COMINUSCULE PARABOLICS OF SIMPLE FINITE DIMENSIONAL LIE SUPERALGEBRAS

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Abstract. We give an explicit classification of the cominuscule parabolic subalgebras of all complex simple finite dimensional Lie superalgebras.

Cominuscule parabolic subalgebras of simple finite dimensional Lie algebras play an important role in representation theory, geometry, and combinatorics. In the complex case the corresponding flag varieties are Hermitian symmetric spaces of compact type [H]. There has been a lot of research on properties of Schubert varieties in those spaces, see Chapter 9 of [BL] for a comprehensive survey. More recently it was proved that many results for Schubert calculus on Grassmannians extend to those [PSo, TY] and that the totally nonnegative part of cominuscule flag varieties can be described much more explicitly than the general case in terms of Le diagrams [P, LW]. The standard Poisson structure on cominuscule flag varieties also has special properties: the (finitely many) orbits of the standard Levi subgroup are complete Poisson submanifolds [BG, GY] and each of them is a quotient of the standard Poisson structure of the Levi subgroup. Finally, one should note that cominuscule parabolics have special properties in numerous other respects.

Parabolic subalgebras of Lie superalgebras are much less well understood. In this paper we address the question of classification of all cominuscule parabolic subalgebras of the complex simple finite dimensional Lie superalgebras. One of the many equivalent definitions of a cominuscule parabolic subalgebra of a simple finite dimensional Lie algebra is that it is a parabolic subalgebra with abelian nilradical. Already the case of $\mathfrak{sl}(m|n)$ presents an intrinsic example of parabolic subalgebras, which should be considered cominuscule in the super sense. The standard $\mathbb{Z}$-grading of $\mathfrak{sl}(m|n)$ gives rise to two maximal parabolic subalgebras whose (pure odd) "nilradicals" are abelian.

In order to define cominuscule parabolics in the super case, we first reexamine the definition of a parabolic subalgebra and nilradical. Unlike the classical (even)

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case there is no uniform definition of either of the two notions in terms of Borel subalgebras and maximal nilpotent ideals, respectively. For the first one we work with root subalgebras. Let $\mathfrak{g}$ be a complex simple finite dimensional Lie superalgebra and $\mathfrak{h}$ be a fixed Cartan subalgebra. Denote by Δ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For $\alpha \in \Delta$ let $\mathfrak{g}^\alpha$ be the corresponding root space. We call a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ a *root subalgebra* if it has the form

$$\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{h}) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha \right) \tag{0.1}$$

for some subset $\Phi \subseteq \Delta$. If Δ is symmetric (i.e. $\Delta = -\Delta$), then we call a proper subset P of Δ a *parabolic set of roots* if

$$\Delta = P \cup (-P) \quad \text{and} \quad \alpha, \beta \in P \quad \text{with} \quad \alpha + \beta \in \Delta \quad \text{implies} \quad \alpha + \beta \in P.$$

If $\Delta \neq -\Delta$, then $P \subsetneq \Delta$ will be called parabolic if $P = \widetilde{P} \cap \Delta$ for some parabolic subset $\widetilde{P}$ of $\Delta \cup (-\Delta)$. We will call a subalgebra of $\mathfrak{g}$ *parabolic* if it is a root subalgebra as in (0.1) for a parabolic subset of roots $\Phi \subsetneq \Delta$ and contains $\mathfrak{h}$. In other words, given a parabolic subset of roots P, then the corresponding parabolic subalgebra of $\mathfrak{g}$ is

$$\mathfrak{p}_P := \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in P} \mathfrak{g}^\alpha \right).$$

For a symmetric root systems Δ and a parabolic set of roots $P \subsetneq \Delta$, we call $L := P \cap (-P)$ the *Levi component* of P, $N^+ := P \backslash (-P)$ the *nilradical* of P, and $P = L \sqcup N^+$ the Levi decomposition of P. In the nonsymmetric case ($\Delta \neq -\Delta$) one cannot use the same formulas, since $N^+ = P \backslash (-P)$ is generally not closed under addition (i.e. $\alpha, \beta \in N^+$, $\alpha + \beta \in \Delta$ does not imply $\alpha + \beta \in N^+$). If $\Delta \neq -\Delta$, then we choose a parabolic subset $\widetilde{P}$ of $\Delta \cap (-\Delta)$ such that $P = \widetilde{P} \cap \Delta$, and define

$$\widetilde{L} = \widetilde{P} \cap (-\widetilde{P}) \quad \text{and} \quad \widetilde{N}^+ = \widetilde{P} \backslash (-\widetilde{P}).$$

We call $L := \widetilde{L} \cap P$ a Levi component of P, $N^+ = \widetilde{N}^+ \cap P$ a nilradical of P, and $P = L \sqcup N^+$ a Levi decomposition of P. We note that in the nonsymmetric case the definition of a Levi component and nilradical of P essentially depends on the choice of a parabolic subset $\widetilde{P}$ of $\Delta \cap (-\Delta)$. We refer the reader to Remarks 1.7 and 3.3 for details.

Let P be a parabolic subset of roots of $\mathfrak{g}$. For a Levi decomposition $P = L \sqcup N^+$ of P we define the subalgebras

$$\mathfrak{l} := \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in L} \mathfrak{g}^\alpha \right) \quad \text{and} \quad \mathfrak{n}^+ := \bigoplus_{\alpha \in N^+} \mathfrak{g}^\alpha$$

of $\mathfrak{p}_P$, and call them a *Levi subalgebra* and *nilradical* of $\mathfrak{p}_P$. This gives rise to the semidirect sum decomposition $\mathfrak{g}_P = \mathfrak{l} \ltimes \mathfrak{n}^+$, which will be called a *Levi decomposition* of $\mathfrak{p}_P$. (Here and below the symbol $\ltimes$ will stand for semi-direct sums of Lie superalgebras.)

We call a parabolic subalgebra $\mathfrak{p}_P$ of $\mathfrak{g}$ *cominuscule* if it has a nilradical $\mathfrak{n}^+$, which is abelian. In this paper we investigate the parabolic subalgebras of all complex simple finite dimensional Lie superalgebras $\mathfrak{g}$. On a case by case basis we classify all of their cominuscule parabolics. One remarkable consequence of our classification is:

Theorem 0.2. *Each cominuscule parabolic subalgebra of a complex simple finite dimensional Lie superalgebra has a unique Levi decomposition.*

For each cominuscule parabolic subalgebra we describe explicitly its Levi subalgebra $\mathfrak{l}$ and the structure of its nilradical $\mathfrak{n}^+$ considered as an $\mathfrak{l}$-module.

We call a parabolic set of roots of $\mathfrak{g}$ cominuscule if $\mathfrak{p}_P$ is a cominuscule parabolic subalgebra of $\mathfrak{g}$. In the classical even case, one approach to comuniscule parabolic subalgebras is through the properties of their root systems. Our treatment is based on the following super version of this approach, which we prove in Proposition 1.16:

If $\mathfrak{g}$ is a simple finite dimensional Lie superalgebra and $\mathfrak{g} \neq S(n), \mathfrak{g} \neq S'(n)$, $\mathfrak{g} \neq \mathfrak{psl}(3|3)$, then a parabolic subset P of Δ is cominuscule if and only if it has a nilradical N^+ such that for every α, β in N^+, $\alpha + \beta \notin \Delta$.

This is deduced from the fact that for all $\mathfrak{g} \neq S(n), \mathfrak{g} \neq S'(n)$, $\mathfrak{g} \neq \mathfrak{psl}(3|3)$:

$$\text{if } \alpha, \beta, \alpha+\beta \in \Delta, \text{ then } [\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \neq 0.$$

(In the super case the root spaces $\mathfrak{g}^\alpha$ can have dimension more than 1 and generally $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \neq \mathfrak{g}^{\alpha+\beta}$.) We classify the cominuscule parabolic subsets P of Δ using the above result and combinatorial arguments for root systems. We prove that a version of the above result is still valid for $\mathfrak{g} = \mathfrak{psl}(3|3)$ and $\mathfrak{g} = S(n), S'(n)$. In this case one has to identify the root systems of $\mathfrak{psl}(3|3)$ and $S(n), S'(n)$ with subsets of the root systems of $\mathfrak{sl}(3|3)$ and $W(n)$, respectively, and compute the sums of roots $\alpha+\beta$ in the root systems of the latter family of Lie superalgebras (see §2.2 and §4.2 for details).

For each simple finite dimensional Lie superalgebra $\mathfrak{g}$ with root system Δ there is a canonical Weyl group that acts on the set of its parabolic subsets of roots and permutes the cominuscule parabolic subsets of Δ. We classify the cominuscule parabolic subsets of roots of $\mathfrak{g}$ up to the action of this Weyl group. For the different types of simple Lie superalgebras the Weyl group is described as follows. If $\mathfrak{g}$ is a classical Lie superalgebra, then $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Its Weyl group $W_{\mathfrak{g}_{\bar{0}}}$ acts in a canonical way on the set of the parabolic subsets of Δ and permutes the cominuscule parabolic subsets of Δ. We classify the cominuscule parabolics of the *basic classical Lie superalgebras* and the *strange classical Lie superalgebras* up to the action of this group in Sections 2 and 3, respectively. The second family of Lie superalgebras contains nonsymmetric root systems Δ. Those are the first ones for which we establish the validity of Theorem 0.2. Finally the classification of cominuscule parabolics of the *Cartan type* Lie superalgebras is carried out in Section 4. In this case $\mathfrak{g}_{\bar{0}}$ has a canonical Levi subalgebra $\mathfrak{l}_{\bar{0}}$. Its Weyl group $W_{\mathfrak{l}_{\bar{0}}}$ acts in a natural way on the set of parabolic subsets of Δ and permutes the cominuscule parabolic subsets of Δ. The latter are classified up to the action of the Weyl group $W_{\mathfrak{l}_{\bar{0}}}$.

To keep the size of the paper down, we will not summarize the results of the classification theorems of Sections 2, 3, and 4. The subsections of those sections are labeled by the corresponding simple Lie superalgebras, so the reader can easily search those results. Another interesting corollary of our classification is that *all cominuscule parabolic sets of roots for simple finite dimensional Lie superalgebras are principal*, i.e. they come from *triangular decompositions* of the root systems (see §1.2 for definitions).

We expect that the class of cominuscule parabolics of simple Lie superalgebras will play an important role, similar to the ones in the even case. In particular, we expect that parabolic induction from such will behave well and that the cominuscule super flag varieties will have many special properties distinguishing them from general super flag varieties.

We finish the introduction with several notational conventions, which will be used throughout the paper. We will denote the standard representations of $\mathfrak{gl}(p)$ and $\mathfrak{gl}(p|q)$ by V^p and $V^{p|q}$, respectively. The same notation will be used for the restrictions of these representations to the subalgebras of $\mathfrak{gl}(p)$ and $\mathfrak{gl}(p|q)$. We will use S^k and $\bigwedge^k$ to denote the kth (super)symmetric power and (super)exterior power, respectively. For a module M, M^* will stand for its dual module. We will follow [P] in our notation for Lie superalgebras, except that we will denote by $S'(n)$ the Lie superalgebras series denoted by $\widetilde{S}(n)$ in [P]. For a Lie superalgebra $\mathfrak{a}$, $\mathfrak{a}'$ will stand for its derived subalgebra $[\mathfrak{a}, \mathfrak{a}]$. Set-theoretic unions will be denoted by $\cup$ and disjoint unions will be denoted by $\sqcup$.

1. Parabolic sets of roots and parabolic subalgebras

This section contains some general facts about parabolic sets of roots and parabolic subalgebras of simple finite dimensional Lie superalgebras $\mathfrak{g}$. We define Levi components and nilradicals of parabolic sets of roots, and use those to define Levi subalgebras and nilradicals of parabolic subalgebras of $\mathfrak{g}$. In the special case of principal parabolic sets of roots those recover the triangular decompositions of $\mathfrak{g}$. We define cominuscule parabolic subalgebras and establish a relationship to the properties of the related parabolic sets of roots.

1.1. **Levi decompositions of parabolic sets of roots.** In what follows, unless otherwise stated $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ will denote a simple finite dimensional Lie superalgebra over $\mathbb{C}$ (see [K] and [Sch] for details). A *Cartan subalgebra* $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ of $\mathfrak{g}$, is by definition a selfnormalizing nilpotent subalgebra. Then $\mathfrak{h}_{\bar{0}}$ is a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, and $\mathfrak{h}_{\bar{1}}$ is the maximal subspace of $\mathfrak{g}_{\bar{1}}$ on which $\mathfrak{h}_{\bar{0}}$ acts nilpotently (see [PS, Proposition 1] for a proof). We denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the *roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$. Thus $\Delta = \{\alpha \in \mathfrak{h}_{\bar{0}}^*,\ \alpha \neq 0 \mid \mathfrak{g}^\alpha \neq 0\}$. For $\bar{\imath} \in \mathbb{Z}/2\mathbb{Z}$ we set $\Delta_{\bar{\imath}} = \{\alpha \in \Delta \mid \mathfrak{g}_{\bar{\imath}}^\alpha \neq 0\}$.

Definition 1.1. *Let Δ be the root system of a simple finite dimensional Lie superalgebra. If $\Delta = -\Delta$, we will call a proper subset P of Δ a* parabolic set of roots *if*

$$\Delta = P \cup (-P) \quad \textit{and} \tag{1.2}$$

$$\alpha, \beta \in P \quad \textit{with} \quad \alpha + \beta \in \Delta \quad \textit{implies} \quad \alpha + \beta \in P. \tag{1.3}$$

If $\Delta \neq -\Delta$, $P \subsetneq \Delta$ will be called a parabolic subset if $P = \widetilde{P} \cap \Delta$ for some parabolic subset $\widetilde{P}$ of $\Delta \cup (-\Delta)$.

Next we define a Levi decomposition of a parabolic set of roots.

Definition 1.4. *Let P be a parabolic set of roots of the root system Δ of a simple finite dimensional Lie superalgebra.*

(1) If $\Delta = -\Delta$ we will call $L := P \cap (-P)$ the Levi component *of P, $N^+ := P \backslash (-P)$* the nilradical *of P, and $P = L \sqcup N^+$* the Levi decomposition *of P.*

(2) If $\Delta \neq -\Delta$, then we choose a parabolic subset $\widetilde{P}$ of $\Delta \cap (-\Delta)$ such that $P = \widetilde{P} \cap \Delta$ and set

$$\widetilde{L} = \widetilde{P} \cap (-\widetilde{P}), \quad \widetilde{N}^+ = \widetilde{P} \backslash (-\widetilde{P}).$$

We call $L := \widetilde{L} \cap P$ a Levi component *of P, $N^+ = \widetilde{N}^+ \cap P$* a nilradical *of P, and $P = L \sqcup N^+$* a Levi decomposition *of P.*

In the nonsymmetric case $\Delta \neq -\Delta$, the definition of a Levi component and nilradical of P essentially depends on the choice of a parabolic subset $\widetilde{P}$ of $\Delta \cap (-\Delta)$ such that $P = \widetilde{P} \cap \Delta$. We provide examples and discuss this further in Remarks 1.7 and 3.3.

The following lemma contains several simple facts for the Levi components and nilradicals of parabolic sets of roots.

Lemma 1.5. *Let P be a parabolic subset of Δ and let $P = \widetilde{P} \cap \Delta$ for some parabolic subset $\widetilde{P}$ of $\Delta \cup (-\Delta)$. Set $\widetilde{L} := \widetilde{P} \cap (-\widetilde{P})$ and $\widetilde{N}^\pm := (\pm\widetilde{P}) \setminus (\mp\widetilde{P})$. Let $L := \widetilde{L} \cap \Delta$ and $N^\pm := \widetilde{N}^\pm \cap \Delta$ be the Levi component and nilradical of P corresponding to $\widetilde{P}$. Then*

$$\Delta \cup (-\Delta) = \widetilde{N}^- \sqcup \widetilde{L} \sqcup \widetilde{N}^+, \quad \Delta = N^- \sqcup L \sqcup N^+,$$

and

$$\widetilde{L} = L \cup (-L), \quad \widetilde{N}^\pm = N^\pm \cup (-N^\mp).$$

If $\alpha_L, \alpha'_L \in L$ and $\alpha_{N^\pm}, \alpha'_{N^\pm} \in N^\pm$ then

(i) $-\alpha_{N^\pm} \in \Delta$ implies $-\alpha_{N^\pm} \in N^\mp$;

(ii) $\alpha_L + \alpha_{N^\pm} \in \Delta$ implies $\alpha_L + \alpha_{N^\pm} \in N^\pm$;

(iii) $\alpha_L + \alpha'_L \in \Delta$ implies $\alpha_L + \alpha'_L \in L$;

(iv) $\alpha_{N^\pm} + \alpha'_{N^\pm} \in \Delta$ *implies* $\alpha_{N^\pm} + \alpha'_{N^\pm} \in N^\pm$.

Proof. The set theoretic identities are easy to deduce and are left to the reader.

Since $P^- = L \sqcup N^- = (\widetilde{L} \sqcup \widetilde{N}^-) \cap \Delta$ is a parabolic subset of Δ, it is sufficient to prove (i)–(iv) for L and N^+. From the definition of L and N^+ it also follows that it is enough to consider the case of $\Delta = -\Delta$, i.e. $\widetilde{P} = P$.

For (i), if $-\alpha_{N^+} \in P$ then $\alpha_{N^+} \in P \cap (-P)$, which is a contradiction. Assume that $\alpha_L + \alpha_{N^+} \in L$ in (ii). Hence $-\alpha_L - \alpha_{N^+} \in L$ which together with $\alpha_L \in L$ imply $\pm\alpha_{N^+} \in P$ leading to a contradiction. If $\alpha_L + \alpha'_L \in N^+$ in (iii), then by (ii), $\alpha_L = -\alpha'_L + (\alpha_L + \alpha'_L) \in N^+$, which is a contradiction. Finally, for (iv), assume that $\alpha_{N^+} + \alpha'_{N^+} \in L$. Then $-\alpha_{N^+} - \alpha'_{N^+} \in L$ and $\alpha_{N^+} \in N^+$ imply $-\alpha'_{N^+} \in L$ by (ii), and again we reach a contradiction. $\square$

1.2. **Principal parabolic sets of roots.** Let V be a finite dimensional real vector space such that $\Delta \subset V \setminus \{0\}$. A partition $\Delta = \Delta^- \sqcup \Delta^0 \sqcup \Delta^+$ is called a *triangular decomposition* of Δ if there exists a functional $\Lambda \in V^*$ such that $\Delta^0 = \Delta \cap \mathrm{Ker}\Lambda$ and $\Delta^\pm = \{\alpha \in \Delta \,|\, \Lambda(\alpha) \gtrless 0\}$. A subset P of Δ is called a *principal parabolic set* if there exists a triangular decomposition $\Delta = \Delta^- \sqcup \Delta^0 \sqcup \Delta^+$ such that $P = \Delta^0 \sqcup \Delta^+$. In such a case we write $P = P(\Lambda)$.

