Quantum cluster algebra structures on quantum nilpotent algebras

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To the memory of Andrei Zelevinsky

Contents

Abstract

All algebras in a very large, axiomatically defined class of quantum nilpotent algebras are proved to possess quantum cluster algebra structures under mild conditions. Furthermore, it is shown that these quantum cluster algebras always equal the corresponding upper quantum cluster algebras. Previous approaches to these problems for the construction of (quantum) cluster algebra structures on (quantized) coordinate rings arising in Lie theory were done on a case by case basis relying on the combinatorics of each concrete family. The results of the paper have a broad range of applications to these problems, including the construction of quantum cluster algebra structures on quantum unipotent groups and quantum double Bruhat cells (the Berenstein–Zelevinsky conjecture), and treat these problems from a unified perspective. All such applications also establish equality between the constructed quantum cluster algebras and their upper counterparts. The proofs rely on Chatters' notion of noncommutative unique factorization domains. Toric frames are constructed by considering sequences of homogeneous prime elements of chains of noncommutative UFDs (a generalization of the construction of Gelfand–Tsetlin subalgebras) and mutations are obtained by altering chains of noncommutative UFDs. Along the way, an intricate (and unified) combinatorial model for the homogeneous prime elements in chains of noncommutative UFDs and their alterations is developed. When applied to special families, this recovers the combinatorics of Weyl groups and double Weyl groups previously used in the construction and categorification of cluster algebras. It is expected that this combinatorial model of sequences of homogeneous prime elements will have applications to the unified categorification of quantum nilpotent algebras.

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CHAPTER 1

Introduction

1.1. Quantum cluster algebras and a general formulation of the main theorem

Cluster algebras were invented by Fomin and Zelevinsky in [**11**] based on a novel construction of producing infinite generating sets via a process of mutation. The initial goal was to set up a combinatorial framework for studying canonical bases and total positivity [**10**]. Remarkably, for the past twelve years cluster algebras and the procedure of mutation have played an important role in a large number of diverse areas of mathematics, including representation theory of finite dimensional algebras, combinatorial and geometric Lie theory, Poisson geometry, integrable systems, topology, commutative and noncommutative algebraic geometry, and mathematical physics. The quantum counterparts of cluster algebras were introduced by Berenstein and Zelevinsky in [**3**]. We refer the reader to the recent surveys [**16, 31, 36, 38**] and the book [**18**] for more information on some of the abovementioned aspects of this theory.

A major direction in the theory of cluster algebras is to prove that important (quantized) coordinate rings of algebraic varieties arising from Lie theory admit (quantum) cluster algebra structures or upper (quantum) cluster algebra structures. For example, the Berenstein–Zelevinsky conjecture [**3**] states that the quantized coordinate rings of double Bruhat cells in all finite dimensional simple Lie groups admit explicit quantum cluster algebra structures. The motivation for this type of problem is that once (quantum) cluster algebra structures are constructed on families of (quantized) coordinate rings, they can then be used in the study of canonical bases of those rings. In the classical case, a cluster algebra structure on the coordinate ring of a variety can be used to investigate its totally positive part.

Going back to the general problem, a second part asks if the constructed upper (quantum) cluster algebra equals the corresponding (quantum) cluster algebra. For example, ten years ago Berenstein, Fomin and Zelevinsky proved [**1**] that the coordinate rings of double Bruhat cells in all simple algebraic groups admit upper cluster algebra structures. Yet it was unknown if these upper cluster algebras equal the corresponding cluster algebras, i.e., if the coordinate rings of double Bruhat cells are actually cluster algebras.

Previous approaches to the above problems relied on a construction of an initial seed and some adjacent seeds in terms of (quantum) minors and related regular functions [**1, 3, 14, 15**]. After that point, two different approaches were followed. The first one, due to Berenstein, Fomin and Zelevinsky [**1**], used the methods of unique factorization domains to prove that the coordinate rings under consideration are upper cluster algebras. It was first applied to coordinate rings of double Bruhat cells [**1**]. This approach was developed further in [**18, 19**] and [**17**]. The second approach was via the construction of a categorification based on concrete combinatorial data from Weyl groups and then to prove that the corresponding (quantum) cluster algebra equals the (quantized) coordinate ring under consideration. This approach is due to Geiß–Leclerc–Schröer [**14, 15**], who applied it to the coordinate rings of the unipotent groups $U_+ \cap w(U_-)$ and the quantum Schubert cell algebras $\mathcal{U}_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ (also called quantum unipotent groups) for symmetric Kac–Moody groups G, where w is a Weyl group element.

In both of the above approaches, one relied on specific data in terms of Weyl group combinatorics for the concrete family of coordinate rings. Moreover, the initial (quantum) seeds were built via a direct construction by considering (quantum) minors.

The goal of this paper is to present a new algebraic approach to quantum cluster algebras based on noncommutative ring theory. We produce a general construction of quantum cluster algebra structures on a broad class of algebras and construct initial clusters and mutations in a uniform and intrinsic way, without ad hoc constructions with quantum minors. We first state the main theorem of the paper in a general form. The following sections contain a precise formulation of it.

Main Theorem I: General Form. *Each algebra in a very large, axiomatically defined class of quantum nilpotent algebras admits a quantum cluster algebra structure. Furthermore, for all such algebras, the latter equals the corresponding upper quantum cluster algebra.*

The theorem has a broad range of applications because many important families of algebras fall within this axiomatic class. In particular, the previously mentioned families are special subfamilies of this class of algebras. Furthermore, the required axioms are easy to verify for additional families of algebras. The proof of the theorem is constructive, so one obtains an explicit quantum cluster algebra structure in each case. Initial clusters are constructed intrinsically as finite sequences of (homogeneous) elements in chains of noncommutative unique factorization domains. Another key feature of the result is that it holds for arbitrary base fields: there are no restrictions on their characteristic and they do not need to be algebraically closed. The proof of the theorem is based on purely ring theoretic arguments which are independent of the characteristic of the field and do not use specialization. Finally, when the methods are applied to algebras arising from quantum groups, the deformation parameter q only needs to be a non-root of unity while the previous methods needed q to be transcendental over $\mathbb{Q}$.

In this paper, we apply the theorem to construct explicit quantum cluster algebra structures on the quantum Schubert cell algebras $\mathcal{U}_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ for all finite dimensional simple Lie algebras $\mathfrak{g}$. (The technique works for all Kac–Moody Lie algebras $\mathfrak{g}$, but the general case requires technicalities which would only increase the size of the paper and obstruct the main idea. Because of this, the minor additional details for infinite dimensional Lie algebras $\mathfrak{g}$ will appear elsewhere.) If $\mathfrak{g}$ is symmetric, this result is due to Geiß, Leclerc and Schröer [**15**]. In this case we obtain the same quantum cluster algebra structure, but under milder assumptions on the base field and the deformation parameter.

In a forthcoming publication [**25**], we will give a proof of the Berenstein–Zelevinsky conjecture [**3**] using the above theorem. We will show that the quantized coordinate rings of all double Bruhat cells are localizations of quantum nilpotent algebras and that applying the above theorem produces precisely the conjectured explicit (upper) quantum cluster algebra structure of Berenstein and Zelevinsky [**3**]. In fact, the final result is stronger than the conjecture in that we prove that in each case the upper quantum cluster algebra coincides with the quantum cluster algebra.

The main theorem of the paper has an analog for a certain (very large, axiomatically defined) class of Poisson structures on polynomial algebras which can be considered as the semiclassical limits of quantum nilpotent algebras. A direct application of that theorem produces explicit cluster algebra structures on unipotent groups of the form $U_+ \cap w(U_-)$ (for any Kac–Moody group G) and on all double Bruhat cells. The latter result also proves that the upper cluster algebras of Berenstein, Fomin and Zelevinsky [**1**] for double Bruhat cells are equal to the corresponding cluster algebras, thus solving one of the abovementioned problems. This will also appear in a forthcoming publication.

1.2. Definition of quantum nilpotent algebras

Next, we proceed with the definition of quantum nilpotent algebras.

Throughout, $\mathbb{K}$ will denote a base field. Its characteristic can be arbitrary, and it need not be algebraically closed. All automorphisms and skew-derivations of $\mathbb{K}$-algebras will be $\mathbb{K}$-linear. For two integers j and k we will denote $[j,k] := \{j, j+1, \dots, k\}$ if $j \leq k$ and $[j,k] := \varnothing$ otherwise.

Consider an iterated Ore extension (or iterated skew polynomial algebra)

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N], \tag{1.1}$$

and denote the intermediate algebras $R_k := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_k; \sigma_k, \delta_k]$ for $k \in [0, N]$. (Thus, $R_0 = \mathbb{K}$ and $R_N = R$.) Our conventions for Ore extensions are detailed in Convention 3.2.

Definition A. An iterated Ore extension R as in (1.1) is called a *Cauchon–Goodearl–Letzter* (*CGL*) *extension* [**33**] if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(i) The elements $x_1, \dots, x_N$ are $\mathcal{H}$-eigenvectors.
(ii) For every $k \in [2, N]$, δ_k is a locally nilpotent σ_k-derivation of R_{k-1}.
(iii) For every $k \in [1, N]$, there exists $h_k \in \mathcal{H}$ such that $\sigma_k = (h_k \cdot)$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

The quantum Schubert cell algebras $\mathcal{U}_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ mentioned earlier, for non-roots of unity $q \in \mathbb{K}^*$, are examples of CGL extensions. We refer the reader to Chapter 9 for details. Particular cases include generic quantized coordinate rings of affine spaces, matrix varieties, symplectic and euclidean spaces, and generic quantized Weyl algebras.

