Reflective prolate-spheroidal operators and the KP/KdV equations

W. Riley Casper[a,1], F. Alberto Grünbaum[b,1], Milen Yakimov[a,1,2], and Ignacio Zurrián[c,1]

[a]Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803; [b]Department of Mathematics, University of California, Berkeley, CA 94720; and [c]Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina

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Commuting integral and differential operators connect the topics of signal processing, random matrix theory, and integrable systems. Previously, the construction of such pairs was based on direct calculation and concerned concrete special cases, leaving behind important families such as the operators associated to the rational solutions of the Korteweg–de Vries (KdV) equation. We prove a general theorem that the integral operator associated to every wave function in the infinite-dimensional adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$ of Wilson always reflects a differential operator (in the sense of *Definition 1* below). This intrinsic property is shown to follow from the symmetries of Grassmannians of Kadomtsev–Petviashvili (KP) wave functions, where the direct commutativity property holds for operators associated to wave functions fixed by Wilson's sign involution but is violated in general. Based on this result, we prove a second main theorem that the integral operators in the computation of the singular values of the truncated generalized Laplace transforms associated to all bispectral wave functions of rank 1 reflect a differential operator. A 90° rotation argument is used to prove a third main theorem that the integral operators in the computation of the singular values of the truncated generalized Fourier transforms associated to all such KP wave functions commute with a differential operator. These methods produce vast collections of integral operators with prolate-spheroidal properties, including as special cases the integral operators associated to all rational solutions of the KdV and KP hierarchies considered by [Airault, McKean, and Moser, *Commun. Pure Appl. Math.* 30, 95–148 (1977)] and [Krichever, *Funkcional. Anal. i Priložen.* 12, 76–78 (1978)], respectively, in the late 1970s. Many examples are presented.

prolate-spheroidal integral operators | reflectivity | rational solutions of the KdV and KP equations | Wilson's adelic Grassmannian

In a pair of ground-breaking works from the late 1940s Claude Shannon laid down the mathematical foundations of communication theory (1, 2). One of the key problems which he raised was "What is the best information that one can infer for a signal $f(t)$ which is time limited to the interval $[-\tau,\tau]$ from knowing its frequencies in the interval $[-\kappa,\kappa]$?" This double-concentration problem leads to the study of the singular values of an operator given by a finite Fourier transform

$$(Ef)(z)=\int_{-\tau}^{\tau} e^{izx} f(x)\,dx, \quad z\in[-\kappa,\kappa].$$

The central issue is the effective computation of the eigenfunctions of the integral operator

$$(EE^{*}f)(z)=2\int_{-\kappa}^{\kappa}\frac{\sin\tau(z-w)}{z-w}f(w)\,dw, \quad z\in[-\kappa,\kappa]. \qquad \textbf{[1]}$$

This problem was beautifully solved by Slepian and Pollak (3) and Landau and Pollak (4) in the early 1960s by showing that the integral operator in Eq. **1** commutes with the differential operator

$$R(z,\partial_z)=\partial_z(\kappa^2-z^2)\partial_z-\tau^2,$$

from which they described the common eigenfunctions via the differential operator. Note that $R(z,\partial_z)$ is the "radial part" of the Laplacian in prolate-spheroidal coordinates, motivating our title. The commuting property was used by Fuchs (5) and Slepian (6) to carry out a detailed analysis of the asymptotics of the eigenvalues of EE^*, while Jimbo et al. (7) showed that its Fredholm determinant is a τ function of Painlevé V.

Remarkably, this commuting property appeared as early as 1907 in the work of Bateman (ref. 8, equations 38–41 accompanied by some differentiation) and later in the classical text by Ince (9). Mehta (10) independently discovered and used it to analyze the Fredholm determinant of the integral operator Eq. **1**, which he then applied to asymptotic problems in random matrices. For recent numerical work on prolate-spheroidal operators see ref. 11; for applications to geophysics see refs. 5–7 and 12.

Slepian (13) found an extension of the time-band limiting analysis to n dimensions. His method was based on passing to polar coordinates and then relying on a different commutativity result. He proved that for integer N the integral operator

$$(\mathcal{E}f)(z)=\int_0^1 J_N(czw)\sqrt{czw}\,f(w)\,dw$$

acting on a subspace of $L^2(0,1;dw)$ with appropriate boundary conditions admits the commuting differential operator

$$\partial_z(1-z^2)\partial_z-c^2z^2+\frac{\frac{1}{4}-N^2}{z^2},$$

where $J_N(x)$ denotes the Bessel function of the first kind.

In the early 1990s Tracy and Widom (14, 15) discovered 1 more remarkable commuting pair of integral and differential

Significance

Pairs of commuting integral and differential operators have been constructed on a case-by-case basis in the analysis of various spectral problems in signal processing, random matrix theory, and integrable systems. We present a unified general construction of commuting pairs based on the intrinsic properties of symmetries of soliton equations. A key ingredient in it is a method proving that the integral operators associated to the points of infinite-dimensional families of solutions of soliton equations canonically "reflect" differential operators. This in turn is used to give examples of interesting commuting pairs.

