_l$ for all $1\le l\le N$, where $Z(\mathcal{T}_{\mathbf{q}})^{\ge 1} := Z(\mathcal{T}_{\mathbf{q}})\cap(\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}})^{\ge 1}$.

Theorem 1.2 is a rigidity result for bi-finite unipotent automorphisms of completed saturated quantum tori in the sense that it asserts that those kinds of automorphisms are only coming from the center of the underlying torus. We derive the Andruskiewitsch–Dumas conjecture from a rigidity theorem for a type of unipotent automorphisms of the algebras $\mathcal{U}_q^+(\mathfrak{g})$. Every strictly dominant integral coweight $\lambda = \sum_{\alpha \in \Pi} m_\alpha \varpi_\alpha^\vee$ of $\mathfrak{g}$ gives rise to a connected $\mathbb{N}$-grading of $\mathcal{U}_q^+(\mathfrak{g})$ obtained by assigning degree $m_\alpha = \langle \lambda, \alpha \rangle$ to E_α, where $\{\varpi_\alpha^\vee \mid \alpha \in \Pi\}$ are the fundamental coweights of $\mathfrak{g}$. For $m \in \mathbb{N}$, denote by $\mathcal{U}_q^+(\mathfrak{g})^{\geq m}$ the space of elements of degree $\geq m$. We call an automorphisms Φ of $\mathcal{U}_q^+(\mathfrak{g})$ λ-unipotent if it satisfies

$$\Phi(E_\alpha) - E_\alpha \in \mathcal{U}_q^+(\mathfrak{g})^{\geq \langle \lambda, \alpha \rangle + 1}, \ \ \forall \alpha \in \Pi.$$

Theorem 1.3. *Let $\mathfrak{g}$ be a simple Lie algebra of rank $r > 1$, $\mathbb{K}$ an arbitrary base field, q a deformation parameter that is not a root of unity. For every strictly dominant integral coweight λ the only λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$ is the identity automorphism.*

Our strategy for the proof of this theorem is as follows. The algebras $\mathcal{U}_q^+(\mathfrak{g})$ are examples of a type of iterated Ore extensions called Cauchon–Goodearl–Letzter extensions and the Cauchon procedure of deleting derivations [13] can be used to embed them into quantum tori, see §2.2 for details. This is not yet sufficient to relate the λ-unipotent automorphisms of $\mathcal{U}_q^+(\mathfrak{g})$ to bi-finite unipotent automorphisms of completed quantum tori. For this we apply a recent result of Geiger and the author [17] stating that one can change the generators of those quantum tori so they become quantum minors in $\mathcal{U}_q^+(\mathfrak{g})$. Recall that a quantum affine space algebra is an algebra with generators $X_1, \ldots, X_N$ and relations as in (1.1). The above mentioned result of [17] leads to a chain of embeddings

$$\mathcal{A} \subset \mathcal{U}_q^+(\mathfrak{g}) \subset \mathcal{T}, \tag{1.4}$$

where $\mathcal{A}$ is a quantum affine space algebra and $\mathcal{T}$ is the corresponding quantum torus (which coincides with the Cauchon quantum torus). In the case when $\mathbb{K}$ has characteristic 0 and q is transcendental over $\mathbb{Q}$ one can also obtain this by applying the results of Geiß–Leclerc–Schröer [18]. Using (1.4) we find a relationship between the λ-unipotent automorphisms of $\mathcal{U}_q^+(\mathfrak{g})$ and the bi-finite unipotent automorphisms of a completion of the (saturated) quantum torus $\mathcal{T}$. Then we apply results from [12, 41] on the normal elements of the algebras $\mathcal{U}_q^+(\mathfrak{g})$ and a theorem for separation of variables for these algebras. These results and Theorem 1.2 are used to prove that every λ-unipotent automorphism Φ of $\mathcal{U}_q^+(\mathfrak{g})$ satisfies

$$\Phi(E_\alpha) = (1 + z_\alpha)E_\alpha, \ \ \forall \alpha \in \Pi$$

for some $z_\alpha \in Z(\mathcal{U}_q^+(\mathfrak{g})) \cap \mathcal{U}_q^+(\mathfrak{g})^{\geq 1}$. Finally, the structure of the torus invariant height one prime ideals of $\mathcal{U}_q^+(\mathfrak{g})$ from [27, 22, 41] is used to establish that $z_\alpha = 0$, $\forall \alpha \in \Pi$. The proof of Theorem 1.3 is given in Section 4. Section 5 contains the proof of the Andruskiewitsch–Dumas conjecture. It is based on Theorem 1.3 and an intermediate classification of the family of automorphisms of $\mathcal{U}_q^+(\mathfrak{g})$ that map the subspace $\mathrm{Span}\{E_\alpha \mid \alpha \in \Pi\}$ to itself.

In Section 6 we prove an extension of Conjecture 1.1, which classifies the automorphism groups of the 2-cocycle twists of the algebras $\mathcal{U}_q^+(\mathfrak{g})$, again in full generality. All proofs in the paper are carried out in such a way so they easily extend to the twisted case. We do not go straight to the twisted case to avoid technicalities, which will obscure the main ideas.

In Section 6 the results on automorphism groups are also applied to obtain a full solution of the isomorphism problem for the family of all algebras obtained by 2-cocycle twists from the algebras $\mathcal{U}_q^+(\mathfrak{g})$ for simple Lie algebras $\mathfrak{g}$. In particular, it is shown that

$$\mathcal{U}_q^+(\mathfrak{g}_1) \cong \mathcal{U}_q^+(\mathfrak{g}_2) \quad \Leftrightarrow \quad \mathfrak{g}_1 \cong \mathfrak{g}_2$$

for all base fields $\mathbb{K}$, non-root of unity q and simple Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$. The idea to apply the results on automorphism groups to this isomorphism problem was suggested by Len Scott.

The methods of this paper have a very broad range of applications to the investigation of automorphism groups of noncommutative algebras. They provide a procedure to deal with individual automorphisms or analyze the full automorphism groups of those algebras R that satisfy

$$\mathcal{A} \subset R \subset \mathcal{T} \tag{1.5}$$

for some quantum torus $\mathcal{T}$ and the corresponding quantum affine space algebra $\mathcal{A}$. This is done by using the above mentioned relationship between the automorphisms of R and the bi-finite unipotent automorphisms of a completion of $\mathcal{T}$, and then applying the rigidity from Theorem 1.2. In its most general form the former relationship is stated in [43, Proposition 3.3] in connection to one such application. There are very large classes of algebras R that satisfy (1.5). For example all quantum cluster algebras. (The above procedure in this case deals with the full automorphism group, not just maps that take clusters to clusters.) In a recent preprint [21] K. Goodearl and the author proved that the property (1.5) is satisfied by all algebras in the large, axiomatically defined class of iterated Ore extensions called Cauchon–Goodearl–Letzter extensions [13, 20]. There are particular families of algebras in the above classes for which the automorphism groups have been of interest. In [43] we apply the methods of this paper to prove the Launois–Lenagan conjecture [32] that for all integers $N \geq 2$ the automorphism group of the algebra $R_q[M_N]$ of quantum matrices of size $N \times N$ is isomorphic to a semidirect product of the torus $\mathbb{T}^{2N-1}$ and a copy of $\mathbb{Z}_2$ corresponding to the transpose automorphism. It was proved [3, 33] for $N = 2$ and 3, and was open for all $N > 3$. Other particular families of algebras to which the procedure is applicable and the automorphism groups have been of interest include quantum groups $R_q[G]$, [26, 23] and the quantum Schubert cell algebras $\mathcal{U}^+[w]$, [14, 29, 24].

Although the above procedure makes sense for commutative algebras R (and thus for classical cluster algebras), in those cases Theorem 1.2 does not produce sufficient restrictions on the possible form of the automorphisms of R. However, there is a Poisson version of Theorem 1.2 about rigidity of automorphisms of Poisson tori. This and some of its applications will be described in another publication.

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2. The algebras $\mathcal{U}_q^\pm(\mathfrak{g})$

2.1. **Quantized universal enveloping algebras.** We will mostly follow the notation of Jantzen's book [24]. Assume that $\mathfrak{g}$ is a complex simple Lie algebra with Weyl group W, set of simple roots Π, and Dynkin diagram Γ. We will identify the set of vertices of Γ with Π. For $\alpha \in \Pi$ denote by ϖ_α and $s_\alpha \in W$ the corresponding fundamental weight and simple reflection. Let $\mathcal{Q}$ and $\mathcal{P}$ be the root and weight lattices of $\mathfrak{g}$. Let $\mathcal{Q}_+ = \mathbb{N}\Pi$ and $\mathcal{P}_+ = \mathbb{N}\{\varpi_\alpha \mid \alpha \in \Pi\}$ be the set of dominant integral weights. The support of $\mu = \sum_{\alpha \in \Pi} m_\alpha \varpi_\alpha \in \mathcal{P}$ is defined by $\mathrm{Supp}(\mu) := \{\alpha \in \Pi \mid m_\alpha \neq 0\}$. Denote by $\mathcal{P}_{++}^\vee = \{\sum m_\alpha \varpi_\alpha^\vee \mid m_\alpha \in \mathbb{Z}_+, \forall \alpha \in \Pi\}$ the set of strictly dominant integral coweights of $\mathfrak{g}$, where $\varpi_\alpha^\vee$ are the fundamental coweights of $\mathfrak{g}$. Let $\langle .,. \rangle$ be the invariant bilinear form on $\mathbb{R}\Pi$ normalized by $\langle \alpha, \alpha \rangle = 2$ for short roots $\alpha \in \Pi$.

Throughout the paper $\mathbb{K}$ will denote a base field (of arbitrary characteristic) and $q \in \mathbb{K}^*$ an element that is not a root of unity. The quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ is the $\mathbb{K}$-algebra with generators $\{K_\alpha^{\pm 1}, E_\alpha, F_\alpha \mid \alpha \in \Pi\}$ and relations [24, §4.3]. Let $\mathcal{U}_q^+(\mathfrak{g})$ and $\mathcal{U}_q^-(\mathfrak{g})$ be the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{E_\alpha \mid \alpha \in \Pi\}$ and $\{F_\alpha \mid \alpha \in \Pi\}$. There is a unique automorphism ω of $\mathcal{U}_q(\mathfrak{g})$ such that

$$\omega(E_\alpha) = F_\alpha, \ \omega(F_\alpha) = E_\alpha, \ \omega(K_\alpha) = K_\alpha^{-1}, \ \ \forall \alpha \in \Pi. \tag{2.1}$$

It restricts to an isomorphism $\omega\colon \mathcal{U}_q^{\pm}(\mathfrak{g}) \to \mathcal{U}_q^{\mp}(\mathfrak{g})$. We will work with the algebra $\mathcal{U}_q^-(\mathfrak{g})$ since we use results from [17, 42] which will need an appropriate reformulation for $\mathcal{U}_q^+(\mathfrak{g})$. The algebra $\mathcal{U}_q^-(\mathfrak{g})$ is the $\mathbb{K}$-algebra with generators $\{F_\alpha \mid \alpha \in \Pi\}$ and the following quantum Serre relations:

$$\sum_{j=0}^{1-a_{\alpha\alpha'}} (-1)^j \begin{bmatrix} 1-a_{\alpha\alpha'} \\ j \end{bmatrix}_{q_\alpha} (F_\alpha)^j F_{\alpha'} (F_\alpha)^{1-a_{\alpha\alpha'}-j} = 0, \ \ \forall \alpha \neq \alpha' \in \Pi, \tag{2.2}$$

where

$$a_{\alpha\alpha'} = 2\langle \alpha, \alpha' \rangle / \langle \alpha, \alpha \rangle \tag{2.3}$$

and $q_\alpha = q^{\langle \alpha,\alpha \rangle/2}$. Here $[m]_q = (q^m - q^{-m})/(q-q^{-1})$ for $m \geq 1$, $[0]_q = 1$, and $[m]_q! = [0]_q \dots [m]_q$, $\begin{bmatrix} m \\ j \end{bmatrix}_q = [m]_q!/[j]_q![m-j]_q!$ for $j \leq m \in \mathbb{N}$. Denote

$$r := \operatorname{rank}(\mathfrak{g}), \ \ N := (\dim \mathfrak{g} - r)/2, \ \text{and} \ \mathbb{T}^r := (\mathbb{K}^*)^r. \tag{2.4}$$

In other words N is the number of positive roots of $\mathfrak{g}$. Then $\operatorname{GK}\dim \mathcal{U}_q^{\pm}(\mathfrak{g}) = N$. (This follows for instance from the iterated Ore extension presentation (2.16) below and the isomorphism $\mathcal{U}_q^+(\mathfrak{g}) \cong \mathcal{U}_q^-(\mathfrak{g})$.) The algebra $\mathcal{U}_q(\mathfrak{g})$ is $\mathcal{Q}$-graded by assigning E_α, F_α, and $K_\alpha^{\pm 1}$ weights α, $-\alpha$, and 0. For $\gamma \in \mathcal{Q}$, the corresponding graded component of $\mathcal{U}_q(\mathfrak{g})$ will be denoted by $\mathcal{U}_q(\mathfrak{g})_\gamma$. The grading gives rise to the following $\mathbb{T}^r$-action on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms:

$$t \cdot u = t^\gamma u, \quad u \in \mathcal{U}_q(\mathfrak{g})_\gamma, \gamma \in \mathcal{Q} \tag{2.5}$$

in terms of the characters

$$t \mapsto t^\gamma := \prod_{\alpha \in \Pi} t_\alpha^{\langle \gamma, \varpi_\alpha \rangle}, \quad t = (t_\alpha)_{\alpha \in \Pi} \in \mathbb{T}^r.$$

2.2. **Cauchon's procedure of deleting derivations and $\mathcal{U}_q^{\pm}(\mathfrak{g})$.** Consider an iterated Ore extension

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \dots [x_N; \sigma_N, \delta_N], \tag{2.6}$$

where for $l \in [2, N]$, σ_l is an automorphism and δ_l is a (left) σ_l-skew derivation of $R_{l-1} := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \dots [x_{l-1}; \sigma_{l-1}, \delta_{l-1}]$. Here and below for $m \leq n \in \mathbb{Z}$, we set $[m, n] = \{m, \dots, n\}$.

Definition 2.1. An iterated Ore extension R given by (2.6) is called a Cauchon–Goodearl–Letzter (CGL) extension if it is equipped with an action of the torus $\mathbb{T}^r := (\mathbb{K}^*)^r$ by algebra automorphisms satisfying the following conditions:

(i) For all $1 \leq k < l \leq N$, $\sigma_l(x_k) = q_{lk} x_k$ for some $q_{lk} \in \mathbb{K}^*$.

(ii) For every $l \in [2, N]$, δ_l is a locally nilpotent σ_l-skew derivation of R_{l-1}.

(iii) The elements $x_1, \dots, x_N$ are $\mathbb{T}^r$-eigenvectors and the set $\{c \in \mathbb{K} \mid \exists t \in \mathbb{T}^r, \ t \cdot x_1 = c x_1\}$ is infinite.

(iv) For every $l \in [2, N]$ there exists $t_l \in \mathbb{T}^r$ such that $t_l \cdot x_l = q_l x_l$ for some $q_l \in \mathbb{K}^*$ which is not a root of unity, and $t_l \cdot x_k = q_{lk} x_k$, $\forall k \in [1, l-1]$ (i.e., $\sigma_l = (t_l \cdot)$ as elements of $\operatorname{Aut}(R_{l-1})$, $\forall l \in [2, N]$).

We note that for all CGL extensions, $\sigma_l \delta_l = q_l \delta_l \sigma_l$, $\forall l \in [2, N]$. In this setting Cauchon [13] iteratively constructed N-tuples of nonzero elements

$$(x_1^{(m)}, \dots, x_N^{(m)})$$

of the division ring of fractions $\operatorname{Fract}(R)$ of R for $m = N+1, \dots, 1$. First,

$$(x_1^{(N+1)}, \dots, x_N^{(N+1)}) := (x_1, \dots, x_N).$$

The other N-tuples are obtained recursively from the formula

$$x_j^{(m)} := \begin{cases} x_j^{(m+1)}, & \text{if } j \geq m \\ \sum_{n=0}^{\infty} \frac{(1-q_m)^{-n}}{(n)_{q_m}!} \left[\delta_m^n \sigma_m^{-n} \left(x_j^{(m+1)}\right)\right] \left(x_m^{(m+1)}\right)^{-n}, & \text{if } j < m \end{cases} \tag{2.7}$$

for $m = N, \dots, 2$. Here $(n)_q! = (0)_q \dots (n)_q$, $(0)_q = 1$, and $(n)_q = (1-q^n)/(1-q)$ for $n \geq 1$. This process is called *Cauchon's procedure of deleting derivations*. The terminology comes from the following fact proved by Cauchon [13]: the subalgebra of $\mathrm{Fract}(R)$ generated by the m-th N-tuple of elements is isomorphic to an iterated Ore extension of the form (2.6), where the derivations $\delta_m, \dots, \delta_N$ are no longer present. Denote the final N-tuple of elements $(\overline{x}_1, \dots, \overline{x}_N) := (x_1^{(2)}, \dots, x_N^{(2)})$ and the subalgebra of $\mathrm{Fract}(R)$ generated those elements and their inverses

$$\mathcal{T} = \mathbb{K}\langle \overline{x}_1^{\pm 1}, \dots, \overline{x}_N^{\pm 1}\rangle. \tag{2.8}$$

Let $q_{ll} = 1$ for $l \in [1, N]$, $q_{kl} = q_{lk}^{-1}$ for $1 \leq k < l \leq N$, and $\mathbf{q} := (q_{kl})_{k,l=1}^N \in M_N(\mathbb{K}^*)$. Cauchon proved [13] that

$$R \subset \mathcal{T} \ \text{ and the map } \ \eta\colon \mathcal{T}_{\mathbf{q}} \to \mathcal{T} \ \text{ given by } \ \eta(X_l) = \overline{x}_l \ \text{ is an isomorphism,} \tag{2.9}$$

recall (1.1). The CGL extension R is called *torsion-free* if the subgroup of $\mathbb{K}^*$ generated by q_{kl} for $1 \leq k < l \leq N$ is torsion-free. In such a case $\mathcal{T}_{\mathbf{q}}$ is a saturated quantum torus as noted in the introduction.