Proposition 1.6. *Every principal parabolic subset P of the set of roots Δ of a simple finite dimensional Lie superalgebra is a parabolic subset of Δ. The set Δ^0 is a Levi component of P and Δ^+ is a nilradical of P.*

Proof. Consider the principal parabolic subset of $\Delta \cup (-\Delta)$:

$$\widetilde{P} = \{\alpha \in \Delta \cup (-\Delta) \,|\, \Lambda(\alpha) \geq 0\}.$$

It is clear that $\widetilde{P}$ is a parabolic subset of $\Delta \cup (-\Delta)$ and that $P = \widetilde{P} \cap \Delta$. This implies the first statement. For the second statement, observe that

$$\widetilde{L} = \widetilde{P} \cap (-\widetilde{P}) = \{\alpha \in \Delta \cup (-\Delta) \,|\, \Lambda(\alpha) = 0\} = \Delta^0 \cup (-\Delta^0)$$

and

$$\widetilde{N}^+ = \widetilde{P} \backslash (-\widetilde{P}) = \{\alpha \in \Delta \cup (-\Delta) \,|\, \Lambda(\alpha) > 0\} = \Delta^+ \cup (-\Delta^-).$$

Therefore $L = \widetilde{L} \cap \Delta = \{\alpha \in \Delta \cup (-\Delta) \,|\, \Lambda(\alpha) = 0\} \cap \Delta = \Delta^0$ and $N^+ = \widetilde{N}^+ \cap \Delta = \{\alpha \in \Delta \cup (-\Delta) \,|\, \Lambda(\alpha) > 0\} \cap \Delta = \Delta^+$. $\square$

Remark 1.7. *A principal parabolic subset P of Δ can have different Levi decompositions when $\Delta \neq -\Delta$. Given P, define the polyhedron*

$$\mathcal{F}(P) = \{\Lambda \in V^* \mid \Lambda(\alpha) \geq 0, \forall \alpha \in P, \ \Lambda(\alpha) < 0, \forall \alpha \in \Delta \backslash P\}. \tag{1.8}$$

(Here and below we use the term polyhedron in the wide sense as a subset of a real vector space, which is an intersection of a finite collection of open and closed half spaces.) There are some immediate consequences of the inequalities in (1.8). *For*

instance, the first condition in (1.8) *implies* $\Lambda(\alpha) = 0$, $\forall \alpha \in P \cap (-P)$. *For each functional* $\Lambda \in \mathcal{F}(P)$ *define*

$$L(\Lambda) = \{\alpha \in \Delta \mid \Lambda(\alpha) = 0\}, \ \ N^+(\Lambda) = \{\alpha \in \Delta \mid \Lambda(\alpha) > 0\}.$$

It follows from Proposition 1.6 that $P = L(\Lambda) \sqcup N^+(\Lambda)$ *is a Levi decomposition of* P, $\forall \Lambda \in \mathcal{F}(P)$. *This Levi decomposition is the same for points* Λ *in the interior of a fixed face of* $\mathcal{F}(P)$, *but differs for points* Λ *that belong to the interiors of different faces of* $\mathcal{F}(P)$. *This is further illustrated in Remark 3.3.*

The converse to the first statement of Proposition 1.6 is true for finite dimensional reductive Lie algebras (see, for example, [Bo, Proposition VI.7.20]). More generally, we have (see [DFG, Proposition 2.10]):

Proposition 1.9. *Let* $\mathfrak{g}$ *be a quasisimple regular Kac-Moody superalgebras and let* P *be a parabolic subset of* Δ. *Then* P *is a principal parabolic subset of* Δ.

The simple finite dimensional Lie superalgebras which are not Lie algebras are: $\mathfrak{sl}(m|n)$ for $m \neq n$, $\mathfrak{psl}(m|m)$, $\mathfrak{osp}(m|2n)$, $D(2,1;\alpha)$, $F(4)$, $G(3)$, $\mathfrak{sp}(n)$, $\mathfrak{psq}(n)$, and the Cartan type superalgebras $W(n)$, $S(n)$, $S'(n)$, and $H(n)$. For the restrictions on the parameters m, n, and α as well as isomorphisms among the superalgebras listed above we refer the reader to [K]. Among those, the quasisimple regular Kac–Moody Lie superalgebras are $\mathfrak{sl}(m|n)$ for $m \neq n$, $\mathfrak{osp}(m|2n)$, $D(2,1;\alpha)$, $F(4)$, and $G(3)$, (see [S2]). Proposition 1.9 applies to them, i.e. all parabolic sets of roots for them are principal.

1.3. **Cominuscule parabolic subalgebras.**

Recall from §1.1 that $\mathfrak{h}$ denotes a fixed Cartan subalgebra of the simple finite dimensional Lie superalgebra $\mathfrak{g}$ and that $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ denotes the corresponding set of roots of $\mathfrak{g}$. For $\alpha \in \Delta$, we will denote by $\mathfrak{g}^\alpha$ the corresponding root space. We will call a subalgebra of $\mathfrak{g}$ a *root subalgebra* if it has the form

$$\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{h}) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha \right) \tag{1.10}$$

for some subset $\Phi \subseteq \Delta$. We will call a root subalgebra of $\mathfrak{g}$ *parabolic* if $\mathfrak{l} \supseteq \mathfrak{h}$ and Φ is a parabolic subset of Δ.

Definition 1.11. *Let* P *be a parabolic subset of the set of roots* Δ *of a simple finite dimensional Lie superalgebra* $\mathfrak{g}$ *with respect to a Cartan subalgebra* $\mathfrak{h}$ *and*

$$\mathfrak{p}_P = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in P} \mathfrak{g}^\alpha \right) \tag{1.12}$$

be the corresponding parabolic subalgebra. Given a Levi decomposition $P = L \sqcup N^+$ of P we define the subalgebras

$$\mathfrak{l} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in L} \mathfrak{g}^\alpha \right) \quad \text{and} \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in N^+} \mathfrak{g}^\alpha \tag{1.13}$$

of $\mathfrak{p}_P$, and call them a Levi subalgebra *and* a nilradical *of $\mathfrak{p}_P$, respectively. The semidirect sum decomposition $\mathfrak{p}_P = \mathfrak{l} \ltimes \mathfrak{n}^+$ will be called* a Levi decomposition *of $\mathfrak{p}_P$.*

In the classical even case ($\mathfrak{g}_{\bar{1}} = 0$) the root system is symmetric $\Delta = -\Delta$ and the above decomposition $\mathfrak{p}_P = \mathfrak{l} \ltimes \mathfrak{n}^+$ is precisely the Levi decomposition of $\mathfrak{p}_P$ for the unique Levi subalgebra $\mathfrak{l}$ containing $\mathfrak{h}$.

The next definition singles out the class of cominuscule parabolic subalgebras.

Definition 1.14. *We call a parabolic subalgebra of a complex simple finite dimensional Lie superalgebra $\mathfrak{g}$* cominuscule*, if has a nilradical which is abelian. A parabolic subset P of the set of roots Δ of $\mathfrak{g}$ will be called cominuscule if the corresponding parabolic subalgebra $\mathfrak{g}_P$ is cominuscule.*

If $\mathfrak{g}$ is a complex simple finite dimensional Lie algebra, then Definition 1.14 singles out exactly the class of cominuscule parabolic subalgebras of $\mathfrak{g}$ which contain the fixed Cartan subalgebra $\mathfrak{h}$.

In the Appendix we will prove the following proposition.

Proposition 1.15. *Let $\mathfrak{g}$ be a simple finite dimensional Lie superalgebra, $\mathfrak{g} \neq S(n), \mathfrak{g} \neq S'(n)$, $\mathfrak{g} \neq \mathfrak{psl}(3|3)$, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and Δ be the corresponding root system. If $\alpha, \beta, \alpha + \beta \in \Delta$, then*

$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \neq 0.$$

We note that in the super case, $\alpha, \beta, \alpha + \beta \in \Delta$ does not imply

$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}.$$

The next proposition generalizes an important property of cominuscule parabolic subalgebras which holds in the classical (even) case. It reduces the problem of classification of cominuscule parabolics of simple finite dimensional Lie superalgebras to a problem for the corresponding root systems and will be extensively used in the paper.

Proposition 1.16. *Let $\mathfrak{g}$ be as in Proposition 1.15. A parabolic subset P of the set of roots of $\mathfrak{g}$ is cominuscule if and only if it has a nilradical N^+ such that for every α, β in N^+, $\alpha + \beta \notin \Delta$,*

Proof. If a parabolic subset of roots has the above property, then for all $\alpha, \beta \in N^+$, $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0$. Therefore the nilradical $\mathfrak{n}^+$ of the parabolic subalgebra $\mathfrak{p}_P$ corresponding to N^+ is abelian, cf. (1.12) and (1.13), and thus $\mathfrak{p}_P$ is cominuscule.

In the other direction, assume that the parabolic subset of roots P is cominuscule. Let N^+ be a nilradical of P such that the corresponding nilradical $\mathfrak{n}^+$ of $\mathfrak{g}$ given by

(1.13) is abelian. If there exist $\alpha, \beta \in N^+$ such that $\alpha + \beta \in \Delta$, then by Proposition 1.15, $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \neq 0$. This is a contradiction since $\mathfrak{g}^\alpha$ and $\mathfrak{g}^\beta$ are subspaces of the nilradical of the parabolic subalgebra $\mathfrak{g}_P$. Therefore for all α, β in N^+, $\alpha + \beta \notin \Delta$, which completes the proof of the proposition. $\square$

Remark 1.17. *The cases $\mathfrak{g} = \mathfrak{psl}(3|3)$, $\mathfrak{g} = S(n)$ and $\mathfrak{g} = S'(n)$ require special attention and modifications of Proposition 1.15 in these cases are established in §2.2 and §4.2, respectively.*

The classification of all cominuscule parabolic subsets established in the next sections implies the following result.

Theorem 1.18. *All cominuscule parabolic subsets of the simple finite dimensional Lie superalgebras are principal.*

1.4. **Passing to subalgebras.** We will need the following lemma for reduction of cominuscule parabolics to certain subalgebras. Its proof is straightforward and will be left to the reader.

Lemma 1.19. *Let $\mathfrak{g}$ be a simple Lie superalgebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{a}$ be subalgebra of $\mathfrak{g}$, which is either a simple superalgebra or an (even) reductive Lie algebra. Assume that $\mathfrak{a} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{a}$ and that $\alpha|_{\mathfrak{a}\cap\mathfrak{h}_{\bar{0}}} \neq \beta|_{\mathfrak{a}\cap\mathfrak{h}_{\bar{0}}}$ for all roots $\alpha \neq \beta$ of $\mathfrak{g}$. Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\Delta_\mathfrak{a}$ be the root system of $\mathfrak{a}$ (considered as a subset of Δ) with respect to $\mathfrak{a} \cap \mathfrak{h}$.*

If P is a parabolic subset of Δ, then $P \cap \Delta_\mathfrak{a}$ is either equal to $\Delta_\mathfrak{a}$ or to a parabolic subset of $\Delta_\mathfrak{a}$. In the latter case, if $P = L \sqcup N^+$ is a Levi decomposition of P, then $P \cap \Delta_\mathfrak{a} = (L \cap \Delta_\mathfrak{a}) \sqcup (N^+ \cap \Delta_\mathfrak{a})$ is a Levi decomposition of $P \cap \Delta_\mathfrak{a}$.

Finally, we will make extensive use of the classification of cominuscule parabolics of classical simple Lie algebras. We recall it below for completeness. We will use the notation of [Bo] for the root spaces of $\mathfrak{sl}(n)$, $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, and $\mathfrak{sp}(2n)$.

Proposition 1.20. *The following list describes the Levi components and nilradicals of all cominuscule parabolic sets of roots for the finite dimensional simple Lie algebras $\mathfrak{g}$ of type A, B, C, and D up to the action of the Weyl group $W_\mathfrak{g}$ of $\mathfrak{g}$.*

(i) $L_{\mathfrak{sl}(n)}(n_0) = \{\varepsilon_i - \varepsilon_j \mid i \neq j \leq n_0 \text{ or } i \neq j > n_0\}$ and $N^+_{\mathfrak{sl}(n)}(n_0) = \{\varepsilon_i - \varepsilon_j \mid i \leq n_0 < j\}$, $1 \leq n_0 \leq n-1$, for $\mathfrak{g} = \mathfrak{sl}(n)$.

(ii) $L_{\mathfrak{so}(2n+1)} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid j > i > 1\}$ and $N^+_{\mathfrak{so}(2n+1)} = \{\varepsilon_1 \pm \varepsilon_j, \varepsilon_1 \mid j > 1\}$ for $\mathfrak{g} = \mathfrak{so}(2n+1)$.

(iii) $L_{\mathfrak{sp}(2n)} = \{\varepsilon_k - \varepsilon_l \mid k \neq l\}$ and $N^+_{\mathfrak{sp}(2n)} = \{\varepsilon_k + \varepsilon_l, 2\varepsilon_k \mid k \neq l\}$ for $\mathfrak{g} = \mathfrak{sp}(2n)$.

(iv) For $\mathfrak{g} = \mathfrak{so}(2n)$, $n \geq 2$:

$$L_{\mathfrak{so}(2n)}(1) = \{\pm\varepsilon_i \pm \varepsilon_j, \mid j > i > 1\} \quad \textit{and} \quad N^+_{\mathfrak{so}(2n)}(1) = \{\varepsilon_1 \pm \varepsilon_j \mid j > 1\},$$

$$L_{\mathfrak{so}(2n)}(n) = \{\varepsilon_i - \varepsilon_j, \mid i \neq j\} \quad \textit{and} \quad N^+_{\mathfrak{so}(2n)}(n) = \{\varepsilon_i + \varepsilon_j \mid i \neq j\},$$

$$\overline{L}_{\mathfrak{so}(2n)}(n) = (\theta L_{\mathfrak{so}(2n)}(n)) \quad \textit{and} \quad \overline{N}^+_{\mathfrak{so}(2n)}(n) = \theta(N^+_{\mathfrak{so}(2n)}(n)),$$

where θ is the involutive automorphism of the Dynkin diagram D_n that preserves $\varepsilon_i - \varepsilon_{i+1}$, $i < n-1$, and interchanges $\varepsilon_{n-1} - \varepsilon_n$ and $\varepsilon_{n-1} + \varepsilon_n$.

For convenience we will also use the notation from Proposition 1.20 (i) for $n_0 = n$, i.e. for $\mathfrak{g} = \mathfrak{sl}(n)$ we set $L_{\mathfrak{sl}(n)}(n) := \Delta$ and $N^+_{\mathfrak{sl}(n)}(n) = \emptyset$.

2. Classification of the cominuscule parabolics of basic classical Lie superalgebras

In this section we classify the cominuscule parabolic subsets of all basic classical Lie superalgebras. Fix such a Lie superalgebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ of it. Denote, as before, the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ by Δ. For this class of superalgebras, the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is a reductive Lie algebra. The Weyl group $W_{\mathfrak{g}_{\bar{0}}}$ of the even part $\mathfrak{g}_{\bar{0}}$ acts on the root system Δ of $\mathfrak{g}$ and thus on the parabolic subsets of Δ. (Each element of the Weyl group of the even part $w \in W_{\mathfrak{g}_{\bar{0}}}$ can be lifted to an automorphism σ_w of $\mathfrak{g}$ stabilizing $\mathfrak{h}$, which induces an automorphism of the set of roots. The later does not depend on the choice of σ_w and by abuse of notation will be denoted by w.) It is obvious that, if a parabolic subset P of Δ is cominuscule, then $w(P)$ is cominuscule for all $w \in W$. We will classify the cominuscule parabolic subsets of Δ up to the action of the Weyl group $W_{\mathfrak{g}_{\bar{0}}}$.

In all proofs P will denote a cominuscule parabolic set of roots in Δ. Since Δ is symmetric, in the definition of a parabolic subset P of Δ there is no need to consider a parabolic subset $\widetilde{P}$ (i.e. $\widetilde{P} = P$), and P has a unique Levi decomposition. The latter is given by $P = L \sqcup N^+$, where the Levi component is $L = P \cap (-P)$ and the nilradical is $N^+ = P \backslash (-P)$. Set $P_{\bar{0}} := P \cap \Delta_{\bar{0}}$, $L_{\bar{0}} := L \cap \Delta_{\bar{0}}$, and $N^+_{\bar{0}} := N^+ \cap \Delta_{\bar{0}}$. By Lemma 1.19, either $P_{\bar{0}} = \Delta_{\bar{0}}$ or $P_{\bar{0}}$ is a parabolic subset of $\Delta_{\bar{0}}$ with Levi component $L_{\bar{0}}$ and nilradical $N^+_{\bar{0}}$. As in Definition 1.11 we will denote by $\mathfrak{l}$ and $\mathfrak{n}^+$ the Levi subalgebra and nilradical of $\mathfrak{p}_P$.

2.1. $\mathfrak{g} = \mathfrak{sl}(m|n)$**,** $m \neq n$**.** In this case $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}$ with $\Delta_{\bar{0}} = \{\varepsilon_i - \varepsilon_j, \delta_k - \delta_l \mid 1 \leq i \neq j \leq m, 1 \leq k \neq l \leq n\}$ and $\Delta_{\bar{1}} = \Delta^+_{\bar{1}} \sqcup \Delta^-_{\bar{1}}$, where $\Delta^{\pm}_{\bar{1}} := \{\pm(\varepsilon_i - \delta_l) \mid 1 \leq i \leq m, 1 \leq l \leq n\}$. We introduce the following sets of roots

$$\begin{aligned}
L_{\mathfrak{sl}(m|n)}(m_0|n_0) \;&:=\; \{\varepsilon_i - \varepsilon_j \mid i, j \leq m_0 \text{ or } i, j > m_0\} \\
&\quad \sqcup \{\delta_k - \delta_l, \mid k, l \leq n_0 \text{ or } k, l > n_0\} \\
&\quad \sqcup \{\pm(\varepsilon_i - \delta_l) \mid i \leq m_0, l \leq n_0 \text{ or } i > m_0, l > n_0\} \text{ and} \\
N^+_{\mathfrak{sl}(m|n)}(m_0|n_0) \;&:=\; \{\varepsilon_i - \varepsilon_j \mid i \leq m_0 < j\} \sqcup \{\delta_k - \delta_l, \mid k \leq n_0 < l\} \\
&\quad \sqcup \{\varepsilon_i - \delta_l, \delta_k - \varepsilon_j \mid i \leq m_0 < j; k \leq n_0 < l\}.
\end{aligned}$$

Set $P_{\mathfrak{sl}(m|n)}(m_0|n_0) := L_{\mathfrak{sl}(m|n)}(m_0|n_0) \sqcup N^+_{\mathfrak{sl}(m|n)}(m_0|n_0)$. Note that, if $m_0 = n_0 = 0$ or $m_0 = m, n_0 = n$, then $P_{\mathfrak{sl}(m|n)}(m_0|n_0) = \Delta$. In all other cases $P_{\mathfrak{sl}(m|n)}(m_0|n_0)$ is proper. It is clear that $\Delta = (-N^+_{\mathfrak{sl}(m|n)}(m_0|n_0)) \sqcup L_{\mathfrak{sl}(m|n)}(m_0|n_0) \sqcup N^+_{\mathfrak{sl}(m|n)}(m_0|n_0)$.