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[1]W.R.C., F.A.G., M.Y., and I.Z. contributed equally to this work.

[2]To whom correspondence may be addressed. Email: yakimov@math.lsu.edu.

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operators associated to the Airy kernel. They effectively used this pair and a modification of the one for the Bessel kernel in their study of the asymptotics of the level spacing distribution functions of the edge scaling limits of the Gaussian unitary ensemble and the Laguerre and Jacobi ensembles. More precisely, Tracy and Widom (14, 15) proved that the integral operator with the Airy kernel

$$\frac{A(z)A'(w)-A'(z)A(w)}{z-w}$$

acting on $L^2(\tau,+\infty;dw)$ admits the commuting differential operator

$$\partial_z(\tau-z)\partial_z - z(\tau-z),$$

where $A(z)$ denotes the Airy function.

All of the above developments fitted into 1 general scheme: Commuting differential operators were constructed for an integral kernel of the form

$$K_\psi(z,w) := \int_{\Gamma_2} \psi(x,z)\psi^*(x,w)\,dx \qquad \textbf{[2]}$$

acting on $L^2(\Gamma_1;dw)$, where Γ_1 and Γ_2 are contours in $\mathbb{C}$, $\psi(x,z)$ is a wave function for the Kadomtsev–Petviashvili (KP) hierarchy, and $\psi^*(x,z)$ is its adjoint wave function. Note in Slepian's Bessel-type example above, we get this kernel form for the square of his integral operator. Many other instances of such commuting pairs were later discovered (16–19), to name a few, and generalized to discrete and matrix-valued settings (20, 21).

The KP and Korteweg–de Vries Hierarchies

The Korteweg–de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

was introduced more than a century ago to model waves on shallow water surfaces. Its complete integrability was established by Miura, Gardner, and Kruskal (22) and Lax (23). A wave function for a solution $u(x,t)$ is a function $\psi(x,z;t)$ satisfying

$$(\partial_x^2 - u(x,t))\psi(x,z;t) = z^2\psi(x,z;t).$$

The KdV equation fits into an infinite system of completely integrable nonlinear partial differential equations in variables $x, t_0, t_1, t_2, \ldots$ known as the KP hierarchy. Alternatively the KdV equation fits into the KdV hierarchy describing KP solutions independent of even times.

The KP hierarchy is an infinite-dimensional integrable system whose wave functions $\psi(x,z)$ are eigenfunctions of differential operators $L(x,\partial_x)$ of higher order and more generally of formal pseudo-differential operators. We refer the reader to van Moerbeke's exposition of the subject (24) from the point of view of evolution on the (infinite-dimensional) Sato's Grassmannian $\mathrm{Gr}^{\mathrm{Sato}}$ and its applications to quantum gravity and intersection theory on moduli spaces of curves via the Kontsevich theorem (25). The latter concerns precisely the solution of the KP hierarchy corresponding to the Airy wave function $\psi_{\mathrm{Ai}}(x,z) = A(x+z)$.

In the late 1970s Airault, McKean, and Moser (26) found a remarkable connection between the (infinite-dimensional) KdV equation and finite-dimensional integrable systems. They proved that any rational solution of the KdV equation that vanishes at infinity has the form

$$u(x,t) = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{(x-x_i(t))^2}$$

and that the KP flow for $t=t_1$ corresponds to the motion of the poles $(x_1(t),\ldots,x_n(t))$ according to the Calogero–Moser system with Hamiltonian $H=\sum_i p_i^2/2 - \sum_{i<j}(x_i-x_j)^{-2}$. Krichever (27) proved that this is true for every rational solution of KP vanishing at infinity and that all solutions of the Calogero–Moser system arise in this way.

Bispectrality and the Adelic Grassmannian

The bispectral problem, posed by Duistermaat and Grünbaum in ref. 28, asks for a classification of all functions $\psi(x,z)$ on a subdomain $\Omega_1\times\Omega_2\subseteq\mathbb{C}^2$ for which there exist 2 differential operators $L(x,\partial_x)$ and $\Lambda(z,\partial_z)$ on Ω_1 and Ω_2 and 2 functions $\theta\colon\Omega_1\to\mathbb{C}$, $f\colon\Omega_2\to\mathbb{C}$, such that

$$L(x,\partial_x)\psi(x,z) = f(z)\psi(x,z),$$
$$\Lambda(z,\partial_z)\psi(x,z) = \theta(x)\psi(x,z).$$

Many important relations of bispectrality to representation theory and algebraic and noncommutative geometry were subsequently found. Early on it was realized that it is advantageous to think of its solutions as wave functions of the KP hierarchy. In this setting ref. 28 provided a classification of the second-order bispectral operators $L(x,\partial_x)$. Half of these come from rational solutions of the KdV equation. The other half consists of the Airy wave function $\psi_{\mathrm{Ai}}(x,z)=A(x+z)$, the Bessel wave functions $\psi_{\mathrm{Be}(\nu)}(x,z)=\sqrt{xy}\,J_\nu(\sqrt{xy})$ for $\nu\in\mathbb{C}\backslash\mathbb{Z}$, and wave functions obtained from them by "master symmetries" of the KdV hierarchy (29).