Denote the braid group of $\mathfrak{g}$ by $\mathcal{B}_{\mathfrak{g}}$ and its standard set of generators by $\{T_\alpha \mid \alpha \in \Pi\}$. Let w_0 be the longest element of the Weyl group W of $\mathfrak{g}$. A word $\mathbf{i} = (\alpha_1, \dots, \alpha_N)$ in the alphabet Π is called a reduced word for w_0 if $s_{\alpha_1} \dots s_{\alpha_N}$ is a reduced expression of w_0, recall (2.4). For such a reduced word $\mathbf{i}$ define

$$w_0(\mathbf{i})_{\leq l} := s_{\alpha_1} \dots s_{\alpha_l}, \ \ l \in [0, N] \tag{2.10}$$

and the Lusztig root vectors

$$F_{\beta_l} := T_{\alpha_1} \dots T_{\alpha_{l-1}}(F_{\alpha_l}), \ \text{ where } \ \beta_l := w_0(\mathbf{i})_{\leq (l-1)} \alpha_{i_l}, \ \ l \in [1, N], \tag{2.11}$$

see [29, §39.3]. Here we use Lusztig's action of $\mathcal{B}_{\mathfrak{g}}$ on $\mathcal{U}_q(\mathfrak{g})$ in the version given in [24, §8.14] by Eqs. 8.14 (2), (3), (7), and (8). We will need the following property (see [24, Proposition 8.20]):

$$\text{if } \ \beta_l = \alpha \in \Pi \ \text{ for some } \ l \in [1, N], \ \text{ then } \ F_{\beta_l} = F_\alpha. \tag{2.12}$$

Given an algebra B, a subalgebra B' of B, $x \in B$, an automorphism σ of B', and a (left) σ-derivation δ of B', we will say that

$$B = B'[x; \sigma, \delta]$$

is an Ore extension presentation of B if the map $\psi\colon B'[y; \sigma, \delta] \to B$ given by $\psi(b') = b'$, $b' \in B'$ and $\psi(y) = x$ is an algebra isomorphism. The Levendorskii–Soibelman straightening law

$$\begin{aligned} F_{\beta_l} F_{\beta_k} - q^{-\langle \beta_l, \beta_k\rangle} F_{\beta_k} F_{\beta_l} \\ = \sum_{\mathbf{m} = (m_{k+1}, \dots, m_{l-1}) \in \mathbb{N}^{l-k-2}} c_{\mathbf{m}} (F_{\beta_{l-1}})^{m_{l-1}} \dots (F_{\beta_{k+1}})^{m_{k+1}}, \ \ c_{\mathbf{m}} \in \mathbb{K}, \end{aligned} \tag{2.13}$$

for $1 \leq k < l \leq N$ (see e.g. [10, Proposition I.6.10]) is used to associate to each reduced word $\mathbf{i}$ for w_0 an iterated Ore extension presentation of $\mathcal{U}_q^-(\mathfrak{g})$. For $l \in [1, N]$ choose an element $t_l \in \mathbb{T}^r$ such that $t_l^{\beta_k} = q^{\langle \beta_k, \beta_l\rangle}$ for all $k \in [1, l]$, cf. (2.5). Let $\mathcal{U}^-[w_0(\mathbf{i})_{\leq l}]$ be the subalgebra of $\mathcal{U}_q^-(\mathfrak{g})$ generated by $F_{\beta_1}, \dots, F_{\beta_l}$ for $l \in [0, N]$. (This is nothing but the quantum Schubert cell algebra of De Concini–Kac–Procesi [14] and Lusztig [29] associated to $w_0(\mathbf{i})_{\leq l}$.) Then $\mathcal{U}^-[w_0(\mathbf{i})_{\leq 0}] = \mathbb{K}$, $\mathcal{U}^-[w_0(\mathbf{i})_{\leq N}] = \mathcal{U}_q^-(\mathfrak{g})$, and for all $l \in [1, N]$ we have the Ore extension presentations

$$\mathcal{U}^-[w_0(\mathbf{i})_{\leq l}] = \mathcal{U}^-[w_0(\mathbf{i})_{\leq (l-1)}][F_{\beta_{l-1}}, \sigma_l, \delta_l].$$

Here

$$\sigma_l := (t_l \cdot) \in \mathrm{Aut}(\mathcal{U}^-[w_0(\mathbf{i})_{\leq (l-1)}]) \tag{2.14}$$

for the element $t_l \in \mathbb{T}^r$ constructed above and the restriction of the action (2.5) to $\mathcal{U}^-[w_0(\mathbf{i})_{\leq (l-1)}]$. The skew derivation δ_l is given by

$$\delta_l(x) := F_{\beta_l} x - q^{\langle \beta_l, \gamma \rangle} x F_{\beta_l}, \quad x \in \mathcal{U}^-[w_0(\mathbf{i})_{\leq (l-1)}]_\gamma, \gamma \in \mathcal{Q}, \tag{2.15}$$

recall (2.13). By composing those presentations, one associates to each reduced word $\mathbf{i}$ for w_0 the iterated Ore extension presentation of $\mathcal{U}_q^-(\mathfrak{g})$

$$\mathcal{U}_q^-(\mathfrak{g}) = \mathbb{K}[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \dots [F_{\beta_N}; \sigma_N, \delta_N]. \tag{2.16}$$

This is a torsion-free CGL extension for the following choice of the coefficients q_{lk}, q_l:

$$q_{lk} = q^{-\langle \beta_l, \beta_k \rangle}, \quad 1 \leq k < l \leq N, \quad q_l = q_{\alpha_l}^{-2}, \quad l \in [1, N], \tag{2.17}$$

see [35].

2.3. **Separation of variables for $\mathcal{U}_q^{\pm}(\mathfrak{g})$ and height one primes.** Recall that a $\mathcal{U}_q(\mathfrak{g})$-module V is called a type one module if it equals the sum of its q-weight spaces defined by

$$V_\mu := \{v \in V \mid K_\alpha v = q^{\langle \mu, \alpha \rangle} v, \ \forall \alpha \in \Pi\}, \ \mu \in \mathcal{P}.$$

The irreducible finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules are parametrized by the dominant integral weights of $\mathfrak{g}$, see [24, Theorem 5.10]. Denote by $V(\lambda)$ the irreducible $\mathcal{U}_q(\mathfrak{g})$-module with highest weight $\lambda \in \mathcal{P}_+$, and fix a highest weight vector v_λ of $V(\lambda)$. The braid group $\mathcal{B}_\mathfrak{g}$ acts on $V(\lambda)$ by [24, Eq. 8.6 (2)]. This action is compatible with the one on $\mathcal{U}_q(\mathfrak{g})$ and in particular satisfies $T_w(V(\lambda)_\mu) = V(\lambda)_{w\mu}$, $\forall w \in W$, $\lambda \in \mathcal{P}_+$, $\mu \in \mathcal{P}$. Because of this, for all $w \in W$ there exists a unique element $\xi_{w,\lambda} \in (V(\lambda)^*)_{-w\lambda}$ such that $\xi_{w,\lambda}(T_{w^{-1}}^{-1} v_\lambda) = 1$, where dual modules are formed using the antipode of the Hopf algebra structure on $\mathcal{U}_q(\mathfrak{g})$. Recall that w_0 denotes the longest element of W. Given $w \in W$, denote the matrix coefficients

$$e_w^\lambda \in (\mathcal{U}_q(\mathfrak{g}))^*, \quad e_w^\lambda(x) := \xi_{w,\lambda}(x T_{w_0^{-1}}^{-1} v_\lambda), \ \ x \in \mathcal{U}_q(\mathfrak{g}) \tag{2.18}$$

called quantum minors in the case when λ is a fundamental weight. We will need their counterparts in $\mathcal{U}_q^-(\mathfrak{g})$. Given $\gamma \in \mathcal{Q}_+ \backslash \{0\}$, denote $n(\gamma) = \dim \mathcal{U}_q^+(\mathfrak{g})_\gamma = \dim \mathcal{U}_q^-(\mathfrak{g})_{-\gamma}$ and fix a pair of dual bases $\{u_{\gamma,j}\}_{j=1}^{n(\gamma)}$ and $\{u_{-\gamma,j}\}_{j=1}^{n(\gamma)}$ of $\mathcal{U}_q^+(\mathfrak{g})_\gamma$ and $\mathcal{U}_q^-(\mathfrak{g})_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [24, Ch. 6]. The universal R-matrix corresponding of $\mathcal{U}_q(\mathfrak{g})$ (without its semisimple part) is given by

$$\mathcal{R} := 1 \otimes 1 + \sum_{\gamma \in \mathcal{Q}_+, \gamma \neq 0} \sum_{j=1}^{n(\gamma)} u_{\gamma,j} \otimes u_{-\gamma,j} \in \mathcal{U}_q^+(\mathfrak{g}) \widehat{\otimes} \mathcal{U}_q^-(\mathfrak{g}). \tag{2.19}$$

Here $\mathcal{U}_q^+(\mathfrak{g}) \widehat{\otimes} \mathcal{U}_q^-(\mathfrak{g})$ denotes the completion of $\mathcal{U}_q^+(\mathfrak{g}) \otimes \mathcal{U}_q^-(\mathfrak{g})$ with respect to the filtration [29, §4.1.1]. There is a unique graded algebra antiautomorphism τ of $\mathcal{U}_q(\mathfrak{g})$ given by

$$\tau(E_\alpha) = E_\alpha, \ \tau(F_\alpha) = F_\alpha, \ \tau(K_\alpha) = K_\alpha^{-1}, \ \ \forall \alpha \in \Pi, \tag{2.20}$$

see [24, Lemma 4.6(b)]. The counterparts of e_w^λ in $\mathcal{U}_q^-(\mathfrak{g})$ are the elements

$$b_w^\lambda := (e_{w,w_0}^\lambda \tau \otimes \mathrm{id}) \mathcal{R}^w \in \mathcal{U}_q^-(\mathfrak{g})_{-(w-w_0)\lambda}, \ \ \lambda \in \mathcal{P}_+. \tag{2.21}$$

They play a key role in the description of the spectra of the algebras $\mathcal{U}_q^-(\mathfrak{g})$, see [42, Theorem 3.1]. A more conceptual way to define them is via a family of homomorphisms which realize the quantum Schubert cell algebras as quotients of quantum function algebras, see [40, Theorem 3.6].

The elements b_1^λ, $\lambda \in \mathcal{P}_+$ are normal elements of $\mathcal{U}_q^-(\mathfrak{g})$

$$b_1^\lambda u = q^{\langle (1+w_0)\lambda, \gamma \rangle} u b_1^\lambda, \ \ \forall u \in \mathcal{U}_q^-(\mathfrak{g})_{-\gamma}, \gamma \in \mathcal{Q}_+, \tag{2.22}$$

and satisfy

$$b_1^{\lambda} b_1^{\lambda'} = b_1^{\lambda'} b_1^{\lambda} = q^{\langle \lambda, (1-w_0)\lambda' \rangle} b_1^{\lambda+\lambda'} \tag{2.23}$$

for $\lambda, \lambda' \in \mathcal{P}_+$, see [41, Eqs. (3.30)-(3.31)]. (Note that for all $\lambda, \lambda' \in \mathcal{P}$, $\langle \lambda, (1-w_0)\lambda' \rangle = \langle \lambda', (1-w_0)\lambda \rangle$.) Denote by $\mathcal{N}_q^-(\mathfrak{g})$ the subalgebra of $\mathcal{U}_q^-(\mathfrak{g})$ generated (and hence spanned) by b_1^{λ}, $\lambda \in \mathcal{P}_+$.

Finally, for a reduced word $\mathbf{i}$ of w_0 define the following subset of $\mathbb{N}^N$:

$$\mathcal{H}_{\mathbf{i}} := \{(j_1, \ldots, j_N) \in \mathbb{N}^N \mid \forall \alpha \in \Pi, \ \exists k \in [1, N] \ \text{ such that } \ \alpha_k = \alpha \ \text{ and } \ j_k = 0\}. \tag{2.24}$$

In other words, for every simple root α we consider the set $\{k \in [1, N] \mid \alpha_k = \alpha\}$ and require that there exists an index k in this set such that $j_k = 0$.

We will need the following results from [41] describing the structure of the algebras $\mathcal{U}_q^-(\mathfrak{g})$ as $\mathcal{N}_q^-(\mathfrak{g})$-modules and the set of height one $\mathbb{T}^r$-prime ideals of $\mathcal{U}_q^-(\mathfrak{g})$.

Theorem 2.2. [41, Theorems 5.1, 5.4 and 6.19, and Proposition 6.9] *For all simple Lie algebras $\mathfrak{g}$, base fields $\mathbb{K}$, and $q \in \mathbb{K}^*$ not a root of unity, the following hold:*

(i) The height one $\mathbb{T}^r$-invariant prime ideals of $\mathcal{U}_q^-(\mathfrak{g})$ are precisely the ideals

$$\mathcal{U}_q^-(\mathfrak{g}) b_1^{\varpi_\alpha}, \ \ \alpha \in \Pi.$$

(ii) All normal elements of $\mathcal{U}_q^-(\mathfrak{g})$ belong to $\mathcal{N}_q^-(\mathfrak{g})$. The algebra $\mathcal{N}_q^-(\mathfrak{g})$ is a polynomial algebra in the generators $\{b_1^{\varpi_\alpha} \mid \alpha \in \Pi\}$, i.e., $\{b_1^{\lambda} \mid \lambda \in \mathcal{P}_+\}$ is a $\mathbb{K}$-basis of $\mathcal{N}_q^-(\mathfrak{g})$.

(iii) The algebra $\mathcal{U}_q^-(\mathfrak{g})$ is a free left (and right) $\mathcal{N}_q^-(\mathfrak{g})$-module with basis

$$\{F_{\beta_N}^{j_N} \ldots F_{\beta_1}^{j_1} \mid (j_1, \ldots, j_N) \in \mathcal{H}_{\mathbf{i}}\}. \tag{2.25}$$

When $\operatorname{char} \mathbb{K} = 0$ and q is transcendental over $\mathbb{Q}$, part (i) of the theorem follows from results of Gorelik [22] and Joseph [27]. Under those conditions on $\mathbb{K}$ and q, part (ii) of the theorem was proved by Caldero in [12]. The second part of the theorem establishes that $\mathcal{N}_q^-(\mathfrak{g})$ is precisely the subalgebra of $\mathcal{U}_q^-(\mathfrak{g})$ generated by all normal elements of $\mathcal{U}_q^-(\mathfrak{g})$. The third part of the theorem can be viewed as a result for separation variables for the algebras $\mathcal{U}_q^-(\mathfrak{g})$, see [41, Section 5].

2.4. **Cauchon's procedure for $\mathcal{U}_q^{\pm}(\mathfrak{g})$ and quantum minors.** Given a reduced word $\mathbf{i} = (\alpha_1, \ldots, \alpha_N)$ for w_0, consider the CGL extension presentation (2.16) of $\mathcal{U}_q^-(\mathfrak{g})$. Denote by $(\overline{F}_{\mathbf{i},1}, \ldots, \overline{F}_{\mathbf{i},N})$ the final N-tuple $(\overline{x}_1, \ldots, \overline{x}_N)$ of the Cauchon procedure of deleting derivations applied to it. Define a successor function $s \colon [1, N] \to [1, N] \sqcup \{\infty\}$ associated to $\mathbf{i}$ by

$$s(l) = \min\{k \mid k > l, \alpha_k = \alpha_l\}, \text{ if } \exists k > l \ \text{ such that } \alpha_k = \alpha_l, \quad s(l) = \infty, \text{ otherwise.} \tag{2.26}$$

Define the quantum minors

$$\Delta_{\mathbf{i},l} := b^{\varpi_{\alpha_l}}_{w_0(\mathbf{i})_{\leq (l-1)}}, \ \ l \in [1, N]. \tag{2.27}$$

The following result from [17] expressing the Cauchon elements $\overline{F}_{\mathbf{i},l}$ in terms of the quantum minors $\Delta_{\mathbf{i},l}$ will be needed later:

Theorem 2.3. [17, Theorem 3.1] *For all simple Lie algebras $\mathfrak{g}$, base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and reduced words $\mathbf{i}$ for w_0, the elements $\overline{F}_{\mathbf{i},1}, \ldots, \overline{F}_{\mathbf{i},N} \in \operatorname{Fract}(\mathcal{U}_q^-(\mathfrak{g}))$ from the Cauchon deleting derivation procedure for the torsion-free CGL presentation (2.16) of $\mathcal{U}_q^-(\mathfrak{g})$ are given by*

$$\overline{F}_{\mathbf{i},l} = \begin{cases} (q_{\alpha_l}^{-1} - q_{\alpha_l})^{-1} \Delta_{\mathbf{i},s(l)}^{-1} \Delta_{\mathbf{i},l}, & \text{if } \ s(l) \neq \infty \\ (q_{\alpha_l}^{-1} - q_{\alpha_l})^{-1} \Delta_{\mathbf{i},l}, & \text{if } \ s(l) = \infty. \end{cases}$$

3. Automorphisms of completed quantum tori

3.1. **Quantum tori.** Let $\mathbb{K}$ be an arbitrary field and $\mathbf{q} = (q_{kl})_{k,l=1}^N \in M_N(\mathbb{K}^*)$ a multiplicatively skew-symmetric matrix. Recall the definition (1.1) of the quantum torus $\mathcal{T}_\mathbf{q}$. Denote

$$X^{(j_1,\dots,j_N)} := X_1^{j_1} \dots X_N^{j_N} \in \mathcal{T}_\mathbf{q}, \ \ (j_1,\dots,j_N) \in \mathbb{Z}^N.$$

Let $\{e_1,\dots,e_N\}$ be the standard basis of $\mathbb{Z}^N$. Thus

$$X^{e_k} = X_k, \ \ k \in [1,N]. \tag{3.1}$$

The quantum torus $\mathcal{T}_\mathbf{q}$ has the $\mathbb{K}$-basis $\{X^f \mid f \in \mathbb{Z}^N\}$. Recall the definition (1.2) of the multiplicative kernel of $\mathbf{q}$. We have

$$Z(\mathcal{T}_\mathbf{q}) = \mathrm{Span}\{X^f \mid f \in \mathrm{Ker}(\mathbf{q})\}. \tag{3.2}$$

It is straightforward to show that each of the following three conditions is equivalent to $\mathcal{T}_\mathbf{q}$ being saturated, recall (1.3):

$$\mathrm{Ker}(\mathbf{q}) = (\mathrm{Ker}(\mathbf{q}) \otimes_\mathbb{Z} \mathbb{R}) \cap \mathbb{Z}^N, \tag{3.3}$$

$$\text{for } \ f \in \mathbb{Z}^N, n \in \mathbb{Z}_+, \ \ X^{nf} \in Z(\mathcal{T}_\mathbf{q}) \Rightarrow X^f \in Z(\mathcal{T}_\mathbf{q}), \tag{3.4}$$

$$\text{for } \ u \in \mathcal{T}_\mathbf{q}, n \in \mathbb{Z}_+, \ \ u^n \in Z(\mathcal{T}_\mathbf{q}) \Rightarrow u \in Z(\mathcal{T}_\mathbf{q}). \tag{3.5}$$

It follows from either of the two conditions (3.4) and (3.5) that the property of a torus $\mathcal{T}_\mathbf{q}$ being saturated is independent of the choice of generators.

An N-tuple $\mathbf{d} = (d_1,\dots,d_N) \in \mathbb{N}^N$ will be called a *degree vector*. Such will be used to define a completion of $\mathcal{T}_\mathbf{q}$ as follows. Define the homomorphism

$$D \colon \mathbb{Z}^N \to \mathbb{Z}, \ \ D(j_1,\dots,j_N) = d_1 j_1 + \dots + d_N j_N$$

and the $\mathbb{Z}$-grading

$$\mathcal{T}_\mathbf{q} = \oplus_{m\in\mathbb{Z}} \mathcal{T}_\mathbf{q}^m, \ \ \mathcal{T}_\mathbf{q}^m = \mathrm{Span}\{X^f \mid D(f) = m\}.$$

Consider the associated valuation $\nu \colon \mathcal{T}_\mathbf{q} \to \mathbb{Z} \sqcup \{\infty\}$ given by

$$\nu(u_m + u_{m+1} + \dots + u_{m'}) := m \ \text{ for } \ u_j \in \mathcal{T}_\mathbf{q}^j, j \in [m,m'], u_m \neq 0.$$

The completion of $\mathcal{T}_\mathbf{q}$ with respect to this valuation is given by

$$\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}} := \{u_m + u_{m+1} + \dots \mid m \in \mathbb{Z}, u_j \in \mathcal{T}_\mathbf{q}^j \text{ for } j \geq m\}.$$

For $m \in \mathbb{Z}$, denote

$$\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq m} := \{u_m + u_{m+1} + \dots \mid u_j \in \mathcal{T}_\mathbf{q}^j \text{ for } j \geq m\} = \{u \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}} \mid \nu(u) \geq m\}.$$

For every graded subalgebra R of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, set $R^{\geq m} = R \cap \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq m}$. It is straightforward to verify that the group of units of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ is given by

$$U(\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}) = \{cX^f + u \mid c \in \mathbb{K}^*, f \in \mathbb{Z}^N, u \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq D(f)+1}\}, \tag{3.6}$$

and that for an element $cX^f + u$ as in the right hand side of (3.6) we have

$$(cX^f + u)^{-1} = c^{-1}X^{-f}(1 + c^{-1}uX^{-f})^{-1} = \sum_{m=0}^{\infty} (-1)^m c^{-m-1} X^{-f} (uX^{-f})^m. \tag{3.7}$$

Definition 3.1. A continuous automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ will be called *unipotent* if

$$\phi(X_k) - X_k \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq d_k+1} \ \text{ for all } k \in [1,N].$$

A unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ will be called *finite* if

$$\phi(X_k) \in \mathcal{T}_\mathbf{q} \ \text{ for all } k \in [1,N].$$

A unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ will be called *bi-finite* if both ϕ and ϕ^{-1} are finite.

Not all finite unipotent automorphisms are bi-finite. In the single parameter case certain finite unipotent automorphisms that are not bi-finite play an important role in the Berenstein–Zelevinsky work on quantum cluster algebras, see [9, Proposition 4.2].