Every nilpotent Lie algebra of dimension m contains a chain of ideals

$$\mathfrak{n} = \mathfrak{n}_m \rhd \mathfrak{n}_{m-1} \rhd \ldots \rhd \mathfrak{n}_1 \rhd \mathfrak{n}_0 = \{0\}$$

with the properties that $\dim(\mathfrak{n}_k/\mathfrak{n}_{k-1}) = 1$ and $[\mathfrak{n}, \mathfrak{n}_k] \subseteq \mathfrak{n}_{k-1}$, for all $1 \leq k \leq m$. For $k \in [1, m]$, let $x_k \in \mathfrak{n}_k \setminus \mathfrak{n}_{k-1}$. This set of elements gives rise to the following iterated Ore extension presentation of $\mathcal{U}(\mathfrak{n})$:

$$\mathcal{U}(\mathfrak{n}) = \mathbb{K}[x_1][x_2; \mathrm{id}, \delta_2] \ldots [x_m; \mathrm{id}, \delta_m].$$

The derivations $\delta_k = \mathrm{ad}_{x_k}$ are locally nilpotent. Thus, all universal enveloping algebras of finite dimensional nilpotent Lie algebras can be presented as iterated

Ore extensions (1.1) with all $\sigma_k = \mathrm{id}$ and δ_k locally nilpotent. One can consider $\mathcal{H} = \{1\}$ acting trivially on them and then all conditions in Definition A will be satisfied except for the second part of (iii) – in this case all eigenvalues λ_k will be equal to 1.

We consider the class of CGL extensions to be the best current definition of quantum nilpotent algebras from a ring theoretic perspective. On the level of presentations, the local nilpotency of δ_k from the classical situation is kept in the definition. The torus $\mathcal{H}$ is used to get hold of the eigenvalues λ_k. The condition that they are non-roots of unity is the main feature of the quantum case. On the level of deeper ring theoretic properties, these algebras exhibit the common property of having only finitely many $\mathcal{H}$- invariant prime ideals [**5, 22**]. This property, first derived from the pioneering works [**26, 27, 29**] on the spectra of quantum groups, has played a key role in the study of spectra of "quantum algebras". From another ring theoretic perspective, all CGL extensions are conjectured to be catenary just as Gabber's theorem for universal enveloping algebras of solvable Lie algebras (though this is currently established only for quantized Weyl algebras [**21**] and quantum Schubert cell algebras [**39**]).

The class of CGL extensions appears to be very large. We do not know any classification results except for very low dimensions [**37**]. As mentioned in Section 1.1, in addition to the quantum Schubert cell algebras, in [**25**] we prove that all quantized coordinate rings of double Bruhat cells are localizations of CGL extensions. When the main theorem is applied to them, these are precisely the localizations with respect to all frozen variables.

The quantum Schubert cell algebras are special members in the class of CGL extensions. The former have the property that their gradings by the character lattices of the torus $\mathcal{H}$ can be specialized to $\mathbb{Z}_{\geq 0}$-gradings that are connected. We provide an example of a CGL extension that does not have this property in Example 3.5. This example can be easily generalized to families of higher dimensional examples.

CGL extensions are very rarely Hopf algebras. Since Drinfeld's definition of quantized universal enveloping algebras [**9**] imposes the condition that they are topological Hopf algebras, we use the term *quantum nilpotent algebras* as opposed to *quantized universal enveloping algebras of nilpotent Lie algebras*. Along these lines one should also note that it is currently unknown whether every CGL extension is a deformation of the universal enveloping algebra of a nilpotent Lie algebra.

All important CGL extensions that we are aware of are *symmetric* in the sense that they are also CGL extensions when the generators $x_1, \ldots, x_N$ are adjoined in the reverse order

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \ldots [x_1; \sigma^*_1, \delta^*_1].$$

(This symmetry condition may only hold for certain orderings of the initial generators.) We refer the reader to Section 3.3 for a detailed discussion of this condition. Here we note that symmetricity of a CGL extension is equivalent to imposing a very mild condition on the action of $\mathcal{H}$ and the following abstract Levendorskii–Soibelman type straightening law:

For all $j < k$, the element $x_k x_j - \lambda_{kj} x_j x_k$ belongs to the unital subalgebra of R generated by $x_{j+1}, \ldots, x_{k-1}$ for some scalar $\lambda_{kj} \in \mathbb{K}^*$.

One easily shows that in the setting of Definition A the scalar λ_{kj} equals the h_k-eigenvalue of x_j, i.e., $h_k \cdot x_j = \lambda_{kj} x_j$.

The class of algebras covered by the main theorem is the class of symmetric CGL extensions satisfying two very mild conditions on the scalars involved.

1.3. Constructing an initial quantum seed and a precise statement of the main result

The main technique that we use to generate quantum clusters in CGL extensions is that of *noncommutative unique factorization domains* (UFDs), as defined by Chatters [**6**]. We briefly recall some background. A nonzero element p of an integral domain R is called prime if it is normal (meaning that $Rp = pR$) and the factor R/pR is a domain. A noetherian domain R is called a UFD if every nonzero prime ideal of R contains a prime element. It is well known that, in the commutative noetherian situation, this definition is equivalent to the more common one. If R is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by algebra automorphisms, then R has a canonical grading by the rational character lattice $X(\mathcal{H})$ of $\mathcal{H}$. Such an algebra R is called an $\mathcal{H}$-UFD if every nonzero $\mathcal{H}$-prime ideal of R contains a homogeneous prime element. As in the commutative situation, two prime elements $p_1, p_2 \in R$ are associates if and only if there exists a unit $a \in R$ such that $p_2 = ap_1$.

Chatters proved [**6**] that the universal enveloping algebra of every solvable Lie algebra over the field of complex numbers is a UFD. Launois, Lenagan and Rigal proved [**33**] that every CGL extension is an $\mathcal{H}$-UFD. (One can view these results as another reason for thinking of CGL extensions as quantum nilpotent algebras.) As noted in Section 1.2, CGL extensions have only finitely many $\mathcal{H}$-prime ideals [**5, 22**], so they have only finitely many homogeneous prime elements up to associates.

Our key method for generating quantum clusters, and the first step towards proving the main theorem, is as follows:

STEP 1. *For an $\mathcal{H}$-UFD R and a chain of $\mathcal{H}$-UFDs*

$$\mathcal{C}: \quad \{0\} \subset R_1 \subsetneq \ldots \subsetneq R_N = R,$$

define the subset

$$P(\mathcal{C}) := \bigcup_{k=1}^{N} \{\textit{homogeneous prime elements of } R_k \textit{ up to associates}\}$$

of R and prove that (under some general assumptions) $P(\mathcal{C})$ is a quantum cluster in R.

From general properties of prime elements, one gets that under mild assumptions the elements of $P(\mathcal{C})$ quasi-commute, i.e., for every $y, y' \in P(\mathcal{C})$, $yy' = \xi y' y$ for some $\xi \in \mathbb{K}^*$. So one of the major points in realizing Step 1 in a particular situation is to ensure that $P(\mathcal{C})$ has the right size, i.e., that for each k, R_k has precisely one homogeneous prime element (up to taking associates) that is not a prime element of some R_i for $i < k$. Furthermore, in order for this to work out the chain $\mathcal{C}$ should be sufficiently fine, e.g., $\operatorname{GK}\dim R_k = k$ for all k. This condition is obviously satisfied for the canonical chain of subalgebras associated to the CGL extension (1.1).

For arbitrary CGL extensions, Step 1 was carried out in [**24**] in full generality. In the statement of this result, for a function $\eta : [1, N] \to \mathbb{Z}$, we will use the canonical predecessor and successor functions $p := p_\eta : [1, N] \to [1, N] \sqcup \{-\infty\}$ and $s := s_\eta : [1, N] \to [1, N] \sqcup \{+\infty\}$ for the level sets of η (see (3.4)–(3.5)).

THEOREM A. [**24**, Theorem 4.3] *Let R be an arbitrary CGL extension of length N. Then there exist a function $\eta : [1, N] \to \mathbb{Z}$ and elements*

$$c_k \in R_{k-1} \ \textit{for all} \ k \in [2, N] \ \textit{with} \ p(k) \neq -\infty$$

such that the elements $y_1, \ldots, y_N \in R$, recursively defined by

$$y_k := \begin{cases} y_{p(k)} x_k - c_k, & \textit{if } p(k) \neq -\infty \\ x_k, & \textit{if } p(k) = -\infty, \end{cases} \tag{1.2}$$

are homogeneous and have the property that for every $k \in [1, N]$,

$$\{y_j \mid j \in [1, k], \, s(j) > k\}$$

is a list of the homogeneous prime elements of R_k up to scalar multiples. The elements $y_1, \ldots, y_N \in R$ with these properties are unique. The function η with these properties is not unique but its level sets are. Furthermore, the function η has the property that $p(k) = -\infty$ if and only if $\delta_k = 0$.

In different words the theorem states that, if $\mathcal{C}_k$ denotes the truncated chain $\{0\} \subset R_1 \subset \ldots \subset R_k$, then in the setting of the theorem

$$P(\mathcal{C}_k) = P(\mathcal{C}_{k-1}) \sqcup \{y_k\}$$

and the prime element y_k of R_k is determined by the linear expression (1.2) with leading term equal to 1 or to a prime element of the previous algebra R_{k-1}. The function η (or more precisely its level sets) keeps track of the leading terms of this recursive formula.