Theorem 2.1. *Every cominuscule parabolic set P of roots of $\mathfrak{sl}(m|n)$, $m \neq n$, is conjugated under the action of the Weyl group $W_{\mathfrak{sl}(m)} \times W_{\mathfrak{sl}(n)}$ of $\mathfrak{g}_{\bar{0}}$ to a unique subset of the form $P_{\mathfrak{sl}(m|n)}(m_0|n_0)$ for some m_0, n_0, such that $0 \leq m_0 \leq m$, $0 \leq n_0 \leq n$, and $(m_0, n_0) \neq (0,0), (m,n)$. For the Levi subalgebras and nilradicals of the corresponding parabolic subalgebras, we have that $\mathfrak{l} \cong \mathfrak{sl}(m_0|n_0) \oplus \mathfrak{sl}(m-m_0|n-n_0) \oplus \mathbb{C}$ and as an $\mathfrak{l}$-module $\mathfrak{n}^+ \cong V^{m_0|n_0} \otimes \left(V^{m-m_0|n-n_0}\right)^*$.*

Recall from the introduction that $V^{p|q}$ denotes the standard representation of $\mathfrak{sl}(p|q)$.

Proof of Theorem 2.1. Let $N^+_{\mathfrak{sl}(m)} := N^+ \cap \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq m\}$ and $N^+_{\mathfrak{sl}(n)} := N^+ \cap \{\delta_k - \delta_l \mid 1 \leq k \neq l \leq n\}$. Obviously, $N^+_{\bar{0}} = N^+_{\mathfrak{sl}(m)} \sqcup N^+_{\mathfrak{sl}(n)}$.

By Proposition 1.9, we know that $P = P(\Lambda)$ is a principal parabolic set of roots determined by some functional Λ. Let $\Lambda(\varepsilon_i - \varepsilon_j) = x_i - x_j, \Lambda(\delta_k - \delta_l) = y_k - y_l$, and $\Lambda(\varepsilon_i - \delta_k) = x_i - y_k$ for some x_i and y_k such that $x_1 + \ldots + x_m + y_1 + \ldots + y_n = 0$.

Case 1: $N^+_{\mathfrak{sl}(m)} \neq \emptyset$ and $N^+_{\mathfrak{sl}(n)} \neq \emptyset$. Proposition 1.20 implies that P is conjugated under the action of the Weyl group $W_{\mathfrak{sl}(m)} \times W_{\mathfrak{sl}(n)}$ to a unique parabolic subset with

$$\begin{aligned} L_{\bar{0}} &= \{\varepsilon_i - \varepsilon_j \mid i,j \leq m_0 \text{ or } i,j > m_0\} \sqcup \{\delta_k - \delta_l \mid k,l \leq n_0 \text{ or } k,l > n_0\} \text{ and} \\ N^+_{\bar{0}} &= \{\varepsilon_i - \varepsilon_j, \delta_k - \delta_l \mid i \leq m_0 < j;\ k \leq n_0 < l\} \end{aligned}$$

for some $1 \leq m_0 \leq m-1$, $1 \leq n_0 \leq n-1$. Let us rename P so it has the above even Levi component and nilradical. Since $\Lambda|_{L_{\bar{0}}} = 0$ and $\Lambda|_{N^+_{\bar{0}}}$ takes positive values we have that $x_1 = \ldots = x_{m_0} = x$, $x_{m_0+1} = \ldots = x_m = x'$, $y_1 = \ldots = y_{n_0} = y$, $y_{n_0+1} = \ldots = y_n = y'$ for some x, y, x', y' such that $x > x'$, $y > y'$, and $m_0 x + (m-m_0)x' + n_0 y + (n-n_0)y' = 0$. By Lemma 1.16 we have that $\varepsilon_{m_0} - \delta_{n_0}$ is not in N^+ and thus $x \leq y$. Indeed, since $\delta_{n_0} - \delta_{n_0+1} \in N^+$, if $\varepsilon_{m_0} - \delta_{n_0} \in N^+$, it would force $\varepsilon_{m_0} - \delta_{n_0+1} = (\varepsilon_{m_0} - \delta_{n_0}) + (\delta_{n_0} - \delta_{n_0+1}) \in N^+$ which contradicts to P being cominuscule. On the other hand, since $\delta_{n_0} - \varepsilon_{m_0} \notin N^+$ we have $y \leq x$, and consequently $x = y$. Similarly, $\varepsilon_{m_0+1} - \delta_{n_0+1}$ and $\delta_{n_0+1} - \varepsilon_{m_0+1}$ are not in N^+ and hence $x' = y'$. The conditions $x = y$, $x' = y'$, and $x > x'$ determine completely P and imply that $L = L_{\mathfrak{sl}(m|n)}(m_0|n_0)$ and $N^+ = N^+_{\mathfrak{sl}(m|n)}(m_0|n_0)$.

Case 2: $N^+_{\mathfrak{sl}(m)} \neq \emptyset$ and $N^+_{\mathfrak{sl}(n)} = \emptyset$. Like in the previous case, P is conjugated under the action of $W_{\mathfrak{sl}(m)} \times W_{\mathfrak{sl}(n)}$ to a unique parabolic subset such that

$$\begin{aligned} L_{\bar{0}} &= \{\varepsilon_i - \varepsilon_j \mid i,j \leq m_0 \text{ or } i,j > m_0\} \sqcup \{\delta_k - \delta_l \mid 1 \leq k \neq l \leq n\} \text{ and} \\ N^+_{\bar{0}} &= \{\varepsilon_i - \varepsilon_j \mid i \leq m_0 < j\} \end{aligned}$$

for some $1 \leq m_0 \leq m-1$. We rename P accordingly. Then from $\Lambda|_{L_{\bar{0}}} = 0$ and $\Lambda|_{N^+_{\bar{0}}} > 0$ we find $x_1 = \ldots = x_{m_0} = x$, $x_{m_0+1} = \ldots = x_m = x'$, $y_1 = \ldots = y_n = y$, for some x, x', y such that $x > x'$ and $m_0 x + (m-m_0)x' + ny = 0$. By Lemma 1.16

we have that $\varepsilon_{m_0+1} - \delta_1$ and $\delta_1 - \varepsilon_{m_0}$ are not in N^+. This leads to $x \geq y \geq x'$. We proceed with three separate subcases. First, if $x = y > x'$ we have

$$\begin{aligned} L &= L_{\bar{0}} \sqcup \{\pm(\varepsilon_i - \delta_l) \mid i \leq m_0, 1 \leq l \leq n\} \text{ and} \\ N^+ &= N^+_{\bar{0}} \sqcup \{\delta_k - \varepsilon_j \mid j > m_0, 1 \leq k \leq n\}. \end{aligned}$$

In this case $L = L_{\mathfrak{sl}(m|n)}(m_0|n)$ and $N^+ = N^+_{\mathfrak{sl}(m|n)}(m_0|n)$, i.e. $n_0 = n$.

Second, if $x > y = x'$, then

$$\begin{aligned} L &= L_{\bar{0}} \sqcup \{\pm(\varepsilon_j - \delta_k) \mid j > m_0, l \leq k \leq n\} \text{ and} \\ N^+ &= N^+_{\bar{0}} \sqcup \{\varepsilon_i - \delta_l \mid i \leq m_0, 1 \leq l \leq n\}. \end{aligned}$$

This leads to $L = L_{\mathfrak{sl}(m|n)}(m_0|0)$ and $N^+ = N^+_{\mathfrak{sl}(m|n)}(m_0|0)$, i.e. $n_0 = 0$.

And third, if $x > y = x'$, we have that $\varepsilon_1 - \delta_k, \delta_k - \varepsilon_m \in N^+$ for every k, so P is not cominuscule.

Case 3: $N^+_{\mathfrak{sl}(m)} = \emptyset$ *and* $N^+_{\mathfrak{sl}(n)} \neq \emptyset$. Similarly to the previous case we verify that P is conjugated under the action of $W_{\mathfrak{sl}(m)} \times W_{\mathfrak{sl}(n)}$ to a unique parabolic subset of the form $P_{\mathfrak{sl}(m|n)}(0|n_0)$ or $P_{\mathfrak{sl}(m|n)}(m|n_0)$.

Case 4: $N^+_{\mathfrak{sl}(m)} = \emptyset$ *and* $N^+_{\mathfrak{sl}(n)} = \emptyset$. In this case

$$L_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq m\} \sqcup \{\delta_k - \delta_l \mid 1 \leq k \neq l \leq n\} \text{ and } N^+_{\bar{0}} = \emptyset.$$

We have that $x_1 = \ldots = x_m = x$ and $y_1 = \ldots = y_n = y$. We easily see that $x \neq y$, which leads to $L = L_{\bar{0}}$ and $\mathfrak{l} = \mathfrak{g}_{\bar{0}}$. Depending on whether $x > y$ or $x < y$, we obtain $N^+ = \{\varepsilon_i - \delta_l \mid 1 \leq i \leq m, 1 \leq l \leq n\}$ or $N^+ = \{\delta_k - \varepsilon_j \mid 1 \leq j \leq m, 1 \leq k \leq n\}$. These cases correspond to $m_0 = m, n_0 = 0$, and $m_0 = 0, n_0 = n$, respectively. □

Remark 2.2. *The cases $m_0 = 0, n_0 = n$ and $m_0 = m, n_0 = 0$ correspond to the "standard" triangular decomposition $\Delta = \Delta^-_{\bar{1}} \sqcup \Delta_{\bar{0}} \sqcup \Delta^+_{\bar{1}}$, namely to $P = \Delta_{\bar{0}} \sqcup \Delta^+_{\bar{1}}$ and $P = \Delta_{\bar{0}} \sqcup \Delta^-_{\bar{1}}$, respectively.*

2.2. $\mathfrak{g} = \mathfrak{psl}(n|n)$**.** In this case $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$. By abuse of notation denote by $\{\varepsilon_1, \ldots, \varepsilon_n, \delta_1, \ldots, \delta_n\}$ the images of the standard basis elements of $\mathfrak{h}^*_{\mathfrak{gl}(n|n)}$ under the canonical projection $\mathfrak{h}^*_{\mathfrak{gl}(n|n)} \mapsto \mathfrak{h}^*_{\mathfrak{sl}(n|n)}$. Thus, $\varepsilon_1 + \ldots + \varepsilon_n = \delta_1 + \ldots + \delta_n$. Then

$$\mathfrak{h}^*_{\mathfrak{psl}(n|n)} = \left\{ \sum_{i=1}^n a_i \varepsilon_i + \sum_{j=1}^n b_j \delta_j \mid \sum_{i=1}^n a_i + \sum_{j=1}^n b_j = 0 \right\}. \tag{2.3}$$

Consider the natural surjective linear map $\mathfrak{h}^*_{\mathfrak{gl}(n|n)} \to \mathfrak{h}^*_{\mathfrak{psl}(n|n)}$ defined in terms of (2.3) by

$$\varepsilon_i \mapsto \varepsilon_i - \sigma, \quad \delta_j \mapsto \delta_j - \sigma, \quad 1 \leq i, j \leq n, \tag{2.4}$$

where $\sigma = \left(\sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} \delta_i\right)/(2n)$. Then, under the map (2.4), the set of root of $\mathfrak{psl}(n|n)$ is identified with the one of $\mathfrak{sl}(n|n)$ (and, hence, of $\mathfrak{gl}(n|n)$ as well). Define

$$\begin{aligned} L_{\mathfrak{psl}(n|n)}(m_0|n_0) &:= \{\varepsilon_i - \varepsilon_j \mid i, j \le m_0 \text{ or } i, j > m_0\} \\ &\sqcup \{\delta_k - \delta_l, \mid k, l \le n_0 \text{ or } k, l > n_0\} \\ &\sqcup \{\pm(\varepsilon_i - \delta_l) \mid i \le m_0, l \le n_0 \text{ or } i > m_0, l > n_0\} \text{ and} \\ N^+_{\mathfrak{psl}(n|n)}(m_0|n_0) &:= \{\varepsilon_i - \varepsilon_j \mid i \le m_0 < j\} \sqcup \{\delta_k - \delta_l, \mid k \le n_0 < l\} \\ &\sqcup \{\varepsilon_i - \delta_l, \delta_k - \varepsilon_j \mid i \le m_0 < j; k \le n_0 < l\}. \end{aligned}$$

Set $P_{\mathfrak{psl}(n|n)}(m_0, n_0) := L_{\mathfrak{psl}(n|n)}(m_0, n_0) \sqcup N^+_{\mathfrak{psl}(n|n)}(m_0, n_0)$. Similarly to the case of $\mathfrak{g} = \mathfrak{sl}(m|n)$, for $m_0 = n_0 = 0$ and $m_0 = n_0 = n$, $P_{\mathfrak{psl}(n|n)}(m_0|n_0) = \Delta$, and for all other pairs (m_0, n_0) $P_{\mathfrak{psl}(n|n)}(m_0|n_0)$ is a proper subset of Δ.

Recall that the Lie superalgebras $\mathfrak{psl}(n|n)$ are not regular Kac-Moody superalgebras. Although there are parabolic subsets of $\mathfrak{psl}(n|n)$ that are not principal parabolic, every parabolic subset P of $\Delta_{\mathfrak{psl}(n|n)}$ is the image of a parabolic subset $\widehat{P}$ of $\Delta_{\mathfrak{gl}(n|n)}$ under the map (2.4), see [DFG, §3]. Since $\mathfrak{gl}(n|n)$ is a quasisimple regular Kac-Moody superalgebra, Proposition 1.9 applies to $\widehat{P}$. Furthermore, using the commutation relations in $\mathfrak{gl}(n|n)$, one can verify that Proposition 1.15 is valid for $\mathfrak{g} = \mathfrak{gl}(n|n)$, $n \ge 2$. Although this proposition fails for $\mathfrak{g} = \mathfrak{psl}(3|3)$ (take for example: $\alpha = \varepsilon_1 - \delta_1$, $\beta = \varepsilon_2 - \delta_2$, $\alpha + \beta = \delta_3 - \varepsilon_3$), we have the following modification.

Lemma 2.5. *Let $\mathfrak{g} = \mathfrak{psl}(3|3)$ and $\alpha, \beta \in \Delta$ be such that $\alpha + \beta \ne 0$. Then $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \ne 0$ if and only if $\alpha + \beta \in \Delta_{\mathfrak{gl}(n|n)}$. Here the root systems of $\mathfrak{h}^*_{\mathfrak{psl}(n|n)}$ and $\mathfrak{h}^*_{\mathfrak{gl}(n|n)}$ are identified via the map (2.4) and the sum $\alpha + \beta$ is taken in $\mathfrak{h}^*_{\mathfrak{gl}(n|n)}$.*

Proof. It is sufficient to show that, under the above conditions on $\alpha, \beta \in \Delta$, there exist $x_\alpha \in \mathfrak{g}^\alpha$ and $x_\beta \in \mathfrak{g}^\beta$ such that $[x_\alpha, x_\beta] \ne 0$. We can choose any nonzero elements of $\mathfrak{gl}(3|3)^\alpha$ and $\mathfrak{gl}(3|3)^\beta$ and consider them as elements of $\mathfrak{g}^\alpha$ and $\mathfrak{g}^\beta$, respectively. Then using the fact that Proposition 1.15 is valid for $\mathfrak{gl}(3|3)$, we complete the proof. $\square$

In view of the above lemma, to obtain a classification of the cominuscule parabolic subsets of $\mathfrak{psl}(n|n)$, one has to modify the proof of Theorem 2.1 for $\mathfrak{g} = \mathfrak{gl}(n|n)$ and then transfer the classification to $\mathfrak{g} = \mathfrak{psl}(n|n)$. The details are left to the reader.

Theorem 2.6. *(i) Let $n > 2$. Every cominuscule parabolic set of roots of $\mathfrak{g} = \mathfrak{psl}(n|n)$ is conjugated under the action of the Weyl group $W_{\mathfrak{sl}(n)} \times W_{\mathfrak{sl}(n)}$ of $\mathfrak{g}_{\bar{0}}$ to a unique subset of the form $P_{\mathfrak{psl}(n|n)}(m_0|n_0)$, such that $0 \le m_0 \le n$, $0 \le n_0 \le n$ and $(m_0, n_0) \ne (0, 0), (n, n)$. Furthermore, for the Levi subalgebras and nilradicals of the corresponding parabolic subalgebras we have $\mathfrak{l} \cong \mathfrak{sl}(m_0|n_0) \oplus \mathfrak{sl}(n - m_0|n - n_0)$ and $\mathfrak{n}^+ \cong V^{m_0|n_0} \otimes \left(V^{n-m_0|n-n_0}\right)^*$ as $\mathfrak{l}$-modules.*

(ii) Every cominuscule parabolic set of roots of $\mathfrak{g} = \mathfrak{psl}(2|2)$ is conjugated under the action of the Weyl group $W_{\mathfrak{sl}(2)} \times W_{\mathfrak{sl}(2)}$ of $\mathfrak{g}_{\bar{0}}$ to $P_{\mathfrak{psl}(2|2)}(1|1)$ with $\mathfrak{l} \cong \mathfrak{sl}(1|1) \oplus \mathfrak{sl}(1|1)$ and $\mathfrak{n}^+ \cong V^{1|1} \otimes \left(V^{1|1}\right)^*$ as an $\mathfrak{l}$-module.

2.3. $\mathfrak{g} = \mathfrak{osp}(2m+1|2n), m \geq 1$. In this case $\mathfrak{g}_{\bar 0} \cong \mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$ with $\Delta_{\bar 0} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \pm\delta_k \pm \delta_l, \pm 2\delta_k \mid 1 \leq i \neq j \leq m, 1 \leq k \neq l \leq n\}$ and $\Delta_{\bar 1} = \{\pm\varepsilon_i \pm \delta_k, \pm\delta_k \mid 1 \leq i \leq m, 1 \leq k \leq n\}$. Set

$$\begin{aligned} L_{\mathfrak{osp}(2m+1|2n)} &:= \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid j > i > 1\} \sqcup \{\pm\delta_k \pm \delta_l, \pm 2\delta_k \mid k \neq l\} \\ &\quad \sqcup \{\pm\delta_k, \pm\varepsilon_i \pm \delta_k, \mid i > 1\} \text{ and} \\ N^+_{\mathfrak{osp}(2m+1|2n)} &:= \{\varepsilon_1 \pm \varepsilon_j, \varepsilon_1 \mid j > 1\} \sqcup \{\varepsilon_1 \pm \delta_k \mid k \geq 1\}. \end{aligned}$$

and $P_{\mathfrak{osp}(2m+1|2n)} := L_{\mathfrak{osp}(2m+1|2n)} \sqcup N^+_{\mathfrak{osp}(2m+1|2n)}$.

