WEAK SPLITTINGS OF QUOTIENTS OF DRINFELD AND HEISENBERG DOUBLES

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Abstract. We investigate the fine structure of the symplectic foliations of Poisson homogeneous spaces. Two general results are proved for weak splittings of surjective Poisson submersions from Heisenberg and Drinfeld doubles. The implications of these results are that the torus orbits of symplectic leaves of the quotients can be explicitly realized as Poisson–Dirac submanifolds of the torus orbits of the doubles. The results have a wide range of applications to many families of real and complex Poisson structures on flag varieties. Their torus orbits of leaves recover important families of varieties such as the open Richardson varieties.

1. Introduction

The geometry of Poisson–Lie groups is well understood, both in the case of the standard Poisson structures on complex simple Lie groups [13, 20] and the general Belavin–Drinfeld Poisson structures [1, 29]. The torus orbits of symplectic leaves in the former case are the double Bruhat cells of the simple Lie group. One of the fundamental results in the theory of cluster algebras is the Berenstein–Fomin–Zelevinsky theorem [2] that their coordinate rings are upper cluster algebras. Recently, the coordinate rings of the SL_n groups, equipped with the Cremmer–Gervais Poisson structures from [1], were also shown to be upper cluster algebras [15]. The motivation for these results is that cluster algebras give rise to Poisson structures by the work of Gekhtman, Shapiro, and Vainshtein [14], and one attempts to go in the opposite direction using Poisson varieties from the theory of quantum groups.

On the other hand, the possible cluster algebra structures on coordinate rings of torus orbits of symplectic leaves of Poisson homogeneous spaces is much less well understood. In the special case of the standard complex Poisson structures on flag varieties, this is precisely the problem of constructing cluster algebra structures on the coordinate rings of the open Richardson varieties. These varieties have been recently studied in [4, 19, 25] in relation to Schubert calculus and total positivity. Chevalier [5] conjectured cluster algebra structures for the Richardson strata in the case when one of the two Weyl group elements is a parabolic Coxeter element. Leclerc [21] generalized this construction and showed that the coordinate ring of each open Richardson variety contains a cluster algebra whose rank is equal to the dimension of the variety. These cluster structures

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come from an additive categorification. Another cluster algebra structure on the Richardson–Lusztig strata of the Grassmannian was conjectured by Muller and Speyer [23] using Postnikov diagrams [24].

In this paper we prove a very general result that realizes torus orbits of symplectic leaves of a large class of Poisson homogeneous spaces as Poisson–Dirac submanifolds of torus orbits of symplectic leaves of Drinfeld and Heisenberg doubles. It applies to many important families of complex and real Poisson structures on flag varieties, double flag varieties, and their generalizations. In the context of cluster algebras, the point of this construction is that the coordinate rings of affine Poisson varieties with conjectured cluster algebra structures are realized as quotients of better understood coordinate rings of Poisson varieties, some of which are already proven to possess cluster algebra structures. The ideals defining these quotients are not Poisson but have somewhat similar properties coming from a notion of "weak splitting of surjective Poisson submersions." The construction of the latter is the main point of the paper.

To explain this in precise terms, we recall that to each point of a Poisson homogeneous space of a Poisson–Lie group Drinfeld [9] associated a Lagrangian subalgebra of the double and proved an equivariance property of this map. Motivated by this construction, Lu and Evens associated to each quadratic Lie algebra (a complex or real Lie algebra $\mathfrak{d}$ equipped with a nondegenerate invariant symmetric bilinear form $\langle .,. \rangle$) the variety of its Lagrangian subalgebras $\mathcal{L}(\mathfrak{d}, \langle .,. \rangle)$ and initiated its systematic study in [10]. This is a singular projective variety.

Fix a connected Lie group D with Lie algebra $\mathfrak{d}$. Given any pair of Lagrangian subalgebras $\mathfrak{g}_\pm$ such that $\mathfrak{d}$ is the vector space direct sum of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ (in other words, given a Manin triple $(\mathfrak{d}, \mathfrak{g}_+, \mathfrak{g}_-)$ with respect to the bilinear form $\langle .,. \rangle$), one defines the r-matrix $r = \frac{1}{2} \sum_j \xi_j \wedge x_j$ where $\{x_j\}$ and $\{\xi_j\}$ is a pair of dual bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$. Using the adjoint action of D on $\mathcal{L}(\mathfrak{d}, \langle .,. \rangle)$, we construct the bivector field $\chi(r)$ on $\mathcal{L}(\mathfrak{d}, \langle .,. \rangle)$. Here and below χ refers to the infinitesimal action associated to a Lie group action. It was proved in [10] that $\chi(r)$ is Poisson. Up to minor technical details, the singular projective Poisson variety

$$(\mathcal{L}(\mathfrak{d}, \langle .,. \rangle), \chi(r))$$

captures the geometry of all Poisson homogeneous spaces of the Poisson–Lie groups integrating the Lie bialgebras $\mathfrak{g}_\pm$. The D-orbits on $\mathcal{L}(\mathfrak{d}, \langle .,. \rangle)$ are compete Poisson submanifolds, i.e., they are unions of symplectic leaves of $\chi(r)$. They have the form $D/N(\mathfrak{l})$ where $\mathfrak{l}$ is a Lagrangian subalgebra of $(\mathfrak{d}, \langle .,. \rangle)$ and $N(\mathfrak{l})$ is the normalizer of $\mathfrak{l}$ in D. The Belavin–Drinfeld Poisson structures [1] are coming from the case when $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ for a complex simple Lie algebra $\mathfrak{g}$. In this case the rank of the Poisson structure $\chi(r)$ at each point of $D/N(\mathfrak{l})$ was computed in [22, Theorem 4.10]. This describes the coarse structure of the symplectic foliations of the spaces $(D/N(\mathfrak{l}), \chi(r))$ or equivalently the variety of Lagrangian subalgebras $(\mathcal{L}(\mathfrak{d}, \langle .,. \rangle), \chi(r))$.

Here we address the problem of describing the fine structure of the symplectic foliations of these spaces. From the point of view of Lie theory and cluster algebras, the most important D-orbits in this picture are the orbits

$$(D/N(\mathfrak{g}_+), \chi(r)) \hookrightarrow (\mathcal{L}(\mathfrak{d}, \langle .,. \rangle), \chi(r)).$$

These Poisson varieties capture all examples of real and complex Poisson structures on flag varieties and double flag varieties that appeared in previous studies, see e.g. [10, 12, 16, 27]. The Poisson varieties $(D/N(\mathfrak{g}_+), \chi(r))$ also have the properties that they are quotients of *Drinfeld and Heisenberg doubles.* Recall that those are the Poisson varieties

$$(D, \pi = L(r) - R(r)) \quad \text{and} \quad (D, \pi' = L(r) + R(r)),$$

respectively. Here and below $R(.)$ and $L(.)$ refer to right and left invariant multivector fields on a Lie group. The Poisson structure π vanishes along the group

$$H := N(\mathfrak{g}_+) \cap N(\mathfrak{g}_-)$$

and as a consequence the left and right action of H on D preserves both Poisson structures π and π'. The corresponding Poisson reductions will be denoted by

$$(D/H, \pi_H) \;\text{ and }\; (D/H, \pi'_H).$$

The canonical projections

$$\mu\colon (D/H, -\pi_H) \to (D/N(\mathfrak{g}_+), \chi(r)) \;\text{ and }\; \mu'\colon (D/H, \pi'_H) \to (D/N(\mathfrak{g}_+), \chi(r)) \tag{1.1}$$

are *surjective Poisson submersions.*

We prove that under certain general assumptions the symplectic leaves of $(D/N(\mathfrak{g}_+), \chi(r))$ can be realized as explicit symplectic submanifolds of the symplectic leaves of the (reduced) *Drinfeld double* $(D/H, -\pi_H)$ or *Heisenberg double* $(D/H, \pi'_H)$ (or even both in some cases). To be more precise, recall [6] that a Poisson manifold (X, π) is *a Poisson–Dirac submanifold admitting a Dirac projection* of a Poisson manifold (M, Π) if X is a submanifold of M, and there exists a subbundle E of $T_X M$ such that

$$E \oplus TX = T_X M \;\text{ and }\; \Pi - \pi \in \Gamma(X, \wedge^2 E).$$

In this framework we find an explicit construction of sections of the surjective Poisson submersion $\mu\colon (D/H, -\pi_H) \to (D/N(\mathfrak{g}_+), \chi(r))$ over each $N(\mathfrak{g}_-)$-orbit whose images are Poisson–Dirac submanifolds of $(D/H, -\pi_H)$. This is the *weak splitting of the first surjective Poisson submersion* in (1.1) from a Drinfeld double. (We refer the reader to Sect. 2 below and [16, Sect. 2] for the definition of the notion and additional background.) As a corollary of the general construction, the symplectic leaves within each $N(\mathfrak{g}_-)$-orbit on $(D/N(\mathfrak{g}_+), \pi_{D/N(\mathfrak{g}_+)})$ are uniformly embedded as symplectic submanifolds of symplectic leaves of $(D/H, -\pi_H)$. Similarly, we construct sections of the second surjective Poisson submersion $\mu'\colon (D/H, \pi'_H) \to (D/N(\mathfrak{g}_+), \chi(r))$ in (1.1) over each $N(\mathfrak{g}_+)$-orbit whose images are Poisson–Dirac submanifolds of $(D/H, \pi'_H)$ admitting a Dirac projection. These constructions of weak splittings work under certain general assumptions, see Theorems 3.5 and 4.2 for details. In Sect. 5 we show that the conditions are satisfied for many important families of Poisson structures.