To construct exchange matrices that go with the quantum clusters from Step 1 (Theorem A), we need to introduce some more notation. The quantum clusters of an algebra generate quantum tori. Recall that a matrix $\mathbf{q} = (q_{kj}) \in M_N(\mathbb{K}^*)$ is called *multiplicatively skew-symmetric* if $q_{kk} = 1$ for all k and $q_{kj} q_{jk} = 1$ for all $k \neq j$. Such a matrix gives rise to the *quantum torus*

$$\mathcal{T}_{\mathbf{q}} := \frac{\mathbb{K}\langle Y_1^{\pm 1}, \ldots, Y_N^{\pm 1} \rangle}{(Y_k Y_j - q_{kj} Y_j Y_k, \ \forall k \neq j)}. \tag{1.3}$$

Given a CGL extension R of length N, we consider the unique multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} \in M_N(\mathbb{K}^*)$ whose entries λ_{kj} for $1 \leq j < k \leq N$ are the h_k-eigenvalues of x_j in the notation of Definition A. Define the (multiplicatively skew-symmetric) bicharacter

$$\Omega_{\boldsymbol{\lambda}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^* \quad \text{by} \quad \Omega_{\boldsymbol{\lambda}}(e_k, e_j) := \lambda_{kj}, \ \ \forall k, j \in [1, N],$$

where $\{e_1, \ldots, e_N\}$ is the standard basis of $\mathbb{Z}^N$. In the setting of Theorem A, define

$$\overline{e}_k := \sum_{n \in \mathbb{Z}_{\geq 0}, \ p^n(k) \neq -\infty} e_{p^n(k)} \in \mathbb{Z}^N, \ \ \forall k \in [1, N].$$

The set $\{\overline{e}_1, \ldots, \overline{e}_N\}$ is another basis of $\mathbb{Z}^N$. We consider the (multiplicatively skew-symmetric) bicharacter

$$\Omega : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^* \quad \text{such that} \quad \Omega(e_k, e_j) := \Omega_{\boldsymbol{\lambda}}(\overline{e}_k, \overline{e}_j), \ \ \forall k, j \in [1, N].$$

The sequence of prime elements in Theorem A produces a quantum cluster of the CGL extension R in the following sense:

For any CGL extension R, the elements $y_1, \dots, y_N \in R$ from Theorem A and their inverses generate a copy of the quantum torus $\mathcal{T}_{\mathbf{q}}$ inside $\mathrm{Fract}(R)$*, where*

$$q_{kj} = \Omega(e_k, e_j) = \Omega_{\boldsymbol{\lambda}}(\overline{e}_k, \overline{e}_j), \ \ \forall k, j \in [1, N]$$

and

$$R \subseteq \mathcal{T}_{\mathbf{q}} \subset \mathrm{Fract}(R).$$

Consider the subset

$$\mathbf{ex} := \{k \in [1, N] \mid s(k) \neq +\infty\}$$

of $[1, N]$. The columns of all $N \times |\mathbf{ex}|$ matrices will be indexed by $\mathbf{ex}$. The set of such matrices with integral entries will be denoted by $M_{N\times\mathbf{ex}}(\mathbb{Z})$.

Main Theorem II: Specific Form. *For every symmetric CGL extension R, the following hold under very mild assumptions on the base field $\mathbb{K}$ and the scalars $\lambda_1, \dots, \lambda_N \in \mathbb{K}^*$:*

(a) *For each $k \in \mathbf{ex}$, there exists a unique vector $b^k = \sum_{l=1}^{N} b_{lk} e_l \in \mathbb{Z}^N$ such that*

$$\Omega(b^k, e_j)^2 = \lambda_{s(k)}^{-\delta_{kj}}, \ \ \forall j \in [1, N]$$

and $y_1^{b_{1k}} \dots y_N^{b_{Nk}}$ is fixed under the $\mathcal{H}$-action. Denote by $\widetilde{B} \in M_{N\times\mathbf{ex}}(\mathbb{Z})$ the matrix with columns $b^k \in \mathbb{Z}^N$, $k \in \mathbf{ex}$.

(b) *The matrix $\widetilde{B}$ is compatible with the quantum cluster $(y_1, \dots, y_N)$ of R in the sense that they define a quantum seed, call it Σ, see* [**3**] *and* §2.3 *for definitions. Let $\mathcal{A}(\Sigma)$ and $\mathcal{U}(\Sigma)$ be the quantum cluster algebra and upper quantum cluster algebra associated to this quantum seed for the set of exchangeable indices* $\mathbf{ex}$ *where none of the frozen cluster variables are inverted.*

(c) *After an explicit rescaling of $y_1, \dots, y_N$, we have the equality*

$$R = \mathcal{A}(\Sigma) = \mathcal{U}(\Sigma).$$

Furthermore, the quantum cluster algebra $\mathcal{A}(\Sigma)$ is generated by finitely many cluster variables.

We refer the reader to Theorem 8.2 for a full statement of the main theorem. The homogeneity condition in part (a) of the theorem can be written in an explicit form taking into account that for each $k \in [1, N]$, y_k is an $\mathcal{H}$-eigenvector with eigenvalue equal to the product of the $\mathcal{H}$-eigenvalues of $x_{p^n(k)}$ for those $n \in \mathbb{Z}_{\geq 0}$ such that $p^n(k) \neq -\infty$ (see Theorem 8.2 (a)).

The quantum cluster algebras in question are built from general (multiparameter) quantum tori by an extension of the Berenstein–Zelevinsky construction [**3**]. The general setting for such quantum cluster algebras is developed in Chapter 2.

We finish this section with the statements of the two mild conditions which are imposed in the Main Theorem. In Proposition 5.8, we prove that

$$\lambda_k = \lambda_l \ \text{ for all } k, l \in [1, N] \text{ such that } \eta(k) = \eta(l) \text{ and } p(k) \neq -\infty,\ p(l) \neq -\infty.$$

Recall also that λ_{kl} denotes the h_k-eigenvalue of x_l for $1 \leq l < k \leq N$. In this setting, the two conditions imposed in the main theorem are:

Condition (A). The base field $\mathbb{K}$ contains square roots $\sqrt{\lambda_{kl}}$ of the scalars λ_{kl} for $1 \leq l < k \leq N$ such that the subgroup of $\mathbb{K}^*$ generated by all of them contains no elements of order 2.

Condition (B). There exist positive integers d_n, $n \in \mathrm{range}(\eta)$, such that

$$\lambda_k^{d_{\eta(l)}} = \lambda_l^{d_{\eta(k)}}$$

for all $k, l \in [1, N]$, $p(k) \neq -\infty$, $p(l) \neq -\infty$.

All symmetric CGL extensions that we are aware of (with the exception of certain very specific two-cocycle twists) satisfy these conditions after a field extension of the base field $\mathbb{K}$. We give a detailed account of this in Remarks 8.5 and 8.6. The first condition is needed when dealing with toric frames as opposed to individual cluster variables. The second condition is needed to ensure that the principal parts of certain exchange matrices that are constructed are skewsymmetrizable. Both issues are minor and the main constructions of quantum clusters and mutations work without imposing those conditions.

1.4. Additional clusters

From now on, R will denote a symmetric CGL extension as in (1.1).

Consider the subset Ξ_N of the symmetric group S_N consisting of all permutations $\tau \in S_N$ such that

$$\tau(k) = \max \tau([1, k-1]) + 1 \text{ or } \tau(k) = \min \tau([1, k-1]) - 1, \ \forall k \in [2, N].$$

It is easy to see that Ξ_N can also be defined as the set of all $\tau \in S_N$ such that $\tau([1, k])$ is an interval for all $k \in [2, N]$. Each symmetric CGL extension has a different CGL extension presentation associated to every element $\tau \in \Xi_N$, of the form

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}] \ldots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}]. \tag{1.4}$$

The proof of this fact and a description of the automorphisms $\sigma''_{\tau(k)}$ and the skew-derivations $\delta''_{\tau(k)}$ are given in Proposition 3.14. This presentation gives rise to a chain of subalgebras of R,

$$\mathcal{C}_\tau : \quad \{0\} \subset R_{\tau,1} \subset \ldots \subset R_{\tau,N} = R,$$

where $R_{\tau,k}$ is the unital subalgebra generated by $x_{\tau(1)}, \ldots, x_{\tau(k)}$. Theorem A associates a quantum cluster

$$(y_{\tau,1}, \ldots, y_{\tau,N}) := P(\mathcal{C}_\tau)$$

to R for each $\tau \in \Xi_N$. Clearly, $\mathrm{id} \in \Xi_N$ and

$$(y_{\mathrm{id},1}, \ldots, y_{\mathrm{id},N}) = P(\mathcal{C}_{\mathrm{id}}) = P(\mathcal{C}) = (y_1, \ldots, y_N).$$

Main Theorem III: Additional Clusters. *Let R be a symmetric CGL extension satisfying the conditions* (A) *and* (B). *For each $\tau \in \Xi_N$, the quantum cluster algebra $\mathcal{A}(\Sigma)$ from Main Theorem* II *has a quantum seed with a set of cluster variables obtained from $(y_{\tau,1}, \ldots, y_{\tau,N})$ by a precise rescaling and permutation* (*see Theorem* 8.2 (a)–(b)). *The exchange matrix $\widetilde{B}_\tau$ for this seed is the unique solution of the equations in part* (a) *of Main Theorem* II *for the bicharacter associated to the CGL extension presentation* (1.4).

In particular, an appropriate scalar multiple of each generator x_k of R is a cluster variable.

The last part of the theorem follows from the first because for each $k \in [1, N]$ there exists an element $\tau \in \Xi_N$ such that $\tau(1) = k$.

1.5. Strategy of the proof

The second step of the proof of the main theorem is to connect the quantum clusters indexed by the elements of Ξ_N by a chain of mutations:

STEP 2. *Let $\tau \neq \tau'$ in Ξ_N be such that for some $k \in [1, N-1]$, $\tau(j) = \tau'(j)$ for all $j \neq k, k+1$ (i.e., the subalgebras in the chains $\mathcal{C}_\tau$ and $\mathcal{C}_{\tau'}$ only differ in position k). We prove that in this setting the following hold for every symmetric CGL extension R:*

(a) *If $\tau(k)$ and $\tau(k+1)$ have different images under the η-function for the presentation* (1.4)*, then the quantum clusters $P(\mathcal{C}_\tau)$ and $P(\mathcal{C}_{\tau'})$ are equal. More precisely,*

$$y_{\tau',j} = y_{\tau,j} \;\; \text{for } j \neq k, k+1 \;\text{ and }\; y_{\tau',k} = y_{\tau,k+1}, \;\; y_{\tau',k+1} = y_{\tau,k}.$$

(b) *If $\tau(k)$ and $\tau(k+1)$ have the same images under the η-function for the presentation* (1.4)*, then the quantum cluster $P(\mathcal{C}_{\tau'})$ of R is a one-step mutation of the quantum cluster $P(\mathcal{C}_\tau)$. More precisely,*

$$y_{\tau',k} = y_{\tau,k}^{-1} \left(\xi_1 \prod_{j \in [1,N],\, c_j > 0} y_{\tau,j}^{c_j} + \xi_2 \prod_{j \in [1,N],\, c_j < 0} y_{\tau,j}^{-c_j} \right) \tag{1.5}$$

for some $\xi_1, \xi_2 \in \mathbb{K}^*$ and an integral vector $(c_1, \ldots, c_N)$ such that $c_k = 0$, and

$$y_{\tau',j} \;\text{ is a scalar multiple of }\; y_{\tau,j} \;\text{ for all }\; j \neq k.$$

This result has a simple proof given in Theorems 4.2, 4.3. It is derived from the uniqueness part of Theorem A and the uniqueness of the decomposition of a normal element of a noncommutative UFD as a product of prime elements. *Thus, in a nutshell, cluster mutation is forced by the uniqueness in Theorem A and the unique decomposition in noncommutative UFDs.*

The simultaneous normalization (rescaling) of all cluster variables $y_{\tau,k}$, $\tau \in \Xi_N$, $k \in [1, N]$ for which all scalars ξ_1 and ξ_2 in (1.5) become equal to 1, turns out to be a rather delicate issue. It is carried out in Chapters 5 and 6 in an explicit form.