Remark 3.2. (i) An automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ is *unipotent* if and only if

$$\phi(u) - u \in \widehat{\mathcal{T}}^{\geq m+1}_{\mathbf{q},\mathbf{d}}, \ \ \forall u \in \widehat{\mathcal{T}}^{\geq m}_{\mathbf{q},\mathbf{d}}, m \in \mathbb{Z}, \tag{3.8}$$

which is also equivalent to

$$\nu\big((\phi - \mathrm{id})u\big) \geq \nu(u) + 1, \ \ \forall u \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}.$$

An endomorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ satisfying (3.8) is a continuous automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. The set of all unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ is a subgroup of the group of all continuous automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$.

(ii) There is an obvious isomorphism between the group of all automorphisms ϕ of $\mathrm{Fract}(\mathcal{T}_{\mathbf{q}})$ such that

$$\phi(X_k) - X_k, \phi^{-1}(X_k) - X_k \in \mathcal{T}^{\geq d_k+1}_{\mathbf{q}}, \ \ \forall k \in [1, N] \tag{3.9}$$

and the group of bi-finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$.

Example 3.3. Denote by $\mathcal{A}_{\mathbf{q}}$ the quantum affine space subalgebra of $\mathcal{T}_{\mathbf{q}}$ generated by $X_1, \ldots, X_N$. In special cases the automorphism groups of such algebras were studied by Alev and Chamarie [3]. Every automorphism ψ of $\mathcal{A}_q$ such that

$$\psi(X_k) - X_k \in \mathcal{A}^{\geq d_k+1}_{\mathbf{q}}, \ \ \forall k \in [1, N] \tag{3.10}$$

uniquely extends to a bi-finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. In particular, this applies to the automorphisms of polynomial algebras (the case $q_{kl} = 1$ for all k, l) satisfying (3.10) for $\mathbf{d} = (1, \ldots, 1)$. Such automorphisms appear in various contexts.

Lemma 3.4. *For all multiplicatively skew-symmetric matrices* $\mathbf{q} = (q_{kl})_{k,l=1}^N$ *and degree vectors* $\mathbf{d}$*, the set of unipotent automorphisms of the completed quantum torus* $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ *is in bijection with the* N*-tuples* $(u_1, \ldots, u_N)$ *of elements of* $\widehat{\mathcal{T}}^{\geq 1}_{\mathbf{q},\mathbf{d}}$ *such that*

$$(1 + u_k)X_k(1 + u_l)X_l = q_{kl}(1 + u_l)X_l(1 + u_k)X_k$$

for all $1 \leq k < l \leq N$.

Proof. If ϕ is a unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, then the N-tuple

$$(\phi(X_1)X_1^{-1} - 1, \ldots, \phi(X_N)X_N^{-1} - 1)$$

satisfies the required property.

For an N-tuple $(u_1, \ldots, u_N)$ with that property, first define

$$\phi(X_k) := (1 + u_k)X_k \ \text{ and } \ \phi(X_k^{-1}) := X_k^{-1}(1 + u_k)^{-1} \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}, \ \ k \in [1, N], \tag{3.11}$$

cf. (3.7). Then extend ϕ to $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ by multiplicativity

$$\phi(X^{j_1e_1+\ldots+j_Ne_N}) := \phi(X_1^{\mathrm{sign}(j_1)})^{|j_1|} \ldots \phi(X_N^{\mathrm{sign}(j_N)})^{|j_N|}, \ \ \forall j_1, \ldots, j_N \in \mathbb{Z},$$

linearity, and continuity. It is straightforward to show that the map ϕ, constructed in this way, is an endomorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, which satisfies (3.8). Thus it is a unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, see Remark 3.2 (i). □

Remark 3.5. The algebra of all continuous endomorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ is closely related to the group of the unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. Every continuous endomorphism ψ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ satisfies

$$\psi(X_k) = c_k(1 + u_k)X^{f_k}, \ \ \forall k \in [1, N] \tag{3.12}$$

for some $f_k \in \mathbb{Z}^N$, $c_k \in \mathbb{K}^*$, $u_k \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}^{\geq 1}$ which have the properties

$$(1+u_k)X^{f_k}(1+u_l)X^{f_l} = q_{kl}(1+u_l)X^{f_l}(1+u_k)X^{f_k}, \ \ \forall k<l \ \text{ and}$$
$$\exists b \in \mathbb{Q}_+ \ \text{ such that } \ D(f_k) = bd_k, \ \ \forall k \in [1,N].$$

(This is easily proved using the continuity of ψ and the fact that all $\psi(X_k)$ should be units of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, cf. (3.6).) Furthermore, every ψ as in (3.12) with c_k, f_k, u_k having the stated properties, uniquely extends to a continuous endomorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$.

Lemma 3.4 implies that among all continuous endomorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, the unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ are precisely the maps (3.12) for which $c_k = 1$ and $f_k = e_k$ for all $k \in [1,N]$.

The next theorem contains the main result in this section.

Theorem 3.6. *Assume that $\mathbb{K}$ is an arbitrary base field, $\mathbf{q} \in M_N(\mathbb{K}^*)$ a multiplicatively skew-symmetric matrix for which the quantum torus $\mathcal{T}_{\mathbf{q}}$ is saturated, and $\mathbf{d} \in \mathbb{Z}_+^N$ a degree vector. Then for every bi-finite unipotent automorphism ϕ of the completed quantum torus $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, there exists an N-tuple*

$$(u_1, u_2, \dots, u_N) \ \text{ of elements of } \ Z(\mathcal{T}_{\mathbf{q}})^{\geq 1}$$

such that $\phi(X_k) = (1+u_k)X_k$ for all $k \in [1,N]$.

In light of Remark 3.2 (ii) this theorem can be restated as follows:

For a saturated quantum torus $\mathcal{T}_{\mathbf{q}}$ and $\mathbf{d} \in \mathbb{Z}_+^N$, every automorphism ϕ of Fract$(\mathcal{T}_{\mathbf{q}})$ *that satisfies* (3.9) *is given by $\phi(X_k) = (1+u_k)X_k$ for some elements $u_1, u_2, \dots, u_N \in Z(\mathcal{T}_{\mathbf{q}})^{\geq 1}$.*

Theorem 3.6 is proved in §3.3. The statement of the theorem does not hold if one replaces bi-finite with finite unipotent automorphisms of completed saturated quantum tori. Counterexamples are provided by the Berenstein–Zelevinsky automorphisms in [9, Proposition 4.2]. Recalling Example 3.3 we note the following:

Corollary 3.7. *Under the above assumptions on* $\mathbf{q}$ *and* $\mathbf{d}$*, for every automorphism ψ of the quantum affine space algebra $\mathcal{A}_{\mathbf{q}}$ satisfying* (3.10) *there exist $u_1, \dots, u_N \in Z(\mathcal{T}_q)^{\geq 1}$ such that $u_k X_k \in \mathcal{A}_{\mathbf{q}}$ and $\psi(X_k) = (1+u_k)X_k$, $\forall k \in [1,N]$.*

Artamonov [6] studied the automorphisms of completions of quantum tori $\mathcal{T}_{\mathbf{q}}$ with respect to maximal valuations $\nu \colon \mathcal{T}_{\mathbf{q}} \backslash \{0\} \to \mathbb{Z}^N$ in a different direction from ours. He considers quantum tori for which the parameters $\{q_{kl} \mid 1 \leq k < l \leq N\}$ form a free subgroup of $\mathbb{K}^*$ of rank $N(N-1)/2$ (such tori appear more rarely and have trivial centers) and deals with all automorphisms as opposed to a special subclass of the unipotent ones.

For the rest of the section we will use the notation in the left hand side of (3.1) for the generators of $\mathcal{T}_{\mathbf{q}}$, which is more instructive in working with quantum tori.

3.2. **Supports and restrictions of unipotent automorphisms.** For an element

$$u = \sum_{f \in \mathbb{Z}^N} c_f X^f \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$$

denote $[u]_f := c_f$. Define its support by

$$\mathrm{Supp}(u) := \{f \in \mathbb{Z}^N \mid [u]_f \neq 0\}.$$

Definition 3.8. Given a finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, we will call the set

$$\begin{aligned}\mathrm{Supp}(\phi) &= \bigcup_{k=1}^{n} \mathrm{Supp}(\phi(X^{e_k})X^{-e_k} - 1) \\ &= \{f \in \mathbb{Z}^N \backslash \{0\} \mid [\phi(X^{e_k})]_{e_k+f} \neq 0 \ \text{ for some } \ k \in [1,N]\}\end{aligned}$$

the *support* of ϕ.

By a strict cone C in $\mathbb{R}^N$ we will mean a set of the form $\mathbb{R}_{\geq 0}X$ for a finite subset $X = \{x_1, \ldots, x_n\}$ of $\mathbb{R}^N$ such that

$$a_1x_1 + \ldots + a_nx_n = 0,\ a_1, \ldots, a_n \in \mathbb{R}_{\geq 0} \Rightarrow a_1 = \ldots = a_n = 0. \tag{3.13}$$

In other words C contains no lines. The support of any finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ lies in a translated half space

$$\mathrm{Supp}(\phi) \subset \{(a_1, \ldots, a_N) \in \mathbb{R}^N \mid d_1a_1 + \ldots + d_Na_N \geq 1\}.$$

Thus for every finite collection $\phi_1, \ldots, \phi_j$ of finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, the set $\mathbb{R}_{\geq 0}(\mathrm{Supp}(\phi_1) \cup \ldots \cup \mathrm{Supp}(\phi_j))$ satisfies the condition (3.13).

Definition 3.9. Given a finite family $\phi_1, \ldots, \phi_j$ of finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, we define its joint cone by

$$\mathrm{Con}(\phi_1, \ldots, \phi_j) = \mathbb{R}_{\geq 0}(\mathrm{Supp}(\phi_1) \cup \ldots \cup \mathrm{Supp}(\phi_j)) \subset \mathbb{R}^N.$$

Lemma 3.10. *(i) If ϕ is a finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and $g \in \mathbb{Z}^N$, then*

$$\mathrm{Supp}(\phi(X^g)X^{-g} - 1) \in \big(\mathbb{N}\,\mathrm{Supp}(\phi)\big)\backslash\{0\}.$$

(ii) If ϕ and ψ are two finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, then

$$\mathrm{Supp}(\phi \circ \psi) \subset \big(\mathbb{N}(\mathrm{Supp}(\phi) \cup \mathrm{Supp}(\psi))\big)\backslash\{0\}$$

and thus $\mathrm{Con}(\phi \circ \psi) \subset \mathrm{Con}(\phi) + \mathrm{Con}(\psi)$.

(iii) For all bi-finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ we have

$$\mathrm{Supp}(\phi^{-1}) \subset \big(\mathbb{N}\,\mathrm{Supp}(\phi)\big)\backslash\{0\} \ \ \text{and} \ \ \mathrm{Supp}(\phi) \subset \big(\mathbb{N}\,\mathrm{Supp}(\phi^{-1})\big)\backslash\{0\}.$$

In particular, $\mathrm{Con}(\phi^{-1}) = \mathrm{Con}(\phi)$.

Proof. (i) The coefficients of $\phi(X^{-e_k})$ are determined from the ones of $\phi(X^{e_k})$ using the equality

$$\phi(X^{-e_k}) = X^{-e_k} \sum_{m=0}^{\infty} \big(1 - \phi(X^{e_k})X^{-e_k}\big)^m, \tag{3.14}$$

cf. (3.7). The case $g = -e_k$ of part (i) follows from this. The general case is obtained by multiplying such expressions for $\phi(X^{\pm e_k})$, $k \in [1, N]$.

Part (ii) is a direct consequence of part (i).

(iii) The coefficients $[\phi^{-1}(X^{e_k})]_{e_k+f}$ are recursively determined for $D(f) = 1, 2, \ldots$ from the coefficients of ϕ by

$$\begin{aligned}[\phi^{-1}(x^{e_k})]_{e_k+f} = -&\sum_{j_1,\ldots,j_N,f_1,\ldots f_N} q_*[\phi(X^{e_k})]_{j_1e_1+\ldots+(j_k+1)e_k+\ldots+j_Ne_N} \\ &\times [\phi^{-1}(X^{j_1e_1})]_{j_1e_1+f_1} \ldots [\phi^{-1}(X^{(j_k+1)e_k})]_{(j_k+1)e_k+f_k} \ldots [\phi^{-1}(X^{j_Ne_N})]_{j_Ne_N+f_N}\end{aligned}$$

for some appropriate elements $q_* \in \mathbb{K}$, where the sum is over $j_1, \ldots, j_N \in \mathbb{Z}$, $f_1, \ldots, f_N \in \mathbb{Z}^N$ such that $\sum_{k=1}^N j_ke_k \in \mathrm{Supp}(\phi)$, $0 < D(f_1), \ldots, D(f_N) < D(f)$, $f_1 + \ldots + f_N + \sum_{k=1}^N j_ke_k = f$. In the right hand side, using (3.14) for $j' \in \mathbb{Z}$ and $f' \in \mathbb{Z}^N$ such that $D(f') > 0$, one expresses $[\phi^{-1}(X^{j'e_k})]_{j'e_k+f'}$ in terms of $[\phi^{-1}(X^{e_k})]_{e_k+g}$ for $g \in \mathbb{Z}^N$, $0 < D(g) \leq D(f') < D(f)$. (Note the finiteness of the sum in the right hand side of the above formula.) Part (iii) of the lemma easily follows from this formula. $\square$

Let C be a strict cone in $\mathbb{R}^N$ and $x \in \mathbb{R}^N\backslash\{0\}$. The ray $\mathbb{R}_{\geq 0}x$ is an *extremal ray* of C, if $x \in C$ and for all $x_1, x_2 \in C$, $x_1 + x_2 \in \mathbb{R}_{\geq 0}v$ implies $x_1, x_2 \in \mathbb{R}_{\geq 0}x$. For a ray $\mathbb{R}_{\geq 0}f$ in $\mathbb{R}^N$ and $u \in \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ denote

$$u|_{\mathbb{R}_{\geq 0}f} = \sum_{g \in (\mathbb{R}_{\geq 0}f \cap \mathbb{Z}^N)} [u]_g X^g.$$

Let ϕ be a finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ with

$$\phi(X^{e_k}) = (1+u_k)X^{e_k}, u_k \in \mathcal{T}_{\mathbf{q}}^{\geq 1}, \ \ k \in [1,N].$$

Let $\mathbb{R}_{\geq 0}f$ be an extremal ray of $\mathrm{Con}(\phi)$. It is straightforward to verify that the N-tuple

$$(u_1|_{\mathbb{R}_{\geq 0}f}, \ldots, u_f|_{\mathbb{R}_{\geq 0}f})$$

of elements of $\mathcal{T}_{\mathbf{q}}^{\geq 1}$ satisfies the conditions of Lemma 3.4. Therefore it defines a finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ which will be denoted by $\phi|_{\mathbb{R}_{\geq 0}f}$. For $f \notin \mathrm{Con}(\phi)$, we set $\phi|_{\mathbb{R}_{\geq 0}f} := \mathrm{id}$.

The proof of the following result is analogous to the proof of Lemma 3.10 and is left to the reader.

Proposition 3.11. *For all completed quantum tori $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and $f \in \mathbb{Z}^N \backslash \{0\}$ the following hold:*

(i) If ϕ is a finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and f is such that $\mathbb{R}_{\geq 0}f$ is an extremal ray of $\mathrm{Con}(\phi)$ *or $f \notin \mathrm{Con}(\phi)$, then*

$$\phi|_{\mathbb{R}_{\geq 0}f}(X^g) = (\phi(X^g)X^{-g})|_{\mathbb{R}_{\geq 0}f} X^g, \ \ \forall g \in \mathbb{Z}^N.$$

(ii) If ϕ and ψ are finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and $\mathbb{R}_{\geq 0}f$ is an extremal ray of $\mathrm{Con}(\phi,\psi)$*, then*

$$(\phi \circ \psi)|_{\mathbb{R}_{\geq 0}f} = \phi|_{\mathbb{R}_{\geq 0}f} \circ \psi|_{\mathbb{R}_{\geq 0}f},$$

cf. Lemma 3.10 (ii).

(iii) If ϕ is a bi-finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and $\mathbb{R}_{\geq 0}f$ is an extremal ray of $\mathrm{Con}(\phi)$*, then $\phi|_{\mathbb{R}_{\geq 0}f}$ is a bi-finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ and*

$$\left(\phi|_{\mathbb{R}_{\geq 0}f}\right)^{-1} = (\phi^{-1})|_{\mathbb{R}_{\geq 0}f},$$

cf. Lemma 3.10 (iii).

3.3. **Bi-finite unipotent automorphisms of completed quantum tori.** Our proof of Theorem 3.6 is based on a result for unipotent automorphisms of completed saturated quantum tori $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ with support not lying in $\mathrm{Ker}(\mathbf{q})$ and on Proposition 3.11. The former is obtained in Proposition 3.14. Proposition 3.12 and Lemma 3.13 contain two auxiliary results for the proof of Proposition 3.14.

For $f \in \mathbb{Z}^N$ denote by $\mu_f \colon \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}} \to \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ the continuous $\mathbb{K}$-linear map, given by

$$\mu_f(X^g) = X^f X^g, \ \ \forall g \in \mathbb{Z}^N.$$

Proposition 3.12. *Let $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ be a completed saturated quantum torus of rank N over an arbitrary field $\mathbb{K}$ as above. For all $f \in \mathbb{Z}^N$, $f \notin \mathrm{Ker}(\mathbf{q})$ and $c \in \mathbb{K}$,*

$$\phi_{f,c}(X^g) := X^g + (1-q_{f,g}^{-1}) \sum_{m=1}^{\infty} c^m (X^f)^m X^g, \ \ g \in \mathbb{Z}^N,$$

where $q_{f,g} := X^f X^g X^{-f} X^{-g} \in \mathbb{K}^*$, defines a unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$.

Proof. Only the fact that $\phi_{f,c}$ is an endomorphism requires a proof since the bijectivity of the map is immediate, see Remark 3.2 (i). It is easy to verify this statement directly. We give another proof that explains the origin of $\phi_{f,c}$. The lemma is effectively a statement for quantum tori defined over the ring $\mathbb{Z}[c, q_{kl}^{\pm 1}, 1 \leq k < l \leq N]$ (where $c, q_{kl}^{\pm 1}$ are independent variables). It is sufficient to prove that $\phi_{f,c}$ is an endomorphism for base fields of characteristic 0. For such, it is obvious that $\psi_{f,c} \colon \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}} \to \widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$, given by

$$\psi_{f,c} = \exp\left(\sum_{m=1}^{\infty} \frac{c^m}{m} \mathrm{ad}_{(X^f)^m} \right),$$

is a unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. It satisfies

$$\psi_{f,c}(X^g) = \exp\left(\sum_{m=1}^{\infty} \frac{c^m}{m}(1-q_{f,g}^{-m})\mu_f^m\right) X^g.$$

The following identity of formal power series over $\mathbb{Q}[q^{\pm 1}, c]$ in the variable z shows that $\psi_{f,c} = \phi_{f,c}$:

$$\exp\Big(\sum_{m=1}^{\infty} c^m(1-q^{-m})z^m/m\Big) = \exp(-\log(1-cz) + \log(1-cq^{-1}z))$$
$$= (1-cq^{-1}z)(1-cz)^{-1} = 1 + (1-q^{-1})\sum_{m=1}^{\infty} c^m z^m, \ \ \forall j \in \mathbb{Z},$$

where $\log(1-z) := -\sum_{m=1}^{\infty} z^m/m$. Therefore $\phi_{f,c}$ is a unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ if the base field has characteristic 0, and by the above reasoning for all base fields. □

Lemma 3.13. *Let $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ be a completed saturated quantum torus over an arbitrary field $\mathbb{K}$ and $f \in \mathbb{Z}^N$ such that $f \notin \mathrm{Ker}(\mathbf{q})$ and $f/j \notin \mathbb{Z}^N$, $\forall j \in \mathbb{Z}_+$. Then for every unipotent automorphism ϕ of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ with $\mathrm{Supp}\,\phi \subseteq \mathbb{N}f$ and $m \in \mathbb{N}$, there exist $c_1, \ldots, c_m \in \mathbb{K}$ such that*

$$\big(\phi(X^g) - \phi_{mf,c_m}\ldots\phi_{f,c_1}(X^g)\big)X^{-g} \in \mathrm{Span}\{X^{(m+1)f}, X^{(m+2)f}, \ldots\}, \ \ \forall g \in \mathbb{Z}^N.$$

In other words, every automorphism ϕ satisfying the conditions in Lemma 3.13 is given by

$$\phi = \ldots \phi_{2f,c_2}\phi_{f,c_1}$$

for some $c_1, c_2, \ldots \in \mathbb{K}$. Note that the infinite (right-to-left) product is well a defined unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$.