Wilson made a deep insight into the bispectral problem (30), providing the concept of classifying bispectral functions $\psi(x,z)$ according to their rank, defined as the greatest common divisor of the orders of all differential operators $L(x,\partial_x)$ having $\psi(x,z)$ as an eigenfunction. For example, in the order 2 classification of ref. 28, the wave functions of the rational solutions of KdV are of rank 1, while the remaining families are of rank 2.

In ref. 30 Wilson classified all bispectral functions $\psi(x,z)$ of rank 1 in terms of an infinite-dimensional sub-Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$ of Sato's Grassamannian $\mathrm{Gr}^{\mathrm{Sato}}$, called the adelic Grassmannian. $\mathrm{Gr}^{\mathrm{ad}}$ consists of those planes $W\in\mathrm{Gr}^{\mathrm{Sato}}$ obtained from the base plane $W_0=\mathbb{C}[z]$ by imposing "adelic-type" conditions at finitely many points. It was shown in ref. 31 that these are precisely the KP wave functions $\psi(x,z)$ such that

$$\psi(x,z) = \frac{1}{p(x)q(z)}P(x,\partial_x)\cdot e^{-xz}$$

and

$$e^{-xz} = \frac{1}{\widetilde{p}(x)\widetilde{q}(y)}\widetilde{P}(x,\partial_x)\cdot\psi(x,z)$$

for some differential operators $P(x,\partial_x)$ and $\widetilde{P}(x,\partial_x)$ with polynomial coefficients and polynomials $p(x)$, $\widetilde{p}(x)$, $q(z)$, $\widetilde{q}(y)$. The orders of the differential operators P and $\widetilde{P}$ will be called degree and codegree of $\psi(x,z)$, respectively.

Wilson (32) completed the circle back to Airault, McKean, and Moser (26) and Krichever (27) by showing that the adelic Grassmannian is the disjoint union of the Calogero–Moser spaces $\mathrm{CM}_n\subset\mathrm{Gr}^{\mathrm{Sato}}$ which are compactifications of the phase spaces of the Calogero–Moser integrable systems on the rational solutions of the KP hierarchy of refs. 26 and 27:

$$\mathrm{Gr}^{\mathrm{ad}} = \bigsqcup_{n\geq 1}\mathrm{CM}_n. \qquad \textbf{[3]}$$

Integral Operators and Points of $\mathrm{Gr}^{\mathrm{ad}}$

Reflectivity. The unifying feature of the diverse lines of research described above is a collection of hand-made examples of integral operators with kernels of the form in Eq. **2** commuting with differential operators, obtained from certain specific wave functions $\psi(x,z)$ in $\mathrm{Gr}^{\mathrm{ad}}$.

For a long time, the examples provided above were the only known examples, and for this reason it was tempting to believe that it was a complete collection of examples. However, this is not true at all! In this paper we give a general solution of the problem that is applicable to the integral operators associated to the wave functions of all points of the adelic Grassmannian. It is based on a conceptual way of constructing the commuting differential operators from bispectral algebras. Our key idea is that the intrinsic property of all of these integral operators is a more general one than a naive commutativity:

Definition 1. *An integral operator T, acting on $L^2(\Gamma)$ for a contour $\Gamma \subset \mathbb{C}$, is said to reflect a differential operator $R(z, \partial_z)$ if*

$$T \circ R(-z, -\partial_z) = R(z, \partial_z) \circ T$$

on a dense subspace of $L^2(\Gamma)$.

In the special case that a wave function $\psi(x,z) \in \mathrm{Gr}^{\mathrm{Sato}}$ satisfies the symmetry condition $\psi(x,z) = \psi(-x,-z)$, this property for the kernel in Eq. **2** reduces to classical commutativity. This happens, for example, in the case of master symmetries (18). However, we will show that even more generally, imaginary rotation arguments transform reflecting pairs to classically commuting ones.