The next step is to extend the clusters $P(\mathcal{C}_\tau)$ to quantum seeds, which amounts to the construction of exchange matrices that are compatible with these quantum clusters. We also need to connect those seeds with cluster mutations. Two elements $\tau, \tau' \in \Xi_N$ will be called adjacent if they satisfy the condition in Step 2, in other words, if there exists $k \in [1, N-1]$ such that

$$\tau' = \tau(k, k+1). \tag{1.6}$$

STEP 3. *Construct exchange matrices $\widetilde{B}_\tau$ for quantum seeds going with the quantum clusters $P(\mathcal{C}_\tau)$, $\tau \in \Xi_N$.*

Show that the seeds corresponding to adjacent pairs $\tau, \tau' \in \Xi_N$ are obtained by one-step mutations from each other.

Let $\tau \in \Xi_N$. Step 2 constructs one column of the desired matrix $\widetilde{B}_\tau$ for each element $\tau' \in \Xi_N$ which is adjacent to τ. More precisely, this is the k-th column of $\widetilde{B}_\tau$ where $k \in [1, N-1]$ is the index from (1.6). Step 3 involves constructing the rest of the matrix $\widetilde{B}_\tau$ and then showing that the exchange matrices for adjacent elements of Ξ_N are related by mutation. One should note that there are not enough adjacent elements to construct the full matrix $\widetilde{B}_\tau$ directly from Step 2.

The realization of Step 3 and the proof of the remaining parts of the main theorem are based on using a chain of successively adjacent elements of Ξ_N of the form

$$\tau_0 = \mathrm{id} \to \tau_1 \to \ldots \to \tau_{M-1} \to \tau_M = w_\circ, \tag{1.7}$$

where $w_\circ$ is the longest element of S_N and $M = N(N-1)/2$ is the length of $w_\circ$. The arrows are drawn for the convenience of the description of the procedure.

We first construct $\widetilde{B} = \widetilde{B}_{\mathrm{id}}$. Each chain (1.7) has the property that

$$\forall k \in [1, N-1], \ \ \exists i \in [1, M-1] \ \ \textit{such that} \ \ \tau_{i+1} = \tau_i(k, k+1). \tag{1.8}$$

This implies that every exchangeable index in $\mathbf{ex} = \{k \in [1, N] \mid s(k) \neq +\infty\}$ *gets mutated when Step* 2 *is applied to the arrows in the chain.*

Let $k \in \mathbf{ex}$. Leaving aside certain technical details concerning a reenumeration of cluster variables, to define the k-th column of $\widetilde{B}$ we choose an index $i \in [1, M-1]$ satisfying (1.8). Step 2 defines the k-th columns of $\widetilde{B}_{\tau_i}$ and $\widetilde{B}_{\tau_{i+1}}$. We construct the k-th column of $\widetilde{B}$ by inverse mutation from the k-th column of $\widetilde{B}_{\tau_i}$ along the chain (1.7). At this point, one has to prove that this is independent of the choice of i. There two problems here. The first is that the integers c_j in (1.5) are not determined from easily accessible ring theoretic data. They are powers in the decomposition of some normal elements as products of primes, but the setting is very general to get hold of them. Secondly, the inverse mutation from such data gets combinatorially overwhelming and one cannot keep track of it in general ring theoretic terms.

We present a very general (ring theoretic) solution to this problem. Firstly, we establish that there is at most one matrix $\widetilde{B}$ that satisfies the conditions in Theorem II (a), using a strong rationality result for the $\mathcal{H}$-primes of CGL extensions [**4**, Theorem II.6.4], a special case of which is that

$$Z(\mathrm{Fract}(R))^{\mathcal{H}} = \mathbb{K},$$

where $Z(.)$ stands for the center of an algebra. We obtain the uniqueness statement in Theorem II (a) by phrasing the conditions for the columns of $\widetilde{B}$ in terms of commutation relations and using the above result. We then use the idea for inverse mutation along the chain (1.7) to prove that a matrix $\widetilde{B}$ with the properties in Theorem II (a) exists.

All other exchange matrices $\widetilde{B}_\tau$ for $\tau \in \Xi_N$ are constructed by mutating $\widetilde{B}$ along a chain which is entirely within Ξ_N, starts with id, ends at τ, and for which each two consecutive elements are adjacent. It is not difficult to see that such a chain exists for every $\tau \in \Xi_N$. Typically, there are many chains with these properties and the independence of $\widetilde{B}_\tau$ from the choice of a chain is proved using one more time the strong rationality result. The last part of Step 3, that the exchange matrices for adjacent elements of Ξ_N are related by mutation, follows from their construction.

At this point, we have a construction of quantum seeds Σ_τ indexed by the elements of Ξ_N and mutations between them. What is left is to establish part (c) of Main Theorem II. This is the last step of the proof of the theorem:

Step 4. *Show that*

$$R = \mathcal{A}(\Sigma) = \mathcal{U}(\Sigma)$$

for $\Sigma := \Sigma_{\mathrm{id}}$, *i.e., that* R *coincides with the quantum cluster algebra and the upper quantum cluster algebra for the quantum seed* Σ.

In the remaining part of this section we sketch how this step is carried out. As previously mentioned, for every $k \in [1, N]$ there is an element $\tau \in \Xi_N$ with $\tau(1) = k$. Since all quantum seeds Σ_τ are mutation equivalent to each other, a scalar multiple of each generator x_k is a cluster variable of $\mathcal{A}(\Sigma)$. Therefore,

$$R \subseteq \mathcal{A}(\Sigma).$$

By the quantum Laurent phenomenon,

$$\mathcal{A}(\Sigma) \subseteq \mathcal{U}(\Sigma).$$

For $\tau \in \Xi_N$, denote by E_τ the multiplicative subset of R generated by all exchangeable cluster variables in Σ_τ. We first prove that E_τ is an Ore set in R. Consider a chain of elements of Ξ_N as in (1.7). We have

$$\mathcal{U}(\Sigma) \subseteq \bigcap_{i=0}^{M} R[E_{\tau_i}^{-1}].$$

Combining all of the above embeddings, we obtain

$$R \subseteq \mathcal{A}(\Sigma) \subseteq \mathcal{U}(\Sigma) \subseteq \bigcap_{i=0}^{M} R[E_{\tau_i}^{-1}]. \tag{1.9}$$

At this point, we use techniques from noncommutative UFDs and iterated Ore extensions to describe the intersection of $R[E_\tau^{-1}]$ and $R[E_{\tau'}^{-1}]$ for two adjacent elements of τ and τ' of Ξ_N. Based on it we prove that

$$\bigcap_{i=0}^{M} R[E_{\tau_i}^{-1}] = R. \tag{1.10}$$

This forces all inclusions in (1.9) to be equalities.

1.6. Organization of the paper and reading suggestions

The paper is organized as follows:

I. Chapter 2 provides the general framework for quantum cluster algebras generalizing the Berenstein–Zelevinsky construction [**3**] to mutations between multiparameter quantum tori (i.e., quantum tori (1.3) for which the scalars q_{kl} are not necessarily integral powers of a single element $q \in \mathbb{K}^*$). This chapter is self contained and can be read independently of [**3**]. Readers who are familiar with [**3**] might look only at the main definitions since the treatment is similar to [**3**]. Chapter 3 reviews the theory of noncommutative UFDs and the material from [**24**] that goes with Theorem A.

II. Chapter 4 carries out Step 2. For readers who are interested in understanding the intrinsic reason for the appearance of cluster mutations in the general framework of quantum nilpotent algebras, we suggest reading the proofs of Theorems 4.2, 4.3.

Chapters 5 and 6 carry out in an explicit form the simultaneous normalization of all cluster variables $y_{\tau,k}$ so that all scalars ξ_1, ξ_2 in part (b) of Step 2 are equal to 1. These chapters are technical and we suggest that readers restrict to the definitions of $y_{[.,.]}$, $u_{[.,.]}$ and $\pi_{[.,.]}$ in Theorem 5.1 (c), Corollary 5.11 and Eq. (6.1), as well as the statement of Proposition 6.3 and the condition (6.13). This information will be sufficient for understanding the statement of the main result.

Chapter 7 proves Eq. (1.10) which essentially carries out Step 4. Readers who are only interested in understanding the main result could skip it.

III. The main result appears in Theorem 8.2. Chapter 8 also carries out Step 3 and the general part of Step 4.

IV. Chapter 10 illustrates how Theorem 8.2 is applied. It proves that the quantum Schubert cell algebras $\mathcal{U}_q(\mathfrak{n}_\pm \cap w(\mathfrak{n}_\mp))$, for all simple Lie algebras $\mathfrak{g}$ and Weyl group elements w, are quantum cluster algebras which in addition coincide with the corresponding upper quantum cluster algebras (Theorem 10.1). Chapter 9 contains the needed details on quantum groups and their relations to quantum nilpotent algebras.

We suggest the following route for readers who are interested in understanding the final result (Theorem 8.2) without the details of its proof:

1. Basics of CGL extensions Sections 3.2–3.3 and the base change in Section 4.2 to include $\sqrt{\lambda_{lj}}$ in $\mathbb{K}$.

2. The setting of Section 8.1 and the statement of Theorem 8.2. We impose conditions (A) and (B), but the unique rescaling is a feature of the algebra. One can read the statement of the theorem knowing only that such a rescaling exists. If one needs to know its precise form, one can return to the definitions of $y_{[.,.]}$, $u_{[.,.]}$ and $\pi_{[.,.]}$ and the condition (6.13) as indicated in II above.

Readers could also consult [**23**] for an abridged version of Theorem 8.2, and its setting and applications.