The above results have a wide range of applications. In the case of the standard Poisson structures on flag varieties we recover the weak splittings from [16]. Double flag varieties arise naturally as closed strata in partitions of wonderful group compactifications [7]. This gives rise to Poisson structures on them that are not products of Poisson structures on each factor [27]. The second splitting result above for Heisenberg doubles is applicable to this family of Poisson varieties. The real forms of a complex simple Lie algebra $\mathfrak{g}$ give rise to real Poisson structures on the related complex flag variety defined in [12]. Again the above second splitting is applicable for this family. Finally, the Delorme's classification result in [8] gives rise to canonical Poisson structures on products of flag varieties for complex simple Lie groups (i.e, flag varieties for a reductive group). Except for some very special cases, these Poisson structures are not products of Poisson structures on the factors. Our weak splittings are applicable for those families too.

The results in the paper are also related to the study of the spectra of the quantizations of the homogeneous coordinate rings of the above mentioned families of varieties. Currently, only the spectra of quantum flag varieties are understood [30]. We expect that a quantum version of our weak splittings of surjective Poisson submersions will be helpful in understanding the spectra of the quantizations of these families of varieties on the basis of the works on spectra of quantum groups [13, 17]. It appears that such quantum weak splittings should be also closely related to the notion of quantum folding of Berenstein and Greenstein [3].

The paper is organized as follows. In Sect. 2 we review the notion of Poisson–Dirac submanifolds of Poisson manifolds, and weak sections and weak splittings of surjective Poisson submersions. In Sect. 3 we prove two general theorems on the construction of weak sections and weak splittings for quotients of Drinfeld doubles. In Sect. 4 similar theorems are proved for quotients of Heisenberg doubles. Sect. 5 contains applications of these theorems.

We finish the introduction with some notation that will be used throughout the paper. Given a group G, $d \in G$ and two subgroups H_1 and H_2 of G, we will denote the H_1-orbit through $dH_2 \in G/H_2$ by

$$H_1 \cdot dH_2 \subset G/H_2$$

(to distinguish it from the double coset $H_1 d H_2 \subset G$).

For a Lie group G, we will denote by G° its identity component. For a smooth manifold X, X° will denote a connected component of X. Given a Lie group G and a subalgebra $\mathfrak{u}$ of its Lie algebra $\mathfrak{g}$, we will denote by $N(\mathfrak{u})$ the normalized of $\mathfrak{u}$ in G with respect to the adjoint action. Finally, recall that a Poisson structure π on a manifold M gives rise to the bundle map $\pi^\sharp \colon T^*M \to TM$, given by

$$\pi^\sharp(\alpha) = \alpha \otimes \mathrm{id}(\pi), \quad \alpha \in T^*_m M, m \in M.$$

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2. Poisson–Dirac submanifolds and weak splittings

In this section we review the notion of weak splitting of a surjective Poisson submersion from [16]. We start by recalling several facts about Poisson–Dirac submanifolds of Poisson manifolds, introduced and studied in [6, 28, 26].

Definition 2.1. A submanifold X of a Poisson manifold (M, Π) is called a *Poisson–Dirac submanifold* if the following conditions are satisfied:

(i) For each symplectic leaf S of (M, Π), the intersection $S \cap X$ is clean (i.e., it is smooth and $T_x(S \cap X) = T_x S \cap T_x X$ for all $x \in S \cap X$), and $S \cap X$ is a symplectic submanifold of $(S, (\Pi|_S)^{-1})$.

(ii) The family of symplectic structures $(\Pi|_S)^{-1}|_{S \cap X}$ is induced by a smooth Poisson structure π on X.

Clearly, in the setting of Definition 2.1, the symplectic leaves of (X, π) are the connected components of the intersections of symplectic leaves of (M, Π) with X.

An important criterion is provided by the following result.

Proposition 2.2. [Crainic, Fernandes] [6] *Assume that X is a submanifold of a Poisson manifold (M, Π) for which there exits a subbundle $E \subset T_X M$ such that*

(i) $E \oplus TX = T_X M$ and

(ii) $\Pi \in \Gamma(X, \wedge^2 TX \oplus \wedge^2 E)$.

Then X is Poisson–Dirac submanifold of (M, Π).

In the setting of Proposition 2.2 the projection of $\Pi|_X$ into $\Gamma(X, \wedge^2 TX)$ along $\wedge^2 E$ is exactly the needed Poisson structure π in Definition 2.1. Poisson–Dirac submanifolds with the property of Proposition 2.2 are called *Poisson–Dirac submanifolds admitting a Dirac projection* by Crainic–Fernandes [6] and quasi-Poisson submanifolds by Vaisman [26]. We will use the former term and call E an associated bundle of the Dirac projection.

In the presence of the condition (i) in Proposition 2.2, the condition (ii) is equivalent [6] to

$$\Pi^\sharp_m((T_m X)^0) \subset E_m, \quad \forall m \in X. \tag{2.1}$$

Here and below for a subspace $V \subseteq T_m M$, V^0 will denote its annihilator subspace in $T^*_m M$.

We continue with the notions of weak sections and weak splittings of surjective Poisson submersions.

Definition 2.3. Assume that (M, Π) and (N, π) are Poisson manifolds, X is a Poisson submanifold of (N, π), and that $p \colon (M, \Pi) \to (N, \pi)$ is a surjective Poisson submersion. A *weak section* of p over X is a smooth map $i \colon X \to M$ such that $p \circ i = \mathrm{id}_X$ and $i(X)$ is a Poisson–Dirac submanifold of (M, Π) with induced Poisson structure $i_*(\pi|_X)$.

In this situation we derive from Proposition 2.2 an explicit realization of all symplectic leaves of $(X, \pi|_X)$ in terms of those of (M, Π):

In the setting of Definition 2.3 one has that each symplectic leaf of $(X, \pi|_X)$ has the form $i^{-1}((i(X) \cap S)^\circ)$ where S is a symplectic leaf of (M, Π). In addition

i realizes explicitly all leaves of $(X, \pi|_X)$ as symplectic submanifolds of symplectic leaves of (M, Π).

Remark 2.4. The following special case of the notion of weak section has an equivalent algebraic characterization which is of particular interest, see [16, Proposition 2.6] for details.

Let $p\colon (M, \Pi) \to (N, \pi)$ be a surjective Poisson submersion, X be an open subset of N, and $i\colon X \to M$ be a smooth map such that $p \circ i = \mathrm{id}_X$ and $i(X)$ is a smooth submanifold of M. In particular, $p^* : (C^\infty(N), \{.,.\}_\pi) \to (C^\infty(M), \{.,.\}_\Pi)$ is a homomorphism of Poisson algebras. Then i is a weak section with associated bundle equal to the tangent bundle to the fibers of p if and only if

$$i^* : (C^\infty(M), \{.,.\}_\Pi) \to (C^\infty(N), \{.,.\}_\pi)$$

is a homomorphism of Poisson $(C^\infty(N), \{.,.\}_\pi)$-modules with respect to the action on the first term coming from p^*.

Definition 2.5. Let (M, Π) and (N, π) be Poisson manifolds and $p\colon (M, \Pi) \to (N, \pi)$ be a surjective Poisson submersion. A *weak splitting* of p is a partition

$$N = \bigsqcup_{a \in A} N_a$$

of (N, π) into complete Poisson submanifolds and a family of weak sections $i_a\colon N_a \hookrightarrow M$ of p (one for each stratum of the partition).

In the category of algebraic varieties we require M and N to be smooth algebraic varieties, X and N_a to be locally closed smooth algebraic subsets, and p, i_a to be algebraic maps.

