CYCLICITY OF LUSZTIG’S STRATIFICATION OF GRASSMANNIANS AND POISSON GEOMETRY

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Abstract. We prove that the standard Poisson structure on the Grassmannian $Gr(k, n)$ is invariant under the action of the Coxeter element $c = (12...n)$. In particular, its symplectic foliation is invariant under $c$. As a corollary, we obtain a second, Poisson geometric proof of the result of Knutson, Lam, and Speyer that the Coxeter element interchanges the Lusztig strata of $Gr(k, n)$. We also relate the main result to known anti-invariance properties of the standard Poisson structures on cominuscule flag varieties.

1. Introduction

For the purpose of the study of canonical bases, Lusztig defined [4] the totally nonnegative part $(G/P)_{≥0}$ of an arbitrary complex flag variety $G/P$. He also constructed an algebro-geometric stratification of $G/P$ and conjectured that intersecting this stratification with $(G/P)_{≥0}$ is producing a cell decomposition of $(G/P)_{≥0}$. This was latter proved by Rietsch in [5]. Both the non-negative part $(G/P)_{≥0}$ and the Lusztig stratification of a flag variety were studied in recent years from many different combinatorial and Lie theoretic points of view.

In a recent work Knutson, Lam, and Speyer proved that the Lusztig stratification of the Grassmannian $Gr(k, n)$ has a remarkable cyclicity property. If $c$ denotes the Coxeter element $(12...n)$ of $S_n$ and the permutation matrix in $GL_n(\mathbb{C})$ which represents it, then $c$ permutes the strata of the Lusztig stratification of $Gr(k, n)$.

In this note we give a Poisson geometric proof of this fact. We also prove a stronger invariance property of a Poisson structure on $Gr(k, n)$. In [2], jointly with Goodearl, we found a Poisson geometric interpretation of the Lusztig stratification of any flag variety $G/P$. For a choice of opposite Borel subgroups $B$ and $B^-$ of $G$ such that $B \subset P$ one defines the standard Poisson structure $\pi_{G/P}$ on $G/P$ which is invariant under the action of the maximal torus $T = B \cap B^-$, see [2] for details. According to [2, Theorem 0.4] the $T$-orbits of symplectic leaves of $\pi_{G/P}$ are exactly the Lusztig strata.

In the case of the complex Grassmannian $Gr(k, n)$ the standard Poisson structure is given by

$$\pi_{k,n} = - \sum_{1 \leq i < j \leq n} \chi(E_{ij}) \wedge \chi(E_{ji})$$

where $\chi: \mathfrak{gl}_n(\mathbb{C}) \rightarrow \text{Vect}(Gr(k, n))$ denotes the induced infinitesimal action from the left action of $GL_n(\mathbb{C})$ on $Gr(k, n)$ and $E_{ij}$ denote the elementary matrices. This Poisson structure is invariant under the action of the maximal torus $T_n$ of diagonal matrices in $GL_n(\mathbb{C})$. For each $w \in S_n$ denote by the same letter the
corresponding permutation matrix in $GL_n(\mathbb{C})$. As before $c$ denotes the permutation matrix corresponding to the Coxeter element $(12\ldots n)$. The main result of this paper is:

**Theorem 1.1.** Multiplication by $c$ is a Poisson automorphism of $(Gr(k, n), \pi_{k,n})$.

It is well known that the action of the permutation matrix $w_o$ corresponding to the longest element of $S_n$ is an anti-Poisson automorphism, see Section 2 for details. Thus Theorem 1.1 implies:

**Corollary 1.2.** The actions of $w_o$, $c$, and $T_n$ generate an action of $I_2(n) \ltimes T_n$ by Poisson and anti-Poisson automorphisms of $(Gr(k, n), \pi_{k,n})$ where $I_2(n)$ denotes the dihedral group of order $2n$.