Theorem 2.7. *Every cominuscule parabolic set of roots P of $\mathfrak{osp}(2m+1|2n)$ is conjugated under the action of the Weyl group $W_{\mathfrak{so}(2m+1)} \times W_{\mathfrak{sp}(2n)}$ of $\mathfrak{g}_{\bar 0}$ to $P_{\mathfrak{osp}(2m+1|2n)}$. For the Levi subalgebra and nilradical of the corresponding parabolic subalgebra, we have $\mathfrak{l} \cong \mathfrak{osp}(2m-1|2n) \oplus \mathbb{C}$ and $\mathfrak{n}^+ \cong V^{2m-1|2n}$ as an $\mathfrak{l}$-module.*

Proof. Let $N^+_{\mathfrak{so}} := N^+ \cap \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq m\}$ and $N^+_{\mathfrak{sp}} := N^+ \cap \{\pm\delta_k \pm \delta_l, \pm 2\delta_k \mid 1 \leq k \neq l \leq n\}$. Obviously, $N^+_{\bar 0} = N^+_{\mathfrak{so}} \sqcup N^+_{\mathfrak{sp}}$.

Case 1: $N^+_{\mathfrak{sp}} \neq \emptyset$. By Proposition 1.20 P is conjugated under the action of the Weyl group of $\mathfrak{sp}(n)$ to a parabolic subset subset with $N^+_{\mathfrak{sp}} = \{\delta_k + \delta_l, 2\delta_k \mid 1 \leq k \neq l \leq n\}$. In particular, δ_k and $2\delta_k$ are in N^+ which by Lemma 1.16 contradicts the assumption that P is cominuscule.

Case 2: $N^+_{\mathfrak{so}} \neq \emptyset$ and $N^+_{\mathfrak{sp}} = \emptyset$. Using Proposition 1.20 again we have that up to the action of the Weyl group of $\mathfrak{so}(2m+1)$,

$$\begin{aligned} L_{\bar 0} &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid j > i > 1 \sqcup \{\delta_k - \delta_l \mid k, l \leq n_0 \text{ or } k, l > n_0\} \text{ and} \\ N^+_{\bar 0} &= \{\varepsilon_i - \varepsilon_j, \delta_k - \delta_l \mid i \leq m_0 < j;\ k \leq n_0 < l\}. \end{aligned}$$

Lemma 1.5 implies that $\pm\delta_k$ and $\pm\varepsilon_i \pm \delta_k$ are in L and thus $\varepsilon_1 \pm \delta_k \in N^+$. Hence $L = L_{\mathfrak{osp}(2m+1|2n)}$ and $N^+ = N^+_{\mathfrak{osp}(2m+1|2n)}$.

Case 3: $N^+_{\mathfrak{so}} = N^+_{\mathfrak{sp}} = \emptyset$. In this case P is not a proper subset of Δ. $\square$

2.4. $\mathfrak{g} = \mathfrak{osp}(1|2n)$. In this case $\mathfrak{g}_{\bar 0} \cong \mathfrak{sp}(2n)$ with $\Delta_{\bar 0} = \{\pm\delta_k \pm \delta_l, \pm 2\delta_k \mid 1 \leq k \neq l \leq n\}$ and $\Delta_{\bar 1} = \{\pm\delta_k \mid 1 \leq k \leq n\}$.

Theorem 2.8. *There are no cominuscule parabolic sets of roots of $\mathfrak{osp}(1|2n)$.*

Proof. If $N^+ \cap \Delta_{\bar 0} \neq \emptyset$, using the same reasoning as in case 1 in the proof of Theorem 2.7 we reach a contradiction. In the case $N^+ \cap \Delta_{\bar 0} = \emptyset$ one easily proves that $P = L = \Delta$. $\square$

2.5. $\mathfrak{g} = \mathfrak{osp}(2m|2n), m > 1$. We have $\mathfrak{g}_{\bar{0}} \cong \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n)$ with $\Delta_{\bar{0}} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\delta_k \pm \delta_l, \pm 2\delta_k \mid 1 \leq i \neq j \leq m, 1 \leq k \neq l \leq n\}$ and $\Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_k \mid 1 \leq i \leq m, 1 \leq k \leq n\}$. Define the following sets of roots:

$$\begin{aligned}
L_{\mathfrak{osp}(2m|2n)}(m) &:= \{\varepsilon_i - \varepsilon_j \mid i \neq j\} \sqcup \{\delta_k - \delta_l \mid k \neq l\} \sqcup \{\pm(\varepsilon_i - \delta_k) \mid i, k \geq 1\},\\
N^+_{\mathfrak{osp}(2m|2n)}(m) &:= \{\varepsilon_i + \varepsilon_j, \mid i \neq j\} \sqcup \{\delta_k + \delta_l, 2\delta_k \mid k \neq l\} \sqcup \{\varepsilon_i + \delta_k \mid i, k \geq 1\},\\
L_{\mathfrak{osp}(2m|2n)}(1) &:= \{\pm\varepsilon_i \pm \varepsilon_j \mid j > i > 1\} \sqcup \{\pm\delta_k \pm \delta_l, \pm 2\delta_k \mid k \neq l\}\\
&\quad \sqcup \{\pm\varepsilon_i \pm \delta_k \mid i > 1, k \geq 1\},\\
N^+_{\mathfrak{osp}(2m|2n)}(1) &:= \{\varepsilon_1 \pm \varepsilon_j \mid j > 1\} \sqcup \{\varepsilon_1 \pm \delta_k \mid k \geq 1\},\\
\overline{L}_{\mathfrak{osp}(2m|2n)}(m) &:= \bar{\theta} L_{\mathfrak{osp}(2m|2n)}(m),\\
\overline{N}^+_{\mathfrak{osp}(2m|2n)}(m) &:= \bar{\theta} N^+_{\mathfrak{osp}(2m|2n)}(m),
\end{aligned}$$

where $\bar{\theta}$ is the involutive automorphism of the Dynkin diagram corresponding to the base $\{\delta_1 - \delta_2, \ldots, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\}$ that interchanges the last two simple roots and preserves all other roots (cf. Proposition 1.20).

Theorem 2.9. *There are three orbits of cominuscule parabolic sets of roots of* $\mathfrak{osp}(2m|2n)$ *under the action of the Weyl group* $W_{\mathfrak{so}(2m)} \times W_{\mathfrak{sp}(2n)}$ *of* $\mathfrak{g}_{\bar{0}}$. *The parabolic subsets* P *with the corresponding Levi superalgebras* $\mathfrak{l}$ *and nilradicals* $\mathfrak{n}^+$ *(considered as* $\mathfrak{l}$*-modules) are listed below.*

(i) $P_{\mathfrak{osp}(2m|2n)}(m) := L_{\mathfrak{osp}(2m|2n)}(m) \sqcup N^+_{\mathfrak{osp}(2m|2n)}(m)$ *with* $\mathfrak{l} \cong \mathfrak{gl}(m|n)$ *and* $\mathfrak{n}^+ \cong \bigwedge^2 V^{m|n}$.

(ii) $P_{\mathfrak{osp}(2m|2n)}(1) := L_{\mathfrak{osp}(2m|2n)}(1) \sqcup N^+_{\mathfrak{osp}(2m|2n)}(1)$ *with* $\mathfrak{l} \cong \mathfrak{osp}(2m-2|2n) \oplus \mathbb{C}$ *and* $\mathfrak{n}^+ \cong V^{2m-2|2n}$.

(iii) $\overline{P}_{\mathfrak{osp}(2m|2n)}(m) = \overline{L}_{\mathfrak{osp}(2m|2n)}(m) \sqcup \overline{N}^+_{\mathfrak{osp}(2m|2n)}(m)$ *with* $\mathfrak{l} \cong \mathfrak{gl}(m|n)$ *and* $\mathfrak{n}^+ \cong \bigwedge^2 V^{m|n}$.

Proof. Let $\Delta_{\mathfrak{so}(2m)} = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq m\}$, $\Delta_{\mathfrak{sp}(2n)} = \{\pm\delta_k \pm \delta_l, 2 \pm \delta_k \mid 1 \leq k \neq l \leq n\}$. Define $L_{\mathfrak{so}} := L \cap \Delta_{\mathfrak{so}(2m)}, N^+_{\mathfrak{so}} := N^+ \cap \Delta_{\mathfrak{so}(2m)}$ and $L_{\mathfrak{sp}} := L \cap \Delta_{\mathfrak{sp}(2n)}, N^+_{\mathfrak{sp}} := N^+ \cap \Delta_{\mathfrak{sp}(2n)}$. Obviously, $L_{\bar{0}} = L_{\mathfrak{so}} \sqcup L_{\mathfrak{sp}}$ and $N^+_{\bar{0}} = N^+_{\mathfrak{so}} \sqcup N^+_{\mathfrak{sp}}$.

By Proposition 1.9, we know that $P = P(\Lambda)$ is a principal parabolic set of roots determined by some functional Λ. Let $\Lambda(\varepsilon_i) = x_i, \Lambda(\delta_k) = y_k$ for some x_i and y_k.

Case 1: $N^+_{\mathfrak{so}} \neq \emptyset$ *and* $N^+_{\mathfrak{sp}} \neq \emptyset$. Proposition 1.20 (iii) implies that in the $W_{\mathfrak{sp}(2n)}$-orbit of P we have a parabolic subset with $L_{\mathfrak{sp}} = \{\delta_k - \delta_l \mid k \neq l\}$ and $N^+_{\mathfrak{sp}} = \{\delta_k + \delta_l, 2\delta_k \mid k \neq l\}$. Next, following Proposition 1.20 (iv), we consider three sub-cases for the $W_{\mathfrak{so}(2m)}$-orbit of P.

Case 1.1: $L_{\mathfrak{so}} = L_{\mathfrak{so}(2m)}(1)$ *and* $N^+_{\mathfrak{so}} = N^+_{\mathfrak{so}(2m)}(1)$. In this case $x_2 = \ldots = x_m = 0$, $x_1 > 0$, and $y_1 = \ldots = y_n > 0$. But then $\varepsilon_2 + \delta_1$ and $-\varepsilon_2 + \delta_1$ are in N^+ and $(\varepsilon_2 + \delta_1) + (-\varepsilon_2 + \delta_1) \in \Delta$, which contradicts the assumption that P is cominuscule.

Case 1.2: $L_{\mathfrak{so}} = L_{\mathfrak{so}(2m)}(m)$ *and* $N^+_{\mathfrak{so}} = N^+_{\mathfrak{so}(2m)}(m)$. Now $x_1 = x_2 = \ldots = x_m = x > 0$ and $y_1 = \ldots = y_n = y > 0$. If $x > y$ then $\varepsilon_1 - \delta_1, 2\delta_1 \in N^+$ with $(\varepsilon_1 - \delta_1) + 2\delta_1 \in \Delta$, which contradict the assumption that P is cominuscule. Similarly if $y > x$ we use that $\delta_1 \pm \varepsilon_1 \in N^+$ and reach a contradiction. It remains to consider $x = y$. In this case one verifies that $L = L_{\mathfrak{osp}(2m|2n)}(m)$ and $N^+ = N^+_{\mathfrak{osp}(2m|2n)}(m)$.

Case 1.3: $L_{\mathfrak{so}} = \bar{\theta} L_{\mathfrak{so}(2m)}(m)$ *and* $N^+_{\mathfrak{so}} = \bar{\theta} N^+_{\mathfrak{so}(2m)}(m)$. This follows from Proposition 1.20 and Case 1.2. We obtain $L = \bar{\theta} L_{\mathfrak{osp}(2m|2n)}(m)$ and $N^+ = \bar{\theta} N^+_{\mathfrak{osp}(2m|2n)}(m)$.

Case 2: $N^+_{\mathfrak{so}} \neq \emptyset$ *and* $N^+_{\mathfrak{sp}} = \emptyset$. In this case $y_1 = \ldots = y_n = 0$. We consider again three subcases for $N^+_{\mathfrak{so}}$.

Case 2.1: $L_{\mathfrak{so}} = L_{\mathfrak{so}(2m)}(1)$ *and* $N^+_{\mathfrak{so}} = N^+_{\mathfrak{so}(2m)}(1)$. We have $x_2 = \ldots = x_m = 0$ and $x_1 > 0$. This leads to $L = L_{\mathfrak{osp}(2m|2n)}(1)$ and $N^+ = N^+_{\mathfrak{osp}(2m|2n)}(1)$.

Case 2.2: $L_{\mathfrak{so}} = L_{\mathfrak{so}(2m)}(m)$ *and* $N^+_{\mathfrak{so}} = N^+_{\mathfrak{so}(2m)}(m)$. In this case $x_1 = \ldots = x_m > 0$. In this case $\varepsilon_1 - \delta_1, \varepsilon_2 + \delta_1 \in N^+$ with $(\varepsilon_1 - \delta_1) + (\varepsilon_2 + \delta_1) \in \Delta$, which leads to a contradiction.

Case 2.3: $L_{\mathfrak{so}} = \bar{\theta} L_{\mathfrak{so}(2m)}(m)$ *and* $N^+_{\mathfrak{so}} = \bar{\theta} N^+_{\mathfrak{so}(2m)}(m)$. Using Case 2.2 and Proposition 1.20(iv) we reach again a contradiction.

Case 3: $N^+_{\mathfrak{so}} = \emptyset$ *and* $N^+_{\mathfrak{sp}} \neq \emptyset$. In this case $x_1 = \ldots = x_m = 0$ and $y_1 = \ldots = y_n > 0$. We now have that $\varepsilon_1 + \delta_1, -\varepsilon_1 + \delta_1 \in N^+$ with $(\varepsilon_1 + \delta_1) + (-\varepsilon_1 + \delta_1) \in \Delta$, again a contradiction.

Case 4: $N^+_{\mathfrak{so}} = N^+_{\mathfrak{sp}} = \emptyset$. In this case $P = L = \Delta$. □

2.6. $\mathfrak{g} = \mathfrak{osp}(2|2n)$**.** We have $\mathfrak{g}_{\bar{0}} \cong \mathbb{C} \oplus \mathfrak{sp}(2n)$ with $\Delta_{\bar{0}} = \{\pm\delta_k \pm \delta_l, \pm 2\delta_k \mid 1 \leq k \neq l \leq n\}$ and $\Delta_{\bar{1}} = \{\pm\varepsilon_1 \pm \delta_k \mid 1 \leq k \leq n\}$. Let us define the following sets of roots:

$$\begin{aligned}
L_{\mathfrak{osp}(2|2n)}(0) &:= \Delta_{\bar{0}}, \\
N^+_{\mathfrak{osp}(2|2n)}(0) &:= \Delta^+_{\bar{1}} = \{\varepsilon_1 \pm \delta_k \mid k \geq 1\}, \\
L_{\mathfrak{osp}(2|2n)}(n) &:= \{\delta_k - \delta_l \mid k \neq l\} \sqcup \{\pm(\varepsilon_1 - \delta_k), \mid k \geq 1\}, \\
N^+_{\mathfrak{osp}(2|2n)}(n) &:= \{\delta_k + \delta_l, 2\delta_k \mid k \neq l\} \sqcup \{\varepsilon_1 + \delta_k \mid k \geq 1\}, \\
\overline{L}_{\mathfrak{osp}(2|2n)}(n) &:= \{\delta_k - \delta_l \mid k \neq l\} \sqcup \{\pm(\varepsilon_1 + \delta_k), \mid k \geq 1\}, \\
\overline{N}^+_{\mathfrak{osp}(2|2n)}(n) &:= \{\delta_k + \delta_l, 2\delta_k \mid k \neq l\} \sqcup \{-\varepsilon_1 + \delta_k \mid k \geq 1\}.
\end{aligned}$$

Note that $\overline{L}_{\mathfrak{osp}(2|2n)}(n) = \tau L_{\mathfrak{osp}(2|2n)}(n)$ and $\overline{N}^+_{\mathfrak{osp}(2|2n)}(n) = \tau N^+_{\mathfrak{osp}(2|2n)}(n)$, where τ is the automorphism of the Dynkin diagram of the base $\{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \delta_n + \varepsilon_1\}$ of Δ that interchanges the last two roots and preserves all other roots (one can think of τ as the diagram automorphism that interchanges ε_1 with $-\varepsilon_1$). Note also that $\tau \Delta^+_{\bar{1}} = -\Delta^+_{\bar{1}}$.

Set $P_{\mathfrak{osp}(2|2n)}(0) := L_{\mathfrak{osp}(2|2n)}(0) \sqcup N^+_{\mathfrak{osp}(2|2n)}(0)$, $P_{\mathfrak{osp}(2|2n)}(n) := L_{\mathfrak{osp}(2|2n)}(0) \sqcup N^+_{\mathfrak{osp}(2|2n)}(n)$, and $\overline{P}_{\mathfrak{osp}(2|2n)}(n) := \overline{L}_{\mathfrak{osp}(2|2n)}(n) \sqcup \overline{N}^+_{\mathfrak{osp}(2|2n)}(n)$.

Theorem 2.10. *There are four orbits of cominuscule parabolic sets of roots of $\mathfrak{g} = \mathfrak{osp}(2|2n)$ under the action of the Weyl group $W_{\mathfrak{sp}(2n)}$ of $\mathfrak{g}_{\bar{0}}$: $P_{\mathfrak{osp}(2|2n)}(0)$, $-P_{\mathfrak{osp}(2|2n)}(0)$,*[1] *$P_{\mathfrak{osp}(2|2n)}(n)$, and $\overline{P}_{\mathfrak{osp}(2|2n)}(n)$. The corresponding Levi subalgebras $\mathfrak{l}$ and nilradicals $\mathfrak{n}^+$ (considered as $\mathfrak{l}$-modules) are given by $\mathfrak{l} \cong \mathfrak{sp}(2n) \oplus \mathbb{C}, \mathfrak{n}^+ \cong V^{2n}$ in the first two cases, and $\mathfrak{l} \cong \mathfrak{sl}(1|n), \mathfrak{n}^+ \cong S^2V^{1|n}$ in the last two.*

Proof. By Proposition 1.9 we have $P = P(\Lambda)$ for some functional Λ. Let $\Lambda(\varepsilon_1) = x, \Lambda(\delta_k) = y_k$, $k = 1, \dots, n$, for some x and y_k in $\mathbb{C}$.

Case 1: $N^+_{\bar{0}} \neq \emptyset$. Due to Proposition 1.20 (iii) we may assume that $L_{\bar{0}} = L_{\mathfrak{sp}(2n)}$ and $N^+_{\bar{0}} = N^+_{\mathfrak{sp}(2n)}$. Thus $y_1 = \ldots = y_n = y > 0$. We proceed with three sub-cases.

Case 1.1: $y > |x|$. In this case $\pm\varepsilon + \delta_1$ are in N^+ and their sum is a root, which is a contradiction.

Case 1.2: $y < |x|$. Similarly to the previous case we reach a contradiction with the assumption that P is cominuscule. For example, if $-x > y > x > 0$ we have that $-\delta_1 - \varepsilon_1$ and $2\delta_1$ are in N^+. The other cases are analogous.

Case 1.3: $y = |x|$. It easily follows that in the case $y = x$ we obtain $P = P_{\mathfrak{osp}(2|2n)}(n)$, while in the case $y = -x$ we find $P = \overline{P}_{\mathfrak{osp}(2|2n)}(n)$.