We have:

Proposition 2.6. *Consider a surjective Poisson submersion $p\colon (M, \Pi) \to (N, \pi)$. Let $N = \bigsqcup_{a \in A} N_a$ and $i_a\colon N_a \to M$, $a \in A$ define a weak splitting of p. Then for all $a \in A$ the following hold:*

(i) Every symplectic leaf of $(N_a, \pi|_{N_a})$ has the form $i_a^{-1}((i_a(N_a) \cap S)^\circ)$ where S is a symplectic leaf of (M, Π).

(ii) Each symplectic leaf S' of $(N_a, \pi|_{N_a})$ is explicitly realized as a symplectic submanifold

$$i_a\colon (S', \pi|_{S'}) \hookrightarrow (S, \Pi|_S)$$

of the unique symplectic leaf S of (M, Π) that contains $i_a(S')$.

All weak sections and weak splittings that we construct in this paper will have the property that their images are Poisson–Dirac submanifolds admitting Dirac projections, i.e. the images will satisfy the conditions in Proposition 2.2.

3. Weak sections of quotients of Drinfeld doubles

We return to the setting of the introduction: Start with a Manin triple $(\mathfrak{d}, \mathfrak{g}_+, \mathfrak{g}_-)$ where $\mathfrak{d}$ is a quadratic Lie algebra with (a fixed) nondegenerate invariant symmetric bilinear form $\langle .,. \rangle$ and $\mathfrak{g}_\pm$ are two Lagrangian subalgebras such that $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces. Let D be a connected Lie group with Lie algebra $\mathfrak{d}$ and $G_\pm$ be the connected subgroups of D with Lie algebras $\mathfrak{g}_\pm$. Fix a pair of dual bases $\{x_j\}$ and $\{\xi_j\}$ of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ with respect to $\langle .,. \rangle$. The standard r-matrix $r = \frac{1}{2} \sum \xi_j \wedge x_j$ gives rise to the Poisson structures

$$\pi = L(r) - R(r) \quad \text{and} \quad \pi' = L(r) + R(r) \tag{3.1}$$

on D. Then (D, π) is a Poisson–Lie group and $G_\pm$ are Poisson–Lie subgroups. Moreover, (D, π) is a Drinfeld double of $(G_\pm, \pi|_{G_\pm})$ and (D, π') is a Heisenberg double of $(G_\pm, \pi|_{G_\pm})$. Finally, $(G_-, -\pi|_{G_-})$ is a dual Poisson–Lie group of $(G_+, \pi|_{G_+})$.

Set for brevity

$$N_\pm := N(\mathfrak{g}_\pm), \ \mathfrak{n}_\pm = \mathrm{Lie}\,(N_\pm) = \mathfrak{n}(\mathfrak{g}_\pm). \tag{3.2}$$

Denote the canonical projections $p_\pm \colon \mathfrak{d} \to \mathfrak{g}_\pm$ along $\mathfrak{g}_\mp$. Identify $\mathfrak{d}^*$ with $\mathfrak{d}$ using the form $\langle .,. \rangle$ and denote by $\alpha(y)$ the right invariant 1-form on D corresponding to $y \in \mathfrak{d} \cong \mathfrak{d}^*$.

The bundle maps $\pi^\sharp \colon T^*D \to TD$ and $(\pi')^\sharp \colon T^*D \to TD$ are given by the following formulas.

Lemma 3.1. *In the above setting, for all $x \in \mathfrak{g}_+$, $\xi \in \mathfrak{g}_-$ and $d \in D$:*

$$\begin{aligned}\pi^\sharp(\alpha_d(x+\xi)) = R_d(x) - L_d(p_+ \mathrm{Ad}_d^{-1}(x+\xi)) \\ = -R_d(\xi) + L_d(p_- \mathrm{Ad}_d^{-1}(x+\xi))\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}(\pi')^\sharp(\alpha_d(x+\xi)) = R_d(x) - L_d(p_- \mathrm{Ad}_d^{-1}(x+\xi)) \\ = -R_d(\xi) + L_d(p_+ \mathrm{Ad}_d^{-1}(x+\xi)).\end{aligned} \tag{3.4}$$

Eq. (3.4) is proved in [11], eqs. (6.2)–(6.4). Eq. (3.3) is analogous.

The Poisson structure π vanishes on

$$H := N_+ \cap N_- = N(\mathfrak{g}_+) \cap N(\mathfrak{g}_-), \tag{3.5}$$

see e.g. [22, Lemma 1.12]. Denote $\mathfrak{h} = \mathrm{Lie}\, H$. Recall that (D, π) is a Poisson Lie group and that both of its regular actions on the Heisenberg double (D, π') (given by $g \cdot g' = gg'$ and $g \cdot g' = g' g^{-1}$) are Poisson. Since $\pi|_H = 0$, the left and right actions of H on D preserve both π and π'. Denote their reductions with respect to the right action of H by

$$\pi_H \ \text{ and } \ \pi'_H \in \Gamma(D/H, \wedge^2 T(D/H)),$$

respectively. Thus the canonical projections

$$\nu \colon (D, \pi) \to (D/H, \pi_H) \ \text{ and } \ \nu' \colon (D, \pi') \to (D/H, \pi'_H)$$

are Poisson.

Since $\mathrm{Lie}\,(N_+) \supseteq \mathfrak{g}_+$, it follows from the definition of the Drinfeld and Heisenberg double Poisson structures (3.1) that

$$\pi_{D/N_+} := \chi(r)$$

is a Poisson structure on D/N_+ and that the standard projections

$$\mu \colon (D, -\pi) \to (D/N_+, \pi_{D/N_+}) \quad \text{and} \quad \mu' \colon (D, \pi') \to (D/N_+, \pi_{D/N_+})$$

are Poisson. Denote by

$$\eta\colon (D/H, -\pi_H) \to (D/N_+, \pi_{D/N_+}) \quad \text{and} \quad \eta'\colon (D/H, \pi'_H) \to (D/N_+, \pi_{D/N_+})$$

the induced surjective Poisson submersions. (They are both Poisson since $\mu = \eta\nu$, $\mu' = \eta'\nu'$, ν, and ν' are surjective Poisson submersions.)

The following proposition will be used in our general construction of weak sections for η:

Proposition 3.2. *Assume that for a given $d \in D$ such that $dHd^{-1} \subset N_-$ there exists a subgroup Q of D with Lie algebra $\mathfrak{q}$ satisfying*

$$\mathfrak{n}_- = \mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{n}_+) + \mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{q}) \quad \text{and} \tag{3.6}$$

$$Q \cap N_+ = H, \ \mathfrak{n}_+ + \mathfrak{q} = \mathfrak{n}_+ + \mathfrak{q}^\perp + \mathrm{Ad}_d^{-1}(\mathfrak{n}_-) = \mathfrak{g}. \tag{3.7}$$

Set

$$G_d := N_- \cap dQd^{-1}. \tag{3.8}$$

Then

$$\widetilde{E}^d \to G_d d, \ \ \widetilde{E}^d_{gd} := R_{gd}(p_+ \mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gd}(\mathfrak{n}_+), \ \ \forall g \in G_d \tag{3.9}$$

is a subbundle of $T_{G_d d}D$ such that

$$\widetilde{E}^d \cap T(G_d d) = L(\mathfrak{h}), \ \ \widetilde{E}^d + T(G_d d) = T_{G_d d} D, \tag{3.10}$$

and

$$\pi_{G_d d} \in \Gamma(G_d d, \wedge^2 \widetilde{E}^d + \wedge^2 T(G_d d)).$$

In (3.10) $L(\mathfrak{h})$ denotes the subbundle of $T(G_d d)$ spanned by left invariant vector fields $L(h)$, $h \in \mathfrak{h}$. Here and below, for a subspace V of $\mathfrak{d}$, we denote $V^\perp = \{z \in \mathfrak{d} \mid \langle z, y \rangle = 0, \forall y \in V\}$.