Remark 1. *The reflection identity of Definition 1 is sensitive to the extension of the differential operator $R(z,\partial_z)$ to $L^2(\Gamma)$, which is not unique, and may hold for a unique choice of this extension. This is a technical point that is often omitted in the classical prolate-spheroidal picture* (33)*.*

First General Theorem—Reflection vs. Commutation. Our first theorem associates to any wave function $\psi(x,z) \in \mathrm{Gr}^{\mathrm{ad}}$ an integral operator T which reflects a differential operator. The reflected differential operator $R(z,\partial_z)$ resides in a natural algebra of differential operators associated to $\psi(x,z)$, called the (right) generalized Fourier algebra, defined in ref. 19 by

$$\mathcal{F}_z(\psi) := \{R : \exists\, L \text{ with } L(x,\partial_x)\psi(x,z) = R(z,\partial_z)\psi(x,z)\}.$$

The differential operators $L(x,\partial_x)$ that appear in the left-hand side also form an algebra, called the (left) generalized Fourier algebra and denoted by $\mathcal{F}_x(\psi)$. The map $L(x,\partial_x) \mapsto R(z,\partial_z)$ defines an algebra antiisomorphism

$$b_\psi : \mathcal{F}_x(\psi) \to \mathcal{F}_z(\psi).$$

The algebras $\mathcal{F}_x(\exp(-xz))$ and $\mathcal{F}_z(\exp(-xz))$ are both equal to the first Weyl algebra and the corresponding map b is closely related to the Fourier transform.

Theorem 1. *For every wave function $\psi(x,z) \in \mathrm{Gr}^{\mathrm{ad}}$, the integral operator T_ψ on $L^2[t,\infty)$ with kernel*

$$K_\psi(z,w) := \int_s^\infty \psi(y,z)\psi^*(y,w)\,dy \tag{4}$$

reflects a (*nonconstant*) *differential operator $R(z,\partial_z) \in \mathcal{F}_z(\psi)$ of order at most $2\min(d_1,d_2)$, where d_1 and d_2 are the degree and codegree of $\psi(x,z)$.*

A key feature of the proof of *Theorem 1*, sketched below, explicitly reduces the problem of finding the operator $R(z,\partial_z)$ to a finite-dimensional linear algebra problem. This in turn provides an effective algorithm for computing the reflected differential operator for all $\psi(x,z) \in \mathrm{Gr}^{\mathrm{ad}}$. In particular, we obtain examples of integral operators commuting with differential operators of orders much higher than can be reasonably found by hand, as shown in *Examples 2* and *5* below.

Wilson's 3 Involutions. In general the operator T defined by *Theorem 1* is not self-adjoint (even formally). In this way we may gain additional insight into the spectra of nonself adjoint integral operators. In connection to Shannon's original questions, we have to be able to detect which operators in *Theorem 1* are of the form EE^* and in particular are self-adjoint. For this we consider the 3 natural involutions of the adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$ introduced by Wilson in ref. 30, along with a fourth involution not previously featured in this context corresponding to Schwartz reflection.

Name	Involution
Adjoint	$a(\psi)(x,z) = \psi^*(x,z) = p(x)^{-1}\widetilde{p}(x)^{-1} \times \widetilde{P}^*(x,\partial_x)\cdot e^{-xz}$
Bispectral	$b(\psi)(x,z) = \psi(z,x)$
Sign	$s(\psi)(x,z) = \psi(-x,-z)$
Schwartz	$c(\psi)(x,z) = \overline{\psi(\overline{x},\overline{z})}$

Note that the adjoint involution was used implicitly in Eq. **2**. Wilson observed that the involutions a, b, and s have the remarkable property that ab is not an involution, but rather

$$(ab)^2 = s. \tag{5}$$

Sketch of the Proof of Theorem 1

Step 1. Another way to phrase Wilson's property in Eq. **5** is that

$$b_{a\psi}(b_\psi^{-1}(R)^*)^*(z,\partial_z) = R(-z,-\partial_z), \quad \forall R \in \mathcal{F}_z(\psi). \tag{6}$$

Consider a differential operator $R_{s,t}(z,\partial_z) \in \mathcal{F}_z(\psi)$ such that both bilinear concomitants

$$\mathcal{C}_{b_\psi^{-1}R_{s,t}}(f,g;s) \quad \text{and} \quad \mathcal{C}_{R_{s,t}}(f,g;-t)$$

are identically 0. We refer the reader to ref. 34 for the definition and properties of bilinear concomitants of differential operators. Applying the identity in Eq. **6** together with integration by parts and the maps b_ψ^{-1} and $b_{a\psi}$, we obtain that such an operator $R_{s,t}(z,\partial_z)$ satisfies

$$\begin{aligned}
&R_{s,t}(z,\partial_z)\cdot K_\psi(z,w)\\
&= \int_s^\infty (R_{s,t}(z,\partial_z)\cdot\psi(x,z))\psi(x,w)^*\,dx\\
&= \int_s^\infty \big(b_\psi^{-1}(R_{s,t})(x,\partial_x)\cdot\psi(x,z)\big)\psi(x,w)^*\,dx\\
&= \int_s^\infty \psi(x,z)\,\big(b_\psi^{-1}(R_{s,t})^*(x,\partial_x)\cdot\psi(x,w)^*\big)\,dx\\
&= \int_s^\infty \psi(x,z)\,\big(b_{a\psi}(b_\psi^{-1}(R_{s,t}))^*(w,\partial_w)\cdot\psi(x,w)^*\big)\,dx\\
&= R_{s,t}^*(-w,-\partial_w)\cdot K_\psi(z,w).
\end{aligned}$$

This identity combined with 1 more integration by parts proves that

$$R_{s,t}(z,\partial_z)\circ T_\psi = T_\psi \circ R_{s,t}(-z,-\partial_z)$$

for the integral operator T_ψ with kernel as in Eq. **4**.