1.7. Notation

We finish the introduction with some notation to be used throughout the paper. For $m, n \in \mathbb{Z}_{>0}$ and a commutative ring D, we will denote by $M_{m,n}(D)$ the set of matrices of size $m \times n$ with entries from D. We will use a multiplicative (exponential) version of the standard matrix multiplication: For $A = (a_{ij}) \in M_{m,n}(\mathbb{Z})$, $B = (b_{jk}) \in M_{n,p}(D)$, and $C = (c_{kl}) \in M_{p,s}(\mathbb{Z})$, denote the matrix

$$ {}^{A}B^{C} := \Big(\prod_{j,k} b_{jk}^{a_{ij} c_{kl}} \Big) \in M_{m,s}(D). $$

The two obvious special cases of this operation are given by

$$ {}^{A}B = \Big(\prod_{j} b_{jk}^{a_{ij}} \Big) \in M_{m,p}(D), \quad B^{C} = \Big(\prod_{k} b_{jk}^{c_{kl}} \Big) \in M_{n,s}(D). $$

Clearly,

$$ {}^{A_1}\big({}^{A}B^{C}\big) = {}^{A_1 A}B^{C} \quad \text{and} \quad \big({}^{A}B^{C}\big)^{C_1} = {}^{A}B^{C C_1} \tag{1.11} $$

for all matrices A_1 and C_1 with integer entries, having the appropriate sizes for which the operations are defined. The transpose map on matrices will be denoted by $A \mapsto A^T$.

Elements of $\mathbb{Z}^N$ will be thought of as column vectors, and the standard basis of $\mathbb{Z}^N$ will be denoted by $\{e_k \mid k \in [1, N]\}$. For $g = \sum_k m_k e_k \in \mathbb{Z}^N$, denote

$$ [g]_+ := \sum_k \max(m_k, 0) e_k, \quad [g]_- := \sum_k \min(m_k, 0) e_k. \tag{1.12} $$

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CHAPTER 2

Quantum cluster algebras

In this chapter we give a definition of quantum cluster algebras and upper quantum cluster algebras (over arbitrary commutative rings) that extends the one of Berenstein and Zelevinsky [**3**]. The construction in [**3**] used uniparameter quantum tori (quantum tori for which the commutation scalars are integral powers of a single element $q \in \mathbb{K}^*$). We extend the construction of [**3**] to mutation of cluster variables coming from general quantum tori. Furthermore, the construction allows for arbitrary frozen variables to be inverted or not. Following the argument of Fomin–Zelevinsky a quantum Laurent phenomenon is established in this generality.

2.1. General quantum tori

Recall that a matrix $\mathbf{q} = (q_{kj}) \in M_N(\mathbb{K}^*)$ is called *multiplicatively skew-symmetric* if $q_{kk} = 1$ and $q_{jk} = q_{kj}^{-1}$ for all $j \neq k$. Such a matrix gives rise to the skew-symmetric bicharacter

$$\Omega_{\mathbf{q}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^* \text{ given by } \Omega_{\mathbf{q}}(e_k, e_j) := q_{kj}, \; \forall k, j \in [1, N], \tag{2.1}$$

i.e.,

$$\Omega_{\mathbf{q}}(f, g) = {}^{f^T}\mathbf{q}^{g}, \quad \forall f, g \in \mathbb{Z}^N. \tag{2.2}$$

(Recall that $e_1, \ldots, e_N$ denote the standard basis vectors of $\mathbb{Z}$ and that all elements of $\mathbb{Z}^N$ are thought of as column vectors.)

A multiplicatively skew-symmetric matrix $\mathbf{q} \in M_N(\mathbb{K}^*)$ gives rise to the quantum torus

$$\mathcal{T}_{\mathbf{q}} := \frac{\mathbb{K}\langle Y_1^{\pm 1}, \ldots, Y_N^{\pm 1} \rangle}{(Y_k Y_j - q_{kj} Y_j Y_k, \; \forall k \neq j)}. \tag{2.3}$$

The subalgebra of $\mathcal{T}_{\mathbf{q}}$

$$\mathcal{A}_{\mathbf{q}} := \mathbb{K}\langle Y_1, \ldots, Y_N \rangle \subset \mathcal{T}_{\mathbf{q}} \tag{2.4}$$

is called a *quantum affine space algebra*. The quantum torus $\mathcal{T}_{\mathbf{q}}$ has the $\mathbb{K}$-basis

$$\{Y^f := Y_1^{m_1} \cdots Y_N^{m_N} \mid f = (m_1, \ldots, m_N)^T \in \mathbb{Z}^N\}. \tag{2.5}$$

Given any subring D of $\mathbb{K}$ that contains the subgroup of $\mathbb{K}^*$ generated by all entries of $\mathbf{q}$, one defines versions of $\mathcal{T}_{\mathbf{q}}$ and $\mathcal{A}_{\mathbf{q}}$ over D. One can either use (2.3) and (2.4) with free algebras over D instead of $\mathbb{K}$, or consider the D-subalgebras of $\mathcal{T}_{\mathbf{q}}$ and $\mathcal{A}_{\mathbf{q}}$ generated by $Y_k^{\pm 1}$, respectively Y_k ($k \in [1, N]$).

2.2. Based quantum tori

For a multiplicatively skew-symmetric matrix $\mathbf{r} := (r_{kj}) \in M_N(\mathbb{K}^*)$ denote

$$\mathbf{r}^{\cdot 2} := (r_{kj}^2) \in M_N(\mathbb{K}^*).$$

The quantum torus $\mathcal{T}_{\mathbf{r}^{\cdot 2}}$ has a $\mathbb{K}$-basis which is obtained by a type of symmetrization of the monomials in (2.5). This basis consists of the elements

$$Y^{(f)} := \mathcal{S}_{\mathbf{r}}(f) Y^f = \mathcal{S}_{\mathbf{r}}(f) Y_1^{m_1} \dots Y_N^{m_N} \quad \text{for} \quad f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N, \tag{2.6}$$

where the symmetrization scalar $\mathcal{S}_{\mathbf{r}}(f) \in \mathbb{K}^*$ is given by

$$\mathcal{S}_{\mathbf{r}}(f) := \prod_{j<k} r_{jk}^{-m_j m_k}. \tag{2.7}$$

We have $Y^{(e_k)} = Y_k$, $\forall k \in [1, N]$. Furthermore, one easily verifies that

$$Y^{(f)} Y^{(g)} = \Omega_{\mathbf{r}}(f, g) Y^{(f+g)}, \quad \forall f, g \in \mathbb{Z}^N, \tag{2.8}$$

recall (2.1). We will call the torus $\mathcal{T}_{\mathbf{r}^{\cdot 2}}$ with the $\mathbb{K}$-basis $\{Y^{(f)} \mid f \in \mathbb{Z}^N\}$ the *based quantum torus* associated to the matrix $\mathbf{r} \in M_N(\mathbb{K}^*)$ (or to the corresponding bicharacter $\Omega_{\mathbf{r}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^*$).

Each $\sigma \in GL_N(\mathbb{Z})$ gives rise to another generating set of $\mathcal{T}_{\mathbf{r}^{\cdot 2}}$, consisting of the elements

$$Y_{\sigma,k} := Y^{(\sigma(e_k))}, \quad k \in [1, N].$$

Define the multiplicatively skew-symmetric matrix

$$\mathbf{r}_\sigma := {}^{\sigma^T}\mathbf{r}^{\sigma} \in M_N(\mathbb{K}^*).$$

It follows from (2.8) that there is a $\mathbb{K}$-algebra isomorphism

$$\psi_\sigma : \mathcal{T}_{\mathbf{r}_\sigma^{\cdot 2}} \to \mathcal{T}_{\mathbf{r}^{\cdot 2}}, \quad \text{given by} \quad \psi_\sigma(Y_k) = Y_{\sigma,k}, \ k \in [1, N]. \tag{2.9}$$

The elements

$$Y_\sigma^{(f)} := \psi_\sigma(Y^{(f)}) = \Big(\prod_{j<k} (\mathbf{r}_\sigma)_{jk}^{-m_j m_k} \Big) Y_{\sigma,1}^{m_1} \dots Y_{\sigma,N}^{m_N} \quad \text{for} \quad f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N$$

also form a basis of $\mathcal{T}_{\mathbf{r}^{\cdot 2}}$. The next proposition shows that this basis coincides with the original basis (2.6) of $\mathcal{T}_{\mathbf{r}^{\cdot 2}}$. In other words, the based quantum torus associated with $\mathbf{r} \in M_N(\mathbb{K}^*)$ is invariant under the canonical action of $GL_N(\mathbb{Z})$.

PROPOSITION 2.1. *For all $\sigma \in GL_N(\mathbb{Z})$ and all multiplicatively skew-symmetric matrices $\mathbf{r} \in M_N(\mathbb{K}^*)$, we have*

$$Y^{(f)} = Y_\sigma^{(\sigma^{-1}(f))}, \quad \forall f \in \mathbb{Z}^N. \tag{2.10}$$

PROOF. For all $f, g \in \mathbb{Z}^N$,

$$\Omega_{\mathbf{r}}(\sigma(f), \sigma(g)) = {}^{(\sigma f)^T}\mathbf{r}^{\sigma g} = {}^{f^T}({}^{\sigma^T}\mathbf{r}^{\sigma})^{g} = \Omega_{\mathbf{r}_\sigma}(f, g).$$

Because of (2.8), to show

$$Y^{(\sigma(f))} = Y_\sigma^{(f)}, \quad \forall f \in \mathbb{Z}^N,$$

it is sufficient to verify the identity for $f = e_k$, $k \in [1, N]$. This is straightforward:

$$Y^{(\sigma(e_k))} = Y_{\sigma,k} = Y_\sigma^{(e_k)}.$$

$\square$

Definition 2.2. Let $\mathcal{F}$ be a division algebra over $\mathbb{K}$. We say that a map

$$M : \mathbb{Z}^N \to \mathcal{F}$$

is a *toric frame* if there exists a multiplicatively skew-symmetric matrix $\mathbf{r} \in M_N(\mathbb{K}^*)$ such that the following conditions are satisfied:

(i) There is an algebra embedding $\varphi : \mathcal{T}_{\mathbf{r}^{\cdot 2}} \hookrightarrow \mathcal{F}$ given by $\varphi(Y_i) = M(e_i)$, $\forall i \in [1, N]$ and $\mathcal{F} = \mathrm{Fract}(\varphi(\mathcal{T}_{\mathbf{r}^{\cdot 2}}))$.