Case 2: $N^+_{\bar{0}} = \emptyset$. In this case $y_1 = \ldots = y_n = 0$, and in particular $L = \Delta_{\bar{0}}$. Depending on the sign of x we have $P = P_{\mathfrak{osp}(2|2n)}(0)$ or $P = -P_{\mathfrak{osp}(2|2n)}(0)$. ☐

2.7. $\mathfrak{g} = D(2,1;\alpha)$**.** We have $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ with $\Delta_{\bar{0}} = \{\pm\gamma_i \mid i = 1,2,3\}$ and $\Delta_{\bar{1}} = \{\frac{1}{2}(\pm\gamma_1 \pm \gamma_2 \pm \gamma_3)\}$. Here $\pm\gamma_i$ denote the roots of the i-th copy of $\mathfrak{sl}(2)$ in $\mathfrak{g}_{\bar{0}}$. In this case we will classify the cominuscule parabolics of $\mathfrak{g}$ up to the action of the group $S(\Delta) = W_{\mathfrak{g}_{\bar{0}}} \ltimes S_3$, where S_3 acts by permutations on $\{\gamma_1, \gamma_2, \gamma_3\}$. (In fact $S(\Delta)$ is the group of automorphisms of Δ.) By considering the action of a larger group than $W_{\mathfrak{g}_{\bar{0}}}$ we avoid a longer list of similarly behaved cominuscule parabolics. We leave to the reader to reconstruct from this the $W_{\mathfrak{g}_{\bar{0}}}$-orbits of cominuscule parabolic subsets of roots of $D(2,1;\alpha)$.

The root system of $\mathfrak{g}$ coincides with the root system of $\mathfrak{osp}(4|2)$ and an explicit isomorphism $\Delta_{\mathfrak{g}} \to \Delta_{\mathfrak{osp}(4|2)}$ is provided by: $\gamma_1 \mapsto \varepsilon_1 + \varepsilon_2$, $\gamma_2 \mapsto \varepsilon_1 - \varepsilon_2$, $\gamma_3 \mapsto 2\delta_1$. Using this equivalence and Theorem 2.9 one easily verifies the following.

Theorem 2.11. *There is only one orbit of cominuscule parabolic sets of roots of $D(1,2;\alpha)$ under the action of $S(\Delta)$: $P_{D(1,2;\alpha)} = L_{D(1,2;\alpha)} \sqcup N_{D(1,2;\alpha)}$, where $L_{D(1,2;\alpha)} =$*

[1]one can certainly consider $-P_{\mathfrak{osp}(2|2n)}(0)$ as $\tau P_{\mathfrak{osp}(2|2n)}(0)$.

$\{\pm\gamma_3, \frac{1}{2}(\pm(\gamma_1-\gamma_2)\pm\gamma_3)\}$ *and* $N^+_{D(1,2;\alpha)} = \{\gamma_1,\gamma_2,\frac{1}{2}(\gamma_1+\gamma_2\pm\gamma_3)\}$. *For the corresponding Levi subalgebra and nilradical we have* $\mathfrak{l} \cong \mathfrak{gl}(2|1)$ *and* $\mathfrak{n}^+ \cong \bigwedge^2 V^{2|1}$ *as an* $\mathfrak{l}$*-module.*

2.8. $\mathfrak{g} = F(4)$**.** We have $\mathfrak{g}_{\bar{0}} \cong \mathfrak{so}(7)\oplus\mathfrak{sl}(2)$ with $\Delta_{\bar{0}} = \{\pm\varepsilon_i\pm\varepsilon_j, \pm\varepsilon_i, \pm\gamma \mid 1\le i\ne j\le 3\}$ and $\Delta_{\bar{1}} = \left\{\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\gamma)\right\}$. Here $\{\pm\varepsilon_i\pm\varepsilon_j,\pm\varepsilon_i \mid 1\le i\ne j\le 3\}$ denote the roots of $\mathfrak{so}(7)$, while $\pm\gamma$ denote the roots of $\mathfrak{sl}(2)$ in $\mathfrak{g}_{\bar{0}}$.

Theorem 2.12. *There are no cominuscule parabolic sets of roots of* $F(4)$*.*

Proof. By Proposition 1.9 we have $P = P(\Lambda)$ for some functional Λ. Let $\Lambda(\varepsilon_i) = x_i, \Lambda(\gamma) = y$, $i = 1,\ldots,3$, for some x_i and y in $\mathbb{C}$.

Case 1: $N^+\cap\{\pm\varepsilon_i\pm\varepsilon_j,\pm\varepsilon_i,\pm\gamma \mid 1\le i\ne j\le 3\}\ne\emptyset$. Due to Proposition 1.20 (ii) we may assume that $N^+\cap\{\pm\varepsilon_i\pm\varepsilon_j,\pm\varepsilon_i,\pm\gamma \mid 1\le i\ne j\le 3\} = N^+_{\mathfrak{so}(5)}$. Thus $x_1>0, x_2 = x_3 = 0$. But then $\frac{1}{2}(\varepsilon_1\pm\varepsilon_2)\in N^+$ and their sum is a root, which implies that P is not cominuscule.

Case 2: $N^+\cap\{\pm\varepsilon_i\pm\varepsilon_j,\pm\varepsilon_i,\pm\gamma \mid 1\le i\ne j\le 3\} = \emptyset$. In this case $x_1 = x_2 = x_3 = 0$. If $y>0$ then $\frac{1}{2}(\varepsilon_1\pm\gamma)\in N^+$ which contradicts again to the fact that P is cominuscule. The case $y<0$ is similar to the case $y>0$, while for $y=0$ we obtain $P=\Delta$. □

2.9. $\mathfrak{g} = G(3)$**.** We have $\mathfrak{g}_{\bar{0}} \cong G_2\oplus\mathfrak{sl}(2)$ with $\Delta_{\bar{0}} = \{\varepsilon_i-\varepsilon_j,\pm\varepsilon_i,\pm\gamma \mid 1\le i\ne j\le 3\}$ and $\Delta_{\bar{1}} = \left\{\pm\frac{\gamma}{2},\pm\varepsilon_i\pm\frac{\gamma}{2} \mid 1\le i\le 3\right\}$. Here $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 0$ and $\{\varepsilon_i-\varepsilon_j,\pm\varepsilon_i \mid 1\le i\ne j\le 3\}$ denote the roots of G_2, while $\pm\gamma$ denote the roots of $\mathfrak{sl}(2)$ in $\mathfrak{g}_{\bar{0}}$.

Theorem 2.13. *There are no cominuscule parabolic sets of roots of* $G(3)$*.*

Proof. By Proposition 1.9 we have $P = P(\Lambda)$ for some functional Λ. Let $\Lambda(\varepsilon_i) = x_i, \Lambda(\gamma) = y$, $i = 1,2$, for some x_i and y in $\mathbb{C}$. Since G_2 has no cominuscule parabolic sets of roots we have that $N^+\cap\{\varepsilon_i-\varepsilon_j,\pm\varepsilon_i \mid 1\le i\ne j\le 3\} = \emptyset$. This implies $x_1 = x_2 = 0$. In the case $y>0$ we have that $\pm\varepsilon_1+\frac{\gamma}{2}\in N^+$ and their sum is a root, which contradicts with the fact that P is cominuscule. The case $y<0$ is similar, while for $y=0$ we obtain $P=\Delta$. □

3. Classification of the cominuscule parabolics of strange classical Lie superalgebras

In this section we classify the cominuscule parabolics of the two strange classical Lie superalgebras $\mathfrak{g} = \mathfrak{psq}(n)$ and $\mathfrak{g} = \mathfrak{sp}(n)$. As in the case of the basic classical Lie superalgebras, for these superalgebras the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is a reductive Lie algebra. Analogously to the previous section, the Weyl group $W_{\mathfrak{g}_{\bar{0}}}$ of $\mathfrak{g}_{\bar{0}}$ acts on the root system Δ of $\mathfrak{g}$ (and thus on $\Delta\cup(-\Delta)$). This action induces an action of $W_{\mathfrak{g}_{\bar{0}}}$ on the set of parabolic subsets of Δ, which preserves the class of cominuscule parabolic subsets of Δ. We will classify the cominuscule parabolic subsets of Δ up to this action of the Weyl group $W_{\mathfrak{g}_{\bar{0}}}$.

In all proofs we will assume that $P = \widetilde{P} \cap \Delta$ is a cominuscule parabolic set of roots for some parabolic subset $\widetilde{P}$ of $\Delta \cup (-\Delta)$ for which the corresponding nilradical $\mathfrak{n}^+$ of $\mathfrak{p}_P$ is abelian, recall Definition 1.4. The root system of $\mathfrak{psq}(n)$ is symmetric (and $P = \widetilde{P}$ for $\mathfrak{g} = \mathfrak{psq}(n)$), while the one of $\mathfrak{sp}(n)$ is not. We will use the notation L, N^+, $\mathfrak{l}$, and $\mathfrak{n}^+$ from Definitions 1.4 and 1.11. Set $P_{\bar{0}} := P \cap \Delta_{\bar{0}}$. By Lemma 1.19, either $P_{\bar{0}} = \Delta_{\bar{0}}$ or $P_{\bar{0}}$ is a parabolic subset of $\mathfrak{g}_{\bar{0}}$ with Levi component $L_{\bar{0}} := L \cap \Delta_{\bar{0}}$ and nilradical $N^+_{\bar{0}} := N^+ \cap \Delta_{\bar{0}}$. We also set $P^- = L \sqcup N^- = (-\widetilde{P}) \cap \Delta$ and $N^- = \widetilde{N}^- \cap \Delta$ where $\widetilde{N}^- = (-\widetilde{P}) \backslash \widetilde{P}$, cf. Lemma 1.5. Then P^- is a parabolic subset of Δ and $P^- = L \sqcup N^-$ is a Levi decomposition of P^-. The corresponding Levi decomposition of the parabolic subalgebra $\mathfrak{p}_{P^-}$ of $\mathfrak{g}$ is $\mathfrak{p}_{P^-} = \mathfrak{l} \ltimes \mathfrak{n}^-$, where $\mathfrak{n}^- := \bigoplus_{\alpha \in N^-} \mathfrak{g}^\alpha$.

3.1. $\mathfrak{g} = \mathfrak{psq}(n)$**.** In this case $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(n)$ with $\Delta_{\bar{0}} = \Delta_{\bar{1}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$. Because Δ coincides with the root system of $\mathfrak{sl}_n$ one can easily modify Proposition 1.20 (i) and obtain the classification of the cominuscule parabolic subalgebras of $\mathfrak{psq}(n)$. For $1 \leq n_0 \leq n-1$, set $L_{\mathfrak{psq}(n)}(n_0) = L_{\mathfrak{sl}(n)}(n_0)$, $N^+_{\mathfrak{psq}(n)}(n_0) = N_{\mathfrak{sl}(n)}(n_0)$, and $P_{\mathfrak{psq}(n)}(n_0) = P_{\mathfrak{sl}(n)}(n_0)$. The details are left to the reader. For the next theorem we introduce some notation. Recall that $\mathfrak{q}(m)$ is the Lie superalgebra of all matrices $X = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where A and B are $m \times m$ matrices. We set $\operatorname{otr}(X) := \operatorname{tr} B$. For n_0, $1 \leq n_0 \leq n-1$, denote

$$\mathfrak{sq}(n_0, n-n_0) := \{(X, Y) \in \mathfrak{q}(n_0) \oplus \mathfrak{q}(n-n_0) \mid \operatorname{otr}(X) + \operatorname{otr}(Y) = 0\},$$

and set $\mathfrak{psq}(n_0, n-n_0) := \mathfrak{sq}(n_0, n-n_0)/(\mathbb{C}\,\mathrm{Id})$. In particular, $\mathfrak{psq}(1, n-1) \simeq \mathfrak{psq}(n-1, 1) \simeq \mathfrak{psq}(n-1)$.

Theorem 3.1. *There are $n-1$ orbits of cominuscule parabolic sets of roots of $\mathfrak{g} = \mathfrak{psq}(n)$ under the action of the Weyl group $W_{\mathfrak{sl}(n)}$ of $\mathfrak{g}_{\bar{0}}$ with representatives $P_{\mathfrak{psq}(n)}(n_0)$, for $1 \leq n_0 \leq n-1$. The corresponding Levi subalgebras $\mathfrak{l}$ and nilradicals (considered as $\mathfrak{l}$-modules) are given by $\mathfrak{l} \cong \mathfrak{psq}(n_0, n-n_0)$ and $\mathfrak{n}^+ \cong V^{n_0|n_0} \otimes (V^{n-n_0|n-n_0})^*$.*

3.2. $\mathfrak{g} = \mathfrak{sp}(n)$**.** We have $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(n)$ with $\Delta_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$ and $\Delta_{\bar{1}} = \{\pm(\varepsilon_i + \varepsilon_j), 2\varepsilon_i \mid 1 \leq i < j \leq n\}$. Since $\Delta \neq -\Delta$, we need to pass to the parabolic subsets of $\Delta \cup -\Delta$. On the other hand $\Delta \cup (-\Delta)$ coincides with the root system of $\mathfrak{sp}(2n)$, hence all parabolic subsets of $\Delta \cup (-\Delta)$ are principal, cf. §1.2. Let us define the following sets of roots:

$$\begin{aligned}
L_{\mathfrak{sp}(n)}(0) &:= \Delta_{\bar{0}}, \\
N^+_{\mathfrak{sp}(n)}(0) &:= \Delta^+_{\bar{1}} = \{\varepsilon_i + \varepsilon_j, 2\varepsilon_i \mid i < j\}, \\
N^-_{\mathfrak{sp}(n)}(0) &:= \Delta^-_{\bar{1}} = \{-\varepsilon_i - \varepsilon_j \mid i < j\}, \\
L_{\mathfrak{sp}(n)}(n_0) &:= \{\varepsilon_i - \varepsilon_j \mid i, j \leq n_0 \text{ or } i, j > n_0\} \sqcup \{\pm(\varepsilon_i + \varepsilon_j) \mid i \leq n_0 < j\}, \\
N^+_{\mathfrak{sp}(n)}(n_0) &:= \{\varepsilon_i - \varepsilon_j \mid i \leq n_0 < j\} \sqcup \{\varepsilon_i + \varepsilon_j \mid i \leq j \leq n_0\} \\
&\quad \sqcup \{-\varepsilon_i - \varepsilon_j \mid i > j > n_0\}, \\
L_{\mathfrak{sp}(n)}(n) &:= \{\varepsilon_i - \varepsilon_j \mid i, j < n\} \sqcup \{\pm(\varepsilon_i + \varepsilon_j), 2\varepsilon_i \mid i < j < n\}, \\
N^+_{\mathfrak{sp}(n)}(n) &:= \{-\varepsilon_n \pm \varepsilon_j \mid j < n\},
\end{aligned}$$

for $1 \leq n_0 \leq n-1$. Set $P_{\mathfrak{sp}(n)}(n_0) := L_{\mathfrak{sp}(n)}(n_0) \sqcup N^+_{\mathfrak{sp}(n)}(n_0)$ for $0 \leq n_0 \leq n$.

Theorem 3.2. *There are $n+2$ orbits of cominuscule parabolic sets of roots of $\mathfrak{g} = \mathfrak{sp}(n)$ under the action of the Weyl group $W_{\mathfrak{sl}(n)}$ of $\mathfrak{g}_{\bar{0}}$ with representatives $P_{\mathfrak{sp}(n)}(n_0)$ for $0 \leq n_0 \leq n$ and $P^-_{\mathfrak{sp}(n)}(0) = -P_{\mathfrak{sp}(n)}(0)$. All cominuscule parabolic sets of roots have unique Levi decompositions and the Levi subalgebras $\mathfrak{l}$ and nilradicals (considered as $\mathfrak{l}$-modules) of the above parabolics are given by:*

- $\mathfrak{l} \cong \mathfrak{sl}(n), \mathfrak{n}^+ \cong S^2 V^n, \mathfrak{n}^- \cong \bigwedge^2 (V^n)^*$ *for* $n_0 = 0$*;*
- $\mathfrak{l} \cong \mathfrak{sl}(n_0|n-n_0), \mathfrak{n}^+ \cong S^2 V^{n_0|n-n_0}$*, for* $1 \leq n_0 \leq n-1$*;*
- $\mathfrak{l} \cong \mathfrak{sp}(n-1) \oplus \mathbb{C}, \mathfrak{n}^+ \cong V^{n-1|n-1}$*, for* $n_0 = n$*.*

Proof. Since $\Delta \cup (-\Delta)$ coincides with the root system of $\mathfrak{sp}(2n)$, all parabolic sets of roots of $\Delta \cup (-\Delta)$ are principal. We consider $\Delta \cup (-\Delta)$ as a subset of a real vector space V with basis $\{\varepsilon_i \mid 1 \leq i \leq n\}$. We have $\widetilde{P} = \widetilde{P}(\Lambda)$ for some functional Λ on V. Let $\Lambda(\varepsilon_i) = x_i$, $1 \leq i \leq n$, for some $x_i \in \mathbb{R}$.

Case 1: $N^+_{\bar{0}} \neq \emptyset$. By Proposition 1.20 we have that, up to the action of the Weyl group $W_{\mathfrak{sl}(n)}$ of $\mathfrak{g}_{\bar{0}}$, $L_{\bar{0}} = L_{\mathfrak{sl}(n)}(n_0)$ and $N^+_{\bar{0}} = N_{\mathfrak{sl}(n)}(n_0)$ for some n_0, $1 \leq n_0 \leq n-1$. Thus $x_1 = \ldots = x_{n_0} = x$, $x_{n_0+1} = \ldots = x_n = y$, for some x and y such that $x > y$. We proceed with three subcases.

Case 1.1: $x + y > 0$. We have that $\varepsilon_1 - \varepsilon_n$ and $\varepsilon_1 + \varepsilon_n$ are in N^+ and their sum is a root, which is a contradiction.

Case 1.2: $x + y = 0$. We have that $x = -y > 0$. In this case $L = L_{\mathfrak{sp}(n)}(n_0)$ and $N^+ = N_{\mathfrak{sp}(n)}(n_0)$.

Case 1.3: $x + y < 0$. If $n_0 < n-1$, then $\varepsilon_1 - \varepsilon_{n-1}$ and $-\varepsilon_1 - \varepsilon_n$ are in N^+ and their sum is a root. It remains to consider the case $n_0 = n-1$. We easily see that $\pm(\varepsilon_i + \varepsilon_j)$ are not in N^+ for every $i, j > 1$. Thus $x = 0$ which leads to $L = L_{\mathfrak{sp}(n)}(n)$ and $N^+ = N^+_{\mathfrak{sp}(n)}(n)$.

Case 2: $N_{\bar{0}}^{+} = \emptyset$. In this case $\Delta_{\bar{0}} \subset L_{\bar{0}}$ and in particular $x_1 = \ldots = x_n = x$. If $x > 0$ we obtain $P = P_{\mathfrak{sp}(n)}(0)$, while for $x < 0$ we have $P = P^{-}_{\mathfrak{sp}(n)}(0)$.

The isomorphisms for the Levi components $\mathfrak{l}$ and the structure of the nilradicals as $\mathfrak{l}$-modules are straightforward and are left to the reader. □

We extend Remark 1.7 with an illustration of the nonuniquenes of Levi decompositions of parabolic sets of roots for the root system of $\mathfrak{sp}(n)$.