The remainder of the proof of *Theorem 1* revolves around demonstrating the existence of a differential operator $R_{s,t}(z,\partial_z)\in\mathcal{F}_z(\psi)$ satisfying the conditions of step 1. Its existence, along with a sharp upper bound on its order, is obtained by algebro-geometric arguments.

Step 2. The operators in the Fourier algebra $\mathcal{F}_x(\psi)$ naturally have a co-order $\operatorname{coord}R(z,\partial_z):=\operatorname{ord}(b_\psi^{-1}R)(x,\partial_x)$. For a pair of nonnegative integers ℓ, m, set

$$\mathcal{F}_z^{\ell,m}(\psi):=\{R\in\mathcal{F}_z(\psi):\operatorname{ord}R\le\ell,\operatorname{coord}R\le m\}.$$

Recall the decomposition in Eq. **3**; let $\psi(x,z)\in\mathrm{CM}_n\subset\mathrm{Gr}^{\mathrm{ad}}$. One shows that $\mathcal{F}_z(\psi)$ is isomorphic to the algebra of differential operators on a rank 1, torsion-free sheaf over the spectral curve of the solution of KP with wave function $\psi(x,z)$. Interpreting n as the differential genus of the sheaf of the curve in the sense of Berest and Wilson (35), and then converting it to the Letzter–Markar–Limanov invariant of the sheaf show that

$$\begin{aligned}\dim\mathcal{F}_z^{\ell,m}(\psi)&=(\ell+1)(m+1)-n\\&\ge(\ell+1)(m+1)-2\min(d_1,d_2)^2\end{aligned}$$

for $\ell,m\ge 2\min(d_1,d_2)-1$.

Step 3. For a differential operator $R(z,\partial_z)$ of order $\le\ell$, the identical vanishing of the concomitant $C_R(f,g;-t)$ is shown to lead to at most $\lceil\ell/2\rceil.\lceil(\ell+1)/2\rceil$ linearly independent (linear) conditions on the coefficients of R and their derivatives.

This estimate, combined with that in step 2, proves the existence of a differential operator $R_{s,t}(z,\partial_z)\in\mathcal{F}_z(\psi)$ satisfying the conditions of Step 1 of order at most $2\min(d_1,d_2)$. ☐

Remark 2.

1) *Wilson's identity on involutions in the Adelic Grassmannian in Eq.* **5** *and its use in step* 1 *are the intrinsic reasons for the appearance of reflectivity in Theorem 1 rather than classical commutativity.*
2) *All previous approaches for constructing commuting pairs of integral and differential operators, like those in refs.* 3, 4, and 13–15, *relied on a by-hand construction of a commuting differential operator. Step* 1 *of the proof is where bispectrality plays a deep role and the operator is constructed from the generalized Fourier algebra* $\mathcal{F}_z(\psi)$.

The Laplace vs. Fourier Pictures

Second General Theorem—the Laplace Picture. Consider a wave function $\psi\in\mathrm{Gr}^{\mathrm{ad}}$. We draw a parallel between the integral operators from *Theorem 1* and those of the form EE^* by considering the following analogs of the Laplace transform and its adjoint:

$$\begin{aligned}L_\psi&:f(x)\mapsto\int_0^\infty\psi(y,z)f(y)\,dy,\\L_\psi^*&:f(z)\mapsto\int_0^\infty\overline{\psi(x,w)}f(w)\,dw.\end{aligned}$$

In the special case that $\psi(x,z)=\exp(-xz)$, the operator L_ψ is precisely the Laplace transform. The time- and band-limited versions of these are (for $z\ge t$)

$$(\mathcal{E}_\psi f)(z)=(\chi_{[t,\infty)}L_\psi\chi_{[s,\infty)}f)(z)=\int_s^\infty\psi(y,z)f(y)\,dy,$$

and (for $x\ge s$)

$$(\mathcal{E}_\psi^* f)(x)=(\chi_{[s,\infty)}L_\psi^*\chi_{[t,\infty)}f)(x)=\int_t^\infty\overline{\psi(x,w)}f(w)\,dw.$$

They give rise to the self-adjoint operator analogous to the one considered by Landau, Pollak, and Slepian (3, 4),

$$\begin{aligned}(\mathcal{E}_\psi\mathcal{E}_\psi^*f)(z)&=\int_t^\infty K_\psi(z,w)f(w)\,dw,\ \text{ where}\\K_\psi(z,w)&=\int_s^\infty\psi(y,z)\overline{\psi(y,w)}\,dy,\end{aligned}$$

viewed as an operator on $L^2(t,\infty)$. Under natural mild conditions on $\psi(x,z)$, *Theorem 1* determines the existence of differential operators reflected by $\mathcal{E}_\psi\mathcal{E}_\psi^*$. For a different situation involving the Laplace transform, see ref. 36.