The condition $dHd^{-1} \subset N_-$ ensures that $G_d d$ is stable under the right action of H. The subbundle $\widetilde{E}^d$ of $T_{G_d d}D$ is equivariant with respect to this action. Indeed, if $h' \in H$, then

$$\begin{aligned} R_{h'} \widetilde{E}^d_{gd} &= R_{gdh'}(p_+ \mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gd} R_{h'}(\mathfrak{n}_+) \\ &= R_{gdh'}(p_+ \mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gdh'}(\mathfrak{n}_+) = \widetilde{E}^d_{gdh'}, \end{aligned} \tag{3.11}$$

because $h' \in H \subset N_-$. Therefore, the pushforward of $\widetilde{E}^d$ to $G_d \cdot dH$ is a subbundle of $T_{G_d \cdot dH}(D/H)$. As an immediate consequence of Proposition 3.2 we obtain the following corollary:

Corollary 3.3. *If, in the above setting, a subgroup Q of D and $d \in D$ satisfy (3.6)–(3.7) and $dHd^{-1} \subset N_-$, then the submanifold $G_d \cdot dH$ of the quotient $(D/H, -\pi_H)$ of the Drinfeld double $(D, -\pi)$ is a Poisson–Dirac submanifold admitting a Dirac projection with associated vector bundle equal to the pushforward $E^d = \nu_*(\widetilde{E}^d)$ of $\widetilde{E}^d$ to $G_d \cdot dH$.*

Proof of Proposition 3.2. Throughout the proof g will denote an element of G_d.

First we prove that $\widetilde{E}^d$ is a subbundle of $T_{G_d d}D$ and that (3.10) holds. Fix $g \in G_d$. We have:

$$\begin{aligned} T_{gd}(G_d d) + \widetilde{E}^d_{gd} &\supset L_{gd}(\mathrm{Ad}_d^{-1}\mathrm{Ad}_g^{-1}(\mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{q})) + L_{gd}(\mathfrak{n}_+) \\ &= L_{gd}(\mathrm{Ad}_d^{-1}(\mathfrak{n}_-) \cap \mathfrak{q} + \mathfrak{n}_+) \supset L_{gd}(\mathrm{Ad}_d^{-1}(\mathfrak{n}_-)) \\ &= L_{gd}(\mathrm{Ad}_d^{-1}\mathrm{Ad}_g^{-1}(\mathfrak{n}_-)) = R_{gd}(\mathfrak{n}_-). \end{aligned}$$

The second inclusion in the chain follows from (3.6). Thus,

$$\begin{aligned} &T_{gd}(G_d d) + \widetilde{E}^d_{gd} \supset R_{gd}(\mathfrak{n}_- + p_+(\mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gd}(\mathfrak{n}_+) \\ &\supset R_{gd}(\mathfrak{n}_- + \mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gd}(\mathfrak{n}_+) \\ &= L_{gd}(\mathrm{Ad}_d^{-1}\mathrm{Ad}_g^{-1}(\mathfrak{n}_-) + \mathrm{Ad}_d^{-1}\mathrm{Ad}_g^{-1}\mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gd}(\mathfrak{n}_+) \\ &= L_{gd}(\mathfrak{n}_+ + \mathfrak{q}^\perp + \mathrm{Ad}_d^{-1}(\mathfrak{n}_-)) = T_{gd}D, \end{aligned}$$

where we used (3.6)–(3.7) and the fact that Q normalizes $\mathfrak{q}^\perp$. Clearly,

$$T_{gd}(G_d d) \supset L_{gd}(\mathfrak{h}) \quad \text{and} \quad \widetilde{E}^d_{gd} \supset L_{gd}(\mathfrak{h}).$$

We claim that

$$\dim(p_+\mathrm{Ad}_d(\mathfrak{q}^\perp)) + \dim \mathfrak{n}_+ + \dim(\mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{q})) = \dim \mathfrak{g} + \dim \mathfrak{h}. \tag{3.12}$$

This implies that $\widetilde{E}^d$ is a subbundle of $T_{G_d d}D$ and that the first equality of (3.10) is satisfied. It also follows from (3.12) that $\widetilde{E}^d$ is the direct sum of the subbundles $R(p_+\mathrm{Ad}_d(\mathfrak{q}^\perp))$ and $L(\mathfrak{n}_+)$ of $T_{G_d d}D$.

Since $\mathfrak{n}_- = \mathfrak{g}_- + \mathfrak{h}$ and $\mathrm{Ad}_d^{-1}(\mathfrak{h}) \subset \mathfrak{h} \subset \mathfrak{q}$, we have

$$\begin{aligned} &\dim(\mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{q})) = \dim \mathfrak{n}_- - \dim \mathfrak{g}_- + \dim(\mathfrak{g}_- \cap \mathrm{Ad}_d(\mathfrak{q})) \\ &= \dim \mathfrak{n}_- - \dim \mathfrak{g}_- + \dim \mathfrak{g} - \dim(\mathfrak{g}_-^\perp + \mathrm{Ad}_d(\mathfrak{q}^\perp)) \\ &= \dim \mathfrak{n}_- - \dim(p_+\mathrm{Ad}_d(\mathfrak{q}^\perp)). \end{aligned}$$

Taking into account that $\dim \mathfrak{n}_+ + \dim \mathfrak{n}_- = \dim \mathfrak{d} + \dim \mathfrak{h}$ leads to (3.12).

Since

$$T_{gd}G_d d = R_{gd}(\mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{q}))$$

and

$$(\mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{q}))^\perp = \mathfrak{n}_-^\perp + \mathrm{Ad}_d(\mathfrak{q}^\perp) \subset \mathfrak{g}_- + p_+(\mathrm{Ad}_d(\mathfrak{q}^\perp)),$$

we have

$$(T_{gd}G_d d)^0 \subset \{\alpha_{gd}(x + \xi) \mid x \in p_+(\mathrm{Ad}_d(\mathfrak{q}^\perp)), \xi \in \mathfrak{g}_-\}.$$

Applying (3.3) gives

$$\pi^\sharp((T_{gd}G_d d)^0) \subset R_{gd}(p_+(\mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gd}(\mathfrak{g}_+) \subset \widetilde{E}^d_{gd}.$$

□

Observe that for $d \in D$ the conditions (3.6)–(3.7) ensure that the product $(N_- \cap dQd^{-1})(N_- \cap dN_+d^{-1})$ is open in N_-. This implies that $(N_- \cap dQd^{-1}) \cdot dN_+$ is open in $N_- \cdot dN_+$ which is a complete Poisson submanifold of $(D/N_+, \pi_{D/N_+})$, [22, Theorem 2.3]. Thus, (3.6)–(3.7) imply that $(N_- \cap dQd^{-1}) \cdot dN_+$ is a Poisson submanifold of $(D/N_+, \pi_{D/N_+})$.

Theorem 3.4. *Assume that for a given $d \in D$ such that $dHd^{-1} \subset N_-$ there exists a subgroup Q of D satisfying the conditions* (3.6)–(3.7)*. Then the smooth map $i \colon G_d \cdot dN_+ \to D/H$ defined by $i(gdN_+) = gdH$ for $g \in G_d := N_- \cap dQd^{-1}$ is a weak section of the surjective Poisson submersion $\eta \colon (D/H, -\pi_H) \to (D/N_+, \pi_{D/N_+})$ over $G_d \cdot dN_+$. Its image is a Poisson–Dirac submanifold of $(D/H, -\pi_H)$ admitting a Dirac projection with associated bundle $E^d := \nu_*(\widetilde{E}^d)$ where $\widetilde{E}^d$ is given by* (3.9)*.*

Proof. It is straightforward to check that i is well defined: If $g_1, g_2 \in G_d$ and $g_1 dN_+ = g_2 dN_+$, then $(g_2)^{-1} g_1 \in N_- \cap d(Q \cap N_+)d^{-1} \subset dHd^{-1}$ because of (3.7). Thus $g_1 dH = g_2 dH$.

For $g \in G_d$, Corollary 3.3 implies that

$$(\pi_H)_{gdH} \in \wedge^2 T_{gdH}(G_d \cdot dH) + \wedge^2 E^d_{gdH}.$$

Observe that E^d_{gdH} contains the tangent space $\nu_*(L_{gd}(\mathfrak{n}_+))$ to the fiber of ν through gdH. Since $\eta_*(-\pi_H) = \pi_{D/N_+}$ and $\nu \circ i = \mathrm{id}_{G_d \cdot dN_+}$, we have that the projection of $(-\pi_H)|_{G_g \cdot dH}$ to $\wedge^2 T(G_d \cdot dH)$ along E^d is $i_*(\pi_{D/N_+}|_{G_d \cdot dN_+})$. This completes the proof of the theorem. □

The next theorem provides a sufficient condition for the existence of a weak splitting of the surjective Poisson submersion in Theorem 3.4.