The Lusztig stratification of the Grassmannian $Gr(k, n)$ is defined as follows, see [4] for details. Let $B$ and $B_-$ be the standard Borel subgroups of $GL_n(\mathbb{C})$ consisting of upper and lower triangular matrices. Denote the maximal parabolic subgroup

$$P_{k,n} = \{(a \ b) \in GL_n(\mathbb{C}) \mid a \in M_{k,k}, b \in M_{k,n-k}, c \in M_{n-k,n-k}\}$$

of $GL_n(\mathbb{C})$ and the induced map

$$q: GL_n(\mathbb{C})/B \to GL_n(\mathbb{C})/P_{k,n} \cong Gr(k, n).$$

The strata in the Lusztig stratification of $Gr(k, n)$ are given by

$$R_{v,w} = q(B_- \cdot vB \cap B \cdot wB), \quad v \in (S_n)_{max}^{S_k \times S_{n-k}}, w \in S_n, v \leq w.$$ 

Here $\leq$ refers to the Bruhat order. We denote by $S_k \times S_{n-k}$ the subgroup of $S_n$ consisting of those $u \in S_n$ such that $u(i) \leq k$ for $i \leq k$ and $u(i) \geq k+1$ for $i \geq k+1$. Finally, $(S_n)_{max}^{S_k \times S_{n-k}}$ denotes the set of maximal length representatives for the cosets $S_n/(S_k \times S_{n-k})$.

The symplectic foliation of a Poisson structure is uniquely determined by it. Thus the $T_n$-orbits of leaves of $\pi_{k,n}$ (which are exactly the Lusztig strata) are an invariant of the pair $(\pi_{k,n}, T_n$-action$)$. Therefore Theorem 1.1 gives a second proof of the result of Knutson, Lam, and Speyer that the action of the Coxeter element $c$ on $Gr(k, n)$ interchanges the Lusztig strata. In fact Theorem 1.1 is equivalent to the stronger statement:

**The action of $c$ on $Gr(k, n)$ restricts to Poisson isomorphisms between various Lusztig strata $(R_{v,w}, \pi_{k,n}|_{R_{v,w}})$ considered as regular Poisson varieties.**

Finally we trace the roots of this phenomenon from a Poisson geometric point of view. It is well known that on any flag variety $G/P$ the standard Poisson structure $\pi_{G/P}$ is anti-invariant under the action of any representative $\hat{w}_o$ of the longest element of the Weyl group $W$ of $G$. If, in addition, $P$ is cominuscule, then [2, Proposition 4.2] implies that the standard Poisson structure on $G/P$ is anti-invariant under the action of any representative $\hat{w}_o^P$ of the longest element of the corresponding parabolic subgroup of $W$. In the special case of the Grassmannian the specific Coxeter element $c$ happens to be a $k$-th root of $\hat{w}_o^P \hat{w}_o$. Thus Theorem 1.1 claiming that the standard Poisson structure on $Gr(k, n)$ is invariant under $c$ is a strengthening of [2, Proposition 4.2]. See Section 2 for more details. We do not know of good Poisson properties of roots of $\hat{w}_o^P \hat{w}_o$ for any other cominuscule
flag varieties.

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2. Proof of Theorem 1.1

Proof of Theorem 1.1. The statement is equivalent to showing that

\[
\sum_{1 \leq i < j \leq n} \chi(\text{Ad}_c(E_{ij})) \wedge \chi(\text{Ad}_c(E_{ji})) = \sum_{1 \leq i < j \leq n} \chi(E_{ij}) \wedge \chi(E_{ji}) = 0;
\]

that is

(2.1) \[ V := \sum_{i=2}^{n} \chi(E_{1i}) \wedge \chi(E_{i1}) = 0. \]

We will check this on the open Schubert cell \( B_{-} \cdot P_{k,n} \subset Gr(k,n) \). Since \( V \) is an algebraic bivector field, this will establish (2.1). Identify