Theorem 2. *For every wave function* $\psi(x,z)$ *in Wilson's adelic Grassmannian, fixed under the involution* ac *of* $\mathrm{Gr}^{\mathrm{ad}}$ (*defined by the table of involutions above*), *the integral operator* $\mathcal{E}_\psi\mathcal{E}_\psi^*$ *reflects a* (*nonconstant*) *differential operator* $R(z,\partial_z)\in\mathcal{F}_z(\psi)$ *of order at most* $2\min(d_1,d_2)$, *where* d_1 *and* d_2 *are the degree and codegree of* $\psi(x,z)$.

Sketch of Proof: From the assumption that $\psi(x,z)$ is fixed under the involution ac, one deduces that $\psi^*(x,z)=\overline{\psi(x,z)}$ for $x,z\in\mathbb{R}$. From this one shows that $\mathcal{E}_\psi\mathcal{E}_\psi^*$ equals the integral operator with kernel K_ψ from *Theorem 1*. ☐

Remark 3. *Under the assumption that* $\psi(x,z)$ *is fixed by* ac, *the reflected operator* $R_{s,t}(z,\partial_z)$ *satisfies the identity* $R_{s,t}^*(z,\partial_z)=R_{s,t}(-z,-\partial_z)$. *In this case, the reflection property may be restated in the form*

$$\mathcal{E}_\psi\mathcal{E}_\psi^*\circ R_{s,t}^*(z,\partial_z)=R_{s,t}(z,\partial_z)\circ\mathcal{E}_\psi\mathcal{E}_\psi^*.$$

Example 1. *Consider the simplest case* $\psi(x,z)=\exp(-xz)$. *The integral operator*

$$(\mathcal{E}_\psi\mathcal{E}_\psi^*f)(z)=\int_t^\infty\frac{\sinh(s(z+w))}{z+w}f(w)\,dw$$

acting on $L^2(t,\infty)$ *reflects the first-order differential operator*

$$R_{s,t}(z,\partial_z)=(z+t)\partial_z+sz.$$

All previous works on this kernel deal with a commuting second-order differential operator.

The wave functions associated to rational solutions (26) of KdV are automatically fixed by the involution a. Additionally, those with real coefficients are fixed by c and thus satisfy the assumptions of *Theorem 2*. These are precisely the bispectral functions in the KdV family in ref. 28 with real coefficients (associated to second-order differential operators of rank 1). There has been a substantial effort since 1986 to find commuting differential operators for the corresponding integral operators, but absolutely no examples have been found beyond the case $\psi(x,z)=\exp(-xz)$ or that in ref. 18. The next example demonstrates how *Theorem 2* resolves this problem.

Example 2. *Let* $r\in\mathbb{R}^*$. *Consider the function*

$$\psi(x,z)=\frac{(x+z^{-1})^3-z^3+r}{x^3+r}e^{-xz},$$

which up to a change of variables is precisely the first nontrivial bispectral function in ref. 28, *equation* 1.39. *The integral operator* $\mathcal{E}_\psi\mathcal{E}_\psi^*$ *has kernel*

$$K_\psi(z,w) = \frac{\psi(s,z)\psi_x(s,w) - \psi_x(s,z)\psi(s,w)}{z^2 - w^2}.$$

By Theorem 2 it reflects a differential operator in $\mathcal{F}_z(\psi)$. Our algorithm produces an operator of order 3, given by

$$\begin{aligned}R_{s,t}(z,\partial_z) = &-(z+t)^2 z\partial_z^3 + \Big(st^3 - 3stz^2 - 2sz^3 - t^3rz^2\\ &-t^2rz^3 - \frac{3}{2}t^2 - 6tz - \frac{9}{2}z^2\Big)\partial_z^2 + \big(s^2t^2z - s^2z^3\\ &-2st^3rz^2 - 2st^2rz^3 - 6stz - 6sz^2 - 2t^3rz\\ &-3t^2rz^2 + 6t^2z^{-1} + 6t - z\big)\partial_z - 6st^3z^{-2} - 3t^2z^{-2}\\ &+ s^3tz^2 - s^2t^3rz^2 - s^2t^2rz^3 - \frac{3}{2}s^2z^2 - 2st^3rz\\ &- 3st^2rz^2 - t^2rz + 3.\end{aligned}$$