Proof of Lemma 3.13. We prove the statement by induction on m, the case $m = 0$ being trivial. Assume its validity for some $m \in \mathbb{N}$, and define

$$\delta(X^g) := \big[\phi(X^g) - \phi_{mf,c_m}\ldots\phi_{f,c_1}(X^g)\big]_{g+(m+1)f} X^{g+(m+1)f}, \ \ g \in \mathbb{Z}^N.$$

Then δ is a derivation of $\mathcal{T}_{\mathbf{q}}$, which must be inner by [36, Corollary 2.3] since $(m+1)f \notin \mathrm{Ker}(\mathbf{q})$, recall Eq. (3.3). Therefore $\delta = c_{m+1}\,\mathrm{ad}_{X^{(m+1)f}}$ for some $c_{m+1} \in \mathbb{K}$. Then

$$X^{-g}\big(\phi(X^g) - \phi_{(m+1)f,c_{m+1}}\ldots\phi_{f,c_1}(X^g)\big) \in \mathrm{Span}\{X^{(m+2)f}, X^{(m+3)f}, \ldots\}, \ \ \forall g \in \mathbb{Z}^N$$

which completes the induction. □

Proposition 3.14. *Assume that $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ is a completed saturated quantum torus of rank N over an arbitrary base field $\mathbb{K}$. If ϕ is a bi-finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ with support in $\mathbb{R}_{\geq 0}f$ for some $f \in \mathbb{Z}^N$ such that $f \notin \mathrm{Ker}(\mathbf{q})$, then ϕ is the identity automorphism.*

Nontrivial finite unipotent automorphisms with support in $\mathbb{R}_{\geq 0}f$ for $f \in \mathbb{Z}^N$, $f \notin \mathrm{Ker}(\mathbf{q})$ which are not bi-finite are constructed in [9, Proposition 4.2].

Proof of Proposition 3.14. We have $\mathbb{R}_{\geq 0}f \cap \mathbb{Z}^N = \mathbb{N}f_0$ for some $f_0 \in \mathbb{Z}^N$. Assume that the statement of the proposition is not correct. Then there exists $k \in [1, N]$ such that

$$\phi(X^{e_k}) = X^{e_k} + c_1 X^{f_0}X^{e_k} + \ldots + c_m X^{mf_0}X^{e_k}$$

for some $m \in \mathbb{Z}_+$, $c_1, \ldots, c_m \in \mathbb{K}$, $c_m \neq 0$. Moreover,

$$\phi^{-1}(X^{e_k}) = X^{e_k} + c'_1 X^{f_0}X^{e_k} + \ldots + c'_j X^{jf_0}X^{e_k}$$

for some $j \in \mathbb{N}$, $c'_1, \ldots, c'_j \in \mathbb{K}$, $c'_j \neq 0$. Lemma 3.13 implies that $\phi^{-1}(X^{f_0}) = X^{f_0}$ and thus

$$\phi^{-1}\phi(X^{e_k}) - c_m c'_j X^{mf_0}X^{jf_0}X^{e_k} \in \mathrm{Span}\{X^{e_k}, \ldots, X^{e_k+(m+j-1)f_0}\}.$$

Therefore $[\phi^{-1}\phi(X^{e_k})]_{e_k+(m+j)f_0} \neq 0$ which contradicts with $\phi^{-1}\phi = \mathrm{id}$ because $m > 0$. □

We now proceed with the proof of Theorem 3.6.

Proof of Theorem 3.6. Assume that ϕ is a bi-finite unipotent automorphism of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$. If $\phi = \mathrm{id}$, we are done. Otherwise, let $\mathbb{R}_{\geq 0} f_1, \ldots, \mathbb{R}_{\geq 0} f_n$ be the extremal rays of $\mathrm{Con}(\phi)$. By Proposition 3.11 (iii), $\phi|_{\mathbb{R}_{\geq 0} f_1}, \ldots, \phi|_{\mathbb{R}_{\geq 0} f_n}$ are bi-finite unipotent automorphisms of $\widehat{\mathcal{T}}_{\mathbf{q},\mathbf{d}}$ that are not equal to the identity automorphism. Proposition 3.14 implies that $f_1, \ldots, f_n \in \mathrm{Ker}(\mathbf{q})$. Therefore

$$\mathrm{Con}(\phi) = \mathbb{R}_{\geq 0} f_1 + \ldots + \mathbb{R}_{\geq 0} f_n \subset \mathrm{Ker}(\mathbf{q}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

It follows from Eq. (3.3) that $\mathrm{Supp}(\phi) \subset \mathrm{Ker}(\mathbf{q})$. The definition of $\mathrm{Supp}(\phi)$ and Eq. (3.2) imply that there exists an N-tuple $(u_1, u_2, \ldots, u_N)$ of elements of $Z(\mathcal{T}_{\mathbf{q}})^{\geq 1}$ such that $\phi(X_k) = (1 + u_k) X_k$, $\forall k \in [1, N]$. This completes the proof the theorem. □

4. Unipotent automorphisms of $\mathcal{U}_q^-(\mathfrak{g})$

4.1. **Statement of the main result.** In this section we carry out the major step of the proof of the Andruskiewitsch–Dumas conjecture. We define unipotent automorphisms of the algebras $\mathcal{U}_q^-(\mathfrak{g})$ in a similar fashion to the case of completed quantum tori, see Definition 4.1 for details. In Theorem 4.2 we prove a rigidity result for them stating that every unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$ is equal to the identity automorphism. The proof of this result is obtained in several reduction stages using the rigidity result from the previous section and Theorems 2.2 and 2.3. The reductions appear in §4.2–4.3 and the proof of Theorem 4.2 is given in §4.4.

Every strictly dominant integral coweight $\lambda = \sum_{\alpha \in \Pi} m_\alpha \varpi_\alpha^\vee \in \mathcal{P}_{++}^\vee$ gives rise to a specialization of the $(-\mathcal{Q}_+)$-grading of the algebra $\mathcal{U}_q^-(\mathfrak{g})$ to an $\mathbb{N}$-grading as follows:

$$\mathcal{U}_q^-(\mathfrak{g}) = \bigoplus_{m \in \mathbb{N}} \mathcal{U}_q^-(\mathfrak{g})^m, \;\; \mathcal{U}_q^-(\mathfrak{g})^m := \bigoplus \{ \mathcal{U}_q^-(\mathfrak{g})_{-\gamma} \mid \gamma \in \mathcal{Q}_+, \langle \lambda, \gamma \rangle = m \}. \tag{4.1}$$

In other words, the generators F_α of $\mathcal{U}_q^-(\mathfrak{g})$ are assigned degrees $m_\alpha = \langle \lambda, \alpha \rangle$. (The graded components in (4.1) depend on the choice of λ, but this dependence will not be explicitly shown for simplicity of the notation as it was done for quantum tori.) This $\mathbb{N}$-grading of $\mathcal{U}_q^-(\mathfrak{g})$ is connected, i.e., $\mathcal{U}_q^-(\mathfrak{g})^0 = \mathbb{K}$. For $m \in \mathbb{N}$, denote

$$\mathcal{U}_q^-(\mathfrak{g})^{\geq m} = \bigoplus_{j \geq m} \mathcal{U}_q^-(\mathfrak{g})^j.$$

For a graded subalgebra R of $\mathcal{U}_q^-(\mathfrak{g})$, set $R^{\geq m} = R \cap \mathcal{U}_q(\mathfrak{g})^{\geq m}$.

Definition 4.1. Given a strictly dominant integral coweight λ, we call an automorphism Φ of $\mathcal{U}_q^-(\mathfrak{g})$ λ-*unipotent* if

$$\Phi(F_\alpha) - F_\alpha \in \mathcal{U}_q^-(\mathfrak{g})^{\geq \langle \lambda, \alpha \rangle + 1} \;\text{ for all }\; \alpha \in \Pi.$$

Every λ-unipotent automorphism Φ of $\mathcal{U}_q^-(\mathfrak{g})$ satisfies

$$\Phi(u) - u \in \mathcal{U}_q^-(\mathfrak{g})^{\geq m+1} \;\text{ for all }\; u \in \mathcal{U}_q^-(\mathfrak{g})^m, m \in \mathbb{N}. \tag{4.2}$$

The set of λ-unipotent automorphisms of $\mathcal{U}_q(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}(\mathcal{U}_q^-(\mathfrak{g}))$ for all $\lambda \in \mathcal{P}_{++}^\vee$.

The following theorem is the major step in our proof of the Andruskiewitsch–Dumas conjecture.

Theorem 4.2. *Let $\mathfrak{g}$ be a simple Lie algebra of rank $r > 1$, $\mathbb{K}$ an arbitrary base field, and q a deformation parameter that is not a root of unity. For every strictly dominant integral coweight λ, the only λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$ is the identity automorphism.*

An analogous statement holds for the algebras $\mathcal{U}_q^+(\mathfrak{g})$ because of the isomorphism $\omega \colon \mathcal{U}_q^\pm(\mathfrak{g}) \to \mathcal{U}_q^\mp(\mathfrak{g})$, see (2.1). Theorem 4.2 trivially holds in the case when $\mathrm{rank}(\mathfrak{g}) = 1$ since $\mathcal{U}_q^-(\mathfrak{sl}_2) = \mathbb{K}[F_\alpha]$.

4.2. **Relations to automorphisms of completed quantum tori.** Let

$$\mathbf{i} = (\alpha_1, \ldots, \alpha_N) \tag{4.3}$$

be a reduced word for the longest element w_0 of W. Recall from §2.3 that we denote by $(\overline{F}_{\mathbf{i},1}, \ldots, \overline{F}_{\mathbf{i},N})$ the final N-tuple of elements of $\mathrm{Fract}(\mathcal{U}_q^-(\mathfrak{g}))$ from the Cauchon deleting derivation procedure for the iterated Ore extension presentation (2.16) of $\mathcal{U}_q^-(\mathfrak{g})$. Denote by $\mathcal{T}(\mathbf{i})$ the subalgebra of $\mathrm{Fract}(\mathcal{U}_q^-(\mathfrak{g}))$ generated by them and their inverses

$$\mathcal{T}(\mathbf{i}) = \mathbb{K}\langle \overline{F}_{\mathbf{i},1}^{\pm 1}, \ldots, \overline{F}_{\mathbf{i},N}^{\pm 1} \rangle \subset \mathrm{Fract}(\mathcal{U}_q^-(\mathfrak{g})),$$

see (2.8). Let $\mathbf{q} \in M_N(\mathbb{K}^*)$ be the multiplicatively skew-symmetric matrix such that

$$q_{kl} = q^{\langle \beta_k, \beta_l \rangle} \ \text{ for } \ 1 \leq k < l \leq N. \tag{4.4}$$

By (2.9) and (2.17) we have the isomorphism of quantum tori

$$\mathcal{T}_{\mathbf{q}} \cong \mathcal{T}(\mathbf{i}), \ \ X_l \mapsto \overline{F}_{\mathbf{i},l}, \ \ l \in [1, N]$$

in the notation of (1.1). We change the generating set $(\overline{F}_{\mathbf{i},1}, \ldots, \overline{F}_{\mathbf{i},N})$ using Theorem 2.3. Recall the definition (2.26) of the successor function $s \colon [1, N] \to [1, N] \cup \{\infty\}$ associated to the reduced word $\mathbf{i}$. For $l \in [1, N]$ denote

$$O(l) = \max\{ m \in \mathbb{N} \mid s^m(j) \neq \infty \},$$

where as usual $s^0 = \mathrm{id}$. Theorem 2.3 implies

$$\Delta_{\mathbf{i},l} = (q_{\alpha_l}^{-1} - q_{\alpha_l})^{O(l)} \overline{F}_{\mathbf{i}, s^{O(l)}(l)} \ldots \overline{F}_{\mathbf{i},l} \tag{4.5}$$

and

$$\mathcal{T}(\mathbf{i}) = \mathbb{K}\langle \Delta_{\mathbf{i},1}^{\pm 1}, \ldots, \Delta_{\mathbf{i},N}^{\pm 1} \rangle \subset \mathrm{Fract}(\mathcal{U}_q^-(\mathfrak{g})).$$

Moreover, from (4.5) we have

$$\Delta_{\mathbf{i},k} \Delta_{\mathbf{i},l} = q^{n'_{kl}} \Delta_{\mathbf{i},l} \Delta_{\mathbf{i},k},$$

where

$$n'_{kl} = \sum_{j=1}^{O(k)} \sum_{m=1}^{O(l)} \mathrm{sign}(s^m(l) - s^j(k)) \langle \beta_{s^j(k)}, \beta_{s^m(l)} \rangle \ \text{ for } \ k, l \in [1, N]. \tag{4.6}$$

Denote by $\mathbf{q}' \in M_N(\mathbb{K})$ the multiplicatively skew-symmetric matrix whose entries are given by $q'_{kl} = q^{n'_{kl}}$. We have the isomorphism

$$\mathcal{T}_{\mathbf{q}'} \cong \mathcal{T}(\mathbf{i}), \ \ X_l \mapsto \Delta_{\mathbf{i},l}, \ \ l \in [1, N]. \tag{4.7}$$

The $\mathcal{Q}$-grading of $\mathcal{U}_q^-(\mathfrak{g})$ gives rise to a $\mathcal{Q}$-grading of the quantum torus $\mathcal{T}(\mathbf{i})$ by assigning

$$\deg \overline{F}_{\mathbf{i},l} := \deg F_{\beta_l} = -\beta_l.$$

Given a strictly dominant integral coweight $\lambda \in \mathcal{P}^\vee_{++}$, this grading can be specialized to a $\mathbb{Z}$-grading

$$\mathcal{T}(\mathbf{i}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{T}(\mathbf{i})^m \ \text{ by setting } \ \deg \overline{F}_{\mathbf{i},l} := \langle \lambda, \beta_l \rangle. \tag{4.8}$$

This grading is compatible with the $\mathbb{N}$-grading (4.1) of $\mathcal{U}_q^-(\mathfrak{g})$

$$\mathcal{U}_q^-(\mathfrak{g})^m \subset \mathcal{T}(\mathbf{i})^m, \ \ \forall m \in \mathbb{N}, \tag{4.9}$$

recall (2.9). Because of (4.5), the $\mathbb{Z}$-grading of $\mathcal{T}(\mathbf{i})$ can be also defined in terms of the generators $\Delta_{\mathbf{i},l}$ by

$$\deg \Delta_{\mathbf{i},l} = \sum_{j=0}^{O(l)} \langle \lambda, \beta_{s^j(l)} \rangle.$$

We associate to $\lambda \in \mathcal{P}^\vee_{++}$ the following degree vector

$$\mathbf{d} = (d_1, \dots, d_N) \in \mathbb{Z}_+^N, \ \ d_l := \sum_{j=0}^{O(l)} \langle \lambda, \beta_{s^j(l)} \rangle, \ \ l \in [1, N] \tag{4.10}$$

for the quantum torus $\mathcal{T}(\mathbf{i}) \cong \mathcal{T}_{\mathbf{q}'}$, recall §3.1. The grading (4.8) is precisely the grading associated to $\mathbf{d}$ as defined in §3.1. Denote by $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$ the corresponding completion of $\mathcal{T}(\mathbf{i})$, which is isomorphic to $\widehat{\mathcal{T}}_{\mathbf{q}', \mathbf{d}}$.

To every λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$ we will associate a bi-finite unipotent automorphism of the completed quantum torus $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$ as follows, recall Definitions 3.1 and 4.1. Let Φ be a unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. By Eqs. (4.2) and (4.9)

$$\Phi(\Delta_{\mathbf{i},l}) = \Delta_{\mathbf{i},l} + u'_l \ \text{ for some } \ u'_l \in \mathcal{U}_q^-(\mathfrak{g})^{\geq d_l + 1} \subseteq \mathcal{T}(\mathbf{i})^{\geq d_l+1}, l \in [1, N].$$

Therefore

$$\Phi(\Delta_{\mathbf{i},l}) = (1 + u_l)\Delta_{\mathbf{i},l} \ \text{ for } \ u_l := u'_l \Delta_{\mathbf{i},l}^{-1} \in \mathcal{T}(\mathbf{i})^{\geq 1}, l \in [1, N].$$

Since Φ is an automorphism of $\mathcal{U}_q^-(\mathfrak{g})$, we obtain from (4.7) that the N-tuple $(u_1, \dots, u_N)$ satisfies the condition in Lemma 3.4. Therefore there exists a unique finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$ given by

$$\phi(\Delta_{\mathbf{i},l}) := \Phi(\Delta_{\mathbf{i},l}) = (1 + u_l)\Delta_{\mathbf{i},l}, \ \ \forall l \in [1, N].$$

We have $\mathcal{U}_q^-(\mathfrak{g}) \subset \mathcal{T}(\mathbf{i}) \subset \widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$ and

$$\Phi|_{\mathcal{U}_q^-(\mathfrak{g})} = \phi. \tag{4.11}$$

Denote by ψ the finite unipotent automorphism of $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$ associated to the inverse unipotent automorphism Φ^{-1} of $\mathcal{U}_q^-(\mathfrak{g})$ by the above construction. It follows from (4.11) that $(\phi \circ \psi)|_{\mathcal{U}_q^-(\mathfrak{g})} = (\psi \circ \phi)|_{\mathcal{U}_q^-(\mathfrak{g})} = \mathrm{id}$. Because $\Delta_{\mathbf{i},l} \in \mathcal{U}_q^-(\mathfrak{g})$, $\forall l \in [1, N]$ and those elements generate $\mathcal{T}(\mathbf{i})$, we have $\phi^{-1} = \psi$. Thus, this construction associates a bi-finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$ to each λ-unipotent automorphism Φ of $\mathcal{U}_q^-(\mathfrak{g})$.

Remark 4.3. We note that one could not achieve the same result using the Cauchon elements $\overline{F}_{\mathbf{i},1}, \dots, \overline{F}_{\mathbf{i},N} \in \mathrm{Fract}(\mathcal{U}_q^-(\mathfrak{g}))$. The key point is that they and their inverses do generate a quantum torus inside $\mathrm{Fract}(\mathcal{U}_q^-(\mathfrak{g}))$, but the Cauchon elements do not belong to $\mathcal{U}_q^-(\mathfrak{g})$ in general (i.e. the first inclusion in (1.5) is not satisfied for the quantum affine space algebra generated by the Cauchon elements). Because of this they cannot be used for the passage to bi-finite unipotent automorphisms of $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$. What we did was to change the Cauchon generators $\overline{F}_{\mathbf{i},1}^{\pm 1}, \dots, \overline{F}_{\mathbf{i},N}^{\pm 1}$ of $\mathcal{T}(\mathbf{i})$ to a new generating set $\Delta_{\mathbf{i},1}^{\pm 1}, \dots, \Delta_{\mathbf{i},N}^{\pm 1}$ of $\mathcal{T}(\mathbf{i})$ in such a way that $\Delta_{\mathbf{i},1}, \dots, \Delta_{\mathbf{i},N} \in \mathcal{U}_q^-(\mathfrak{g})$. Because of these two properties, this new set of generators can be used to establish the connection to bi-finite unipotent automorphisms of completed quantum tori as indicated.

We use the above relationship and Theorem 3.6 to obtain our first reduction for the proof of Theorem 4.2.

Recall that the longest element w_0 of the Weyl group W gives rise to the involution

$$\alpha \mapsto \widetilde{\alpha} := -w_0(\alpha) \tag{4.12}$$

of Π, which is an element of $\mathrm{Aut}(\Gamma)$.