(2.2) \[ M_{n-k,k} \cong B_{-} \cdot P_{k,n} \subset Gr(k,n), \quad X \mapsto \left( \frac{I_k}{X} \begin{smallmatrix} 0 \\ I_{n-k} \end{smallmatrix} \right) \cdot P_{k,n} \]

where \( M_{n-k,k} \) denotes the space of \((n-k) \times k\) complex matrices. Applying [1, eq. (3.17)] we get that under (2.2)

(2.3) \[ \chi(E_{1,i+k}) \mapsto - \sum_{p=1}^{n-k} \sum_{q=1}^{k} x_{pi} x_{iq} \frac{\partial}{\partial x_{pq}}, \quad \text{for } i = 1, \ldots, n-k. \]

It is obvious that

(2.4) \[ \chi(E_{i+k,1}) \mapsto \frac{\partial}{\partial x_{i1}}, \quad \text{for } i = 1, \ldots, n-k. \]

Let \( 1 \leq i, j \leq k \). Then

\[
\text{Ad}_{\exp(sE_{ij})} \left( \begin{smallmatrix} I_k & 0 \\ X & I_{n-k} \end{smallmatrix} \right) = \left( \begin{smallmatrix} I_k & 0 \\ X - s \sum_{p=1}^{n-k} x_{pi} E_{pj} & I_{n-k} \end{smallmatrix} \right),
\]

which implies that under (2.2)

(2.5) \[ \chi(E_{i,j}) \mapsto - \sum_{p=1}^{n-k} x_{pi} \frac{\partial}{\partial x_{pj}}. \]
The summation in (2.1) can be taken from \( i = 1 \) since \( \chi(E_{11}) \wedge \chi(E_{11}) = 0 \). Applying (2.3), (2.4), and (2.5) we obtain that under the identification (2.2)

\[
V|_{B_{-}P_{k,n}} \to \sum_{i=1}^{k} \sum_{p=1}^{n-k} \sum_{q=1}^{k} x_{pi} x_{qi} \frac{\partial}{\partial x_{pi}} \wedge \frac{\partial}{\partial x_{qi}} - \sum_{i=1}^{n-k} \sum_{p=1}^{k} \sum_{q=1}^{k} x_{pi} x_{q1} \frac{\partial}{\partial x_{pi}} \wedge \frac{\partial}{\partial x_{q1}} = 0.
\]

This implies (2.1) and the statement of the Theorem. □

For an arbitrary complex simple group \( G \) and a maximal parabolic subgroup \( P \), one defines the standard Poisson structure

\[
\pi_{G/P} = -\chi (r_G)
\]

on the flag variety \( G/P \) induced from a compatible triangular decomposition of \( G \) (a pair of Borel subgroups \( B \) and \( B_- \), such that \( B \cap B_- = T \) is a maximal torus of \( G \) and \( B \subset P \)), see e.g. [2]. Here \( \chi : \wedge^2 \text{Lie}(G) \to \Gamma(TG/P, G/P) \) denotes the induced action from the infinitesimal action of \( \text{Lie}(G) \). The standard \( r \)-matrix \( r_G \in \wedge^2 \text{Lie}(G) \) obtained from the triangular decomposition of \( G \) is given by:

\[
r_G = \sum_{\alpha \in \Delta_+} e_\alpha \wedge f_\alpha
\]

where \( e_\alpha \) and \( f_\alpha \) are appropriately normalized root vectors of \( \text{Lie}(G) \) and \( \Delta_+ \) is the set of positive roots of \( G \), cf. [2]. It is obvious that the action of any representative \( \tilde{w}_o \) of the longest element of the Weyl group \( W \) of \( G \) on \( (G/P, \pi_{G/P}) \) is anti-Poisson, since \( \text{Ad}_{\tilde{w}_o} \) interchanges \( e_\alpha \) and \( f_\alpha \).