(ii) For all $f \in \mathbb{Z}^N$, $M(f) = \varphi(Y^{(f)})$.

In other words, a toric frame is an embedding of a based quantum torus into a division algebra $\mathcal{F}$ such that $\mathcal{F}$ equals the classical quotient ring of the image.

For a toric frame $M : \mathbb{Z}^N \to \mathcal{F}$, there exists a unique matrix $\mathbf{r} = (r_{kj}) \in M_N(\mathbb{K}^*)$ with the properties of the definition, and it is determined by M by

$$r_{kj} = M(e_k)M(e_j)M(e_k + e_j)^{-1}, \quad \forall 1 \le k < j \le N.$$

It will be called *the matrix of the toric frame* and will be denoted by $\mathbf{r}(M) := \mathbf{r}$.

Suppose that $M : \mathbb{Z}^N \to \mathcal{F}$ is a toric frame, with matrix $\mathbf{r}$ and algebra embedding $\varphi : \mathcal{T}_{\mathbf{r}^{\cdot 2}} \hookrightarrow \mathcal{F}$. Let $\sigma \in GL_N(\mathbb{Z})$, and let $\psi_\sigma : \mathcal{T}_{\mathbf{r}_\sigma^{\cdot 2}} \to \mathcal{T}_{\mathbf{r}^{\cdot 2}}$ be the isomorphism (2.9). Then $\varphi\psi_\sigma : \mathcal{T}_{\mathbf{r}_\sigma^{\cdot 2}} \to \mathcal{F}$ is an algebra embedding such that $\varphi\psi_\sigma(Y_i) = M\sigma(e_i)$ for all $i \in [1, N]$. Moreover, using (2.10) we see that $\varphi\psi_\sigma(Y^{(f)}) = M\sigma(f)$ for all $f \in \mathbb{Z}^N$. Thus,

$$M\sigma : \mathbb{Z}^N \to \mathcal{F} \ \text{ is a toric frame with matrix } \ \mathbf{r}_\sigma = {}^{\sigma^T}\mathbf{r}^{\sigma}. \tag{2.11}$$

2.3. Compatible pairs

We fix in addition a positive integer n such that $n \le N$ and a subset $\mathbf{ex} \subseteq [1, N]$ of cardinality n. The indices in $\mathbf{ex}$ will be called *exchangeable*, and those in $[1, N] \backslash \mathbf{ex}$ *frozen*.

The entries of all $N \times n$, $n \times N$, and $n \times n$ matrices will be indexed by the sets $N \times \mathbf{ex}$, $\mathbf{ex} \times N$, and $\mathbf{ex} \times \mathbf{ex}$, respectively. For a matrix $\widetilde{B} = (b_{kj}) \in M_{N\times n}(\mathbb{Z})$, its $\mathbf{ex} \times \mathbf{ex}$ submatrix will be denoted by B and will be called the *principal part* of $\widetilde{B}$. The columns of $\widetilde{B}$ will be denoted by $b^j \in \mathbb{Z}^N$, $j \in \mathbf{ex}$. Recall that B is said to be *skew-symmetrizable* provided there exist positive integers d_k, $k \in \mathbf{ex}$ such that $d_k b_{kj} = -d_j b_{jk}$ for all $k, j \in \mathbf{ex}$.

Let $\mathbf{r} \in M_N(\mathbb{K}^*)$ be a multiplicatively skew-symmetric matrix and $\widetilde{B} = (b_{kj}) \in M_{N\times n}(\mathbb{Z})$. Define the matrix $\widetilde{\mathbf{t}} := {}^{\widetilde{B}^T}\mathbf{r} \in M_{n\times N}(\mathbb{K}^*)$. Its entries are given by

$$t_{kj} := \Omega_{\mathbf{r}}(b^k, e_j) = \left({}^{\widetilde{B}^T}\mathbf{r}\right)_{kj} = {}^{(b^k)^T}\mathbf{r}^{e_j} = \prod_{l=1}^{N} r_{lj}^{b_{lk}}, \ \ \forall k \in \mathbf{ex}, \ j \in [1, N]. \tag{2.12}$$

The pair $(\mathbf{r}, \widetilde{B})$ is *compatible* if the following two conditions are satisfied:

$$t_{kj} = 1, \ \ \forall k \in \mathbf{ex}, \ j \in [1, N], \ k \ne j \quad \text{and} \tag{2.13}$$

$$t_{kk} \ \text{ are not roots of unity}, \ \ \forall k \in \mathbf{ex}. \tag{2.14}$$

Denote the $\mathbf{ex} \times \mathbf{ex}$ submatrix of $\widetilde{\mathbf{t}}$ by $\mathbf{t}$. The condition (2.13) implies that

$$\mathbf{t}^B = \widetilde{\mathbf{t}}^{\widetilde{B}}. \tag{2.15}$$

PROPOSITION 2.3. *If the pair* $(\mathbf{r}, \widetilde{B})$ *is compatible, then* $\widetilde{B}$ *has full rank and the matrix*

$$\mathbf{t}^B = (t_{kk}^{b_{kj}})_{k,j\in\mathbf{ex}} \in M_{n\times n}(\mathbb{K}^*)$$

is multiplicatively skew-symmetric.

PROOF. First we show that $\widetilde{B}$ has full rank. If this is not the case, then $\mathrm{Ker}(\widetilde{B}) \cap \mathbb{Z}^n \neq 0$. Let $\sum_{j\in\mathbf{ex}} a_j e_j \in \mathrm{Ker}(\widetilde{B}) \cap \mathbb{Z}^n$ be a nonzero vector. Applying (2.13), we obtain

$$t_{jj}^{a_j} = \prod_k t_{kj}^{a_k} = \prod_{k,l} r_{lj}^{b_{lk}a_k} = 1, \quad \forall j \in \mathbf{ex}.$$

The condition (2.14) implies $a_j = 0$ for all $j \in \mathbf{ex}$, which is a contradiction.

Since $\mathbf{t}^B = \widetilde{\mathbf{t}}^{\widetilde{B}} = {}^{\widetilde{B}^T}\mathbf{r}^{\widetilde{B}}$, cf. (2.15), the second statement follows at once from the fact that $\mathbf{r}$ is multiplicatively skew-symmetric. □

Unlike the uniparameter case [**3**, Proposition 3.3], compatibility of $(\mathbf{r}, \widetilde{B})$ does not in general imply that B is skew-symmetrizable. The next lemma describes an instance when this condition appears naturally.

LEMMA 2.4. *Assume that* $(\mathbf{r}, \widetilde{B})$ *is a compatible pair. If there exist positive integers* d_k*,* $k \in \mathbf{ex}$ *such that*

$$t_{kk}^{d_j} = t_{jj}^{d_k}, \quad \forall j, k \in \mathbf{ex}, \tag{2.16}$$

then the principal part of $\widetilde{B}$ *is skew-symmetrizable via these* d_k*, i.e.,* $d_k b_{kj} = -d_j b_{jk}$ *for all* $k, j \in \mathbf{ex}$*.*

PROOF. By Proposition 2.3, $t_{kk}^{b_{kj}} = t_{jj}^{-b_{jk}}$ for all $j, k \in \mathbf{ex}$. Combining this with (2.16) leads to

$$t_{jj}^{d_k b_{kj}} = t_{kk}^{b_{kj} d_j} = t_{jj}^{-d_j b_{jk}}.$$

Thus $d_k b_{kj} = -d_j b_{jk}$ because of (2.14). □

REMARK 2.5. The condition (2.16) is satisfied if there exists a non-root of unity $q \in \mathbb{K}^*$ such that each $t_{kk} = q^{m_k}$ for some $m_k \in \mathbb{Z}_{>0}$. Then one can set $d_k = m_k$.

2.4. Mutation of compatible pairs

We proceed with the definition of mutation of a compatible pair $(\mathbf{r}, \widetilde{B})$ in direction $k \in \mathbf{ex}$. The entries (b'_{ij}) of the mutation $\mu_k(\widetilde{B})$ of the second matrix are given by the Fomin–Zelevinsky formula [**11**]:

$$b'_{ij} := \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases} \tag{2.17}$$

By [**1**, Eq. (3.2)], the matrix $\mu_k(\widetilde{B})$ is also given by

$$\mu_k(\widetilde{B}) = E_\epsilon \widetilde{B} F_\epsilon$$

for both choices of signs $\epsilon = \pm$, where $E_\epsilon = E_\epsilon^{\widetilde{B}}$ and $F_\epsilon = F_\epsilon^{\widetilde{B}}$ are the $N \times N$ and $n \times n$ matrices with entries given by

$$(E_\epsilon)_{ij} = \begin{cases} \delta_{ij}, & \text{if } j \neq k \\ -1, & \text{if } i = j = k \\ \max(0, -\epsilon b_{ik}), & \text{if } i \neq j = k \end{cases}$$
$$(F_\epsilon)_{ij} = \begin{cases} \delta_{ij}, & \text{if } i \neq k \\ -1, & \text{if } i = j = k \\ \max(0, \epsilon b_{kj}), & \text{if } i = k \neq j. \end{cases} \tag{2.18}$$

We refer the reader to [**11, 3**] for a detailed discussion of the properties of the mutation of the matrix $\widetilde{B}$. Finally, we define the mutation in direction k of the first matrix of the compatible pair $(\mathbf{r}, \widetilde{B})$ by

$$\mu_k(\mathbf{r}) := {}^{E_\epsilon^T}\mathbf{r}^{E_\epsilon}. \tag{2.19}$$

This is a multiplicative version of [**3**, Eq. (3.4)].

Proposition 2.6. *Let $(\mathbf{r}, \widetilde{B})$ be a compatible pair.*

(a) *The matrix $\mu_k(\mathbf{r})$, defined in* (2.19)*, does not depend on the choice of sign $\epsilon = \pm$.*

(b) *Assume also that the principal part of $\widetilde{B}$ is skew-symmetrizable. Then the pair $(\mu_k(\mathbf{r}), \mu_k(\widetilde{B}))$ is compatible, the principal part of $\mu_k(\widetilde{B})$ is skew-symmetrizable, and the $\widetilde{\mathbf{t}}$-matrix of this pair coincides with the one of the pair $(\mathbf{r}, \widetilde{B})$.*

We define the *mutation in direction $k \in \mathbf{ex}$* of the compatible pair $(\mathbf{r}, \widetilde{B})$ to be the compatible pair $(\mu_k(\mathbf{r}), \mu_k(\widetilde{B}))$.