Proposition 4.4. *Let $\mathfrak{g}$, $\mathbb{K}$, $q \in \mathbb{K}^*$, and $\lambda \in \mathcal{P}^\vee_{++}$ be as in Theorem 4.2. Assume that $\alpha \in \Pi$ and $\mathbf{i} = (\alpha_1, \dots, \alpha_N)$ is a reduced word for w_0 such that*

$$\alpha_N = \widetilde{\alpha}. \tag{4.13}$$

If Φ is a λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$, then there exists

$$z_\alpha \in Z(\mathcal{T}(\mathbf{i}))^{\geq 1}$$

such that

$$\Phi(F_\alpha) = (1 + z_\alpha) F_\alpha.$$

Proof. Since $q \in \mathbb{K}^*$ is not a root of unity, the subgroup of $\mathbb{K}^*$ generated by all entries of the matrix $\mathbf{q}'$ is torsion-free. Therefore the quantum tori $\mathcal{T}(\mathbf{i})$ and $\mathcal{T}_{\mathbf{q}'}$ are saturated, recall (4.7). We apply Theorem 3.6 to the bi-finite unipotent automorphism ϕ of $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{d})$ associated to Φ. It implies that

$$\Phi(\Delta_{\mathbf{i},N})\Delta_{\mathbf{i},N}^{-1} = \phi(\Delta_{\mathbf{i},N})\Delta_{\mathbf{i},N}^{-1} = 1 + z_\alpha$$

for some $z_\alpha \in Z(\mathcal{T}(\mathbf{i}))^{\geq 1}$. Theorem 2.3 and Eq. (2.7) imply

$$(q_{\alpha_N}^{-1} - q_{\alpha_N})^{-1} \Delta_{\mathbf{i},N} = \overline{F}_{\mathbf{i},N} = F_{\beta_N}.$$

Since $\beta_N = s_1 \dots s_{N-1}(\alpha_N) = -w_0 s_{\alpha_N}(\alpha_N) = \widetilde{\alpha}_N = \alpha$, it follows from Eq. (2.12) that

$$F_{\beta_N} = F_\alpha. \tag{4.14}$$

Therefore $\Phi(F_\alpha) = (1 + z_\alpha) F_\alpha$. □

Remark 4.5. One can rephrase the arguments from the first part of this subsection (the proof of Eq. (4.7)) in a shorter form, which only uses [21, Proposition 3.3] and [9, Theorem 10.1], and does not use [13]. This proof has the disadvantage that the passage from unipotent automorphisms of $\mathcal{U}_q^-(\mathfrak{g})$ to bi-finite unipotent automorphisms of completed quantum tori misses the conceptual point described in Remark 4.3. For the sake of completeness we sketch this shorter route. It follows from Theorem 10.1 in [9] that

$$\Delta_{\mathbf{i},k}\Delta_{\mathbf{i},l} = q^{n'_{kl}} \Delta_{\mathbf{i},l}\Delta_{\mathbf{i},k}, \quad \forall k, l \in [1, N] \tag{4.15}$$

for some $n'_{kl} \in \mathbb{Z}$, see [21, §3.3] for details. By [21, Proposition 3.3] for all $l \in [1, N]$,

$$\Delta_{\mathbf{i},l} = (q_{\alpha_l}^{-1} - q_{\alpha_l}) \Delta_{\mathbf{i},s(l)} F_{\beta_l} \mod \mathcal{U}_q^-(\mathfrak{g})_{\mathbf{i},[l+1,N]}, \qquad \text{if } s(l) \neq \infty,$$
$$\Delta_{\mathbf{i},l} = (q_{\alpha_l}^{-1} - q_{\alpha_l}) F_{\beta_l} \mod \mathcal{U}_q^-(\mathfrak{g})_{\mathbf{i},[l+1,N]}, \qquad \text{if } s(l) = \infty,$$

where $\mathcal{U}_q^-(\mathfrak{g})_{\mathbf{i},[l+1,N]}$ denotes the unital subalgebra of $\mathcal{U}_q^-(\mathfrak{g})$ generated by $F_{\beta_{l+1}}, \dots, F_{\beta_N}$. (Recall §2.4 for the definition of $s \colon [1, N] \to [1, N] \sqcup \{\infty\}$.) The above three facts and the Levendorskii–Soibelman straightening law (2.13) imply that the $\mathbb{K}$-subalgebra of $\mathrm{Fract}(\mathcal{U}_q^-(\mathfrak{g}))$ generated by $\Delta_{\mathbf{i},1}^{\pm 1}, \dots, \Delta_{\mathbf{i},N}^{\pm 1}$ is isomorphic to a quantum torus and that the integers n'_{kl} are given by (4.6). The second fact follows by comparing leading terms in (4.15) with respect to PBW bases and using (2.13). This establishes the isomorphism from (4.7) and the embedding $\mathcal{U}_q^-(\mathfrak{g}) \subset \mathcal{T}(\mathbf{i})$, and we can proceed with the other arguments as described above. In the case when the base field $\mathbb{K}$ has characteristic 0 and $q \in \mathbb{K}$ is transcendental over $\mathbb{Q}$ the isomorphism from (4.7) and the embedding $\mathcal{U}_q^-(\mathfrak{g}) \subset \mathcal{T}(\mathbf{i})$ also follow from the work of Geiß, Leclerc and Schröer [18].

4.3. **Second reduction step for Theorem 4.2.** Consider the involution (4.12) of Π. Denote its fixed point set by $\Pi^0 = \{\alpha \in \Pi \mid -w_0(\alpha) = \alpha\}$. Choose a set of base points Π^+ of its 2-element orbits. Let $\Pi^- = -w_0(\Pi^+)$. Then we have the decomposition

$$\Pi = \Pi^0 \sqcup \Pi^+ \sqcup \Pi^-.$$

The kernel $\mathrm{Ker}(1 + w_0) := \{\lambda \in \mathcal{P} \mid (1 + w_0)\lambda = 0\}$ is given by

$$\mathrm{Ker}(1 + w_0) = \mathbb{Z}\{\varpi_\alpha \mid \alpha \in \Pi^0\} \oplus \mathbb{Z}\{\varpi_\alpha + \varpi_{\widetilde{\alpha}} \mid \alpha \in \Pi^+\}. \tag{4.16}$$

It follows from the second statement in Theorem 2.2 (ii) that the subalgebra

$$\mathcal{C}_q^-(\mathfrak{g}) = \mathbb{K}\langle b_1^{\varpi_{\alpha'}}, b_1^{\varpi_\alpha} b_1^{\varpi_{\widetilde{\alpha}}} \mid \alpha' \in \Pi^0, \alpha \in \Pi^+ \rangle \subset \mathcal{N}_q^-(\mathfrak{g})$$

is a polynomial algebra over $\mathbb{K}$ in the generators

$$\{b_1^{\varpi_{\alpha'}} \mid \alpha' \in \Pi^0\} \sqcup \{b_1^{\varpi_\alpha} b_1^{\varpi_{\widetilde{\alpha}}} \mid \alpha \in \Pi^+\}. \tag{4.17}$$

We show in Lemma 4.7 that $\mathcal{C}_q^-(\mathfrak{g}) = Z(\mathcal{U}_q^-(\mathfrak{g}))$. The following is the second reduction step in our proof of Theorem 4.2:

Proposition 4.6. *Let $\mathfrak{g}$, $\mathbb{K}$, $q \in \mathbb{K}^*$, and $\lambda \in \mathcal{P}_{++}^\vee$ be as in Theorem 4.2. If Φ is a λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$, then there exist elements*

$$z_\alpha \in \mathcal{C}_q^-(\mathfrak{g})^{\geq 1} \;\; \textit{for} \;\; \alpha \in \Pi$$

such that

$$\Phi(F_\alpha) = (1 + z_\alpha) F_\alpha, \;\; \forall \alpha \in \Pi.$$

Before we proceed with the proof of Proposition 4.6 we obtain several auxiliary results. Denote the multiplicative subset

$$\Omega(\mathfrak{g}) = \mathbb{K}^* \{ b_1^\lambda \mid \lambda \in \mathcal{P}_+ \}$$

consisting of normal elements of $\mathcal{U}_q^-(\mathfrak{g})$, recall (2.22). Consider the localizations

$$\mathcal{N}_q^-(\mathfrak{g})^\sharp := \mathcal{N}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}] \subset \mathcal{U}_-^q[\Omega(\mathfrak{g})^{-1}].$$

The second statement in Theorem 2.2 (ii) implies that $\mathcal{N}_q^-(\mathfrak{g})^\sharp$ is a Laurent polynomial algebra over $\mathbb{K}$ in the generators $\{ b_1^{\varpi_\alpha} \mid \alpha \in \Pi \}$. Denote by $\mathcal{C}_q^-(\mathfrak{g})^\sharp$ the localization of $\mathcal{C}_q^-(\mathfrak{g})$ by the multiplicative subset generated by the elements (4.17). This localization is a Laurent polynomial algebra over $\mathbb{K}$ in the generators (4.17).

Lemma 4.7. *In the above setting*

$$Z(\mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}]) = \mathcal{C}_q^-(\mathfrak{g})^\sharp \tag{4.18}$$

and

$$Z(\mathcal{U}_q^-(\mathfrak{g})) = \mathcal{C}_q^-(\mathfrak{g}). \tag{4.19}$$

Lemma 4.7 was obtained by Caldero [11] in the case when $\mathbb{K} = \mathbb{C}(q)$.

Proof. Each $\mu \in \mathcal{P}$ can be uniquely represented as $\mu = \lambda_+ - \lambda_-$, where $\lambda_\pm \in \mathcal{P}_+$ and $\operatorname{Supp} \lambda_+ \cap \operatorname{Supp} \lambda_- = \emptyset$. Denote

$$b_1^\mu := (b_1^{\lambda_+})^{-1} b_1^{\lambda_-} \in \mathcal{N}_q^-(\mathfrak{g})^\sharp. \tag{4.20}$$

By (2.22), for all $\mu \in \mathcal{P}$

$$b_1^\mu u = q^{\langle (1+w_0)\mu, \gamma \rangle} u b_1^\mu, \;\; \forall\, u \in \mathcal{U}_q^-(\mathfrak{g})_{-\gamma}, \gamma \in \mathcal{Q}_+.$$

The first statement in Theorem 2.2 (ii) implies that $Z(\mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}])$ is equal to the span of those b_1^μ such that $\mu \in \operatorname{Ker}(1 + w_0)$. Eq. (4.18) now follows from Eq. (4.16). By Theorem 2.2 (iii), $\mathcal{U}_q^-(\mathfrak{g})$ is a free left $\mathcal{N}_q^-(\mathfrak{g})$-module in which $\mathcal{N}_q^-(\mathfrak{g})$ is a direct summand. This implies $\mathcal{C}_q^-(\mathfrak{g})^\sharp \cap \mathcal{U}_q^-(\mathfrak{g}) = \mathcal{C}_q^-(\mathfrak{g})$ and the validity of (4.19). ∎

Given $\alpha \in \Pi$ and a reduced word $\mathbf{i}$ for w_0 as in (4.3), denote

$$l(\alpha, \mathbf{i}) = \min\{ l \in [1, N] \mid \alpha_l = \alpha \}.$$

From the definition [24, Eq. 8.6 (2)] of the braid group action on $V(\varpi_\alpha)$ it follows at once that $T_{w_0(\mathbf{i})^{-1}_{\leq (l(\alpha,\mathbf{i})-1)}} v_{\varpi_\alpha} = v_{\varpi_\alpha}$ which implies

$$\Delta_{\mathbf{i}, l(\alpha, \mathbf{i})} = b_1^{\varpi_\alpha}, \forall \alpha \in \Pi,$$

recall §2.3 for notation. Therefore for all reduced words $\mathbf{i}$ for w_0, we have

$$\mathcal{C}_q^-(\mathfrak{g})^\sharp \subset \mathcal{N}_-^q(\mathfrak{g})^\sharp \subset \mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}] \subset \mathcal{T}(\mathbf{i}) \subset \operatorname{Fract}(\mathcal{U}_q(\mathfrak{g})).$$

Let z be a Laurent monomial in $\Delta_{\mathbf{i},1}, \dots, \Delta_{\mathbf{i},N}$ which belongs to the center of $\mathcal{T}(\mathbf{i})$. Define the ideal

$$I := \{ r \in \mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}] \mid zr \in \mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}] \}$$

of $\mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}]$. It is nonzero since $\Delta_{\mathbf{i},1}, \ldots, \Delta_{\mathbf{i},N} \in \mathcal{U}_q^-(\mathfrak{g})$. Furthermore, I is $\mathbb{T}^r$-stable since z is an eigenvector of $\mathbb{T}^r$ and the base field $\mathbb{K}$ is infinite. By [42, Theorem 3.1 (b)], $\mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}]$ is a $\mathbb{T}^r$-simple algebra, so $I = \mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}]$ and $z \in Z(\mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}])$. It follows from (3.2) that

$$Z(\mathcal{U}_q^-(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}]) = Z(\mathcal{T}(\mathbf{i})).$$

Lemma 4.7 implies

$$Z(\mathcal{T}(\mathbf{i})) = \mathcal{C}_q^-(\mathfrak{g})^\sharp \text{ for all reduced words } \mathbf{i} \text{ for } w_0. \tag{4.21}$$

This fact was earlier obtained by Bell and Launois in [8, Proposition 3.3] using a result of De Concini and Procesi [15, Lemma 10.4 (b)] describing $\mathrm{Ker}(\mathbf{q})$ for the matrix (4.4), recall (3.2). The advantage of the above proof is that it trivially extends to the multiparameter case, see Section 6.

Lemma 4.8. *In the above setting, for all $\alpha \in \Pi$, the module $\mathcal{N}_q^-(\mathfrak{g}) F_\alpha$ is a direct summand of $\mathcal{U}_q^-(\mathfrak{g})$ considered as a left $\mathcal{N}_q^-(\mathfrak{g})$-module. In particular, if $u \in \mathcal{N}_q^-(\mathfrak{g})^\sharp$ and $uF_\alpha \in \mathcal{U}_q^-(\mathfrak{g})$, then $u \in \mathcal{N}_q^-(\mathfrak{g})$.*

Proof. Let $\mathbf{i}$ be a reduced word for w_0 as in (4.3) such that $\alpha_N = \widetilde{\alpha}$. This implies $F_{\beta_N} = F_\alpha$, see (4.14). The first part of the lemma will follow from Theorem 2.2 (iii) once we show

$$(0, \ldots, 0, 1) \in \mathcal{H}_{\mathbf{i}},$$

recall (2.24) for the definition of the set $\mathcal{H}_{\mathbf{i}}$. This is in turn equivalent to saying that there exists $l < N$ such that $\alpha_l = \alpha_N$. Assume the opposite. Then $w_0 s_{\alpha_N}$ belongs to the parabolic subgroup W_{α_N} of W generated by all simple reflections except s_{α_N}. This is a contradiction since it implies that the longest element of W_{α_N} has length $N-1$, which is impossible if $r = \mathrm{rank}(\mathfrak{g}) > 1$. It proves the first statement of the lemma. The second statement is a direct consequence of the first one. □

Proof of Proposition 4.6. Let Φ be a λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. By Proposition 4.4 and Eq. (4.21) there exist elements

$$z_\alpha \in (\mathcal{C}_q^-(\mathfrak{g})^\sharp)^{\geq 1} \text{ for } \alpha \in \Pi$$

such that $\Phi(F_\alpha) = (1 + z_\alpha) F_\alpha$, $\forall \alpha \in \Pi$. Lemma 4.8 implies that $z_\alpha \in \mathcal{C}_q^-(\mathfrak{g})^\sharp \cap \mathcal{N}_q^-(\mathfrak{g}) = \mathcal{C}_q^-(\mathfrak{g})$ because $\mathcal{N}_q^-(\mathfrak{g})$ is a polynomial algebra. Hence

$$z_\alpha \in \mathcal{C}_q^-(\mathfrak{g})^{\geq 1}, \ \ \forall \alpha \in \Pi,$$

which completes the proof of the proposition. □

4.4. **Proof of Theorem 4.2.** In this subsection we complete the proof of the triviality of all unipotent automorphisms of the algebras $\mathcal{U}_q^-(\mathfrak{g})$.

Proof of Theorem 4.2. Let λ be a strictly dominant integral coweight and Φ a λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. Proposition 4.6 implies that there exist elements

$$z_\alpha \in \mathcal{C}_q^-(\mathfrak{g})^{\geq 1} \text{ for } \alpha \in \Pi$$

such that $\Phi(F_\alpha) = (1 + z_\alpha) F_\alpha$, $\forall \alpha \in \Pi$. For $\gamma = \sum_\alpha m_\alpha \alpha \in \mathcal{Q}_+$ denote

$$z_\gamma := \prod_{\alpha \in \Pi} (1 + z_\alpha)^{m_\alpha} - 1 \in \mathcal{C}_q^-(\mathfrak{g})^{\geq 1}. \tag{4.22}$$

Since $\mathcal{U}_q^-(\mathfrak{g})$ is generated by the set $\{F_\alpha \mid \alpha \in \Pi\}$, we have

$$\Phi(u) = (1 + z_\gamma) u, \ \ \forall\, x \in \mathcal{U}_q^-(\mathfrak{g})_{-\gamma}, \gamma \in \mathcal{Q}_+.$$

In particular,

$$\Phi(b_1^{\varpi_\alpha}) = (1 + z_{(1 - w_0)\varpi_\alpha}) b_1^{\varpi_\alpha}, \ \ \forall \alpha \in \Pi,$$

recall (2.21). Therefore

$$\Phi(\mathcal{U}_q^-(\mathfrak{g})b_1^{\varpi_\alpha}) \subseteq \mathcal{U}_q^-(\mathfrak{g})b_1^{\varpi_\alpha}, \ \ \forall \alpha \in \Pi. \tag{4.23}$$

By Theorem 2.2 (iii), $\mathcal{U}_q^-(\mathfrak{g})b_1^{\varpi_\alpha}$ are height one prime ideals of $\mathcal{U}_q^-(\mathfrak{g})$. Hence we have equalities in (4.23) because Φ is an automorphism. This implies that there exist elements $x_\alpha \in \mathcal{U}_q^-(\mathfrak{g})$ for $\alpha \in \Pi$ such that

$$b_1^{\varpi_\alpha} = x_\alpha(1 + z_{(1-w_0)\varpi_\alpha})b_1^{\varpi_\alpha}, \ \ \forall \alpha \in \Pi.$$

Furthermore, the algebras $\mathcal{U}_q^-(\mathfrak{g})$ are domains whose groups of units are reduced to scalars because they are iterated Ore extensions. Thus $u_\alpha(1 + z_{(1-w_0)\varpi_\alpha}) = 1$ and $1 + z_{(1-w_0)\varpi_\alpha} \in \mathbb{K}^*$, $\forall \alpha \in \Pi$. It follows from Eq. (4.22) that $z_{(1-w_0)\varpi_\alpha} = 0$, $\forall \alpha \in \mathcal{P}_+$. From Eq. (4.22) we obtain

$$z_{(1-w_0)\mu} = 0, \ \ \text{i.e.,} \ \ \Phi|_{\mathcal{U}_q^-(\mathfrak{g})_{-(1-w_0)\mu}} = \mathrm{id}, \ \ \forall \mu \in \mathcal{P}_+.$$

Choose μ to be equal to the highest root $\sum_{\alpha\in\Pi} m_\alpha \alpha \in \mathcal{P}_+$ of $\mathfrak{g}$. Taking into account that $(1 - w_0)\mu = 2\mu$ leads to

$$\prod_{\alpha\in\Pi}(1 + z_\alpha)^{2m_\alpha} = 1.$$

If the highest term of z_α with respect to the $\mathbb{N}$-grading (4.1) of $\mathcal{U}_q^-(\mathfrak{g})$ is in degree n_α, then the left hand side has a nontrivial component in degree $\sum_{\alpha\in\Pi} 2m_\alpha n_\alpha$. The right hand side lies in degree 0. Therefore $n_\alpha = 0$ for all $\alpha \in \Pi$ because $m_\alpha > 0$, $\forall \alpha \in \Pi$. Hence we obtain $z_\alpha = 0$, $\forall \alpha \in \Pi$ and $\Phi = \mathrm{id}$. □

5. Proof of the Andruskiewitsch–Dumas conjecture

5.1. **Statement of the main result.** Here we complete the proof of the Andruskiewitsch–Dumas conjecture. This result is stated in Theorem 5.1. Its proof relies on Theorem 4.2 and on a classification result for a type of automorphisms of $\mathcal{U}_q^-(\mathfrak{g})$ which we call *linear*, see Definition 5.2.