Denote the Levi factor of \( P \) containing \( T \) by \( L \), and the longest element of the subgroup of \( W \) corresponding to \( L \) by \( w^P_o \). Let \( \tilde{w}^P_o \) be any representative of \( w^P_o \) in the normalizer of \( T \).

Recall that among several equivalent definitions/characterizations of cominuscule parabolic subgroups: a parabolic subgroup \( P \) of \( G \) is cominuscule if and only if its unipotent radical is abelian. According to [2, Proposition 4.2], if \( P \) cominuscule, then \( \pi_{G/P} \) is also given by

\[
\pi_{G/P} = -\chi (r_L).
\]

where \( r_L \in \wedge^2 \text{Lie}(L) \) is the standard \( r \)-matrix of \( L \). Thus the action of \( \tilde{w}^P_o \) on \( (G/P, \pi_{G/P}) \) is anti-Poisson as well. So:

**Proposition 2.1.** For any cominuscule parabolic subgroup \( P \) of a complex simple Lie group \( G \), the action of \( \tilde{w}^P_o \tilde{w}_o \) on \( (G/P, \pi_{G/P}) \) is Poisson.

In the special case of the Grassmannian \( \text{Gr}(k, n) \)

\[
w^P_o w_o = c^k
\]

for the particular Coxeter element \( c \). Taking powers of this product, we see that the action of \( c^{\text{gcd}(k,n)} \) on \( \text{Gr}(k, n) \) is Poisson. In the case when \( k \) and \( n \) are relatively prime this gives yet another proof of Theorem 1.1. One could argue that Theorem 1.1 holds because it is true for relatively prime \( k \) and \( n \), and its
statement (cf. also its proof) is independent of the numerical properties of $k$ and $n$.

Conceptually the invariance of $\pi_{k,n}$ under $c$ is the result of a two step process:
1. From Proposition 2.1 one has the invariance of $\pi_{G/P}$ under the product $\dot{w}_o^P \dot{w}_o$ of the longest elements of the Weyl groups of $G$ and the Levi subgroup $L$, for arbitrary cominuscule flag variety $G/P$.
2. In the case of the Grassmannian the Coxeter element $c$ which is a $k$-th root of $\dot{w}_o^P \dot{w}_o$ acts by Poisson automorphisms of $(\operatorname{Gr}(k,n), \pi_{k,n})$ as well.

The special property of the Coxeter element $c = (12\ldots n)$ is that the other Coxeter elements of $S_n$ are not roots of $\dot{w}_o^P \dot{w}_o$.

3. Corollaries

The symplectic foliation of a Poisson manifold $(M, \pi)$ is an invariant of it. Similarly if a group $H$ act on $(M, \pi)$ by Poisson automorphisms the partition of $M$ into $H$-orbits of symplectic leaves is an invariant of $(M, \pi)$ considered as a Poisson $H$-space. Since the partition of $\operatorname{Gr}(k,n)$ into $T_n$-orbits of symplectic leaves of $(\operatorname{Gr}(k,n), \pi_{k,n})$ is exactly the Lusztig stratification of $\operatorname{Gr}(k,n)$ due to [2, Theorem 0.4] and $c$ normalizes $T_n$, Theorem 1.1 implies the following Theorem of Knutson, Lam, and Speyer:

**Theorem 3.1.** The action of the permutation matrix corresponding to the Coxeter element $c = (12\ldots n)$ on $\operatorname{Gr}(k,n)$ permutes the strata $R_{v,w} = q(B_\cdot vB \cap B \cdot wB)$ of the Lusztig stratification.

As we pointed out, in addition, when $c$ maps one Lusztig stratum $R_{v_1,w_1}$ to another $R_{v_2,w_2}$, it matches the regular Poisson structures $\pi_{k,n}|_{R_{v_1,w_1}}$ and $\pi_{k,n}|_{R_{v_2,w_2}}$.

Finally let us point out that all constructions and invariance properties are valid over the reals since all constructions are derived from the real split form $\operatorname{GL}_n(\mathbb{R})$.

**References**


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