Proof of Proposition 2.6. We follow the argument of the proof of [**3**, Proposition 3.4] phrased in terms of multiplicative expressions.

(a) As in [**3**, Eqs. (3.5) and (3.7)], $E_\epsilon^2 = I_N$, $F_\epsilon^2 = I_n$ and (a correction) E_+E_- equals the $N \times N$ matrix G with columns

$$g_j = e_j + \delta_{jk} b^k, \ \ j \in [1, N].$$

Because of (1.11), part (a) is equivalent to saying that ${}^{G^T}\mathbf{r}^G = \mathbf{r}$, which is verified as follows (using (2.13)–(2.14)):

$$\begin{aligned} \left({}^{G^T}\mathbf{r}^G\right)_{ij} &= {}^{e_i^T + \delta_{ik}(b^k)^T}\mathbf{r}^{e_j + \delta_{jk}b^k} = \Omega_\mathbf{r}(e_i + \delta_{ik}b^k, e_j + \delta_{jk}b^k) \\ &= \Omega_\mathbf{r}(e_i, e_j) t_{kk}^{\delta_{ik}\delta_{jk}} t_{kk}^{-\delta_{ik}\delta_{jk}} = \Omega_\mathbf{r}(e_i, e_j) = r_{ij}, \ \ \forall i, j \in [1, N]. \end{aligned}$$

(b) We have

$${}^{\mu_k(\widetilde{B})^T}(\mu_k(\mathbf{r})) = {}^{F_\epsilon^T \widetilde{B}^T E_\epsilon^T E_\epsilon^T}\mathbf{r}^{E_\epsilon} = {}^{F_\epsilon^T \widetilde{B}^T}\mathbf{r}^{E_\epsilon} = {}^{F_\epsilon^T}\widetilde{\mathbf{t}}^{E_\epsilon}.$$

The second statement in Proposition 2.3 and the fact that for all $i, j \in \mathbf{ex}$, b_{ij} and $-b_{ji}$ have the same signs (which follows from the skew-symmetrizability assumption) imply

$$\widetilde{\mathbf{t}}^{E_\epsilon} = {}^{F_\epsilon^T}\widetilde{\mathbf{t}}.$$

Therefore

$${}^{\mu_k(\widetilde{B})^T}(\mu_k(\mathbf{r})) = {}^{F_\epsilon^T}\widetilde{\mathbf{t}}^{E_\epsilon} = \widetilde{\mathbf{t}}$$

and the pair $(\mu_k(\mathbf{r}), \mu_k(\widetilde{B}))$ satisfies the conditions (2.13)–(2.14). Furthermore, the $\widetilde{\mathbf{t}}$-matrices of the compatible pairs $(\mathbf{r}, \widetilde{B})$ and $(\mu_k(\mathbf{r}), \mu_k(\widetilde{B}))$ are equal. The principal part of $\mu_k(\widetilde{B})$, namely $\mu_k(B)$, is skew-symmetrizable (for the same choice of positive integers d_j, $j \in \mathbf{ex}$ as for B), by the observations in [**11**, Proposition 4.5]. $\square$

2.5. Quantum seeds and mutations

Definition 2.7. A *quantum seed* of a division algebra $\mathcal{F}$ is a pair $(M, \widetilde{B})$ consisting of a toric frame $M : \mathbb{Z}^N \to \mathcal{F}$ and an integer matrix $\widetilde{B} \in M_{N \times n}(\mathbb{Z})$ satisfying the following two conditions:

(i) The pair $(\mathbf{r}(M), \widetilde{B})$ is compatible, where $\mathbf{r}(M)$ is the matrix of M.
(ii) The principal part of $\widetilde{B}$ is skew-symmetrizable.

The elements $M(e_1), \dots, M(e_N) \in \mathcal{F}$ will be called *cluster variables* of the seed $(M, \widetilde{B})$. The cluster variables $M(e_k)$, $k \in \mathbf{ex}$ will be called *exchangeable* and the $M(e_k)$ for $k \in [1, N] \backslash \mathbf{ex}$ *frozen.*

The following lemma is an analog of [**3**, Proposition 4.2]. It will be used in defining mutations of quantum seeds.

Lemma 2.8. *Assume that* $\mathbf{r} \in M_N(\mathbb{K}^*)$ *is a multiplicatively skew-symmetric matrix, and that* $k \in [1, N]$ *and* $g \in \mathbb{Z}^N$ *are such that* $\Omega_{\mathbf{r}}(g, e_j) = 1$ *for* $j \neq k$ *and* $\Omega_{\mathbf{r}}(g, e_k)$ *is not a root of unity. For each* $\epsilon = \pm$*, there exists a unique automorphism* $\rho_{g,\epsilon} = \rho^{\mathbf{r}}_{g,\epsilon}$ *of* $\mathrm{Fract}(\mathcal{T}_{\mathbf{r}^{\cdot 2}})$ *such that*

$$\rho_{g,\epsilon}(Y^{(e_j)}) = \begin{cases} Y^{(e_k)} + Y^{(e_k + \epsilon g)}, & \textit{if } j = k \\ Y^{(e_j)}, & \textit{if } j \neq k. \end{cases} \tag{2.20}$$

Proof. Since $\Omega_{\mathbf{r}}(e_k + \epsilon g, e_j) = \Omega_{\mathbf{r}}(e_k, e_j)$ for $j \neq k$, we have

$$\big(Y^{(e_k)} + Y^{(e_k+\epsilon g)}\big) Y^{(e_j)} = r_{kj}^2 Y^{(e_j)} \big(Y^{(e_k)} + Y^{(e_k+\epsilon g)}\big), \ \ \forall j \neq k.$$

Hence, there exists a $\mathbb{K}$-algebra homomorphism $\rho_{g,\epsilon} : \mathcal{T}_{\mathbf{r}^{\cdot 2}} \to \mathrm{Fract}(\mathcal{T}_{\mathbf{r}^{\cdot 2}})$ satisfying (2.20). Denote $g = \sum_j n_j e_j$. We have $\Omega_{\mathbf{r}}(g, e_k)^{n_k} = \Omega_{\mathbf{r}}(g, g) = 1$. Thus $n_k = 0$ because $\Omega_{\mathbf{r}}(g, e_k)$ is assumed to be a non-root of unity. This implies that $\rho_{g,\epsilon}(Y^{(g)}) = Y^{(g)}$. Consequently, $\rho_{g,\epsilon}$ is the identity on the set $G := \mathbb{K}[Y^{(g)}] \backslash \{0\}$. Now G is an Ore set in $\mathcal{T}_{\mathbf{r}^{\cdot 2}}$ (observe that $Y^{(f)} G = G Y^{(f)}$ for all $f \in \mathbb{Z}^N$), and so $\rho_{g,\epsilon}$ extends to a $\mathbb{K}$-algebra endomorphism of $\mathcal{T}_{\mathbf{r}^{\cdot 2}} G^{-1}$.

Similarly, there exists a $\mathbb{K}$-algebra endomorphism ρ' of $\mathcal{T}_{\mathbf{r}^{\cdot 2}} G^{-1}$ such that

$$\rho'(Y^{(e_j)}) = \begin{cases} \big(1 + \Omega_{\mathbf{r}}(g, e_k)^{-\epsilon} Y^{(\epsilon g)}\big)^{-1} Y^{(e_k)}, & \text{if } j = k \\ Y^{(e_j)}, & \text{if } j \neq k. \end{cases}$$

Now $\rho_{g,\epsilon} \rho'(Y^{(e_j)}) = \rho' \rho_{g,\epsilon}(Y^{(e_j)}) = Y^{(e_j)}$ for all $j \in [1, N]$. Therefore $\rho_{g,\epsilon}$ and ρ' are inverse automorphisms of $\mathcal{T}_{\mathbf{r}^{\cdot 2}} G^{-1}$, so they extend to inverse automorphisms of $\mathrm{Fract}(\mathcal{T}_{\mathbf{r}^{\cdot 2}})$. The uniqueness of $\rho_{g,\epsilon}$ is clear, because the elements $Y^{(e_j)}$, $j \in [1, N]$ generate $\mathrm{Fract}(\mathcal{T}_{\mathbf{r}^{\cdot 2}})$ as a division algebra. $\square$

Straightforward calculations yield that

$$\rho^{\mathbf{r}}_{g,\epsilon}(Y^{(f)}) = \begin{cases} P^{\mathbf{r}, m_k}_{g,\epsilon,+} Y^{(f)}, & \text{if } m_k \geq 0 \\ \big(P^{\mathbf{r}, -m_k}_{g,\epsilon,-}\big)^{-1} Y^{(f)}, & \text{if } m_k < 0 \end{cases} \tag{2.21}$$

for $f = (m_1, \dots, m_N)^T \in \mathbb{Z}^N$, where

$$P^{\mathbf{r},m}_{g,\epsilon,\pm} := \prod_{i=1}^{m} \big(1 + \Omega_{\mathbf{r}}(g, e_k)^{\mp\epsilon(2i-1)} Y^{(\epsilon g)}\big), \ \ \forall m \in \mathbb{Z}_{\geq 0}.$$

Unlike the uniparameter case [**3**, Eq. (4.9)], here $\rho_{-g,-\epsilon}$ equals $\rho_{g,\epsilon}$ rather than $\rho_{g,\epsilon}^{-1}$. This stems from the fact that in the general situation, there is no analog of the minimal positive values $d(-)$ used in [**3**, Subsec. 4.2].