Theorem 3.5. *Assume that Q is a subgroup of D with Lie algebra $\mathfrak{q}$ such that there exists a set of representatives $\mathcal{D}$ for the (N_-, N_+)-double cosets of D satisfying*

$$N_- = (N_- \cap dQd^{-1})(N_- \cap dN_+ d^{-1}) \quad \text{and} \tag{3.13}$$

$$dHd^{-1} \subset N_-, \; Q \cap N_+ = H, \; \mathfrak{n}_+ + \mathfrak{q} = \mathfrak{n}_+ + \mathfrak{q}^\perp + \mathrm{Ad}_d^{-1}(\mathfrak{n}_-) = \mathfrak{g} \tag{3.14}$$

for all $d \in \mathcal{D}$. Then the partition

$$D/N_+ = \sqcup_{d \in \mathcal{D}} (N_- \cap dQd^{-1}) \cdot dN_+ \tag{3.15}$$

and the family of smooth maps

$$i_d \colon (N_- \cap dQd^{-1}) \cdot dN_+ \to D/H, \quad i_d(gdN_+) := gdH, \;\; \forall g \in N_- \cap dQd^{-1}$$

is a weak splitting of the surjective Poisson submersion

$$\eta \colon (D/H, -\pi_H) \to (D/N_+, \pi_{D/N_+}).$$

In addition, the images of i_d are Poisson–Dirac submanifolds of $(D/H, -\pi_H)$ admitting Dirac projections with associated bundles $E^d := \nu_*(\widetilde{E}^d)$ for the bundles $\widetilde{E}^d$ given by (3.9)*.*

Proof. The condition (3.13) implies that

$$(N_- \cap dQd^{-1}) \cdot dN_+ = N_- \cdot dN_+.$$

It follows from the definition of the set $\mathcal{D}$ that (3.15) defines a partition of G/N_+. This equality also implies that each stratum of the partition is a complete Poisson submanifold of $(D/N_+, \pi_{D/N_+})$, because this is a property of all N_+-orbits on D/N_+. The rest of the theorem follows from Theorem 3.4. □

Proposition 2.6 (i) implies that in the setting of Theorem 3.4 each symplectic leaf of $(D/N_+, \pi_{D/N_+})$ inside the stratum $(N_- \cap dQd^{-1}) \cdot dN_+ = G_d \cdot dN_+$ is of the form

$$i_d^{-1}((i_d(G_d \cdot dN_+) \cap S)^\circ)$$

for a symplectic leaf of S of $(D/H, -\pi_H)$. Furthermore, by Proposition 2.6 (ii) each symplectic leaf of $(D/N_+, \pi_{D/N_+})$ is explicitly realized as a symplectic submanifold of a symplectic leaf of $(D/H, -\pi_H)$ via one of the maps i_d.

Remark 3.6. In Proposition 3.2, and Theorems 3.4 and 3.5 one can replace $N_\pm$ with any pair of subgroups $N'_\pm$ of D such that $N^\circ_\pm \subset N'_\pm \subset N_\pm$. The corresponding statements hold true for $H := N'_+ \cap N'_-$. Their proofs are analogous and are left to the reader.

Remark 3.7. If the group D and the subgroup Q in Theorems 3.4 and 3.5 are algebraic, then the constructed weak sections and splittings are algebraic.

4. Weak sections of quotients of Heisenberg doubles

In this section we prove results for quotients of Heisenberg doubles that are similar to the results from the previous section for quotients of Drinfeld doubles. We use the setting and notation of the previous section. Using the second part of Lemma 3.1 one proves the following analog of Corollary 3.3 and Theorem 3.4 for Heisenberg doubles. We omit its proof since it is analogous to the case of Drinfeld doubles.

Theorem 4.1. *Let d and a subgroup Q of D with Lie algebra $\mathfrak{q}$ satisfy*

$$\mathfrak{n}_+ = \mathfrak{n}_+ \cap \mathrm{Ad}_d(\mathfrak{n}_+) + \mathfrak{n}_+ \cap \mathrm{Ad}_d(\mathfrak{q}) \ \text{ and} \tag{4.1}$$

$$dHd^{-1} \subset N_+, \ Q \cap N_+ = H, \ \mathfrak{n}_+ + \mathfrak{q} = \mathfrak{n}_+ + \mathfrak{q}^\perp + \mathrm{Ad}_d^{-1}(\mathfrak{n}_+) = \mathfrak{g}. \tag{4.2}$$

Set $G'_d = N_+ \cap dQd^{-1}$. Then the submanifold $G'_d \cdot dH$ of the quotient $(D/H, \pi'_H)$ of the Heisenberg double (D, π') is a Poisson–Dirac submanifold admitting a Dirac projection with associated vector bundle equal to the the pushforward $F^d := \nu'_*(\widetilde{F}^d)$ of the vector bundle

$$\widetilde{F} \to G'_d d, \ \ \widetilde{F}_{gd} := R_{gd}(p_- \mathrm{Ad}_d(\mathfrak{q}^\perp)) + L_{gd}(\mathfrak{n}_+), \ \ \forall g \in G'_d. \tag{4.3}$$

In addition, the map $i \colon G'_d \cdot dN_+ \to D/H$ defined by $i(gdN_+) = gdH$ for $g \in G'_d$ is a weak section of the surjective Poisson submersion $\eta' \colon (D/H, \pi'_H) \to (D/N_+, \pi_{D/N_+})$ over $G'_d \cdot dN_+$.

As in the previous section the theorem implies the following:

Theorem 4.2. *Let Q be a subgroup of D with Lie algebra $\mathfrak{q}$ for which there exists a set of representatives $\mathcal{D} \subset N(H)$ for the (N_+, N_+)-double cosets of D satisfying*

$$N_+ = (N_+ \cap dQd^{-1})(N_+ \cap dN_+d^{-1}) \tag{4.4}$$

and (4.2) *for all $d \in \mathcal{D}$. Then the partition*

$$D/N_+ = \sqcup_{d \in \mathcal{D}} (N_+ \cap dQd^{-1}) \cdot dN_+$$

and the family of maps

$$i'_d \colon (N_+ \cap dQd^{-1}) \cdot dN_+ \to D/H, \ \ i'_d(gdN_+) := gdH, \ \ \forall g \in N_+ \cap dQd^{-1}$$

provide a weak splitting of the surjective Poisson submersion $\eta' \colon (D/H, \pi'_H) \to (D/N_+, \pi_{D/N_+})$. *In addition, the images of* i'_d *are Poisson–Dirac submanifolds of* $(D/H, \pi'_H)$ *admitting Dirac projections with associated bundles* $F^d := \nu'_*(\widetilde{F}^d)$ *for the bundles* $\widetilde{F}^d$ *given by* (4.3) *with* $G'_d = N_+ \cap dQd^{-1}$.

In light of Proposition 2.6, Theorem 4.2 provides an explicit realization of the symplectic leaves of $(D/N_+, \pi_{D/N_+})$ as symplectic submanifolds of the symplectic leaves of the Heisenberg double $(D/H, \pi'_H)$.

As in the case of Drinfeld doubles, in Theorems 4.1 and 4.2 one can replace $N_\pm$ with any pair of subgroup $N'_\pm$ of D such that $N^\circ_\pm \subset N'_\pm \subset N_\pm$ in which case one sets $H = N'_+ \cap N'_-$. The proofs of those slightly more general statements are analogous.

If the groups D and Q are algebraic, the above constructed weak sections and weak splittings are also algebraic.

5. Applications to flag varieties

This section contains applications of the results from the previous two sections to Poisson structures on flag varieties. Subsections 5.1–5.4 deal with complex algebraic Poisson structures. There we construct weak splittings for complex surjective Poisson submersions from Drinfeld and Heisenberg doubles to flag varieties, double flag varieties and certain natural multi-flag generalizations. In §5.5 we give applications to real algebraic Poisson structures on flag varieties.

We note that all of the weak splittings that are constructed in this section provide (via Proposition 2.6) explicit realizations of the symplectic leaves of Poisson structures on flag varieties as symplectic submanifolds of symplectic leaves of Drinfeld and Heisenberg doubles.

Throughout the section G will denote an arbitrary connected complex simple Lie group with Lie algebra $\mathfrak{g}$. The Killing form on $\mathfrak{g}$ will be denoted by $\langle ., . \rangle$. We fix a pair of opposite Borel subgroups $B_\pm$ of G and the corresponding maximal torus $T := B_+ \cap B_-$. Let $U_\pm$ be the unipotent radicals of $B_\pm$. Set

$$\mathfrak{h} := \mathrm{Lie}\, T, \quad \mathfrak{b}_\pm := \mathrm{Lie}\, B_\pm, \quad \text{and} \quad \mathfrak{u}_\pm := \mathrm{Lie}\, U_\pm.$$

Denote the Weyl group of G by W. For each $w \in W$, fix a representative $\dot{w}$ in the normalizer of T in G.