Third General Theorem—The Fourier Picture. By performing a 90° rotation in the complex variable z, we move from the Laplace transform picture to the Fourier transform picture. We prove that in this way one can convert the reflected differential operators in the Laplace picture to commuting differential operators in the Fourier picture. Specifically, we replace the operators L_ψ and L_ψ^* with their Fourier counterparts

$$F_\psi : f(x) \mapsto \int_{-\infty}^{\infty} \psi(y,-iz)f(y)\,dy,$$
$$F_\psi^* : f(z) \mapsto \int_{-\infty}^{\infty} \overline{\psi(x,-iw)}f(w)\,dw.$$

In the special case $\psi(y,z) = \exp(-yz)$, the operator F_ψ is the Fourier transform. We define the time- and band-limited operators E_ψ and E_ψ^* similarly to $\mathcal{E}_\psi$ and $\mathcal{E}_\psi^*$:

$$(E_\psi f)(z) = (\chi_{[t,\infty)}F_\psi \chi_{[s,\infty)}f)(z) = \int_s^\infty \psi(y,-iz)f(y)\,dy,$$
$$(E_\psi^* f)(x) = (\chi_{[s,\infty)}F_\psi^* \chi_{[t,\infty)}f)(x) = \int_t^\infty \overline{\psi(x,-iw)}f(w)\,dw.$$

The self-adjoint operator

$$(E_\psi E_\psi^* f)(z) = \int_s^\infty \int_t^\infty \psi(y,-iz)\overline{\psi(y,-iw)}f(w)\,dw\,dy$$

acting on $L^2(t,\infty)$ no longer has a simple kernel expression as above since the relevant integral does not converge outright, but can be given sense as a distribution. Even so, the method of proof of *Theorem 1* applies, giving us a certain relationship between an integral and differential operators. Serendipitously, due to the change in sign with complex conjugation, in this case we obtain a strict commutativity relation.

Theorem 3. *For every wave function $\psi(x,z)$ in Wilson's adelic Grassmannian, fixed under the involution ac of $\mathrm{Gr}^{\mathrm{ad}}$, the integral operator $E_\psi E_\psi^*$ commutes with the differential operator $R_{s,it}(-iz,i\partial_z)$, where $R_{s,t}(z,\partial_z)$ is the corresponding differential operator in Theorem 2 (its coefficients are rational functions in z,s,t).*

In particular, we obtain that $E_\psi E_\psi^*$ commutes with the self-adjoint operator $R_{s,it}(-iz,i\partial_z)R_{s,it}^*(-iz,i\partial_z)$.

Sketch of Proof: One repeats step 1 of the proof of *Theorem 1* to show that $R_{s,it}(-iz,i\partial_z)$ commutes with $E_\psi E_\psi^*$ for every differential operator $R(z,\partial_z) \in \mathcal{F}_z(\psi)$ for which both bilinear concomitants

$$\mathcal{C}_{b_\psi^{-1}R_{s,it}(-iz,i\partial_z)}(f,g;s) \quad \text{and} \quad \mathcal{C}_{R_{s,it}(-iz,i\partial_z)}(f,g;it)$$

are identically 0. The operator $R_{s,t}(z,\partial_z)$ from *Theorem 2* has these properties because its coefficients are rational functions in z,s,t. This is proved by an analysis of the structure of the algebra $\mathcal{F}_z(\psi)$. □

Note that for analytic reasons one cannot deduce *Theorem 3* from *Theorem 2* by an elementary change of variables.

Theorems 1–3 form the foundation for our study of the asymptotics of the eigenvalues of the integral operators associated to the wave functions of the rational solutions of the KP equation and the numerical properties of the associated eigenfunctions (which greatly generalize the prolate-spheroidal wave functions).

Example 3. *Consider the case $\psi(x,z) = \exp(-xz)$ as in Example 1. The self-adjoint integral operator is given by*

$$(E_\psi E_\psi^* f)(z) = \int_s^\infty \int_t^\infty e^{iy(z-w)}f(w)\,dw\,dy.$$

The eigenvalues of this operator are precisely the singular values of the semiinfinite time-band limiting of the Fourier transform. This integral operator commutes with the first-order differential operator

$$R_{s,it}(-iz,i\partial_z) = (z-t)\partial_z - isz$$

obtained from the differential operator in Example 1. As a consequence we obtain that $E_\psi E_\psi^*$ commutes with the self-adjoint second-order differential operator

$$\begin{aligned}-R_{s,it}(-iz,i\partial_z)R_{s,it}^*(-iz,i\partial_z) = &\,\partial_z(z-t)^2\partial_z\\ &-is\{z(z-t),\partial_z\},\end{aligned}$$

where here $\{\cdot,\cdot\}$ denotes the anticommutator bracket.