Recall that Γ denotes the Dynkin diagram of $\mathfrak{g}$. The automorphism group of the directed graph Γ is denoted by $\mathrm{Aut}(\Gamma)$. One has the embeddings

$$\Upsilon^\pm \colon \mathbb{T}^r \rtimes \mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\mathcal{U}_q^\pm(\mathfrak{g})).$$

To the pair $(t, \theta) \in \mathbb{T}^r \rtimes \mathrm{Aut}(\Gamma)$, where $t = (t_\alpha)_{\alpha\in\Pi}$, one associates the automorphisms $\Upsilon^\pm_{(t,\theta)} \in \mathrm{Aut}(\mathcal{U}_q^\pm(\mathfrak{g}))$ given by

$$\Upsilon^+_{(t,\theta)}(E_\alpha) = t \cdot E_{\theta(\alpha)} = t_{\theta(\alpha)}E_{\theta(\alpha)}, \tag{5.1}$$

$$\Upsilon^-_{(t,\theta)}(F_\alpha) = t \cdot F_{\theta(\alpha)} = t^{-1}_{\theta(\alpha)}F_{\theta(\alpha)} \tag{5.2}$$

for the $\mathbb{T}^r$-action (2.5). The following theorem proves the Andruskiewitsch–Dumas conjecture:

Theorem 5.1. *For all simple Lie algebras $\mathfrak{g}$ of rank $r > 1$, base fields $\mathbb{K}$, and deformation parameters $q \in \mathbb{K}^*$ that are not roots of unity, the maps*

$$\Upsilon^\pm \colon \mathbb{T}^r \rtimes \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\mathcal{U}_q^\pm(\mathfrak{g}))$$

are group isomorphisms.

The key point of the theorem is the surjectivity of the maps $\Upsilon^\pm$. Because of the isomorphism ω from (2.1) the plus and minus cases are equivalent. For $\mathrm{rank}(\mathfrak{g}) = 1$, one has $\mathcal{U}_q^-(\mathfrak{g}) = \mathbb{K}[F_\alpha]$ and $\mathrm{Aut}(\mathcal{U}_q^-(\mathfrak{g})) \cong \mathbb{K} \rtimes \mathbb{K}^*$. The theorem is proved in §5.3. First, we consider the following special type of automorphisms of $\mathcal{U}_q^-(\mathfrak{g})$.

Definition 5.2. We call an automorphism Φ of $\mathcal{U}_q^-(\mathfrak{g})$ *linear* if

$$\Phi(F_\alpha) \in \mathrm{Span}\{F_{\alpha'} \mid \alpha' \in \Pi\}, \ \ \forall \alpha \in \Pi.$$

The set of all linear automorphisms of $\mathcal{U}_q^-(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}(\mathcal{U}_q^-(\mathfrak{g}))$. In the next subsection we prove the following classification of linear automorphisms of $\mathcal{U}_q^-(\mathfrak{g})$.

Proposition 5.3. *Let $\mathfrak{g}$ be a simple Lie algebra, $\mathbb{K}$ and arbitrary base field, and $q \in \mathbb{K}^*$ not a root of unity. All linear automorphism of $\mathcal{U}_q^-(\mathfrak{g})$ are of the form $\Upsilon^-_{(t,\theta)}$ for some $t \in \mathbb{T}^r$ and $\theta \in \mathrm{Aut}(\Gamma)$.*

5.2. **Linear automorphisms of $\mathcal{U}_q^-(\mathfrak{g})$.** Before we proceed with the proof of Proposition 5.3, we establish an auxiliary lemma. For a linear automorphism Φ of $\mathcal{U}_q^-(\mathfrak{g})$ given by $\Phi(F_\alpha) = \sum_{\alpha' \in \Pi} c_{\alpha\alpha'} F_{\alpha'}$, $c_{\alpha\alpha'} \in \mathbb{K}$, denote

$$\chi(\Phi, \alpha) = \{\alpha' \in \Pi \mid c_{\alpha\alpha'} \neq 0\}. \tag{5.3}$$

Lemma 5.4. *Assume that, in the setting of Proposition 5.3, Φ is a linear automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. Then the following hold:*

(i) For all $\alpha, \alpha' \in \Pi$ such that $a_{\alpha\alpha'} = -1$ (recall (2.3)) we have

$$\chi(\Phi, \alpha) \cap \chi(\Phi, \alpha') = \emptyset \;\; \text{and} \;\; |\chi(\Phi, \alpha)| = 1.$$

(ii) If there exist an element θ of the symmetric group S_Π and scalars $t'_\alpha \in \mathbb{K}^*$ for $\alpha \in \Pi$ such that

$$\Phi(F_\alpha) = t'_\alpha F_{\theta(\alpha)}, \;\; \forall \alpha \in \Pi, \tag{5.4}$$

then $\theta \in \mathrm{Aut}(\Gamma)$ and $\Phi = \Upsilon^-_{(t,\theta)}$, where $t = (t_\alpha)_{\alpha \in \Pi} \in \mathbb{T}^r$ is given by $t_\alpha = t^{-1}_{\theta^{-1}(\alpha)}$.

Denote by $\mathcal{F}^-(\mathfrak{g})$ the free $\mathbb{K}$-algebra in the generators $\{F_\alpha \mid \alpha \in \Pi\}$. It is $(-\mathcal{Q}_+)$-graded by assigning weight $-\alpha$ to F_α. For $\alpha \neq \alpha' \in \Pi$ denote by $R_{\alpha,\alpha'}$ the the expression in the left hand side of Eq. (2.2) considered as an element of $\mathcal{F}^-(\mathfrak{g})$. Let $\mathcal{I}_q^-(\mathfrak{g})$ be the (graded) two sided ideal of $\mathcal{F}^-(\mathfrak{g})$ generated by all such elements $R_{\alpha,\alpha'}$. Denote by $\mathcal{I}_q^-(\mathfrak{g})_{-\gamma}$ its graded component of weight $-\gamma$ for $\gamma \in \mathcal{Q}_+$. We have a canonical isomorphism $\mathcal{F}^-(\mathfrak{g})/\mathcal{I}_q^-(\mathfrak{g}) \cong \mathcal{U}_q^-(\mathfrak{g})$. By abuse of notation we will denote by the same letter the canonical lifting of a linear automorphism Φ of $\mathcal{U}_q^-(\mathfrak{g})$ to an automorphism of the free algebra $\mathcal{F}^-(\mathfrak{g})$ which preserves $\mathcal{I}_q^-(\mathfrak{g})$.

Proof of Lemma 5.4. (i) Let $\alpha_0 \in \chi(\Phi, \alpha) \cap \chi(\Phi, \alpha')$. The component of $\Phi(R_{\alpha,\alpha'})$ in $\mathcal{F}_q^-(\mathfrak{g})_{-3\alpha_0}$ is

$$(1 - q_\alpha)(1 - q_\alpha^{-1}) F^3_{\alpha_0} \neq 0,$$

yet obviously $\mathcal{I}_q^-(\mathfrak{g})_{-3\alpha_0} = 0$, $\forall \alpha_0 \in \Pi$. This proves the first fact in part (i). For the second fact in (i), assume that $\alpha_1, \alpha_2 \in \chi(\Phi, \alpha)$ and $\alpha_3 \in \chi(\Phi, \alpha')$ for three distinct simple roots $\alpha_1, \alpha_2, \alpha_3$. The component of $\Phi(R_{\alpha,\alpha'})$ of weight $-\alpha_1 - \alpha_2 - \alpha_3$ is

$$F_{\alpha_3}(F_{\alpha_1}F_{\alpha_2} + F_{\alpha_2}F_{\alpha_1}) - (q_\alpha + q_\alpha^{-1})(F_{\alpha_1}F_{\alpha_3}F_{\alpha_2} + F_{\alpha_2}F_{\alpha_3}F_{\alpha_1}) + (F_{\alpha_1}F_{\alpha_2} + F_{\alpha_2}F_{\alpha_1})F_{\alpha_3}.$$

This again leads to a contradiction since $\mathcal{I}_q^-(\mathfrak{g})_{-\alpha_1-\alpha_2-\alpha_3}$ is spanned by the elements $[E_{\alpha_i}, E_{\alpha_j}]E_{\alpha_k}$ and $E_{\alpha_k}[E_{\alpha_i}, E_{\alpha_j}]$ for all permutations ijk of 123 such that α_i and α_j are not connected vertices of Γ, (i.e., $a_{\alpha_i\alpha_j} = 0$).

Part (ii) easily follows by examining in a similar way the nonzero elements $\Phi(R_{\alpha,\alpha'}) \in \mathcal{I}_q^-(\mathfrak{g})_{(1-a_{\alpha\alpha'})\theta(\alpha)-\theta(\alpha')}$ for those $\alpha, \alpha' \in \Pi$ such that $a_{\alpha\alpha'} = 0$ or -1. □

Proof of Proposition 5.3. First, we assume that $\mathfrak{g}$ is a simple Lie algebra which is not of type B_r for $r \geq 3$. In this case every linear automorphism of $\mathcal{U}_q^-(\mathfrak{g})$ satisfies the condition in Lemma 5.4 (ii). If $\mathfrak{g}$ is simply laced or of types F_4 or C_r for $r \geq 3$, this follows from the second fact in Lemma 5.4 (i) because for these root systems for every simple root α of $\mathfrak{g}$ there exists a simple root α' such that $a_{\alpha\alpha'} = -1$. For root systems of rank 2 this follows from the first fact in Lemma 5.4 (i). Thus for root systems different from B_r, $r \geq 3$ the proposition follows from Lemma 5.4 (ii).

Now, assume that $\mathfrak{g}$ is of type B_r for some $r \geq 3$. Denote the short simple root of $\mathfrak{g}$ by α_r and the long simple roots of $\mathfrak{g}$ by $\alpha_1, \dots, \alpha_{r-1}$ (enumerated consecutively along Γ so that α_{r-1}

is adjacent to α_r). The second fact in Lemma 5.4 (i) implies that there exist an element θ of the symmetric group S_Π and scalars $t'_{\alpha_j} \in \mathbb{K}^*$, $j \in [1,r]$, $t''_{\alpha_j} \in \mathbb{K}$, $j \in [1, r-1]$ such that

$$\Phi(F_{\alpha_j}) = t'_{\alpha_j} F_{\theta(\alpha_j)}, \quad \forall j \in [1, r-1] \quad \text{and} \tag{5.5}$$

$$\Phi(F_{\alpha_r}) = t'_{\alpha_r} F_{\theta(\alpha_r)} + \sum_{j=1}^{r-1} t''_{\alpha_j} F_{\theta(\alpha_j)}. \tag{5.6}$$

By considering the nonzero elements $\Phi(R_{\alpha_j,\alpha_{j'}}) \in \mathcal{I}_q^-(\mathfrak{g})_{(1-a_{\alpha_j\alpha_{j'}})\theta(\alpha_j)-\theta(\alpha_{j'})}$ for $j \neq j' \in [1, r-1]$ as in Lemma 5.4 (ii) and using that $r \geq 3$ one obtains $\theta(\alpha_r) = \alpha_r$. Set $\Phi_0(F_{\alpha_j}) = t'_{\alpha_j} F_{\theta(\alpha_j)}$ for $j \in [1,r]$. Consider the strictly dominant integral coweight $\lambda = \sum_{j=1}^r n_j \varpi^\vee_{\alpha_j} \in \mathcal{P}^\vee_{++}$ for $n_1 = \ldots = n_{r-1} = 2$ and $n_r = 1$. Then $\Phi(F_{\alpha_j}) - \Phi_0(F_{\alpha_j}) \in \mathcal{U}_q^-(\mathfrak{g})^{\geq n_j+1}$, $\forall j \in [1,r]$ for the $\mathbb{N}$-grading of $\mathcal{U}_q^-(\mathfrak{g})$ corresponding to λ, see §4.1. For graded reasons Φ_0 defines a linear automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. Lemma 5.4 (ii) implies $\theta \in \mathrm{Aut}(\Gamma)$ and so $\theta = \mathrm{id}$. Hence $\Phi_0 = \Upsilon^-_{(t,\mathrm{id})}$, where $t = (t_\alpha)_{\alpha\in\Pi}$ is given by $t_\alpha = (t'_\alpha)^{-1}$. Eqs. (5.5)–(5.6) imply that $(\Upsilon^{-1}_{(t,\mathrm{id})})^{-1}\Phi(F_{\alpha_r})$ is a λ-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. It follows from Theorem 4.2 that $\Phi = \Upsilon^-_{(t,\mathrm{id})}$. □

5.3. **Proof of the main theorem.** In the remaining part of this section we will use the $\mathbb{N}$-grading of $\mathcal{U}_q^-(\mathfrak{g})$ associated to

$$\lambda = \rho^\vee = \sum_{\alpha\in\Pi} \varpi^\vee_\alpha \in \mathcal{P}^\vee_{++},$$

recall §4.1.

The following result is due to Launois [31, Proposition 2.3]. We give a slightly different proof which extends to the twisted case under weaker assumptions on the twisting cocycle. The lemma can be also proved using the method of Alev, Andruskiewitsch, and Dumas [2, Proposition 1.2].

Lemma 5.5. *For all simple Lie algebras $\mathfrak{g}$ of rank $r > 1$, base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, and automorphisms Φ of $\mathcal{U}_q(\mathfrak{g})$, we have*

$$\Phi(F_\alpha) \in \mathcal{U}_q^-(\mathfrak{g})^{\geq 1}.$$

Proof. Launois and Lenagan proved the following fact in [32, Proposition 3.2] (see also [31, Corollary 2.4]):

Let $R = \oplus_{m\in\mathbb{N}} R^m$ be a connected $\mathbb{N}$-graded $\mathbb{K}$-algebra generated in degree one by $x_i \in A^1$, $i = 1, \ldots, n$ such that for each i there exists $y_i \in R$ with $x_i y_i = q_i y_i x_i$ for some $q_i \in \mathbb{K}^*$, $q_i \neq 1$. Then for each automorphism Φ of R we have $\Phi(x_i) \in R^{\geq 1}$.

The algebra $\mathcal{U}_q^-(\mathfrak{g})$ satisfies the above property which implies the validity of the proposition. To see this, for each $\alpha \in \Pi$ choose $\alpha' \in \Pi$ such that $a_{\alpha\alpha'} \neq 0$ (recall (2.3)) and define

$$x_{\alpha\alpha'} = \sum_{j=0}^{-a_{\alpha\alpha'}} (-q_\alpha)^j \begin{bmatrix} -a_{\alpha\alpha'} \\ j \end{bmatrix}_{q_\alpha} (F_\alpha)^j F_{\alpha'} (F_\alpha)^{-a_{\alpha\alpha'}-j}.$$

It follows from the quantum Serre relations (2.2) that

$$x_{\alpha\alpha'} F_\alpha = q_\alpha^{-1} F_\alpha x_{\alpha\alpha'}$$

and we have $q_\alpha \neq 1$. □

Proof of Theorem 5.1. Let $\Phi \in \mathrm{Aut}(\mathcal{U}_q^-(\mathfrak{g}))$. Lemma 5.5 implies $\Phi(F_\alpha) \in \mathcal{U}_q^-(\mathfrak{g})^{\geq 1}$, $\forall \alpha \in \Pi$. For $\alpha \in \Pi$, let $\Phi_0(F_\alpha) \in \mathcal{U}_q^-(\mathfrak{g})^1 = \mathrm{Span}\{F_{\alpha'} \mid \alpha' \in \Pi'\}$ be the unique element such that

$$\Phi(F_\alpha) - \Phi_0(F_\alpha) \in \mathcal{U}_q^-(\mathfrak{g})^{\geq 2}.$$

For graded reasons, Φ_0 extends to a linear automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. Applying Proposition 5.3 we obtain that there exist $\theta \in \mathrm{Aut}(\Gamma)$ and $t \in \mathbb{T}^r$ such that $\Phi_0 = \Upsilon^-_{(t,\theta)}$. Therefore $(\Upsilon^-_{(t,\theta)})^{-1}\Phi$ is a $\rho^\vee$-unipotent automorphism of $\mathcal{U}_q^-(\mathfrak{g})$. It follows from Theorem 4.2 that $\Phi = \Upsilon^-_{(t,\theta)}$. This completes the proof of the minus case of the theorem. The plus case follows by applying the isomorphism (2.1). □

6. The multiparameter case

6.1. **Statement of main result.** In this section we extend Theorem 5.1 to a classification of the automorphism groups of the 2-cocycle twists of all algebras $\mathcal{U}_q^\pm(\mathfrak{g})$. This result is stated in Theorem 6.2, which is proved in §6.3. The main step is a rigidity result for the unipotent automorphisms of the twisted algebras proved in Theorem 6.4. Finally, §6.4 contains a classification of the isomorphisms between all algebras obtained by 2-cocycle twists from the algebras $\mathcal{U}_{q,\mathbf{p}}^\pm(\mathfrak{g})$ for simple Lie algebras $\mathfrak{g}$. This is done in Theorem 6.2. In particular, it is shown that

$$\mathcal{U}_q^\pm(\mathfrak{g}_1) \cong \mathcal{U}_q^\pm(\mathfrak{g}_2) \quad \Leftrightarrow \quad \mathfrak{g}_1 \cong \mathfrak{g}_2$$

for all base fields $\mathbb{K}$, non-root of unity q and simple Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$.

Let R be a $\mathbb{K}$-algebra graded by an abelian group C, $R = \oplus_{c\in C} R_c$. For a 2-cocycle $\mathbf{p} \in Z^2(C, \mathbb{K}^*)$, define [7] a new algebra structure on the $\mathbb{K}$-vector space R by twisting the product in R as follows:

$$u_1 * u_2 = \mathbf{p}(c_1, c_2) u_1 u_2, \quad c_i \in C, u_i \in R_{c_i}, i = 1, 2.$$

The twisted algebra, to be denoted by $R_{\mathbf{p}}$, is canonically C-graded. Artin, Schelter, and Tate [7] proved that up to a graded isomorphism $R_{\mathbf{p}}$ only depends on the cohomology class of $\mathbf{p}$. They also proved that, if C is a free abelian group, then

$$\mathbf{r}\colon C \times C \to \mathbb{K}^* \ \text{ given by } \ \mathbf{r}(c_1, c_2) = \mathbf{p}(c_1, c_2)\mathbf{p}(c_2, c_1)^{-1}, \ c_1, c_2 \in C \tag{6.1}$$

is a multiplicatively skew-symmetric group bicharacter and the cohomology classes $H^2(C, \mathbb{K}^*)$ are classified by multiplicatively skew-symmetric square matrices of size equal to the rank of C (obtained by restricting $\mathbf{r}$ to a minimal set of generators of C).

Given $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$, denote by $\mathcal{U}_{q,\mathbf{p}}^\pm(\mathfrak{g})$ the associated 2-cocycle twist of $\mathcal{U}_q^\pm(\mathfrak{g})$ for the $\mathcal{Q}$-grading from §2.1. The isomorphism (2.1) defines an isomorphism of the twisted algebras

$$\omega\colon \mathcal{U}_{q,\mathbf{p}}^\pm(\mathfrak{g}) \to \mathcal{U}_{q,\mathbf{p}}^\mp(\mathfrak{g}) \tag{6.2}$$

because of the above mentioned property of $\mathbf{r}$. The algebra $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g})$ can be described as the $\mathbb{K}$-algebra with generators $\{F_\alpha \mid \alpha \in \Pi\}$ and relations

$$\sum_{j=0}^{1-a_{\alpha\alpha'}} (-\mathbf{r}(\alpha', \alpha))^j \begin{bmatrix} 1 - a_{\alpha\alpha'} \\ j \end{bmatrix}_{q_\alpha} (F_\alpha)^j F_{\alpha'} (F_\alpha)^{1-a_{\alpha\alpha'}-j} = 0, \ \ \forall \alpha \neq \alpha' \in \Pi, \tag{6.3}$$

recall (2.2).

For every 2-cocycle $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ denote by $G_{\mathbf{p}}$ the subgroup of $\mathbb{K}^*$ generated by the set

$$q^2 \cup \{q^{-\langle \alpha, \alpha' \rangle} \mathbf{r}(\alpha, \alpha') \mid \alpha \neq \alpha' \in \Pi\}.$$

If one chooses a linear ordering $<$ on Π, then the group $G_{\mathbf{p}}$ is also generated by the above elements for the pairs with $\alpha < \alpha'$.