5.1. The simplest applications of Theorems 3.4 and 4.1 are to weak splittings for Poisson structures on flag varieties.

Recall the standard Manin triple

$$\mathfrak{d} := \mathfrak{g} \oplus \mathfrak{h}, \ \mathfrak{g}_\pm := \{(x_\pm + h, \pm h) \mid x_\pm \in \mathfrak{u}_\pm, h \in \mathfrak{h}\}$$

with respect to the invariant bilinear form on $\mathfrak{d}$ given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle := \langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle, \quad \forall x_i \in \mathfrak{g}, y_i \in \mathfrak{h}.$$

The Drinfeld and Heisenberg double Poisson structures on $D := G \times T$ will be denoted by π and π', respectively. In this case $N_\pm = N(\mathfrak{g}_\pm) = B_\pm \times T$ and $H = N_+ \cap N_- = T \times T$. The reductions of $-\pi$ and π' to $G/T \cong (G \times T)/(T \times T) = D/H$ will be denoted by $-\pi_T$ and π'_T. Both structures reduce to the same Poisson structure on the flag variety $G/B_+ \cong (G \times T)/(B_+ \times T) = D/N_+$ called the standard Poisson structure. The latter will be denoted by π_{G/B_+}.

It is easy to verify that the group $Q = N_- = B_- \times T$ and the set $\mathcal{D} = \{(\dot{w}, 1) \mid w \in W\}$ satisfy the conditions in Theorem 3.5 for the above choice of D and $\mathfrak{g}_\pm$. This implies the following result of Goodearl and the author proved in [16, Theorem 3.2]. For its statement we need to introduce some additional notation. Denote the vector bundle

$$\widetilde{E}_w \to \big((B_- \cap wB_-w^{-1}) \times T\big)(\dot{w}, 1)$$

with fibers

$$\widetilde{E}^w_g := R_g\big(p_+ \mathrm{Ad}_{(\dot{w},1)}(\mathfrak{u}_- \oplus 0)\big) + L_g(\mathfrak{b}_+ \oplus \mathfrak{h})$$

for $g \in \big((B_- \cap wB_-w^{-1}) \times T\big)(\dot{w}, 1)$ where $p_+ \colon \mathfrak{d} \to \mathfrak{g}_+$ is the projection along $\mathfrak{g}_-$. Here the direct sum notation is used to denote subspaces of $\mathfrak{g} \oplus \mathfrak{h}$ identified with $\mathrm{Lie}\,(G \times T)$. Denote the canonical projection

$$\nu \colon G \times T \to (G \times T)/(T \times T) \cong G/T. \tag{5.1}$$

By Corollary 3.3 the pushforward $E^w := \nu_*(\widetilde{E}^w)$ is a well defined vector bundle over $B_- \cdot wT \subset G/T$.

Theorem 5.1. [16] *For all connected complex simple Lie groups G, the partition of the full flag variety G/B_+ into Schubert cells*

$$G/B_+ = \sqcup_{w \in W} B_- \cdot wB_+$$

and the family of maps

$$i_w \colon B_- \cdot wB_+ \to G/T, \;\; i_w(b_- \dot{w} B_+) := b_- \dot{w} T, \;\; \forall b_- \in B_- \cap wB_-w^{-1}$$

define a weak splitting of the surjective Poisson submersion

$$(G/T, -\pi_T) \to (G/B_+, \pi_{G/B_+})$$

from a Drinfeld double to the flag variety. Furthermore, the image of each map i_w is a Poisson–Dirac submanifold of $(G/T, -\pi_T)$ admitting a Dirac projection with associated bundle E^w defined above.

Theorem 4.2 for weak splittings of quotients of Heisenberg doubles is also applicable to flag varieties to obtain a weak splitting for the surjective Poisson submersion

$$(G/T, -\pi'_T) \to (G/B_+, \pi_{G/B_+}).$$

It is easy to verify that the conditions of Theorem 4.2 are satisfied by for the same group $Q = N_- = B_- \times T$, set $\mathcal{D} = \{(\dot{w}, 1) \mid w \in W\}$ and the current choice of D and $\mathfrak{g}_\pm$. Applying the theorem leads to the following result:

Theorem 5.2. *For all connected complex simple Lie groups G, the partition of the full flag variety G/B_+ into Schubert cells*

$$G/B_+ = \sqcup_{w \in W} B_+ \cdot wB_+$$

and the family of maps

$$i'_w \colon B_+ \cdot wB_+ \to G/T, \;\; i'_w(b_+ wB_+) = b_+ \dot{w} T, \;\; b_+ \in B_+ \cap wB_-w^{-1}$$

is a weak splitting of the surjective Poisson submersion

$$(G/T, \pi'_T) \to (G/B_+, \pi_{G/B_+})$$

from a Heisenberg double to the flag variety. The image of each map i'_w *is a Poisson–Dirac submanifold of* $(G/T, \pi'_T)$ *admitting a Dirac projection with associated bundle* $\nu_*(\widetilde{F}^w)$ *for the pushforward bundle* $\nu_*(\widetilde{F}^w)$ *with respect to* (5.1) *where*

$$\widetilde{F}^w \to \big((B_+ \cap wB_-w^{-1}) \times T\big)(\dot{w}, 1)$$

is the vector bundle with fibers

$$\widetilde{F}^w_g := R_g\big(p_- \mathrm{Ad}_{(\dot{w},1)}(\mathfrak{u}_- \oplus 0)\big) + L_g(\mathfrak{b}_+ \oplus \mathfrak{h})$$

for $g \in \big((B_+ \cap wB_-w^{-1}) \times T\big)(\dot{w}, 1)$ *and* $p_- \colon \mathfrak{d} \to \mathfrak{g}_-$ *is the projection along* $\mathfrak{g}_+$*.*

Because of of Proposition 2.6, Theorems 5.1 and 5.2 provide an explicit realization of the symplectic leaves of the flag varieties $(G/B_+, \pi_{G/B_+})$ as symplectic submanifolds of the symplectic leaves of Drinfeld and Heisenberg doubles.

We note that the partitions into Schubert cells in Theorems 5.1 and 5.2 are with respect to opposite Borel subgroups. The two results can be derived from each other. Let $w_\circ$ be the longest element of W. The equivalence is shown using the facts that the translation action of $\dot{w}_\circ$ on $(G/B_+, \pi_{G/B_+})$ is anti-Poisson, and the left translation action of $(\dot{w}_\circ, 1)$ on $G \times T$ interchanges π and π'.

5.2. Next, we consider the standard Manin triple

$$\mathfrak{d} := \mathfrak{g} \oplus \mathfrak{g}, \;\; \mathfrak{g}_+ := \{(x_+ + h, x_- - h) \mid x_\pm \in \mathfrak{u}_\pm, h \in \mathfrak{h}\}, \quad \mathfrak{g}_- := \{(x, x) \mid x \in \mathfrak{g}\} \tag{5.2}$$

with respect to the invariant bilinear form on $\mathfrak{d}$

$$\langle (x_1, y_1), (x_2, y_2) \rangle := \langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle, \;\; x_i, y_i \in \mathfrak{g}.$$

Let $D := G \times G$. For this setting we have

$$N_+ = N(\mathfrak{g}_+) = B_+ \times B_-, \;\; N_-^\circ = N(\mathfrak{g}_-)^\circ = G_\Delta, \;\text{ and }\; N_+ \cap N_-^\circ = T_\Delta$$

where G_Δ and T_Δ are the diagonal subgroups of $G \times G$ and $T \times T$, respectively.

The Drinfeld and Heisenberg double Poisson structures on $G \times G$ will be again denoted by π and π'. The group T_Δ acts on the left and right on $(G \times G, \pi)$ and $(G \times G, \pi')$ by Poisson automorphisms. The reductions of π and π' to $(G \times G)/T_\Delta$ will be denoted by π_{T_Δ} and π'_{T_Δ}. The pushforwards of $-\pi_{T_\Delta}$ and π'_{T_Δ} to $(G \times G)/N_+ \cong G/B_+ \times G/B_-$ are well defined and are equal to each other. Denote the resulting Poisson structure by π_{df}. The Poisson manifold

$$(G/B_+ \times G/B_-, \pi_{df})$$

is the double flag variety studied in [27]. Theorem 3.5 cannot be applied to the Poisson submersion $((G \times G)/T_\Delta, -\pi_{T_\Delta}) \to (G/B_+ \times G/B_-, \pi_{df})$, but Theorem 4.2 can be applied for the following choice of a group Q and a set $\mathcal{D}$:

$$Q = (U_- \times U_+)T_\Delta, \quad \mathcal{D} = \{(\dot{w}, \dot{v}) \mid w, v \in W\},$$