Remark 3.3. *Denote the subset of roots*

$$P := \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \sqcup \{2\varepsilon_i \mid 1 \le i \le n\} \subset \Delta.$$

Let us identify V^* with $\mathbb{R}^n$, where $\Lambda \in V^* \mapsto (\Lambda(\varepsilon_1), \ldots, \Lambda(\varepsilon_n))$. The polyhedron $\mathcal{F}(P)$ defined in Remark 1.7 is given by

$$\mathcal{F}(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > \ldots > x_n \ge 0\}.$$

In particular, $\mathcal{F}(P)$ is nonempty and P is a principal parabolic subset of Δ. Furthermore, $\mathcal{F}(P)$ has two faces with interiors

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > \ldots > x_n > 0\} \text{ and } \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > \ldots > x_n = 0\}.$$

The corresponding Levi components are given by

$$L = \emptyset \text{ and } L = \{2\varepsilon_n\},$$

respectively. One easily generalizes this example to show that for every parabolic subset P of the root system of $\mathfrak{sp}(n)$, the polyhedron $\mathcal{F}(P)$ has at most two faces. (The small number of faces is due to the fact that $\Delta\backslash(-\Delta)$ has only n roots.) The corresponding Levi decompositions of P are easily described in a similar fashion.

4. Classification of cominuscule parabolics of Cartan type Lie superalgebras

In this section we classify the cominuscule parabolic sets of roots of all Cartan type Lie superalgebras $\mathfrak{g}$. Each such Lie superalgebra has a natural Cartan subalgebra $\mathfrak{h}$. The corresponding root system will be denoted by Δ. In each case the even part $\mathfrak{g}_{\bar{0}}$ has a natural Levi subalgebra $\mathfrak{l}_{\bar{0}}$ such that $\mathfrak{l}_{\bar{0}} \cap \mathfrak{h} = \mathfrak{h}_{\bar{0}}$. The action of the Weyl group $W_{\mathfrak{l}_{\bar{0}}}$ of the Levi subalgebra $\mathfrak{l}_{\bar{0}}$ on the root system of $\mathfrak{l}_{\bar{0}}$ extends to an action of $W_{\mathfrak{l}_{\bar{0}}}$ on Δ. For each parabolic subset P of Δ with a Levi decomposition $P = L \sqcup N^+$ and $w \in W_{\mathfrak{l}_{\bar{0}}}$, $w(P)$ is a parabolic subset of Δ and $w(P) = w(L) \sqcup w(N^+)$ is a Levi decomposition of $w(P)$. Furthermore, P is cominuscule if and only if $w(P)$ is cominuscule. Our classification amounts to classifying the orbits of cominuscule parabolic subsets of Δ under the action of the Weyl group $W_{\mathfrak{l}_{\bar{0}}}$ of the Levi subalgebra $\mathfrak{l}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$.

Given a pair of integers $j \le k$, set $[j, k] := \{j, \ldots, k\}$.

4.1. $\mathfrak{g} = W(n)$. Let $\mathfrak{g} := W(n) = W(\xi_1, \dots, \xi_n)$ be the Lie superalgebra consisting of the superderivations of the Grassmann algebra $\bigwedge(n) = \bigwedge(\xi_1, \dots, \xi_n)$. The elements of $W(n)$ have the form $\sum_{i=1}^n p_i(\xi_1, \dots, \xi_n) \frac{\partial}{\partial \xi_i}$, where $p_i \in \bigwedge(n)$ and $\frac{\partial}{\partial \xi_i}$ are the derivations of $\bigwedge(n)$ such that $\frac{\partial}{\partial \xi_l}(\xi_l) = \delta_{il}$, for all $1 \leq l \leq n$. The standard Cartan subalgebra of $W(n)$ is

$$\mathfrak{h} = \operatorname{Span}\left\{ \xi_i \frac{\partial}{\partial \xi_i} \;\middle|\; 1 \leq i \leq n \right\}.$$

Both $\bigwedge(n)$ and $W(n)$ have natural gradings: $\bigwedge(n) = \bigoplus_{k=0}^n \bigwedge(n)_k$ and $W(n) = \bigoplus_{k=-1}^{n-1} W(n)_k$, where

$$\bigwedge(n)_k := \{p(\xi_1, \dots, \xi_n) \mid \deg p = k\} \text{ and } W(n)_k := \{\sum_{l=1}^n p_i \frac{\partial}{\partial \xi_i} \mid \deg p_i = k+1\}.$$

In particular, $W(n)_0 \cong \mathfrak{gl}(n)$. Set $\bigwedge(n)_{\geq j} := \bigoplus_{k \geq j} \bigwedge(n)_k$ and $W(n)_{\geq j} := \bigoplus_{k \geq j} W(n)_k$. The Lie algebra $\mathfrak{g}_{\bar{0}}$ has the Levi subalgebra

$$\mathfrak{l}_{\bar{0}} = W(n)_0 \supset \mathfrak{h}_{\bar{0}}$$

and nilradical

$$\mathfrak{n}_{\bar{0}}^+ = \bigoplus_{k \geq 1} W(n)_{2k}.$$

The Weyl group $W_{\mathfrak{l}_{\bar{0}}}$ is isomorphic to the symmetric group S_n. Its action on the root lattice of $\mathfrak{l}_{\bar{0}}$ extends to actions on $\bigwedge(n)$ and $W(n)$ by Lie algebra automorphisms: for $\sigma \in S_n$, $\sigma(\xi_i) = x_{\sigma(i)}$, $\sigma(\partial/\partial \xi_i) = \partial/\partial \xi_{\sigma(i)}$.

For $1 \leq i \leq n$ denote

$$\varepsilon_i \in \mathfrak{h}^*, \quad \varepsilon_i \left(\xi_l \frac{\partial}{\partial \xi_l} \right) = \delta_{il}.$$

Given $I = \{i_1, \dots, i_k\} \subseteq [1, n]$ and $j \in [1, n]$ such that $j \notin I$, we set

$$\varepsilon_{I,j} := \varepsilon_{i_1} + \ldots + \varepsilon_{i_k} - \varepsilon_j.$$

For $I = \{i_1, \dots, i_k\} \subsetneq [1, n]$, set

$$\varepsilon_I := \varepsilon_{i_1} + \ldots + \varepsilon_{i_k}.$$

The root system of $\mathfrak{g} = W(n)$ is

$$\Delta = \{\varepsilon_{I,j} \mid I \subseteq [1, n], j \in [1, n], j \notin I\} \sqcup \{\varepsilon_I \mid I \subsetneq [1, n]\}.$$

The corresponding root spaces are

$$\mathfrak{g}^{\varepsilon_{I,j}} = \operatorname{Span}\left\{ \xi_{i_1} \dots \xi_{i_k} \frac{\partial}{\partial \xi_j} \right\}$$

for $I = \{i_1, \dots, i_k\} \subseteq [1, n]$, $j \in ([1, n] \backslash I)$ and

$$\mathfrak{g}^{\varepsilon_I} = \operatorname{Span}\left\{ \xi_{i_1} \dots \xi_{i_k} \xi_l \frac{\partial}{\partial \xi_l} \;\middle|\; l \in ([1, n] \backslash I) \right\}$$

for $I = \{i_1, \dots, i_k\} \subsetneq [1, n]$.

Consider the subalgebra of $\mathfrak{g} = W(n)$

$$\mathfrak{s} := \mathrm{Span}\left\{ \frac{\partial}{\partial \xi_i}, \xi_i \frac{\partial}{\partial \xi_j}, \xi_i \sum_{l=1}^{n} \xi_l \frac{\partial}{\partial \xi_l} \;\Big|\; 1 \le i \ne j \le n \right\} \supset \mathfrak{h}.$$

Its root system, considered as a subsystem of Δ, is given by

$$\Delta_{\mathfrak{s}} = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\} \sqcup \{\pm\epsilon_i \mid 1 \le i \le n\}.$$

We have the isomorphism

$$\mathfrak{s} \cong \mathfrak{sl}(1|n), \quad \xi_i \frac{\partial}{\partial \xi_j} \mapsto E_{i+1,j+1} - \delta_{ij} E_{11},\; \xi_i \sum_{l=1}^{n} \xi_l \frac{\partial}{\partial \xi_l} \mapsto E_{i+1,1},\; \frac{\partial}{\partial \xi_i} \mapsto E_{1,i+1}, \tag{4.1}$$

for $1 \le i \ne j \le n$.

Recall that the root system of $\mathfrak{sl}(1|n)$ is

$$\{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\} \sqcup \{\pm(\delta_1 - \epsilon_i) \mid 1 \le i \le n\}.$$

(We interchanged the roles of ε and δ from §2.1 in order to align that notation to the standard one for $W(n)$.) The root system of $\mathfrak{sl}(1|n)$ is identified with $\Delta_{\mathfrak{s}}$ via the isomorphism (4.1) and $\delta_1 \mapsto 0$.

In order to describe the cominuscule parabolic sets of roots of $W(n)$, we introduce the following sets

$$\begin{aligned}
L_{W(n)}(n_0) &= \{\varepsilon_I \mid I \subseteq [1, n_0]\} \sqcup \{\varepsilon_{I,j} \mid I \subseteq [1, n_0], j \in [1, n_0], j \notin I\} \\
&\quad \sqcup \{\varepsilon_{I \sqcup \{i\}, j} \mid I \subseteq [1, n_0], i \ne j \in [n_0 + 1, n]\}, \\
N^+_{W(n)}(n_0) &= \{\varepsilon_{I,j} \mid I \subseteq [1, n_0], j \in [n_0 + 1, n]\}, \\
N^-_{W(n)}(n_0) &= \{\varepsilon_I \mid I \subseteq [1, n], I \not\subseteq [1, n_0]\} \\
&\quad \sqcup \{\varepsilon_{I,j} \mid I \subseteq [1, n], I \not\subseteq [1, n_0], j \in [1, n_0], j \notin I\} \\
&\quad \sqcup \{\varepsilon_{I,j} \mid I \subseteq [1, n], j \in [n_0 + 1, n], j \notin I, |I \cap [n_0 + 1, n]| \ge 2\},
\end{aligned}$$

where $0 \le n_0 < n$ and $|S|$ denotes the cardinality of a finite set S. The sets

$$P_{W(n)}(n_0) = L_{W(n)}(n_0) \sqcup N^+_{W(n)}(n_0) \text{ and } P^-_{W(n)}(n_0) = L_{W(n)}(n_0) \sqcup N^-_{W(n)}(n_0) \tag{4.2}$$

are principal parabolic subsets of Δ with respect to the functionals Λ_{n_0} and $-\Lambda_{n_0}$, where

$$\Lambda_{n_0}(\varepsilon_i) = 0 \text{ for } i \in [1, n_0], \;\; \Lambda_{n_0}(\varepsilon_i) = -1 \text{ for } i \in [n_0 + 1, n]. \tag{4.3}$$

In particular, $\Delta = N^+(n_0) \sqcup L(n_0) \sqcup N^-(n_0)$ and the decompositions in (4.2) are Levi decompositions. Using Proposition 1.20, it is straightforward to verify that the sets $P_{W(n)}(n_0)$ are cominuscule, while the set $P^-_{W(n)}(n_0)$ is cominuscule if and only if $n_0 = n - 1$. Denote by $\mathfrak{l}_{W(n)}(n_0)$ and $\mathfrak{n}^{\pm}_{W(n)}(n_0)$ the root subalgebras of $W(n)$ corresponding to the sets of roots $L_{W(n)}(n_0)$ and $N^{\pm}_{W(n)}(n_0)$, and such that

$\mathfrak{l}_{W(n)}(n_0) \supset \mathfrak{h}$ and $\mathfrak{n}^{\pm}(n_0) \cap \mathfrak{h} = 0$, recall (1.10). For $1 \leq k \leq l \leq n$ denote the subalgebra

$$\mathfrak{gl}[k,l] = \operatorname{Span}\left\{ \xi_i \frac{\partial}{\partial \xi_j} \;\middle|\; k \leq i, j \leq l \right\}$$

of $W(n)$. Clearly $\mathfrak{gl}[k,l] \cong \mathfrak{gl}(l-k)$.

Lemma 4.4. *For all $0 \leq n_0 \leq n-1$ we have the isomorphism of Lie superalgebras*

$$\mathfrak{l}_{W(n)}(n_0) \cong W(\xi_1, \ldots, \xi_{n_0}) \ltimes \left(\bigwedge(\xi_1, \ldots, \xi_{n_0}) \otimes \mathfrak{gl}[n_0+1, n] \right),$$

where the second term represents the Lie superalgebra which is the tensor product of a supercommutative algebra and a Lie superalgebra, and $W(\xi_1, \ldots, \xi_{n_0})$ acts on $\bigwedge(\xi_1, \ldots, \xi_{n_0})$ by derivations. Moreover, we have the isomorphisms of $\mathfrak{l}_{W(n)}(n_0)$-modules

$$\begin{aligned}
\mathfrak{n}^{+}_{W(n)}(n_0) &\cong \bigwedge(\xi_1, \ldots, \xi_{n_0}) \otimes (V^{n-n_0})^* \;\; \textit{and} \\
\mathfrak{n}^{-}_{W(n)}(n_0) &\cong W(\xi_1, \ldots, \xi_{n_0}) \otimes \bigwedge(\xi_{n_0+1}, \ldots, \xi_n)_{\geq 1} \\
&\quad \oplus \bigwedge(\xi_1, \ldots, \xi_{n_0}) \otimes W(\xi_{n_0+1}, \ldots, \xi_n)_{\geq 1}.
\end{aligned}$$

For the module structure of the right hand sides we use the adjoint actions of the chain of Lie subalgebras $\mathfrak{gl}[n_0+1, n] \subset W(\xi_{n_0+1}, \ldots, \xi_n)_{\geq 1} \subset W(\xi_{n_0+1}, \ldots, \xi_n)$ and the (left) multiplication action of the supercommutative algebra $\bigwedge(\xi_1, \ldots, \xi_{n_0})$ on $W(\xi_1, \ldots, \xi_{n_0})$ and itself. The symbol $(V^{n-n_0})^*$ denotes the dual of the vector representation of $\mathfrak{gl}[n_0+1, n]$.

The proof of of Lemma 4.4 amounts to a direct computation of the root algebras $\mathfrak{l}_{W(n)}(n_0)$ and $\mathfrak{n}^{\pm}_{W(n)}(n_0)$ and is left to the reader. The next result classifies and describes the cominuscule parabolic sets of roots of $W(n)$.

Theorem 4.5. *There are $n+1$ orbits of cominuscule parabolic subsets of the root system of $W(n)$ under the action of the Weyl group $W_{\mathfrak{l}_{\bar{0}}} \cong S_n$. The parabolic sets $P = P_{W(n)}(n_0)$, $0 \leq n_0 \leq n-1$, and $P = P^{-}_{W(n)}(n-1)$ provide representatives of those orbits. They have unique Levi decompositions given by*

$$P_{W(n)}(n_0) = L_{W(n)}(n_0) \sqcup N_{W(n)}(n_0),\; 0 \leq n_0 \leq n-1 \;\; \textit{and} \tag{4.6}$$

$$P^{-}_{W(n)}(n-1) = L_{W(n)}(n-1) \sqcup N^{-}_{W(n)}(n-1). \tag{4.7}$$

The Levi components $\mathfrak{l}_{W(n)}(n_0)$ of the corresponding parabolic subalgebras of $W(n)$ and their nilradicals $\mathfrak{n}^{\pm}_{W(n)}(n_0)$ considered as $\mathfrak{l}_{W(n)}(n_0)$-modules are given by Lemma 4.4

Proof. Assume that P is a cominuscule parabolic subset of Δ, and that $P = L \sqcup N^+$ is a Levi decomposition of P. First we show that $\Delta_{\mathfrak{s}} \not\subseteq P$. Indeed, if $\Delta_{\mathfrak{s}} \subseteq P$, then

$\pm\varepsilon_i \in L$ for all $i \in [1,n]$. This implies that $\Delta = P$, which contradicts to the condition that P is a proper subset of Δ.

Therefore by Lemma 1.19, $P \cap \Delta_{\mathfrak{s}}$ is a cominuscule parabolic subset of $\Delta_{\mathfrak{s}}$. Observe that $\mathfrak{s}_{\bar{0}} = \mathfrak{l}_{\bar{0}}$, thus $W_{\mathfrak{s}_{\bar{0}}} \cong W_{\mathfrak{l}_{\bar{0}}} \cong S_n$. Using Theorem 2.1 and the isomorphism (4.1) we obtain that there exist integers $0 \le m_0 \le 1$ and $0 \le n_0 \le n$, $(m_0, n_0) \ne (0,0)$, $(1,n)$, such that P is conjugated under the action of $W_{\mathfrak{l}_{\bar{0}}}$ to a cominuscule parabolic subset of Δ such that

$$L \cap \Delta_{\mathfrak{s}} = L_{\mathfrak{s}}(m_0,n_0) \text{ and } N^+ \cap \Delta_{\mathfrak{s}} = N^+_{\mathfrak{s}}(m_0,n_0), \tag{4.8}$$

where

$$L_{\mathfrak{s}}(m_0,n_0) = \{\varepsilon_i - \varepsilon_j \mid i,j \le n_0 \text{ or } i,j > n_0\}, \tag{4.9}$$
$$\sqcup\{\pm\varepsilon_i \mid i \le n_0 \text{ if } m_0 = 1, i > n_0 \text{ otherwise}\}$$
$$N^+_{\mathfrak{s}}(0,n_0) = \{\varepsilon_i - \varepsilon_j \mid i \le n_0 < j\} \sqcup \{\varepsilon_i \mid i \le n_0\}, \tag{4.10}$$
$$N^+_{\mathfrak{s}}(1,n_0) = \{\varepsilon_i - \varepsilon_j \mid i \le n_0 < j\} \sqcup \{-\varepsilon_j \mid j > n_0\}. \tag{4.11}$$

We conjugate P by an element of $W_{\mathfrak{l}_{\bar{0}}}$ so that (4.8) holds.

We first consider the case $m_0 = 0$. If $n_0 > 1$ then from (4.8) and (4.10) we obtain $\varepsilon_1, \varepsilon_2 \in N^+$. Since $\varepsilon_1 + \varepsilon_2 \in \Delta$, Proposition 1.20 leads to a contradiction. Thus $n_0 = 1$. Lemma 1.5 implies that

$$L \supseteq \{\varepsilon_{I,j} \mid I \subseteq [2,n], j \in [2,n], j \notin I\} \sqcup \{\varepsilon_I \mid I \subseteq [2,n]\}, \tag{4.12}$$
$$N^+ \supseteq \{\varepsilon_{\{1\}\sqcup I,j} \mid I \subseteq [2,n], j \in [2,n], j \notin I\} \sqcup \{\varepsilon_{\{1\}\sqcup I} \mid I \subseteq [2,n]\}, \tag{4.13}$$

where w_0 is the longest element of $W_{\mathfrak{l}_{\bar{0}}} \cong S_n$. If any of these inclusions are strict, then P contains an element of the form $\varepsilon_{I,1}$, where $I \subseteq [2,n]$, $1 \notin I$. Since $\pm\epsilon_j \in P$ for all $j \in [2,n]$, this would imply that $-\epsilon_1 \in P$. Therefore $P = \Delta$, because $\epsilon_1 \in P$. This is a contradiction. Thus both inclusion (4.12) and (4.13) are equalities, which implies that $w_0(L) = L_{W(n)}(n-1)$ and $w_0(N^+) = N^-_{W(n)}(n-1)$, where w_0 is the longest element of $W_{\mathfrak{l}_{\bar{0}}} \cong S_n$.