Definition 6.1. A 2-cocycle $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ will be called torsion-free if the subgroup $G_{\mathbf{p}}$ of $\mathbb{K}^*$ is torsion-free.

Note that, if $\mathbf{p}$ is torsion-free, then $q \in \mathbb{K}^*$ is not a root of unity. On the other hand if $q \in \mathbb{K}^*$ is not a root of unity then the trivial cocycle $\mathbf{p}$ is torsion-free.

Denote

$$\mathrm{Aut}(\Gamma, \mathbf{p}) = \{\theta \in \mathrm{Aut}(\Gamma) \mid \mathbf{r}(\theta(\alpha), \theta(\alpha')) = \mathbf{r}(\alpha, \alpha'), \ \ \forall \alpha, \alpha' \in \Pi\}.$$

We have an embedding $\Upsilon^{\pm}\colon \mathbb{T}^r \rtimes \mathrm{Aut}(\Gamma, \mathbf{p}) \hookrightarrow \mathrm{Aut}(\mathcal{U}^{\pm}_{q,\mathbf{p}}(\mathfrak{g}))$, where for $(t, \theta) \in \mathbb{T}^r \rtimes \mathrm{Aut}(\Gamma, \mathbf{p})$ the automorphism $\Upsilon^{\pm}_{(t,\theta)} \in \mathrm{Aut}(\mathcal{U}^{\pm}_{q,\mathbf{p}}(\mathfrak{g}))$ is given by (5.1)–(5.2).

Theorem 6.2. *For every simple Lie algebra $\mathfrak{g}$ of rank $r > 1$, base field $\mathbb{K}$, $q \in \mathbb{K}^*$, and a torsion-free 2-cocycle $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ satisfying*

$$q_\alpha \mathbf{r}(\alpha, \alpha'), q_\alpha^{-1} \mathbf{r}(\alpha, \alpha') \neq 1, \ \forall \alpha, \alpha' \in \Pi \ \text{ such that } \ a_{\alpha\alpha'} = -1, \tag{6.4}$$

the map

$$\Upsilon^{\pm}\colon \mathbb{T}^r \rtimes \mathrm{Aut}(\Gamma, \mathbf{p}) \to \mathrm{Aut}(\mathcal{U}^{\pm}_{q,\mathbf{p}}(\mathfrak{g}))$$

is a group isomorphism.

The special case of $\mathfrak{g} = \mathfrak{so}_5$ of this theorem was obtained by Tang [39]. Because of the isomorphism (6.2), it is sufficient to prove the theorem in the minus case.

We finish this subsection with a result which explains the origin of the torsion-free condition from Definition 6.1. Let $\mathbf{i} = (\alpha_1, \dots, \alpha_N)$ be a reduced word for the longest element w_0 of W. All automorphisms and skew derivations in the iterated Ore extension presentation (2.16) are graded. Thus for all $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$, we have the iterated Ore extension presentation

$$\mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g}) = \mathbb{K}[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \dots [F_{\beta_N}; \sigma_N, \delta_N], \tag{6.5}$$

where σ_l and δ_l are still given by (2.14) and (2.15) but this time $t_l \in \mathbb{T}^r$ are such that $t_l^{\beta_k} = q^{\langle \beta_l, \beta_k \rangle} \mathbf{r}(\beta_l, \beta_k)^{-1}$, $\forall k \in [1, l]$. If $q \in \mathbb{K}^*$ is not a root of unity, this is a CGL extension for the following choice of the elements q_{lk} and $q_l \in \mathbb{K}^*$ (recall Definition 2.1):

$$q_{lk} = q^{-\langle \beta_l, \beta_k \rangle} \mathbf{r}(\beta_l, \beta_k), \ 1 \leq k < l \leq N \ \text{ and } \ q_l = q_{\alpha_l}^{-2}, \ l \in [1, N]. \tag{6.6}$$

Proposition 6.3. *Let $\mathfrak{g}$ be a simple Lie algebra of rank $r > 1$. For all 2-cocycles $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ and reduced words $\mathbf{i}$ for the longest element w_0 of W, the group $G_{\mathbf{p}}$ is precisely the subgroup of $\mathbb{K}^*$ generated by the elements $q_{lk} \in \mathbb{K}^*$, $1 \leq k < l \leq N$ given by* (6.6).

In particular, $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ is a torsion-free cocycle if and only if the iterated Ore extension presentation (6.5) *of $\mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g})$ associated to one reduced word $\mathbf{i}$ for w_0 (and thus to every reduced word $\mathbf{i}$ for w_0) is is a torsion-free CGL extension.*

Proof. Denote by $G_{\mathbf{i}}$ the subgroup of $\mathbb{K}^*$ generated by the elements q_{lk}, $1 \leq k < l \leq N$ in Eq. (6.6). First we show that $G_{\mathbf{p}} \subseteq G_{\mathbf{i}}$. Assume that $\alpha, \alpha' \in \Pi$ and α comes before α' in the ordering

$$\beta_1, \dots, \beta_N \tag{6.7}$$

from Eq. (2.11) of the positive roots of $\mathfrak{g}$. If α and α' are not connected with an edge in Γ, then $\langle \alpha, \alpha' \rangle = 0$ and $q^{-\langle \alpha, \alpha' \rangle} \mathbf{r}(\alpha, \alpha') = (q^{-\langle \alpha', \alpha \rangle} \mathbf{r}(\alpha', \alpha))^{-1} \in G_{\mathbf{i}}$. If they are connected by an edge, then the root $\alpha + \alpha'$ of $\mathfrak{g}$ is listed between α and α' in (6.7) since the ordering (6.7) is convex and

$$q^{-\langle \alpha, \alpha \rangle} = (q^{-\langle \alpha + \alpha', \alpha \rangle} \mathbf{r}(\alpha + \alpha', \alpha))(q^{-\langle \alpha', \alpha \rangle} \mathbf{r}(\alpha', \alpha))^{-1} \in G_{\mathbf{i}},$$
$$q^{-\langle \alpha', \alpha' \rangle} = (q^{-\langle \alpha', \alpha + \alpha' \rangle} \mathbf{r}(\alpha', \alpha + \alpha'))(q^{-\langle \alpha', \alpha \rangle} \mathbf{r}(\alpha', \alpha))^{-1} \in G_{\mathbf{i}}.$$

Thus $q^2 \in G_{\mathbf{i}}$ and $q^{-\langle \alpha, \alpha' \rangle} \mathbf{r}(\alpha, \alpha') = q^{-2\langle \alpha, \alpha' \rangle} (q^{-\langle \alpha', \alpha \rangle} \mathbf{r}(\alpha', \alpha))^{-1} \in G_{\mathbf{i}}$. Hence $G_{\mathbf{p}} \subseteq G_{\mathbf{i}}$. For the opposite inclusion, we fix a linear ordering $<$ on Π. If $\beta_l = \sum_{\alpha \in \Pi} m_\alpha \alpha$ and $\beta_k = \sum_{\alpha \in \Pi} n_\alpha \alpha$ for $k < l$, then

$$q^{-\langle \beta_l, \beta_k \rangle} \mathbf{r}(\beta_l, \beta_k) = \prod_{\alpha \in \Pi} q^{-m_\alpha n_\alpha \langle \alpha, \alpha \rangle} \prod_{\alpha < \alpha' \in \Pi} q^{-2 m_{\alpha'} n_\alpha \langle \alpha, \alpha' \rangle} \left(q^{-\langle \alpha, \alpha' \rangle} \mathbf{r}(\alpha, \alpha') \right)^{m_\alpha n_{\alpha'} - m_{\alpha'} n_\alpha} \in G_{\mathbf{p}}.$$

This completes the proof of the proposition. □

The above argument shows that the group $G_{\mathbf{p}}$ can be also characterized as the subgroup of $\mathbb{K}^*$ generated by all elements of the form $q^{-\langle \beta, \beta' \rangle} \mathbf{r}(\beta, \beta')$, where β and β' run over all positive roots of $\mathfrak{g}$.

6.2. **Unipotent automorphisms of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$.** Each strictly dominant integral coweight $\lambda = \sum_{\alpha\in\Pi} m_\alpha \varpi^\vee_\alpha$ gives rise to a specialization of the $(-\mathcal{Q}_+)$-grading of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ to a connected $\mathbb{N}$-grading by setting $\deg F_\alpha := m_\alpha = \langle \lambda, \alpha \rangle$ for $\alpha \in \Pi$. We will denote the corresponding graded components by $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})^m$, $m \in \mathbb{N}$. Analogously to the untwisted case we will call an automorphism Φ of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ λ-unipotent if

$$\Phi(F_\alpha) - F_\alpha \in \mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})^{\geq \langle \lambda, \alpha \rangle + 1}, \ \ \forall \alpha \in \Pi.$$

Theorem 6.4. *Let $\mathfrak{g}$ be a Lie algebra of rank $r > 1$, $\mathbb{K}$ and arbitrary base field, $q \in \mathbb{K}^*$, and $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ a torsion-free 2-cocycle. Every λ-uniponet automorphism Φ of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ for a strictly dominant integral coweight λ is equal to the identity automorphism.*

Proof. Let $\mathbf{i}$ be a reduced word for w_0. It follows from Proposition 6.3 that (6.5) is a torsion–free CGL extension presentation of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$. Denote by $(\overline{F}_{\mathbf{i},1}, \ldots, \overline{F}_{\mathbf{i},N})$ the final N-tuple from the Cauchon deleting derivation procedure applied to it. Let $\mathbf{q}$ denote the multiplicatively skew-symmetric $N \times N$ matrix whose entries q_{lk}, $1 \leq k < l \leq N$ satisfy eq. (6.6). Then we have an isomorphism of quantum tori

$$\mathcal{T}_\mathbf{q} \cong \mathcal{T}(\mathbf{i}, \mathbf{p}) := \mathbb{K}\langle \overline{F}^{\pm 1}_{\mathbf{i},1}, \ldots \overline{F}^{\pm 1}_{\mathbf{i},N} \rangle \subseteq \mathrm{Fract}(\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})) \ \text{ given by } \ X_l \mapsto \overline{F}_{\mathbf{i},l}, l \in [1, N]$$

recall (1.1). Since the CGL extension presentation (6.5) is torsion-free, the quantum torus $\mathcal{T}_\mathbf{q}$ is saturated, see §2.2. Eq. (4.5) is a graded equality in $\mathcal{U}^-_q(\mathfrak{g})$ and thus it holds in $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ after an appropriate rescaling. Thus

$$\Delta_{\mathbf{i},1}, \ldots, \Delta_{\mathbf{i},N}$$

is a generating set of the quantum torus $\mathcal{T}(\mathbf{i}, \mathbf{p})$. Recall from §3.1 that the property of a quantum torus being saturated does not depend on the choice of its generators. We use the degree vector $\mathbf{d}$ from (4.10) to define a $\mathbb{Z}$-grading on $\mathcal{T}(\mathbf{i}, \mathbf{p})$ and to form a completion as in §3.1. This completion will be denoted by $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{p}, \mathbf{d})$. Analogously to §4.1, to every λ-unipotent automorphism Φ of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ we associate a bi-finite unipotent automorphism of the completed saturated quantum torus $\widehat{\mathcal{T}}(\mathbf{i}, \mathbf{p}, \mathbf{d})$.

Analogs of Propositions 4.4 and 4.6 hold under very mild modifications. Denote by $\mathcal{N}^-_{q,\mathbf{p}}(\mathfrak{g})$ the subalgebra of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ generated (and spanned) by b^λ_1, $\lambda \in \mathcal{P}_+$. It is a quantum affine space algebra with generators $b^{\varpi_\alpha}_1$, $\alpha \in \Pi$ and relations $b^{\varpi_\alpha}_1 b^{\varpi_{\alpha'}}_1 = \mathbf{r}((1-w_0)\varpi_\alpha, (1-w_0)\varpi_{\alpha'}) b^{\varpi_{\alpha'}}_1 b^{\varpi_\alpha}_1$, $\forall \alpha, \alpha' \in \Pi$, recall (2.22). Consider the localization $\mathcal{N}^-_{q,\mathbf{p}}(\mathfrak{g})^\sharp := \mathcal{N}^-_{q,\mathbf{p}}(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}]$ where $\Omega(\mathfrak{g}) = \{ b^\lambda_1 \mid \lambda \in \mathcal{P}_+\}$ and the elements b^μ_1, $\mu \in \mathcal{P}$ in it given by (4.20). For all $\mu \in \mathcal{P}$, we have

$$b^\mu_1 u = q^{\langle (1+w_0)\mu, \gamma \rangle} \mathbf{r}((1-w_0)\mu, \gamma) u b^\mu_1, \ \ \forall \, u \in \mathcal{U}^-_q(\mathfrak{g})_{-\gamma}, \gamma \in \mathcal{Q}_+.$$

This property and the first part of Theorem 2.2 (ii) imply

$$Z(\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})[\Omega(\mathfrak{g})^{-1}]) = \mathcal{C}^-_{q,\mathbf{p}}(\mathfrak{g})^\sharp := \{ b^\mu_1 \mid \mu \in \mathcal{P}, \ q^{\langle (1+w_0)\mu, \gamma \rangle} \mathbf{r}((1-w_0)\mu, \gamma) = 1, \ \forall \gamma \in \mathcal{Q} \}.$$

The argument of the proof of Eq. (4.21) gives

$$Z(\mathcal{T}(\mathbf{i}, \mathbf{p})) = \mathcal{C}^-_{q,\mathbf{p}}(\mathfrak{g})^\sharp.$$

For graded reasons it follows from Theorem 2.2 (iii) that $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ is a free left and right $\mathcal{N}^-_{q,\mathbf{p}}(\mathfrak{g})$-module with basis (2.25). Analogously to the proofs of Lemmas 4.7 and 4.8 and Proposition 4.6 we obtain

$$Z(\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})) = \mathcal{C}^-_{q,\mathbf{p}}(\mathfrak{g}) := \mathrm{Span}\{ b^\mu_1 \mid \mu \in \mathcal{P}_+, \ q^{\langle (1+w_0)\mu, \gamma \rangle} \mathbf{r}((1-w_0)\mu, \gamma) = 1, \ \forall \gamma \in \mathcal{Q} \}$$

and then show that for every λ-unipotent automorphism $\Phi \in \mathrm{Aut}(\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g}))$ there exist elements $z_\alpha \in \mathcal{C}^-_{q,\mathbf{p}}(\mathfrak{g})^{\geq 1}$ for $\alpha \in \Pi$ such that

$$\Phi(F_\alpha) = (1 + z_\alpha) F_\alpha, \ \ \forall \alpha \in \Pi. \tag{6.8}$$

Recall Theorem 2.2 (i). The argument of the proof of [19, Theorem 4.1] of Goodearl and Lenagan shows that $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g}) b^{\varpi_\alpha}_1$ are height one prime ideals of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ for all $\alpha \in \Pi$. Using this, analogously to §4.4 one shows that $z_\alpha = 0$, $\forall \alpha \in \Pi$. This completes the proof of the theorem. $\square$

6.3. **Proof of Theorem 6.2.** To each semisimple Lie algebra $\mathfrak{g}$ one can attach a $\mathbb{K}$-algebra $\mathcal{U}_q^-(\mathfrak{g})$ analogously to §2.1 using the normalized W-invariant bilinear form on $\mathbb{R}\Pi$ such that $\langle \alpha, \alpha \rangle = 2$ for all short simple roots α of $\mathfrak{g}$. We will need those algebras for induction purposes. For a subset $\Pi' \subset \Pi$ and $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$, denote by $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g}_{\Pi'})$ the subalgebra of $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g})$ generated by F_α for $\alpha \in \Pi'$. An algebra automorphism $\Phi \in \mathrm{Aut}(\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g}_{\Pi'}))$ will be called linear if $\Phi(F_\alpha) \subseteq \mathrm{Span}\{F_{\alpha'} \mid \alpha' \in \Pi'\}$, $\forall \alpha \in \Pi'$. We will use the notation from Eq. (5.3) for those.

Given $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$, for $c \in \mathbb{K}^*$ denote

$$\Pi^c = \{\alpha \in \Pi \mid \exists \alpha' \in \Pi \ \text{ such that } \ a_{\alpha\alpha'} = 0 \ \text{ and } \ \mathbf{r}(\alpha', \alpha) = c\},$$

recall (2.3). Denote

$$\Pi^* = \cup_{c \in \mathbb{K}^*, c \neq 1} \Pi^c, \ \text{ and } \ \Pi^\bullet = \Pi \backslash \Pi^*.$$

The following lemma is proved analogously to Lemma 5.4.

Lemma 6.5. *For all semisimple Lie algebras $\mathfrak{g}$, base fields $\mathbb{K}$, $q \in \mathbb{K}^*$, $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$, and linear automorphisms Φ of $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g})$ the following hold:*

(i) If $\alpha, \alpha' \in \Pi$ are such that $a_{\alpha\alpha'} = 0$ and $\mathbf{r}(\alpha', \alpha) \neq 1$ then $\chi(\Phi, \alpha) \cap \chi(\Phi, \alpha') = \emptyset$.

(ii) If $\alpha \in \Pi^c$ for some $c \neq 1$, then $\chi(\alpha, \Phi) \subseteq \Pi^c$.

(iii) If the condition (6.4) is satisfied, then for all $\alpha, \alpha' \in \Pi$ such that $a_{\alpha\alpha'} = -1$ we have $\chi(\Phi, \alpha) \cap \chi(\Phi, \alpha') = \emptyset$. If, in addition, for such a pair (α, α') we have $\chi(\Phi, \alpha) \subseteq \Pi^\bullet$, then $|\chi(\Phi, \alpha)| = 1$.

(iv) If there exist an element θ of the symmetric group S_Π and scalars $t'_\alpha \in \mathbb{K}^*$ for $\alpha \in \Pi$ such that

$$\Phi(F_\alpha) = t'_\alpha F_{\theta(\alpha)}, \ \ \forall \alpha \in \Pi,$$

then $\theta \in \mathrm{Aut}(\Gamma, \mathbf{p})$ and $\Phi = \Upsilon^-_{(t,\theta)}$, where $t = (t_\alpha)_{\alpha \in \Pi} \in \mathbb{T}^r$ is given by $t_\alpha = t^{-1}_{\theta^{-1}(\alpha)}$.

Next we prove an extension of Proposition 5.3 to the twisted case.