Example 4. *Consider the wave function*

$$\psi(x,z) = \frac{(x+z^{-1})^3 - z^3 + r}{x^3 + r}e^{-xz}$$

from Example 2. The associated integral operator $E_\psi E_\psi^*$ commutes with the third-order differential operator $R := R_{s,it}(-iz,i\partial_z)$, where $R_{s,t}(z,\partial_z)$ is the differential operator in Example 2. Its formal adjoint is $R^* = -R + s^2t^2 + 4$, so that $E_\psi E_\psi^*$ commutes with the sixth-order self-adjoint operator

$$-RR^* = R^2 - (s^2t^2 + 4)R.$$

Simultaneous Reflectivity/Commutativity

The proof of *Theorem 1* produces a large algebra of reflected operators rather than a single one, because the argument can be applied to the full Fourier algebra $F_z(\psi)$ of $\psi \in \mathrm{Gr}^{\mathrm{ad}}$. This can be used to prove the existence of universal operators which are simultaneously reflected by (or commute with) finite-dimensional collections of integral operators.

Theorem 4.

1) *Consider any finite collection of wave functions $\{\psi_k(x,z) : 1 \le k \le n\} \in \mathrm{Gr}^{\mathrm{ad}}$ and let T_k be the associated integral operators as in Theorem 1 for the same values of s and t. There exists a nonconstant differential operator in $\bigcap_k \mathcal{F}_z(\psi_k)$ simultaneously reflected by each of the integral operators T_k for all k.*

2) *If, in addition, all wave functions* $\psi_k(x,z)$ *are fixed under the involution* ac *of* $\mathrm{Gr}^{\mathrm{ad}}$, *then there exists a differential operator* $R^{univ}_{s,t}(z,\partial_z)$ *which is simultaneously reflected by all integral operators spanned by* $\mathcal{E}_j\mathcal{E}_k^*$ *for* $1\le j,k\le n$. *This differential operator has rational coefficients in* z,s,t *and* $\widetilde{R}_{s,t}(z,\partial_z):=R^{univ}_{s,it}(-iz,i\partial_z)$ *commutes with all integral operators* $E_jE_k^*$ *for* $1\le j,k\le n$.

In the situation of part 2 all integral operators $E_jE_k^*$, $1\le j,k\le n$, commute with the self-adjoint operator $\widetilde{R}_{s,t}(z,\partial_z)\widetilde{R}^*_{s,t}(z,\partial_z)$. Furthermore, since the Fourier algebra of $\exp(-xz)$ is just the algebra of differential operators with polynomial coefficients, we can force all of the coefficients of $\widetilde{R}_{s,t}(z,\partial_z)$ to be polynomials in z.

Example 5. *Consider the pair of wave functions* $\{\psi_1(x,z),\psi_2(x,z)\}$ *with* $\psi_n(x,z)=K_n(xz)\sqrt{xz}$ *for* $K_n(z)$ *the modified Bessel function of the second kind. Thus by Theorem 4 there should exist a self-adjoint differential operator* $\widetilde{R}_{s,t}(z,\partial_z)$ *in* $\mathcal{F}_z(\psi_1)\cap\mathcal{F}_z(\psi_2)$ *with polynomial coefficients which commutes with the integral operators* $E_kE_j^*$ *defined by the wave functions* $\psi_k(x,z)$ *for* $k=1,2$. *Note also that* $\widetilde{R}_{s,t}(z,\partial_z)$ *will commute with the integral operator* EE^* *associated with the wave function from Example 2 for any* r, *since this operator will be a linear combination of the* $E_kE_j^*$*s. Using our algorithm for Theorem 1, we obtain an operator of order 6 of the form*

$$\widetilde{R}_{s,t}(z,\partial_z)=\sum_{m=0}^{3}\partial_z^m f_m(z)\partial_z^m,$$

where

$$f_0(z)=\frac{z^2(3s^6t^3-54s^4t)}{6}+s^6z^5-\frac{3s^6tz^4}{2}+12s^4z^3,$$
$$f_1(z)=(z-t)\left(3s^4z^4-3s^4tz^3+12\,s^2z^2+9s^2tz-9s^2t^2\right),$$
$$f_2(z)=(z-t)^2\left(3s^2z^3-\frac{3s^2tz^2}{2}+12t\right),$$
$$f_3(z)=(z-t)^3z^2.$$

Concluding Remarks

We have presented a unified general construction of commuting pairs based on the intrinsic properties of symmetries of soliton equations. It has not escaped our notice that the specific connection we have described between commuting integral and differential operators and solutions of the KdV equation, in particular the critical role of the reflecting property in these classical problems, opens up avenues of broad applications of integrable systems to spectral analysis of integral operators, going far beyond sinc, Bessel, and Airy kernels. Additionally, the constructed pairs of commuting integral and differential operators may have a role to play in random matrix theory.

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