Now let $m_0 = 1$. We will show that $L = L_{W(n)}(n_0)$ and $N^+ = N^+_{W(n)}(n_0)$. First we prove that

$$L \supseteq L_{W(n)}(n_0) \text{ and } N^+ \supseteq N^+_{W(n)}(n_0). \tag{4.14}$$

Since for all $i \in [1,n_0]$, $\pm\varepsilon_i \in L$, Lemma 1.5 (iii) implies that for all $I \subseteq [1,n_0]$, $\varepsilon_I \in L$, and for all $I \subseteq [1,n_0]$ and $j \in [1,n_0]$, $j \notin I$, $\varepsilon_{I,j} = \varepsilon_I - \epsilon_j \in L$. Similarly for all $I \subseteq [1,n_0]$ and $i \ne j \in [n_0+1,n]$, $\varepsilon_{I\sqcup\{i\},j} = \varepsilon_I + (\varepsilon_i - \epsilon_j) \in L$. This proves the first inclusion in (4.14).

Since $\varepsilon_I \in L$ for $I \subseteq [1,n_0]$ and by (4.11) $-\epsilon_j \in N^+$ for $j \in [n_0+1,n]$, Lemma 1.5 (ii) implies that $\varepsilon_{I,j} \in N^+$, under the same conditions on I and j. This proves the second inclusion in (4.14).

To prove that the inclusions in (4.14) are equalities, we need to show that

$$P \cap N^-_{W(n)}(n_0) = \emptyset. \tag{4.15}$$

First we show that

$$\varepsilon_I \notin P, \ \text{ for } \ I \subseteq [1,n], I \not\subseteq [1,n_0]. \tag{4.16}$$

Assume the opposite. Then $-\varepsilon_j \in N^+$, $\forall j \in I \cap [n_0+1,n]$ and Lemma 1.5 (ii), (iv) imply that $\varepsilon_{I\cap[1,n_0]} = \varepsilon_I + \sum_{j \in I\cap[n_0+1,n]}(-\varepsilon_j) \in N^+$. Since we already showed that $\varepsilon_{I\cap[1,n_0]} \in L$, this contradicts with $L \cap N^+ = \emptyset$.

Next we prove that $\varepsilon_{I,j} \notin P$ for $I \subseteq [1,n]$, $I \not\subseteq [1,n_0]$, $j \in [1,n_0]$. If this is not the case, then $\varepsilon_j \in L$, $\forall j \in [1,n_0]$ and (1.2) imply that $\varepsilon_I = \varepsilon_{I,j} + \varepsilon_j \in P$, which contradicts with (4.16).

Finally, we prove that $\varepsilon_{I,j} \notin P$ for $I \subseteq [1,n]$, $j \in [n_0+1,n]$ such that $j \notin I$ and $|I \cap [n_0+1,n]| \geq 2$. Assume the opposite and choose $i \in I \cap [n_0+1,n]$. Then $\varepsilon_j - \varepsilon_i \in L$, $\forall j \in [n_0+1,n]$ and (1.3) imply that $\varepsilon_{I\backslash\{i\}} = \varepsilon_{I,j} + (\varepsilon_j - \varepsilon_i) \in P$, which again contradicts with (4.16) since $I\backslash\{i\} \not\subseteq [1,n_0]$.

This proves that each cominuscule parabolic set of roots for $W(n)$ is conjugated under the action of the Weyl group $W_{\mathfrak{l}_{\bar{0}}}$ to one of the sets

$$P = P_{W(n)}(n_0), \ 0 \leq n_0 \leq n-1, \ \text{ and } \ P = P^-_{W(n)}(n-1), \tag{4.17}$$

and that those parabolic subsets are principal and have unique Levi decompositions given by (4.6)–(4.7). It remains to show that none of those parabolic sets are in the same $W_{\mathfrak{l}_{\bar{0}}}$-orbit. Since $\mathfrak{l}_{\bar{0}} = \mathfrak{s}_{\bar{0}}$, the set of roots $\Delta_{\mathfrak{s}}$ is stable under the action of $W_{\mathfrak{l}_{\bar{0}}}$. If two parabolic sets of roots of $W(n)$, P and P' are conjugated under $W_{\mathfrak{l}_{\bar{0}}}$, then $P' \cap \Delta_{\mathfrak{s}} \in W_{\mathfrak{l}_{\bar{0}}}(P \cap \Delta_{\mathfrak{s}})$. Since

$$\begin{aligned} P_{W(n)}(n_0) \cap \Delta_{\mathfrak{s}} &= L_{\mathfrak{s}}(1,n_0) \sqcup N^+_{\mathfrak{s}}(1,n_0), \ 1 \leq n_0 \leq n-1, \\ P^-_{W(n)}(n-1) \cap \Delta_{\mathfrak{s}} &= w_0(L_{\mathfrak{s}}(1,n-1) \sqcup N^+_{\mathfrak{s}}(0,n-1)), \end{aligned}$$

where w_0 is the longest element of $W_{\mathfrak{l}_{\bar{0}}} \cong S_n$, Theorem 2.1 implies that none of the parabolic sets (4.17) are in the same $W_{\mathfrak{l}_{\bar{0}}}$-orbit. □

Remark 4.18. *Following the proof of Theorem 4.5 we may define alternatively the sets $L_{W(n)}(n_0)$, $N_{W(n)}(n_0)$, and $N^-_{W(n)}(n_0)$ as the unique sets L, N^+, N^- that satisfy the properties of Lemma 1.16 and such that $\pm\varepsilon_i \in L$, $1 \leq i \leq n_0$, $\varepsilon_j \in N^-$, $-\varepsilon_j \in N^+$, $n_0 < j \leq n$.*

The proof of Theorem 4.5 implies the following result.

Corollary 4.19. *If $n_0 > 1$, then the cominuscule parabolic subset of roots $P_{\mathfrak{sl}(n_0)}(n_0)$ of $W(n)_0$ has a unique cominuscule parabolic $W(n)$-extension: $P_{W(n)}(n_0)$. The cominuscule parabolic subset $P_{\mathfrak{sl}(n_0)}(1)$ has two distinct extensions: $P_{W(n)}(1)$ and $w_0 P^-_{W(n)}(n-1)$. The cominuscule parabolic subsets of roots $P_{\mathfrak{sl}(1|n)}(1|n_0)$ of $\mathfrak{s} \cong \mathfrak{sl}(1|n)$ have no cominuscule parabolic $W(n)$-extensions for $n_0 > 1$. In all other cases, $P_{\mathfrak{sl}(1|n)}(m_0|n_0)$*

has a unique cominuscule parabolic $W(n)$-extension $\widehat{P}_{\mathfrak{sl}(1|n)}(m_0|n_0)$: $\widehat{P}_{\mathfrak{sl}(1|n)}(0|1) = P^-_{W(n)}(n-1)$ and $\widehat{P}_{\mathfrak{sl}(1|n)}(1|n_0) = P_{W(n)}(n_0)$ for $0 \leq n_0 \leq n-1$.

Remark 4.20. *There are two important particular cases in Theorem 4.5. The first such case is $n_0 = 0$ when we have $\mathfrak{l} = W(n)_0$, $\mathfrak{n}^+ = W(n)_{-1}$ and $\mathfrak{n}^- = W(n)_{\geq 1}$. The other case is $n_0 = n-1$ which corresponds to $\mathfrak{l} \cong W(n-1) \ltimes (\mathfrak{gl}(1) \otimes \bigwedge(n-1))$, and $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are isomorphic to $\bigwedge(n-1) \otimes V^*$ and $W(n-1) \otimes V$, respectively, where V is the standard one dimensional $\mathfrak{gl}(1)$-module.*

Theorem 4.5 also implies:

Remark 4.21. *All cominuscule parabolic sets of roots of $W(n)$ are principal parabolic subsets with functionals defined in (4.3).*

4.2. $\mathfrak{g} = S(n)$ **and** $\mathfrak{g} = S'(n)$**.** There are two associative superalgebras of differential forms defined over $\bigwedge(n)$, namely $\Omega(n)$ and $\Theta(n)$. The superalgebra $\Omega(n)$ has generators $d\xi_1, \dots, d\xi_n$ and defining relations $d\xi_i \circ d\xi_j = d\xi_j \circ d\xi_i$, $\deg d\xi_i = \bar{0}$, while the superalgebra $\Theta(n)$ has generators $\theta\xi_1, \dots, \theta\xi_n$ and relations $\theta\xi_i \wedge \theta\xi_j = -\theta\xi_j \wedge \theta\xi_i, \deg \theta\xi_i = \bar{1}$. Note that the differentials d and θ are derivations of degree $\bar{1}$ and $\bar{0}$ respectively. Let $\mu_n := \theta\xi_1 \wedge \dots \wedge \theta\xi_n$ be the standard volume form in $\Omega(n)$ and $\mu_n' := (1 + \xi_1 \dots \xi_n)\mu_n$.

Every derivation D of $W(n)$ and every automorphism Φ of $\bigwedge(n)$ extend uniquely to a derivation $\widetilde{D}^d$ and an automorphism $\widetilde{\Phi}^d$ (respectively, $\widetilde{D}^\theta$ and $\widetilde{\Phi}^\theta$) of $\Omega(n)$ (resp., $\Theta(n)$) so that $[\widetilde{D}^d, d] = 0$ and $[\widetilde{\Phi}^d, d] = 0$ (resp., $\widetilde{D}^\theta\theta f - \theta\widetilde{D}^\theta f = 0$ and $\widetilde{\Phi}^\theta\theta f - \theta\widetilde{\Phi}^\theta f = 0$ for every $f \in \bigwedge(n)$). We denote by $S(n)$ the Lie superalgebra $\{D \in W(n) \mid \widetilde{D}^\theta(\mu_n) = 0\}$ and by $S'(n)$ the Lie superalgebra $\{D \in W(n) \mid \widetilde{D}^\theta(\mu_n') = 0\}$. Since $S'(2k+1) \cong S(2k+1)$ we consider $S'(n)$ only for even numbers n. In explicit terms we have:

$$
\begin{aligned}
S(n) &= \operatorname{Span}\left\{ \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_j} \mid f \in \bigwedge(n), 1 \leq i,j \leq n \right\}, \\
S'(n) &= \operatorname{Span}\left\{ (1 - \xi_1 \dots \xi_n)\left(\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_j} \right) \mid f \in \bigwedge(n), i,j = 1, \dots, n \right\}.
\end{aligned}
$$

In particular, $S(n)_0 = S'(n)_0 \cong \mathfrak{sl}(n)$. Set $S(n)_{\geq j} := \bigoplus_{i \geq j} S(n)_i$. Note that $S'(n)$ is not a graded Lie subalgebra of $W(n)$, but it has a filtration induced by the filtration $\{W(n)_{\geq j}\}_j$ of $W(n)$. The corresponding graded superalgebra is isomorphic to $S(n)$.

We fix the Cartan subalgebra of $S(n)$ to be $\mathfrak{h}_{S(n)} = \mathfrak{h}_{W(n)} \cap S(n)$ and set $\mathfrak{h}_{S'(n)} = \mathfrak{h}_{S(n)}$ for even n. The root systems of $S(n)$ and $S'(n)$ coincide and can be described as follows. Denote by $\iota_S : S(n) \to W(n)$ the natural inclusion. Let $\overline{\Delta}_{S(n)}$ be the set obtained from $\Delta_{W(n)}$ by removing the n roots $\varepsilon_{[1,n]\setminus\{i\}}$, $1 \leq i \leq n$:

$$\overline{\Delta}_{S(n)} = \{\varepsilon_{I,j} \mid I \subseteq [1,n], j \in [1,n], j \notin I\} \sqcup \{\varepsilon_I \mid |I| \leq n-2\}.$$

The kernel of the restriction map $\iota_S^*|_{\mathfrak{h}^*_{W(n)}} : \mathfrak{h}^*_{W(n)} \to \mathfrak{h}^*_{S(n)}$ equals $\mathbb{C}(\varepsilon_1 + \ldots + \varepsilon_n)$. Throughout this subsection, we will identify the root system $\Delta_{S(n)}$ with $\overline{\Delta}_{S(n)}$ via ι_S^*. In particular, by abuse of notation, for a root $\alpha \in \overline{\Delta}_{S(n)}$, we will write α instead of $\iota_S^*(\alpha)$. All sums of roots of $\Delta_{S(n)}$ will be computed in $\mathfrak{h}^*_{W(n)}$ via this identification.

The root spaces of $S(n)$ are described as follows:

$$S(n)^{\varepsilon_{I,j}} = \operatorname{Span}\left\{\xi_{i_1} \ldots \xi_{i_k} \frac{\partial}{\partial \xi_j}\right\}$$

for $I = \{i_1, \ldots, i_k\} \subseteq [1, n]$, $j \notin I$ and

$$S(n)^{\varepsilon_I} = \operatorname{Span}\left\{\xi_{i_1} \ldots \xi_{i_k}\left(\xi_l \frac{\partial}{\partial \xi_l} - \xi_m \frac{\partial}{\partial \xi_m}\right) \;\Big|\; l, m \in ([1, n]\backslash I)\right\}$$

for $I = \{i_1, \ldots, i_k\} \subsetneq [1, n]$.

On the other hand, the root spaces of $S'(n)$ coincide with those of $S(n)$ except for $S'(n)^{-\varepsilon_j}$, $j = 1, \ldots, n$. For the latter root spaces we have

$$S'(n)^{-\varepsilon_j} = \operatorname{Span}\left\{(1 - \xi_1 \ldots \xi_n)\frac{\partial}{\partial \xi_j}\right\}.$$

The following lemma can be proved using the explicit description of $\mathfrak{g}^\alpha$ with the same reasoning as in the proof of Proposition 1.15 for $\mathfrak{g} = W(n)$.

Lemma 4.22. *Let $\mathfrak{g} = S(n)$ or $\mathfrak{g} = S'(n)$, and let $\alpha, \beta \in \Delta_{S(n)}$ be such that $\alpha+\beta \neq 0$. Then*

(i) $\left[\mathfrak{g}^\alpha, \mathfrak{g}^\beta\right] \neq 0$ if and only if $\alpha + \beta \in \Delta_{W(n)}$ for $\mathfrak{g} = S(n)$;

(ii) $\left[\mathfrak{g}^\alpha, \mathfrak{g}^\beta\right] \neq 0$ if and only if $\alpha + \beta \in \Delta_{W(n)}$ or $\alpha = -\varepsilon_i$, $\beta = -\varepsilon_j$, some $i \neq j$, for $\mathfrak{g} = S'(n)$.

Recall that we identify $\Delta_{S(n)}$ and $\overline{\Delta}_{S(n)}$ and the sum $\alpha + \beta$ in (i) and (ii) is taken in $\mathfrak{h}^*_{W(n)}$.

We define $L_{S(n)}(n_0)$, $N^+_{S(n)}(n_0)$, $N^-_{S(n)}(n_0)$, $P_{S(n)}(n_0)$, and $P^-_{S(n)}(n_0)$ with the same formulas that we used for the corresponding sets in §4.1.

Contrary to $W(n)$, $S(n)$ does not have a subalgebra whose root system is $\{\pm\varepsilon_i, \varepsilon_i - \varepsilon_j \mid i \neq j\}$. However, we may define a monomorphism $\iota_W : W(n) \to S(n+1)$ by

$$\xi_{i_1} \ldots \xi_{i_k} \frac{\partial}{\partial \xi_j} \mapsto \xi_{i_1} \ldots \xi_{i_k} \frac{\partial}{\partial \xi_j} \;\text{ and }\; \xi_{i_1} \ldots \xi_{i_k} \xi_j \frac{\partial}{\partial \xi_j} \mapsto \xi_{i_1} \ldots \xi_{i_k}\left(\xi_j \frac{\partial}{\partial \xi_j} - \xi_{n+1}\frac{\partial}{\partial \xi_{n+1}}\right),$$

for $j \notin \{i_1, \ldots, i_k\}$.

The Lie algebra $\mathfrak{l}_{\bar{0}} := S(n)_0 = S'(n)_0$ is a Levi subalgebra of $S(n)_{\bar{0}}$ and $S'(n)_{\bar{0}}$ containg $\mathfrak{h}_{S(n)}$ and $\mathfrak{h}_{S'(n)}$. The Weyl group $W_{\mathfrak{l}_{\bar{0}}}$ is isomorphic to the symmetric group S_n. It acts on $S(n)$ and $S'(n)$ by Lie algebra automorphisms and on the corresponding root systems, as follows. The automorphisms $\sigma \in S_n$ from §4.1 leave invariant the volume forms μ_n and μ'_n. Thus each $\sigma \in S_n$ preserves $S(n)$ and $S'(n)$. The corresponding action of S_n on the root systems of $S(n)$ and $S'(n)$ is simply the restriction of the action of S_n from $\Delta_{W(n)}$ to $\overline{\Delta}_{S(n)}$. The following theorem classifies the cominuscule parabolic subsets of roots of $S(n)$ and shows that each of them can be obtained by restricting a cominuscule parabolic subset of roots of $W(n)$.

Theorem 4.23. *(i) There are $n+1$ orbits of cominuscule parabolic subsets of the root system of $S(n)$ under the action of the Weyl group $W_{\mathfrak{l}_{\bar{0}}} \cong S_n$. The parabolic subsets $P_{S(n)}(n_0)$, $0 \leq n_0 \leq n-1$, and $P = P^-_{S(n)}(n-1)$ provide representatives of these orbits. Each cominuscule parabolic subalgebra of $S(n)$ has a unique Levi decomposition. The Levi subalgebra $\mathfrak{l}$ and the niradicals $\mathfrak{n}^+$ and $\mathfrak{n}^-$ of the above cominuscule parabolic subalgebras can be obtained by intersecting the corresponding subalgebras of $W(n)$ described in Theorem 4.5 with $S(n)$.*

(ii) The Lie superalgebra $S'(n)$ has no cominuscule parabolic subsets.

Proof. We start with $\mathfrak{g} = S(n)$. Let P be a cominuscule parabolic subset of Δ and $P = L \sqcup N^+$ be a Levi decomposition of P for which the coresponding niradical $\mathfrak{n}^+$ of $\mathfrak{p}_P$ is abelian, recall Definition 1.11. Let N^- be as in Lemma 1.5.

For simplicity of the notation in this proof we set $\Delta = \Delta_{S(n)}$ and $\Delta_{\mathfrak{sl}(n)} = \Delta_{S(n)_0}$. Assume first that $N^+ \cap \Delta_{\mathfrak{sl}(n)} \neq \emptyset$. Then by Proposition 1.20, there exits n_0, $1 \leq n_0 \leq n-1$, such that after conjugating P by an element $\sigma \in S_n$ we have $L \cap \Delta_{\mathfrak{sl}(n)} = L_{\mathfrak{sl}(n)}(n_0)$ and $N^+ \cap \Delta_{\mathfrak{sl}(n)} = N^+_{\mathfrak{sl}(n)}(n_0)$. We proceed with a case-by-case verification using Lemma 4.22.