Proposition 6.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r, $\mathbb{K}$ an arbitrary base field, $q \in \mathbb{K}^*$, and $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ a torsion-free 2-cocycle satisfying (6.4). Then every linear automorphism Φ of $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g})$ is of the form $\Upsilon^-_{(t,\theta)}$ for some $\theta \in \mathrm{Aut}(\Gamma, \mathbf{p})$ and $t \in \mathbb{T}^r$.*

Proof. First, we claim that for every $\alpha \in \Pi^*$ there exists a subset $\Pi' \subseteq \Pi^*$ containing α such that Φ restricts to a linear automorphism of $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g}_{\Pi'})$ and

$$\mathbf{r}(\alpha_1, \alpha_2) = 1, \ \ \forall \alpha_1, \alpha_2 \in \Pi' \ \text{ such that } \ a_{\alpha_1\alpha_2} = 0. \tag{6.9}$$

Choose $c \in \mathbb{K}^*$, $c \neq 1$ such that $\alpha \in \Pi^c$. By Lemma 6.5 (ii), Φ restricts to a linear automorphism of $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g}_{\Pi^c})$. If $\Pi' = \Pi^c$ satisfies (6.9), this proves the claim. Otherwise we continue recursively by using $\mathfrak{g}_{\Pi^c}$ in place of $\mathfrak{g}$. Analogously to the proof of Proposition 5.3, the claim and Lemma 6.5 (iii) imply that there exist $\theta^* \in S_{\Pi^*}$ and $t^* = (t'_\alpha)_{\alpha \in \Pi^*} \in \mathbb{T}^{|\Pi^*|}$ such that

$$\Phi(F_\alpha) = t'_\alpha F_{\theta^*(\alpha)}, \ \ \forall \alpha \in \Pi^*. \tag{6.10}$$

Let

$$\Phi(F_\alpha) = \sum_{\alpha' \in \Pi^\bullet} c_{\alpha\alpha'} F_{\alpha'} + \sum_{\alpha'' \in \Pi^*} c_{\alpha\alpha''} F_{\alpha''}, \ \ \alpha \in \Pi^\bullet.$$

for some $c_{\alpha\alpha'}, c_{\alpha\alpha''} \in \mathbb{K}$. It follows from (6.10) and the form of the quantum Serre relation (6.3) that

$$\Phi^\bullet(F_\alpha) := \sum_{\alpha' \in \Pi^\bullet} c_{\alpha\alpha'} F_{\alpha'}, \ \ \alpha \in \Pi^\bullet$$

extends to a linear automorphism of $\mathcal{U}_{q,\mathbf{p}}^-(\mathfrak{g}_{\Pi^\bullet})$. Analogously to the proof of Proposition 5.3, using Lemma 6.5 (iii) one obtains that there exist $\theta^\bullet \in S_{\Pi^\bullet}$ and $t^\bullet = (t''_\alpha) \in \mathbb{T}^{|\Pi^\bullet|}$ such that

$$\Phi^\bullet(F_\alpha) = t''_\alpha F_{\theta^\bullet(\alpha)}, \ \ \forall \alpha \in \Pi^\bullet.$$

Consider the $\mathbb{N}$-grading of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ associated to the strictly dominant integral coweight $\lambda = \sum_{\alpha\in\Pi} n_\alpha \varpi^\vee_\alpha$, where $n_\alpha = 1$ if $\alpha \in \Pi^\bullet$ and $n_\alpha = 2$ if $\alpha \in \Pi^*$. For graded reasons it follows that

$$\Phi_0(F_\alpha) := \begin{cases} t'_\alpha F_{\theta^\bullet(\alpha)}, & \text{if } \alpha \in \Pi^\bullet \\ t''_\alpha F_{\theta^*(\alpha)}, & \text{if } \alpha \in \Pi^* \end{cases}$$

extends to a linear automorphism of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ such that

$$\Phi(F_\alpha) - \Phi_0(F_\alpha) \in \mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})^{\geq n_\alpha+1}, \ \forall \alpha \in \Pi. \tag{6.11}$$

Lemma 6.5 (iv) implies that $\Phi_0 = \Upsilon^-_{(t,\theta)}$ for some $\theta \in \mathrm{Aut}(\Gamma, \mathbf{p})$ and $t \in \mathbb{T}^r$. By Eq. (6.11), $(\Upsilon^-_{(t,\theta)})^{-1}\Phi$ is a λ-unipotent automorphism of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ and by Theorem 6.4 $\Phi = \Upsilon^-_{(t,\theta)}$. □

The last step before the proof of Theorem 6.2 is an extension of Lemma 5.5 to the twisted case.

Lemma 6.7. *Let $\mathfrak{g}$ be a simple Lie algebra, $\mathbb{K}$ an arbitrary base field, $q \in \mathbb{K}^*$ not a root of unity, and $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$. For all automorphisms Φ of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ we have*

$$\Phi(F_\alpha) \in \mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})^{\geq 1}$$

with respect to the $\mathbb{N}$-grading of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$ associated to $\lambda = \rho^\vee = \sum_{\alpha\in\Pi} \varpi^\vee_\alpha$.

Proof. For $\alpha \neq \alpha'$ denote

$$x^{\pm}_{\alpha\alpha'} = \sum_{j=0}^{-a_{\alpha\alpha'}} (-\mathbf{r}(\alpha', \alpha) q_\alpha^{\pm 1})^j \begin{bmatrix} -a_{\alpha\alpha'} \\ j \end{bmatrix}_{q_\alpha^{\pm 1}} (F_\alpha)^j F_{\alpha'} (F_\alpha)^{-a_{\alpha\alpha'}-j}.$$

It follows from (6.3) that

$$x^{\pm}_{\alpha\alpha'} F_\alpha = \mathbf{r}(\alpha', \alpha) q_\alpha^{\mp a_{\alpha\alpha'}} F_\alpha x_{\alpha\alpha'}.$$

If $a_{\alpha\alpha'} \neq 0$, then either $\mathbf{r}(\alpha', \alpha) q_\alpha^{a_{\alpha\alpha'}} \neq 1$ or $\mathbf{r}(\alpha', \alpha) q_\alpha^{-a_{\alpha\alpha'}} = 1$ because q is not a root of unity. This establishes the lemma analogously to the proof of Lemma 5.5. □

Proof of Theorem 6.2. Because of the isomorphism from Eq. (6.2) it is sufficient to prove the minus case of the theorem. Lemma 6.7 implies that every automorphism $\Phi \in \mathrm{Aut}(\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g}))$ satisfies $\Phi(F_\alpha) \in \mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})^{\geq 1}$ with respect to $\mathbb{N}$-grading from §6.2 corresponding to the strictly dominant integral coweight $\lambda = \rho^\vee = \sum_{\alpha\in\Pi} \varpi^\vee_\alpha$. For each $\alpha \in \Pi$ there exists a unique element $\Phi_0(F_\alpha) \in \mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})^1$ such that

$$\Phi(F_\alpha) - \Phi_0(F_\alpha) \in \mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})^{\geq 2}. \tag{6.12}$$

For graded reasons Φ_0 extends to a linear automorphism of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$. Hence, by Proposition 6.6, $\Phi_0 = \Upsilon^-_{(t,\theta)}$ for some $\theta \in \mathrm{Aut}(\Gamma, \mathbf{p})$ and $t \in \mathbb{T}^r$. Moreover, it follows from (6.12) that $\Phi_0^{-1}\Phi$ is a $\rho^\vee$-unipotent automorphism of $\mathcal{U}^-_{q,\mathbf{p}}(\mathfrak{g})$. Theorem 6.4 implies that $\Phi = \Phi_0$ and thus $\Phi = \Upsilon^-_{(t,\theta)}$. □

Remark 6.8. It is easy to see that the proof of Theorem 6.2 easily extends to the more general statement for all semisimple Lie algebras $\mathfrak{g}$ not having a simple direct summand of rank 1. For simplicity of the exposition we only considered the simple case. The details of the semisimple case are left to the reader.

6.4. **The isomorphism problem for the algebras $\mathcal{U}^{\pm}_{q,\mathbf{p}}(\mathfrak{g})$.** The following theorem applies the results on automorphism groups from this and the previous sections to the isomorphism problem for the class of algebras of the form $\mathcal{U}^{\pm}_{q,\mathbf{p}}(\mathfrak{g})$ for simple Lie algebras $\mathfrak{g}$. The idea to apply results on automorphism groups to this isomorphism problem was suggested to us by Len Scott.

Theorem 6.9. *Let $\mathfrak{g}_i$, $i = 1, 2$ be two simple Lie algebras with root lattices $\mathcal{Q}_i$, Dynkin diagrams Γ_i and set of simple roots Π_i (considered as the set of vertices of Γ_i). Let $\mathbb{K}$ be an arbitrary base field, $q \in \mathbb{K}^*$, and $\mathbf{p}_i \in Z^2(\mathcal{Q}_i, \mathbb{K}^*)$ be two 2-cocycles satisfying* (6.4) *and such that $G_{\mathbf{p}_1} G_{\mathbf{p}_2}$ is a torsion-free subgroup of $\mathbb{K}^*$. Then*

$$\mathcal{U}^{\pm}_{q,\mathbf{p}_1}(\mathfrak{g}_1) \cong \mathcal{U}^{\pm}_{q,\mathbf{p}_2}(\mathfrak{g}_2) \tag{6.13}$$

if and only if

$$\begin{gathered} \mathfrak{g}_1 \cong \mathfrak{g}_2 \ \textit{ and there exists a graph isomorphism } \ \theta\colon \Gamma_1 \to \Gamma_2 \\ \textit{such that } \ \mathbf{r}_2(\theta(\alpha), \theta(\alpha')) = \mathbf{r}_1(\alpha, \alpha'), \ \ \forall \alpha, \alpha' \in \Pi_1. \end{gathered} \tag{6.14}$$

Here, as in (6.1)*, we set $\mathbf{r}_i(\alpha, \alpha') := \mathbf{p}(\alpha, \alpha')\mathbf{p}(\alpha', \alpha)^{-1}$ for $\alpha, \alpha' \in \Pi_i$.*

Corollary 6.10. *For all simple Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, base fields $\mathbb{K}$ and a non-root of unity $q \in \mathbb{K}^*$,*

$$\mathcal{U}^{\pm}_{q}(\mathfrak{g}_1) \cong \mathcal{U}^{\pm}_{q}(\mathfrak{g}_2) \quad \textit{if and only if} \quad \mathfrak{g}_1 \cong \mathfrak{g}_2.$$

Proof of Theorem 6.9. It is obvious that (6.14) implies (6.13). The Gelfand–Kirillov dimensions of $\mathcal{U}_{q,\mathbf{p}}(\mathfrak{g}_i)$ are equal to the number of positive roots of $\mathfrak{g}_i$ because of the iterated Ore extension presentation (6.5). Therefore (6.13) $\Rightarrow$ (6.14) in the case when the rank of one of the algebras $\mathfrak{g}_i$ is equal to 1. Thus all we need to show is that

$$(6.13) \Rightarrow (6.14) \ \text{ in the case when } \operatorname{rank}(\mathfrak{g}_1), \operatorname{rank}(\mathfrak{g}_2) > 1.$$

Because of the isomorphism (6.2) we can restrict to the minus case.

Set $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Its root lattice, Dynkin diagram and set of simple roots are given by $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$, $\Gamma = \Gamma_1 \sqcup \Gamma_2$ and $\Pi = \Pi_1 \sqcup \Pi_2$. There is a unique cocycle $\mathbf{p} \in Z^2(\mathcal{Q}, \mathbb{K}^*)$ such that $\mathbf{p}(\alpha, \alpha') := \mathbf{p}_i(\alpha, \alpha')$ if $\alpha, \alpha' \in \mathcal{Q}_i$ for some $i = 1, 2$ and $\mathbf{p}(\alpha, \alpha') = 1$ if $\alpha \in \mathcal{Q}_1, \alpha' \in \mathcal{Q}_2$ or $\alpha \in \mathcal{Q}_2, \alpha' \in \mathcal{Q}_1$.

The conditions on $\mathbf{p}_1$ and $\mathbf{p}_2$ are equivalent to saying that $\mathbf{p}$ is a torsion-free 2-cocycle that satisfies (6.4). We also have a canonical isomorphism

$$\mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g}) \cong \mathcal{U}^{-}_{q,\mathbf{p}_1}(\mathfrak{g}_1) \otimes \mathcal{U}^{-}_{q,\mathbf{p}_2}(\mathfrak{g}_2).$$

Let $\Psi\colon \mathcal{U}^{-}_{q,\mathbf{p}_1}(\mathfrak{g}_1) \to \mathcal{U}^{-}_{q,\mathbf{p}_2}(\mathfrak{g}_2)$ be an algebra isomorphism. Following Kimmerle [28], consider the involutive automorphism

$$\Phi \in \operatorname{Aut}(\mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g})), \quad \text{given by } \ \Phi(u_1 \otimes u_2) := \Psi^{-1}(u_2) \otimes \Psi(u_1), \ \ \forall u_i \in \mathcal{U}_{q,\mathbf{p}_i}(\mathfrak{g}_i). \tag{6.15}$$

It is easy to see that the statement of Lemma 6.7 holds for every semisimple Lie algebra $\mathfrak{g}$ without simple direct summands of rank 1. It follows from Lemma 6.7 that $\Phi(F_\alpha) \in \mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g})^{\geq 1}$ $\forall \alpha \in \Pi$ with respect to $\mathbb{N}$-grading from §6.2 corresponding to $\lambda = \rho^\vee = \sum_{\alpha \in \Pi} \varpi^\vee_\alpha$. As in the proof of Theorem 6.2, for each $\alpha \in \Pi$ there exists a unique element $\Phi_0(F_\alpha) \in \mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g})^1$ such that

$$\Phi(F_\alpha) - \Phi_0(F_\alpha) \in \mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g})^{\geq 2}.$$

For graded reasons Φ_0 extends to a linear automorphism of $\mathcal{U}^{-}_{q,\mathbf{p}}(\mathfrak{g})$. Applying Proposition 6.6, we obtain that $\Phi_0 = \Upsilon^{-}_{(t,\theta_0)}$ for some $\theta_0 \in \operatorname{Aut}(\Gamma, \mathbf{p})$ and $t \in \mathbb{T}^{|\Pi|}$. The definition (6.15) of Φ implies that $\theta_0(\Gamma_1) = \Gamma_2$ and $\theta_0(\Gamma_2) = \Gamma_1$. From the definitions of $\operatorname{Aut}(\Gamma, \mathbf{p})$ and $\mathbf{p}$ we obtain

$$\mathbf{r}_2(\theta_0(\alpha), \theta_0(\alpha')) = \mathbf{r}_1(\alpha, \alpha'), \ \ \forall \alpha, \alpha' \in \Pi_1.$$

Thus (6.14) holds for $\theta := \theta_0|_{\Gamma_1}$, which completes the proof of the theorem. $\square$

References

[1] J. Alev, *Un automorphisme non modéré de $\mathcal{U}(\mathfrak{g}_3)$*, Comm. Algebra **14** (1986), no. 8, 1365–1378.

[2] J. Alev, N. Andruskiewitsch, and F. Dumas, *On algebras with few automorphisms*, preprint 2006.

[3] J. Alev and M. Chamarie, *Dérivations et automorphismes de quelques algèbres quantiques*, Comm. Algebra **20** (1992), no. 6, 1787–1802.

[4] J. Alev and F. Dumas, *Rigidité des plongements des quotients primitifs minimaux de $U_q(sl(2))$ dans l'algèbre quantique de Weyl-Hayashi*, Nagoya Math. J. **143** (1996), 119–146.

[5] N. Andruskiewitsch and F. Dumas, *On the automorphisms of $U_q^+(\mathfrak{g})$*, In: Quantum groups, 107–133, IRMA Lect. Math. Theor. Phys. 12, Eur. Math. Soc., Zürich, 2008, ArXiv:math.QA/0301066.

[6] V. A. Artamonov, *Valuations on quantum division rings*, Special issue dedicated to Alexei Ivanovich Kostrikin, Comm. Algebra **29** (2001), no. 9, 3889–3904.

[7] M. Artin, W. Schelter, and J. Tate, *Quantum deformations of* GL_n, Comm. Pure Appl. Math. **44** (1991), no. 8-9, 879–895.

[8] J. P. Bell and S. Launois, *On the dimension of H-strata in quantum algebras*, Algebra Number Theory **4** (2010), no. 2, 175–200.

[9] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Adv. Math **155** (2005), no. 2, 405–455.

[10] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2002.

[11] Ph. Caldero, *Sur le centre de $U_q(\mathfrak{n}^+)$*, Festschrift on the occasion of the 65th birthday of Otto Krötenheerdt, Beiträge Algebra Geom. **35** (1994), no. 1, 13–24.

[12] Ph. Caldero, *Étude des q-commutations dans l'algèbre $U_q(\mathfrak{n}^+)$*, J. Algebra **178** (1995), no. 2, 444–457.

[13] G. Cauchon, *Effacement des dérivations et spectres premiers d'algébres quantiques,* J. Algebra **260** (2003), no.2, 476–518.

[14] C. De Concini, V. Kac, and C. Procesi, *Some quantum analogues of solvable Lie groups,* In: Geometry and analysis (Bombay, 1992), pp. 41–65, Tata Inst. Fund. Res., Bombay, 1995.

[15] C. De Concini and C. Procesi, *Quantum groups*, In: D-modules, representation theory, and quantum groups (Venice, 1992), 31–140, Lect. Notes in Math., 1565, Springer, Berlin, 1993.

[16] O. Fleury, *Automorphismes de $\breve{U}_q(\mathfrak{b}^+)$*, Beiträge Algebra Geom. **38** (1997), no. 2, 343–356.

[17] J. Geiger and M. Yakimov, *Quantum Schubert cells via representation theory and ring theory*, preprint, ArXiv:1203.3780, to appear in Michigan Math. J.

[18] C. Geiß, B. Leclerc, and J. Schröer, *Cluster structures on quantized coordinate rings*, Selecta Math. **19** (2013), no. 2, 337–397.

[19] K. R. Goodearl and T. H. Lenagan, *Quantum determinantal ideals*, Duke Math. J. **103** (2000), no. 1, 165–190.

[20] K. R. Goodearl and E. S. Letzter, *The Dixmier–Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras,* Trans. Amer. Math. Soc. **352** (2000), no. 3, 1381–1403.

[21] K. R. Goodearl and M. T. Yakimov, *From quantum Ore extensions to quantum tori via noncommutative UFDs*, preprint, ArXiv:1208.6267.

[22] M. Gorelik, *The prime and the primitive spectra of a quantum Bruhat cell translate,* J. Algebra **227** (2000), no. 1, 211–253.

[23] T. J. Hodges, T. Levasseur, and M. Toro, *Algebraic structure of multiparameter quantum groups*, Adv. Math. **126** (1997), no. 1, 52–92.

[24] J. C. Jantzen, *Lectures on quantum groups,* Grad. Studies in Math., 6. Amer. Math. Soc., Providence, RI, 1996.

[25] A. Joseph, *A wild automorphism of $U\,sl(2)$*, Math. Proc. Cambridge Philos. Soc. **80** (1976), no. 1, 61–65.

[26] A. Joseph, *On the prime and primitive spectra of the algebra of functions on a quantum group*, J. Algebra **169** (1994), no. 2, 441–511.

[27] A. Joseph, *Sur les idéaux génériques de l'agèbre des fonctions sur un groupe quantique*, C. R. Acad. Sci. Paris Sér. I. Math. **321** (1995), no. 2, 135–140.

[28] W. Kimmerle, *Variations of the Zassenhaus conjecture*, in: DMV Seminar band **18**, Group Rings and Class Groups, Roggenkamp/Taylor, 1992.

[29] G. Lusztig, *Introduction to quantum groups*, Progr. Math. 110, Birkhäuser, 1993.

[30] S. Launois, *Primitive ideals and automorphism group of $U_q^+(B_2)$*, J. Algebra Appl. **6** (2007), no. 1, 21–47.

[31] S. Launois, *On the automorphism groups of q-enveloping algebras of nilpotent Lie algebras*, In: Proc. Workshop "From Lie Algebras to Quantum Groups", CIM publ., vol. 28 (2007), 125–143.

[32] S. Launois and T. H. Lenagan, *Primitive ideals and automorphisms of quantum matrices*, Algebr. Represent. Theory **10** (2007), no. 4, 339–365.

[33] S. Launois and T. H. Lenagan, *Automorphisms of quantum matrices*, Glasgow Math. J. **55**A (2013), 89–100.

[34] S. Launois and S. A. Lopes, *Automorphisms and derivations of $U_q(\mathfrak{sl}_4^+)$*, J. Pure Appl. Algebra **211** (2007), no. 1, 249–264.

[35] A. Mériaux and G. Cauchon, *Admissible diagrams in $U_q^w(\mathfrak{g})$ and combinatoric properties of Weyl groups*, Represent. Theory **14** (2010), 645–687.

[36] J. M. Osborn and D. S. Passman, *Derivations of skew polynomial rings*, J. Algebra **176** (1995), no. 2, 417–448.

[37] L. Rigal, *Spectre de l'algèbre de Weyl quantique*, Beiträge Algebra Geom. **37** (1996), no. 1, 119–148.

[38] I. P. Shestakov and U. U. Umirbaev, *The tame and the wild automorphisms of polynomial rings in three variables*, J. Amer. Math. Soc. **17** (2004), no. 1, 197–227

[39] X. Tang, *The prime ideal stratification and the automorphism group of $U_{r,s}^+(B_2)$*, preprint, ArXiv:1109.2640.

[40] M. Yakimov, *A classification of H-primes of quantum partial flag varieties*, Proc. Amer. Math. Soc. **138** (2010), 1249–1261.

[41] M. Yakimov, *On the spectra of quantum groups*, preprint, ArXiv:1106.3821, to appear in Mem. Amer. Math. Soc.

[42] M. Yakimov, *Spectra and catenarity of multiparameter quantum Schubert cells*, Glasgow Math J. **55**A (2013), 169–194.

[43] M. Yakimov, *The Launois–Lenagan conjecture*, J. Algebra **392** (2013), 1–9.

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