and the above D and $\mathfrak{g}_\pm$. This gives a weak splitting of the surjective Poison submersion $((G\times G)/T_\Delta, \pi'_{T_\Delta}) \to (G/B_+ \times G/B_-, \pi_{df})$. We will need the following notation to state the result. Denote the projection

$$\nu' \colon G\times G \to (G\times G)/T_\Delta. \tag{5.3}$$

Consider the vector bundle

$$\widetilde{F}^{w,v} \to \big((U_+ \cap wU_-w^{-1}) \times (U_- \cap vU_+v^{-1})\big)(\dot{w},\dot{v})T_\Delta$$

with fibers

$$\widetilde{F}^{w,v}_g := R_g(p_- \mathrm{Ad}_{(\dot{w},\dot{v})}(\mathfrak{u}_- \oplus \mathfrak{u}_+ + \mathfrak{t}_{a\Delta})) + L_g(\mathfrak{b}_+ \oplus \mathfrak{b}_-) \tag{5.4}$$

for $g \in \big((U_+ \cap wU_-w^{-1}) \times (U_- \cap vU_+v^{-1})\big)(\dot{w},\dot{v})T_\Delta$ where the direct sum notation is used for subspaces of $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}$ identified with $\mathrm{Lie}\,(G\times G)$, $p_- \colon \mathfrak{d} \to \mathfrak{g}_-$ denotes the projection along $\mathfrak{g}_+$, and $\mathfrak{t}_{a\Delta}$ is the antidiagonal of $\mathfrak{t}\oplus\mathfrak{t}$.

Theorem 5.3. *For all connected complex simple Lie groups G, the partition of the double flag variety into $B_+\times B_-$-Schubert cells*

$$G/B_+ \times G/B_- = \sqcup_{w,v\in W} B_+\cdot wB_+ \times B_- \cdot vB_-$$

and the family of maps

$$i'_{w,v} \colon B_+\cdot wB_+ \times B_-\cdot vB_- \to (G\times G)/T_\Delta$$

given by

$$i'_{w,v}(u_+wB_+, u_-vB_-) := (u_+\dot{w}, u_-\dot{v})T_\Delta$$

for all $u_+ \in U_+ \cap wU_-w^{-1}$ and $u_- \in U_- \cap vU_+v^{-1}$ is a weak splitting of the surjective Poisson submersion

$$((G\times G)/T_\Delta, \pi'_\Delta) \to (G/B_+ \times G/B_-, \pi_{df})$$

from a Heisenberg double to the double flag variety. The image of each map $i'_{w,v}$ is a Poisson–Dirac submanifold of $((G\times G)/T_\Delta, \pi'_\Delta)$ admitting a Dirac projection with associated vector bundle $\nu'_*(\widetilde{F}^{w,v})$, cf. (5.3) *and* (5.4).

5.3. All results in Sect. 5 on weak splittings for full flag varieties have analogs to partial flag varieties. We will provide full details in the case of double partial flag varieties. The generalizations of the results in §5.1, 5.4, and 5.5 are analogous.

For a subset I of simple roots of $\mathfrak{g}$ denote by $P^I_\pm \supseteq B_\pm$ the corresponding parabolic subgroups of G and by W_I the subgroup of the Weyl group. Let W^I be the sets of minimal length representatives for the cosets in W/W_I.

Fix two subsets I_1, I_2 of simple roots of $\mathfrak{g}$. The pushforward of π_{df} under the canonical projection

$$G/B_+ \times G/B_- \to G/P^{I_1}_+ \times G/P^{I_2}_-$$

is a well defined Poisson structure since $P^{I_1}_+ \times P^{I_2}_-$ is a Poisson–Lie subgroup of $(G\times G, \pi)$. Denote this pushforward by π^{I_1,I_2}_{df}. The map

$$(G/B_+ \times G/B_-, \pi_{df}) \to (G/P^{I_1}_+ \times G/P^{I_2}_-, \pi^{I_1,I_2}_{df})$$

is a surjective Poisson submersion and its restrictions

$$(B_+\cdot wB_+ \times B_-\cdot vB_-, \pi_{df}) \to (B_+\cdot wP^{I_1}_+ \times B_-\cdot vP^{I_2}_-, \pi^{I_1,I_2}_{df})$$

are Poisson isomorphisms for all $w \in W^{I_1}$, $v \in W^{I_2}$. (Similar Poisson isomorphisms are constructed in the settings of §5.1, 5.4, and 5.5. This produces the generalizations of those results to the cases of partial flag varieties.) Taking inverses of the above Poisson isomorphisms and composing them with the maps $i'_{w,v}$ in Theorem 5.3 leads to the following result:

Corollary 5.4. *For all connected complex simple Lie groups G and subsets of simple roots I_1 and I_2, the partition of the corresponding double partial flag variety into $B_+ \times B_-$-Schubert cells*

$$G/P_+^{I_1} \times G/P_-^{I_2} = \sqcup_{w \in W^{I_1}, v \in W^{I_2}} B_+ \cdot wP_+^{I_1} \times B_- \cdot vP_-^{I_2}$$

and the family of maps

$$j'_{w,v} \colon B_+ \cdot wP_+^{I_1} \times B_- \cdot vP_-^{I_2} \to (G \times G)/T_\Delta$$

given by

$$j'_{w,v}(u_+ wP_+^{I_1}, u_- vP_-^{I_2}) := (u_+ \dot{w}, u_- \dot{v}) T_\Delta$$

for all $u_+ \in U_+ \cap wU_- w^{-1}$ and $u_- \in U_- \cap vU_+ v^{-1}$ is a weak splitting of the surjective Poisson submersion

$$((G \times G)/T_\Delta, \pi'_\Delta) \to (G/P_+^{I_1} \times G/P_-^{I_2}, \pi_{df}^{I_1, I_2})$$

from a Heisenberg double to the double partial flag variety. The image of each map $j'_{w,v}$ (which is the same as the image of the map $i'_{w,v}$) is a Poisson–Dirac submanifold of $((G \times G)/T_\Delta, \pi'_\Delta)$ admitting a Dirac projection with associated vector bundle $\nu'_*(\widetilde{F}^{w,v})$, see (5.3) *and* (5.4)*.*

5.4. The results in §5.1–5.2 can be generalized to a very large class of Poisson structures on multiple flag varieties. Those are Cartesian products of flag varieties for complex simple Lie groups (i.e., flag varieties for reductive Lie groups) with Poisson structures which in general are not products of Poisson structures on the factors. Since the arguments are similar, we only state the results leaving the details to the reader.

We start with any reductive Lie algebra $\mathfrak{d}$ and an invariant bilinear form $\langle ., . \rangle$ on it. All Lagrangian subalgebras and Manin triples in this situation were classified by Delorme [8] up to the action of the adjoint group of $\mathfrak{d}$. Let $\overline{\mathfrak{b}}_\pm$ be a pair of opposite Borel subalgebras of $\mathfrak{d}$ and $\overline{\mathfrak{t}} := \overline{\mathfrak{b}}_+ \cap \overline{\mathfrak{b}}_-$ be the corresponding Cartan subalgebra of $\mathfrak{d}$. Denote by $\overline{\mathfrak{u}}_\pm$ the nilradicals of $\overline{\mathfrak{b}}_\pm$. Let D be a connected reductive Lie group with Lie algebra $\mathfrak{d}$, and $\overline{B}_\pm$ and $\overline{T}$ be its Borel subgroups and maximal torus corresponding to $\overline{\mathfrak{b}}_\pm$ and $\overline{\mathfrak{t}}$.

Consider any Manin triple

$$(\mathfrak{d}, \mathfrak{g}_+, \mathfrak{g}_-) \quad \text{such that} \quad \mathfrak{g}_+ \subset \overline{\mathfrak{b}}_+. \tag{5.5}$$

The results of Delorme imply that, after a conjugation by an element of $\overline{B}_+$, one has

$$\mathfrak{g}_+ = \overline{\mathfrak{u}}_+ + \mathfrak{g}_+ \cap \overline{\mathfrak{t}}$$

(in particular, $N(\mathfrak{g}_+) = \overline{B}_+$) and the group $H := N(\mathfrak{g}_+) \cap N(\mathfrak{g}_-)^\circ$ satisfies

$$H = \overline{B}_+ \cap N(\mathfrak{g}_-)^\circ = \overline{T} \cap N(\mathfrak{g}_-)^\circ. \tag{5.6}$$

One can write an explicit formula for the subgroup H of the maximal torus $\overline{T}$ in terms of generalized Belavin–Drinfeld triples in the setting of [8]. We leave the details to the reader since this requires extra notation. In the rest we will assume that the conjugation by an element of $\overline{B}_+$ is performed so that the above conditions are satisfied.