Case 1: $\varepsilon_1 \in N^+$. Then $\varepsilon_j = \varepsilon_1 + (\varepsilon_j - \varepsilon_1) \in N^+$ for every $j \leq n_0$. In particular, P is not cominuscule if $n_0 > 1$ since $\varepsilon_1 + \varepsilon_j \in \Delta_{W(n)}$, cf. Lemma 4.22. If $n_0 = 1$, then $\varepsilon_j \in L$ for every $j > 1$. Indeed, if $\varepsilon_j \notin L$, then either $\varepsilon_j \in N^+$ or $-\varepsilon_j \in N^+$, which together with $\varepsilon_1 \in N^+$ leads to a contradiction since $\varepsilon_1 \pm \varepsilon_j \in \Delta_{W(n)}$. From here it is not difficult to verify that $P = w_0 P^-_{S(n)}(n-1)$.

Case 2: $\varepsilon_1 \in N^-$. For $j \leq n_0$, $(\varepsilon_j - \varepsilon_1) \in L$, and for $j > n_0$, $(\varepsilon_j - \varepsilon_1) \in N^-$. Thus for all j, $\varepsilon_j = \varepsilon_1 + (\varepsilon_j - \varepsilon_1) \in N^-$ and $-\varepsilon_j \in N^+$. It is easy to conclude from here that $P = P_{S(n)}(0)$, which is a contradiction to $N^+ \cap \Delta_{\mathfrak{sl}(n)} \neq \emptyset$.

Case 3: $\varepsilon_1 \in L$. Now we have $\varepsilon_j = \varepsilon_1 + (\varepsilon_j - \varepsilon_1) \in L$ for every $j \leq n_0$. If $\varepsilon_n \in L$, then we $\varepsilon_i = \varepsilon_n + (\varepsilon_i - \varepsilon_n) \in L$ for all $i > n_0$. Thus $\varepsilon_n \in L$ implies $\varepsilon_j \in L$ for all j and hence $L = P = \Delta$, which is a contradiction. If $\varepsilon_n \in N^+$, then $\varepsilon_1 = \varepsilon_n + (\varepsilon_1 - \varepsilon_n) \in N^+$, which is again a contradiction. It remains to consider the case when $\varepsilon_n \in N^-$. Then $\varepsilon_j \in N^-$ for $j > n_0$. From here one easily verifies that $P = P_{S(n)}(n_0)$.

Now we assume $N^+ \cap \Delta_{\mathfrak{sl}(n)} = \emptyset$, and hence $\Delta_{\mathfrak{sl}(n)} \subset L$. Then either all ε_j are in N^+ or all ε_j are in N^-. (Otherwise there will exist two indices $i \neq j$ such that $\varepsilon_i, -\varepsilon_j \in N^+$, which contradicts to the assumption that P is cominuscule since $\varepsilon_i - \varepsilon_j \in \Delta_{W(n)}$, cf. Lemma 4.22.) Using once again the assumption that P is cominuscule and Lemma 4.22 we rule out the case $\varepsilon_j \in N^+$, $\forall j$ since $\varepsilon_i + \varepsilon_j \in \Delta_{W(n)}$, $\forall i \neq j$. Therefore $\varepsilon_j \in N^-$ for all j, which leads to $P = P_{S(n)}(0)$.

The isomorphisms for the Levi subalgebras and nilradicals of the cominuscule parabolic subalgebras obtained in this way are analogous to the $W(n)$ case. The case $\mathfrak{g} = S'(2l)$ follows from $\mathfrak{g} = S(2l)$ and is left to the reader. □

Remark 4.24. *The explicit isomorphisms for $\mathfrak{l}$, $\mathfrak{n}^+$, and $\mathfrak{n}^-$ are analogous to the $W(n)$ case but are rather lengthy and will be omitted. In the particular case $n_0 = 0$, we have $\mathfrak{l} = S(n)_0$, $\mathfrak{n}^+ = S(n)_{-1}$ and $\mathfrak{n}^- = S(n)_{\geq 1}$, while for $n_0 = n-1$, $\mathfrak{l} = \iota_W(W(\xi_1,\ldots,\xi_{n-1}))$, $\mathfrak{n}^+ \cong \bigwedge(\xi_1,\ldots,\xi_{n-1}) \otimes V^*$ and $\mathfrak{n}^- \cong S(\xi_1,\ldots,\xi_{n-1}) \otimes V$, where V denotes the standard $\mathfrak{gl}[n-1,n]$-representation.*

Remark 4.25. *One can prove Theorem 4.5 using the same reasoning as in the proof of Theorem 4.23. The advantage of the present proof of Theorem 4.5 is that it provides a valuable connection between the cominuscule parabolics of $\mathfrak{sl}(1|n)$ and $W(n)$ (cf. Corollary 4.19) which is not obvious otherwise.*

Remark 4.26. *All cominuscule parabolic sets of roots of $S(n)$ are principal parabolic subsets with functionals defined in (4.3).*

4.3. $\mathfrak{g} = H(n)$. The Hamiltonian finite dimensional Lie superalgebras are defined by $\widetilde{H}(n) := \{D \in W(n) \mid \widetilde{D}^d \omega_n = 0\}$ and $H(n) := [\widetilde{H}(n), \widetilde{H}(n)]$. In explicit form:

$$\begin{aligned}\widetilde{H}(n) &= \operatorname{Span}\left\{ D_f := \sum_i \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_i} \mid f \in \bigwedge(n), f(0) = 0,\ 1 \leq i,j \leq n \right\},\\ \widetilde{H}(n) &= H(n) \oplus \mathbb{C} D_{\xi_1\ldots\xi_n}.\end{aligned}$$

If we consider $\bigwedge(n)$ as a Poisson superalgebra, then the map $\mathcal{D} : \bigwedge(n) \to \widetilde{H}(n)$, $f \mapsto D_f$, is a surjective homomorphism of Lie superalgebras with $\ker \mathcal{D} = \mathbb{C}$. In particular, $[D_f, D_g] = D_{\{f,g\}}$ where $\{f,g\} := (-1)^{\deg f} \sum_{i=1}^n \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}$.

The Lie superalgebra $H(n)$ is a graded subalgebra of $W(n)$. Set $H(n)_k = H(n) \cap W(n)_k$, $-1 \leq k \leq n-1$. We have $H(n)_0 \cong \mathfrak{so}(n)$. Every Cartan subalgebra $\mathfrak{h}$ of $H(n)$ has a nilpotent part. An explicit description of such subalgebras can be found in Appendix A of [GP2]. We fix such $\mathfrak{h}$ for which $\mathfrak{h} \cap H(n)_0$ equals $\operatorname{Span}\{D_{\xi_i\xi_{i+l}} \mid i = 1,\ldots,[n/2]\}$. The root system of $H(n)$ is given by $\Delta = \{\varepsilon_I - \varepsilon_J \mid I, J \subset [1,[n/2]], I \cap J = \emptyset\}$, where $\varepsilon_I = \sum_{i\in I} \varepsilon_i$ and the arithmetic of ε_i's is the same as in the case of $W(n)$. All roots vanish on $\mathfrak{h}'_{\bar{0}}$. Denote by $\Delta_{\mathfrak{so}(n)}$ the subset of roots corresponding to $H(n)_0$.

The Lie algebra $H(n)_0$ is a Levi subalgebra of $\mathfrak{g}_{\bar{0}}$. As in the previous two subsections, every element w of the Weyl group of $H(n)_0$ can be extended to a Weyl automorphism σ_w of $H(n)$. This induces an action of $W_{H(n)}$ on Δ and on the set of parabolic subsets of Δ. We define

$$\begin{aligned} L_{H(n)} &:= \{\varepsilon_I - \varepsilon_J \mid I, J \subset [1,[n/2]], I \cap J = \emptyset, 1 \notin I, 1 \notin J\},\\ N^+_{H(n)} &:= \{\varepsilon_I - \varepsilon_J \mid I, J \subset [1,[n/2]], I \cap J = \emptyset, 1 \in I\},\end{aligned}$$

and $P_{H(n)} := L_{H(n)} \sqcup N^+_{H(n)}$. One easily checks that $\Delta = (-N^+_{H(n)}) \sqcup L_{H(n)} \sqcup N^+_{H(n)}$ and that this is a triangular decomposition with respect to the functional Λ given by $\Lambda(\varepsilon_1) = 1$ and $\Lambda(\varepsilon_i) = 0$ for $i > 1$. Therefore $P_{H(n)}$ is a principal parabolic set of roots and its Levi decomposition is $P_{H(n)} = L_{H(n)} \sqcup N^+_{H(n)}$. Using Proposition 1.15 one verifies that $P_{H(n)}$ is a cominuscule parabolic set of roots.

Theorem 4.27. *Let $\mathfrak{g} = H(n)$, $n \geq 5$, and $l = [n/2]$. The set of all cominuscule parabolic subsets of the root system of $H(n)$ forms a single orbit under the action of the Weyl group $W_{H(n)_0} \cong W_{\mathfrak{so}(n)}$. The parabolic subset $P_{H(n)}$ provides a representative of this orbit. Moreover, the following isomorphisms hold for the corresponding Levi subalgebra $\mathfrak{l}$ and nilradical $\mathfrak{n}^+$ (considered as an $\mathfrak{l}$-module): $\mathfrak{l} \cong H(n-2) \otimes \bigwedge(1) \oplus \mathbb{C}^2$, $\mathfrak{n}^+ \cong \widetilde{H}(n-2) \oplus \mathbb{C}$.*

Proof. Let P be a cominuscule parabolic subset of Δ with Levi decomposition $P = L \sqcup N^+$. If $N^+ \cap \Delta_{\mathfrak{so}(n)} = \emptyset$, then one easily proves that that $P = L = \Delta$. Assume that $N^+ \cap \Delta_{\mathfrak{so}(n)} \neq \emptyset$. We proceed with a case-by-case verification checking for which i, ε_i is in P.

Case 1: $n = 2l+1$. It follows from Proposition 1.20 (ii) that one can conjugate P by an element of $W_{H(n)_0}$ so that $L \cap \Delta_{\mathfrak{so}(2l+1)} = L_{\mathfrak{so}(2l+1)}$ and $N^+ \cap \Delta_{\mathfrak{so}(2l+1)} = N^+_{\mathfrak{so}(2l+1)}$. This implies that $\varepsilon_1 \in N^+$ and $\varepsilon_i \in L$ for $i > 1$. Now one easily obtains that $P = P_{H(2l+1)}$.

Case 2: $n = 2l$. Following Proposition 1.20 (iv) we proceed with three subcases.

Case 2.1: $P \cap \Delta_{\mathfrak{so}(2l)} = L_{\mathfrak{so}(2l)}(1) \sqcup N^+_{\mathfrak{so}(2l)}(1)$. With the aid of Proposition 1.16 we easily find that $\varepsilon_1 \in N^+$ and $\varepsilon_i \in L$, for all $i > 1$. Thus $P = P_{H(2l)}$.

Case 2.2: $P \cap \Delta_{\mathfrak{so}(2l)} = L_{\mathfrak{so}(2l)}(l) \cup N^+_{\mathfrak{so}(2l)}(l)$. Using again Proposition 1.16 we verify that $\varepsilon_i \in N^+$ for all $i = 1, 2, ..., l$. Since $l > 2$, this contradicts to P being cominuscule.

Case 2.3: $P \cap \Delta_{\mathfrak{so}(2l)} = L_{\mathfrak{so}(2l)}(l-1) \cup N^+_{\mathfrak{so}(2l)}(l-1)$. We reach a contradiction in a similar fashion to the previous subcase.

The isomorphisms for the Levi subalgebra and nilradical follow from the explicit description of the root spaces of $H(n)$, see e.g. [GP2, Appendix A]. More precisely, we have:

$$
\begin{aligned}
\widetilde{\mathfrak{l}} &= \mathcal{D}\Big(\bigwedge(\xi_2, \ldots, \xi_l, \xi_{l+2}, \ldots, \xi_n) \oplus \xi_1\xi_{l+1} \bigwedge(\xi_2, \ldots, \xi_l, \xi_{l+2}, \ldots, \xi_n)\Big) \\
\widetilde{\mathfrak{n}}^+ &\cong \widetilde{H}(\xi_2, \ldots, \xi_l, \xi_{l+2}, \ldots, \xi_n) \oplus \mathbb{C}D_{\eta_1},
\end{aligned}
$$

where $\eta_1 := \frac{1}{\sqrt{2}}(\xi_1 + \sqrt{-1}\xi_{l+1})$, and $\widetilde{\mathfrak{l}}$ and $\widetilde{\mathfrak{n}}^+$ denote the corresponding to the sets of roots $L_{H(n)}$ and $N^+_{H(n)}$ subalgebras of $\widetilde{H}(n)$.

$\square$

Remark 4.28. *Theorem 4.27 implies that all cominuscule parabolic sets of roots of $H(n)$ are principal. One should note though that not every parabolic subset of roots of $H(n)$ is principal as shown in* [DFG, §3].

Remark 4.29. *Due to the fact that for every root $\varepsilon_I - \varepsilon_J$ of $H(n)$, the intersection $H(n)_i \cap H(n)^{\varepsilon_I - \varepsilon_J}$ is nontrivial for more than one index i, the subalgebra $H(n)_{-1} \oplus H(n)_0$ is not a cominuscule parabolic subalgebra of $H(n)$ according to our definition. Some authors studied versions of parabolic subalgebras of simple finite dimensional Lie superalgebras, which are not root subalgebras but at the same time are more restrictive than our definition in different respects (see for example* [IO] *which deals with $\mathbb{Z}$-graded Lie superalgebras). It will be interesting to study cominuscule subalgebras in those frameworks. For instance $H(n)_{-1} \oplus H(n)_0$ will be an example of such a subalgebra.*

Appendix

In this Appendix we prove Proposition 1.15. We need to show that if $\alpha, \beta, \alpha+\beta \in \Delta$, then $\left[\mathfrak{g}^\alpha, \mathfrak{g}^\beta\right] \neq 0$. For all classical Lie superalgebras $\mathfrak{g}$, except $\mathfrak{g} = \mathfrak{sp}(n), \mathfrak{psq}(n)$, the proposition follows from Proposition 2.5.5(e) in [K]. For the two remaining cases of classical Lie superalgebras and for $\mathfrak{g} = W(n), H(n)$, it is sufficient to give examples of elements $x_\alpha \in \mathfrak{g}^\alpha$ and $x_\beta \in \mathfrak{g}^\beta$ such that $[x_\alpha, x_\beta] \neq 0$. For $\mathfrak{g} = \mathfrak{sp}(n), \mathfrak{g} = \mathfrak{psq}(n)$, any nonzero x_α, x_β will work, see [FSS].

Let now $\mathfrak{g} = W(n)$. Using the explicit description of $\mathfrak{g}^\alpha$ provided in §4.1, we provide examples considering three cases for α and β: (i) $\{\alpha = \varepsilon_I, \beta = \varepsilon_{I'}\}$, (ii) $\{\alpha = \varepsilon_{I,j}, \beta = \varepsilon_{I'}, j \notin I\}$, (iii) $\{\alpha = \varepsilon_{I,j}, \beta = \varepsilon_{I',j'}, j \notin I, j' \notin I'\}$. We only consider case (iii) as the other two are similar. Since $\alpha+\beta \in \Delta$, we may assume that $j \in I'$. Then for $I = \{i_1, \ldots, i_k\}$ and $I' = \{i'_1, \ldots, i'_l\}$, $i'_1 = j$, we choose $x_\alpha = \xi_{i_1} \ldots \xi_{i_k} \frac{\partial}{\partial \xi_j}$ and $x_\beta = \xi_{i'_1} \ldots \xi_{i'_l} \frac{\partial}{\partial \xi_{j'}}$. Then $[x_\alpha, x_\beta]$ will contain the nonzero homogeneous summand $\xi_{i_1} \ldots \xi_{i_k} \xi_{i'_2} \ldots \xi_{i'_l} \frac{\partial}{\partial \xi_{j'}}$, and hence is nonzero.

Assume now that $\mathfrak{g} = H(n)$. The description of the root spaces $\mathfrak{g}^\alpha$ and the graded root spaces $\mathfrak{g}^\alpha \cap \mathfrak{g}_i$ can be found in the Appendix A of [GP2]. The examples of x_α and x_β are given according to that description on a case-by-case basis. Consider the case $n = 2l$, $\alpha = \varepsilon_I - \varepsilon_J$. Let $\beta = \varepsilon_{I'} - \varepsilon_{J'}$. For convenience we will treat I, J, I', J' both as ordered tuples and as subsets of $[1, n]$. Note that $I \cap J = I \cap I' = I' \cap J' = J \cap J' = \emptyset$. We first consider the case when either $I \cap J' \neq \emptyset$ or $I' \cap J \neq \emptyset$. Let $x_\alpha = D_{\eta_I \eta_{\widehat{J}}}$ and $x_\beta = D_{\eta_{I'} \eta_{\widehat{J'}}}$, where $\eta_i := \frac{1}{\sqrt{2}}\left(\xi_i + \sqrt{-1}\xi_{i+l}\right)$, $\eta_{i+l} := \frac{1}{\sqrt{2}}\left(\xi_i - \sqrt{-1}\xi_{i+l}\right)$, $1 \leq i \leq n$, and $\eta_I = \eta_{i_1} \ldots \eta_{i_k}$, $\widehat{I} = (i_1 + l, \ldots, i_k + l)$ whenever $I = (i_1, \ldots, i_k)$, $1 \leq i_1 < \ldots < i_k \leq l$. Then $[x_\alpha, x_\beta] = D_F$, where $F = \{\eta_I \eta_{\widehat{J}}, \eta_{I'} \eta_{\widehat{J'}}\}$. Here $\{f, g\} = (-1)^{\deg f} \left(\sum_{i=1}^l \frac{\partial f}{\partial \eta_{i+l}} \frac{\partial g}{\partial \eta_i} + \sum_{i=1}^l \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \eta_{i+l}} \right)$. Using the assumption that $I \cap J' \neq \emptyset$ or $I' \cap J \neq \emptyset$, we verify that $[x_\alpha, x_\beta] \neq 0$. Now consider the case $I \cap J' = I' \cap J = \emptyset$. Because $\alpha+\beta \in \Delta$ we also have $I \cap I' = J \cap J' = \emptyset$. Take

$k \in J'$. Then $k \notin I, k \notin J$, so we may choose $x_\alpha = D_{\eta_I \eta_k \eta_{\widehat{J}} \eta_{k+l}}$ and again $x_\beta = D_{\eta_{I'} \eta_{\widehat{J'}}}$. We have that $\{\eta_I \eta_k \eta_{\widehat{J}} \eta_{k+l}, \eta_{I'} \eta_{\widehat{J'}}\} \neq 0$ and hence $[x_\alpha, x_\beta] \neq 0$. The case $n = 2l + 1$ is treated analogously. This completes the proof of Proposition 1.15. □

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