Denote the Drinfeld and Heisenberg double Poisson structures on D corresponding to a Manin triple of the type (5.5) by π and π'. By the general facts in Sect. 3, the left and right regular actions of H on (G, π') are Poisson. Denote the reduction $(D/H, \pi'_H)$ for the right action and the surjective Poisson submersion

$$\nu' \colon (D, \pi') \to (D/H, \pi'_H). \tag{5.7}$$

The Drinfeld and Heisenberg Poisson structures π and π' descend to the same Poisson structure on the multiple flag variety $D/N(\mathfrak{g}_+) = D/\overline{B}_+$ which will be denoted by $\pi_{D/\overline{B}_+}$. The Poisson structures in §5.1–5.2 are special cases of this construction when

$$D = G \times T \quad \text{or} \quad D = G \times G$$

for a complex simple Lie group G and a maximal torus T of G.

The canonical projection

$$(D/H, \pi'_H) \to (D/\overline{B}_+, \pi_{D/\overline{B}_+})$$

is Poisson, because $\overline{B}_+ = N(\mathfrak{g}_+)$ is a Poisson–Lie subgroup of (D, π). Denote the connected subgroups of D with Lie algebras $\overline{\mathfrak{u}}_\pm$ by $\overline{U}_\pm$. Let $\overline{W}$ be the Weyl group of D. For each of $w \in \overline{W}$, fix a representative $\dot{w}$ in the normalizer of the maximal torus $\overline{T}$ of D.

A simple computation shows that the conditions of Theorem 4.2 are satisfied for the group $Q = H\overline{U}_-$ and the set $\mathcal{D} = \{\dot{w} \mid w \in \overline{W}\}$. We have:

Theorem 5.5. *For all Manin triples for a connected reductive algebraic group D of the form* (5.5)*, the partition of the multiple flag variety $D/\overline{B}_+$ into Schubert cells*

$$D/\overline{B}_+ = \sqcup_{w \in \overline{W}} \overline{B}_+ \cdot w \overline{B}_+$$

and the family of maps

$$i'_w \colon \overline{B}_+ \cdot w\overline{B}_+ \to D/H, \ \ i'_w(u_+ w \overline{B}_+) := u_+ \dot{w} H, \ \ \forall u_+ \in \overline{U}_+ \cap w \overline{U}_- w^{-1}$$

(recall (5.6)*) is a weak splitting of the surjective Poisson submersion*

$$(D/H, \pi'_H) \to (D/\overline{B}_+, \pi_{D/\overline{B}_+})$$

from a Heisenberg double to the multiple flag variety. The image of each map i'_w is a Poisson–Dirac submanifold of $(D/H, \pi'_H)$ admitting a Dirac projection with associated bundle $\nu'_*(\widetilde{F}^w)$ for the pushforward with respect to (5.7) *of the vector bundle*

$$\widetilde{F}^w \to (\overline{U}_+ \cap w \overline{U}_- w^{-1}) \dot{w} H$$

with fibers

$$\widetilde{F}^w_g := R_g\big(p_- \mathrm{Ad}_{\dot{w}}(\overline{\mathfrak{u}}_- + \mathfrak{h}^\perp)\big) + L_g(\overline{\mathfrak{b}}_+)$$

for $g \in (\overline{U}_+ \cap w \overline{U}_- w^{-1}) \dot{w} H$. Here $\mathfrak{h}^\perp$ denotes the orthogonal complement to $\mathfrak{h} := \mathrm{Lie}\, H$ in $\overline{\mathfrak{t}}$ with respect to $\langle ., . \rangle$, and $p_- \colon \mathfrak{d} \to \mathfrak{g}_-$ is the projection along $\mathfrak{g}_+$.

5.5. All results in §5.1–5.4 remain valid when all complex groups are replaced with their real split forms. These provide many examples of weak splittings of surjective Poisson submersions to real flag varieties.

We continue with certain non-split analogs of the results in §5.1 and 5.4 which concern the real Poisson structures on complex flag varieties introduced by Foth and Lu in [12]. Let $\mathfrak{g}$ be a complex simple Lie algebra and G a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Consider $\mathfrak{g}$ as a real quadratic Lie algebra with the nondegenerate bilinear form

$$x, y \in \mathfrak{g} \mapsto \operatorname{Im}\langle x, y\rangle \in \mathbb{R}. \tag{5.8}$$

Each Vogan diagram v for $\mathfrak{g}$ gives rise to a complex conjugate linear involution τ_v of $\mathfrak{g}$ and to the real form $\mathfrak{g}_v := \mathfrak{g}^{\tau_v}$ of $\mathfrak{g}$, see [18, 12] for details. The map τ_v is defined using a choice of root vectors of $\mathfrak{g}$. We will denote by $\mathfrak{t}$ the corresponding Cartan subalgebra of $\mathfrak{g}$ (which is stable under τ_v) and by $\mathfrak{u}_\pm$ the nilradicals of the corresponding Borel subalgebras of $\mathfrak{g}$. The following is a (real) Manin triple:

$$(\mathfrak{d} := \mathfrak{g}, \mathfrak{g}_+ := \mathfrak{t}^{-\tau_v} + \mathfrak{u}_+, \mathfrak{g}_- := \mathfrak{g}_v),$$

see [12, Sect. 2]. Let $D := G$ considered as a real Lie group. We have

$$N_+ = N(\mathfrak{g}_+) = B_+ \quad \text{and} \quad N_-^\circ = N(\mathfrak{g}_-) = G_v$$

where G_v is the connected subgroup of G with Lie algebra $\mathfrak{g}_v$ and B_+ is the Borel subgroup of G with Lie algebra $\mathfrak{t} + \mathfrak{u}_+$. The group $H := N_+ \cap N_-^\circ$ is the connected subgroup of T with Lie algebra

$$\operatorname{Lie} H = \mathfrak{t}^{\tau_v}. \tag{5.9}$$

Denote once again the associated Drinfeld and Heisenberg double Poisson structures on G by π_v and π'_v. We have $\pi|_H = 0$, and thus H acts by Poisson automorphisms on (G, π') on the left and right. Denote the reduced Poisson structure on G/H by $\pi'_{H,v}$ and the Poisson projection

$$\nu' \colon (G, \pi'_v) \to (G/H, \pi'_{H,v}). \tag{5.10}$$

The pushforwards of $-\pi_v$ and π'_v under the canonical projection $G \to G/B_+$ are well defined and are equal to each other, because $B_+ = N(\mathfrak{g}_+)$. Denote the corresponding real Poisson structure on the complex flag variety G/B_+ by $\pi_{G/B_+,v}$. It has very interesting properties, for example the intersections of the orbits of the Borel subgroup B_+ and the real form G_v are regular complete Poisson submanifolds. Theorem 4.2 is applicable to construct a weak splitting of the real surjective Poisson submersion

$$(G/H, \pi'_{H,v}) \to (G/B_+, \pi_{G/B_+,v}).$$

Once again it is easy to verify that the group $Q = HU_-$ and the set $\mathcal{D} = \{\dot{w} \mid w \in W\}$ satisfy the conditions of Theorem 4.2. This leads to the following result:

Theorem 5.6. *For all connected, simply connected complex simple groups G and Vogan diagrams v, the partition of the flag variety G/B_+ into Schubert cells*

$$G/B_+ = \sqcup_{w \in W} B_+ \cdot wB_+$$

and the family of maps

$$i'_w \colon B_+ \cdot wB_+ \to G/H, \;\; i'_w(u_+ wB_+) = u_+ \dot{w} H, \;\; u_+ \in U_+ \cap wU_- w^{-1}$$

is a weak splitting of the real surjective Poisson submersion

$$(G/H, \pi'_{H,v}) \to (G/B_+, \pi_{G/B_+,v})$$

from a Heisenberg double to the flag variety, where the group H is given by (5.9)*. The image of each map i'_w is a Poisson–Dirac submanifold of $(G/H, \pi'_H)$ admitting a Dirac projection with associated bundle $\nu'_*(\widetilde{F}^w)$ for the pushforward with respect to* (5.10) *of the vector bundle*

$$\widetilde{F}^w \to (U_+ \cap wU_-w^{-1})\dot{w}H$$

with fibers

$$\widetilde{F}^w_g := R_g\big(p_- \mathrm{Ad}_{\dot{w}}(\mathfrak{t}^{\tau_v} + \mathfrak{u}_-)\big) + L_g(\mathfrak{b}_+)$$

for $g \in (U_+ \cap wU_-w^{-1})\dot{w}H$. Here $p_- \colon \mathfrak{g} \to \mathfrak{g}_-$ denotes the projection along $\mathfrak{g}_+$.

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