For the rest of this section, we assume that $(M, \widetilde{B})$ is a quantum seed of $\mathcal{F}$ and denote $\mathbf{r} := \mathbf{r}(M)$. We also fix $k \in \mathbf{ex}$. Since $E_\epsilon \in GL_N(\mathbb{Z})$, (2.11) shows that

$$ME_\epsilon : \mathbb{Z}^N \to \mathcal{F} \ \text{ is a toric frame with matrix } \ \mu_k(\mathbf{r}) = {}^{E_\epsilon^T}\mathbf{r}^{E_\epsilon}. \tag{2.22}$$

By abuse of notation, we will identify $\mathcal{F}$ with $\mathrm{Fract}(\mathcal{T}_{\mu_k(\mathbf{r})\cdot 2})$ via the embedding $\mathcal{T}_{\mu_k(\mathbf{r})\cdot 2} \hookrightarrow \mathcal{F}$ associated to this toric frame. From (2.13) and the fact that $E_\epsilon b^k = b^k$, it follows that

$$\Omega_{\mu_k(\mathbf{r})}(b^k, e_j) = \Omega_{\mathbf{r}}(b^k, E_\epsilon e_j) = t_{jj}^{-\delta_{jk}}, \ \ \forall j \in [1, N].$$

Applying the condition (2.14) and Lemma 2.8, we obtain the automorphisms $\rho_{b^k,\epsilon} = \rho^{\mu_k(\mathbf{r})}_{b^k,\epsilon} \in \mathrm{Aut}(\mathcal{F})$. When applying these automorphisms, any term $Y^{(f)} = M(f)$ in an expression such as (2.20) or (2.21) must be replaced by $ME_\epsilon(f)$.

Proposition 2.9. *Assume that $(M, \widetilde{B})$ is a quantum seed and $k \in \mathbf{ex}$.*

(a) *The map $\rho_{b^k,\epsilon} ME_\epsilon : \mathbb{Z}^N \to \mathcal{F}$ is a toric frame of $\mathcal{F}$ and is independent of the choice of sign $\epsilon = \pm$. The matrix of this toric frame equals $\mu_k(\mathbf{r}(M))$.*

(b) *The pair $(\rho_{b^k,\epsilon} ME_\epsilon, \mu_k(\widetilde{B}))$ is a quantum seed of $\mathcal{F}$.*

Proof. (a) The fact that $\rho_{b^k,\epsilon} ME_\epsilon : \mathbb{Z}^N \to \mathcal{F}$ is a toric frame of $\mathcal{F}$ is immediate because $E_\epsilon \in GL_N(\mathbb{Z})$ and $\rho_{b^k,\epsilon} \in \mathrm{Aut}(\mathcal{F})$. The last fact and (2.22) imply that its matrix is $\mu_k(\mathbf{r})$. By (2.20),

$$\rho_{b^k,\epsilon} ME_\epsilon(e_j) = ME_\epsilon(e_j) = M(e_j), \ \ \forall j \neq k \tag{2.23}$$

and

$$\begin{aligned} \rho_{b^k,\epsilon} ME_\epsilon(e_k) &= ME_\epsilon(e_k) + ME_\epsilon(e_k + \epsilon b^k) = M(E_\epsilon e_k) + M(E_\epsilon e_k + \epsilon b_k) \\ &= M(-e_k + [-\epsilon b^k]_+) + M(-e_k + [-\epsilon b^k]_+ + b^k) \\ &= M(-e_k + [b^k]_+) + M(-e_k - [b^k]_-), \end{aligned} \tag{2.24}$$

cf. (1.12). Hence, the toric frame $\rho_{b^k,\epsilon} ME_\epsilon$ is independent of the choice of sign $\epsilon = \pm$.

(b) By Proposition 2.6 (b), the pair $(\mathbf{r}(\rho_{b^k,\epsilon} ME_\epsilon), \mu_k(\widetilde{B})) = (\mu_k(\mathbf{r}), \mu_k(\widetilde{B}))$ is compatible, and the principal part of $\mu_k(\widetilde{B})$ is skew-symmetrizable. □

Definition 2.10. Define the *mutation in direction $k \in \mathbf{ex}$* of the quantum seed $(M, \widetilde{B})$ of a division algebra $\mathcal{F}$ to be the quantum seed $(\mu_k(M), \mu_k(\widetilde{B}))$, where $\mu_k(M) := \rho_{b^k,\epsilon} ME_\epsilon = \rho^{\mu_k(\mathbf{r})}_{b^k,\epsilon} ME_\epsilon^{\widetilde{B}}$.

Corollary 2.11. *For all quantum seeds $(M, \widetilde{B})$ of $\mathcal{F}$ and $k \in \mathbf{ex}$, the following hold:*

(a) *The matrix of the toric frame* $\mu_k(M)$ *equals* $\mu_k(\mathbf{r}(M))$. *This toric frame satisfies*

$$\begin{aligned} \mu_k(M)(e_j) &= M(e_j), \ \ \forall j \neq k, \\ \mu_k(M)(e_k) &= M(-e_k + [b^k]_+) + M(-e_k - [b^k]_-), \end{aligned} \tag{2.25}$$

cf. (1.12).

(b) *Mutation is involutive:*

$$\mu_k^2(M, \widetilde{B}) = (M, \widetilde{B}).$$

Proof. Part (a) follows from Proposition 2.9 (a) and (2.23)–(2.24). Turning to part (b), we use the independence of signs (Proposition 2.9(a)) to write

$$\mu_k^2(M) = \rho^{\mu_k^2(\mathbf{r})}_{\mu_k(\widetilde{B})^k, -\epsilon} \big(\rho^{\mu_k(\mathbf{r})}_{b^k, \epsilon} M E_\epsilon^{\widetilde{B}} \big) E_{-\epsilon}^{\mu_k(\widetilde{B})} = \rho^{\mathbf{r}}_{-b^k, -\epsilon} \rho^{\mu_k(\mathbf{r})}_{b^k, \epsilon} M E_\epsilon^{\widetilde{B}} E_\epsilon^{\widetilde{B}},$$

and we apply (2.21) to see that $\mu_k^2(M) = M$ (it suffices to check this on the $Y^{(e_j)}$, $j \in [1, N]$). Finally, we recall the well known fact [**11**, p. 510] that $\mu_k^2(\widetilde{B}) = \widetilde{B}$. $\square$

Two quantum seeds $(M, \widetilde{B})$ and $(M', \widetilde{B}')$ of a division algebra $\mathcal{F}$ will be called *mutation-equivalent* if they can be obtained from each other by a sequence of mutations. Definition 2.10 implies at once:

Corollary 2.12. *If* $(M, \widetilde{B})$ *and* $(M', \widetilde{B}')$ *are mutation-equivalent quantum seeds of* $\mathcal{F}$, *then there exists* $\sigma \in GL_N(\mathbb{Z})$ *and* $\psi \in \mathrm{Aut}(\mathcal{F})$ *such that*

$$M' = \psi M \sigma : \mathbb{Z}^N \to \mathcal{F}.$$

In particular, the following subgroups of $\mathbb{K}^*$ *are equal:*

$$\langle \Omega_{\mathbf{r}(M)}(f, g) \mid f, g \in \mathbb{Z}^N \rangle = \langle \Omega_{\mathbf{r}(M')}(f, g) \mid f, g \in \mathbb{Z}^N \rangle.$$

The first part of Corollary 2.12 is one of the conditions in the Berenstein–Zelevinsky definition [**3**] of toric frame in the case of uniparameter quantum tori.

The following well known equivariance of mutation of pairs will be needed in Chapter 8. For a group G we denote by $X(G)$ its character lattice. For a G-eigenvector u, χ_u will denote its G-eigenvalue.

Lemma 2.13. *Let* $\mathcal{F}$ *be a division algebra and* G *a group acting on* $\mathcal{F}$ *by algebra automorphisms. Assume that* $(M, \widetilde{B})$ *is a quantum seed for* $\mathcal{F}$ *such that* $M(f)$ *is a* G*-eigenvector for* $f \in \mathbb{Z}^N$ *(or, which is the same, for* $f = e_1, \dots, e_N$*) and* $\chi_{M(b^j)} = 1$ *for all columns* b^j *of* $\widetilde{B}$. *Then all mutations* $(\mu_k(M), \mu_k(\widetilde{B}))$, $k \in \mathbf{ex}$ *have the same properties.*

Proof. Fix $k \in \mathbf{ex}$ and denote for brevity $M' := \mu_k(M)$ and $\widetilde{B}' := \mu_k(\widetilde{B})$. The entries and columns of $\widetilde{B}'$ will be accordingly denoted by b'_{lj} and $(b')^j$. Since $\chi_{M(b^k)} = 1$, we have $\chi_{M(-e_k + [b^k]_+)} = \chi_{M(-e_k - [b^k]_-)}$. Hence, $M'(e_k)$ is a G-eigenvector and

$$\chi_{M'(e_k)} = \chi_{M(-e_k + [b^k]_+)} = \chi_{M(-e_k - [b^k]_-)}. \tag{2.26}$$

Of course, $M'(e_j) = M(e_j)$ is a G-eigenvector for all $j \neq k$.

For the second condition, we have $\chi_{M'((b')^k)} = \chi_{M(-b^k)} = 1$ since $b'_{kk} = 0$. Now, let $j \neq k$. If $b_{kj} = 0$, then $\chi_{M'((b')^j)} = \chi_{M(b^j)} = 1$. If $b_{kj} \neq 0$, denote

$\epsilon = \mathrm{sign}(b_{kj})$. By (2.17), $(b')^j = b^j + \epsilon b_{kj}[b^k]_\epsilon - 2b_{kj}e_k$, and using (2.26) we obtain

$$\begin{aligned}\chi_{M'((b')^j)} &= \chi_{M(b^j+\epsilon b_{kj}[b^k]_\epsilon - b_{kj}e_k)}\chi_{M'(-b_{kj}e_k)}\\ &= \chi_{M(b^j+\epsilon b_{kj}[b^k]_\epsilon - b_{kj}e_k)}\chi_{M(b_{kj}e_k - \epsilon b_{kj}[b^k]_\epsilon)} = \chi_{M(b^j)} = 1. \qquad \square\end{aligned}$$

2.6. Quantum cluster algebras and the Laurent phenomenon

Let $(M, \widetilde{B})$ be a quantum seed of a division algebra $\mathcal{F}$ over $\mathbb{K}$. Fix a subset $\mathbf{inv} \subseteq [1,N]\backslash\mathbf{ex}$. This set will be used to determine which frozen variables (indexed by $[1,N]\backslash\mathbf{ex}$) will be inverted in the definition of quantum cluster algebras. Let D be a unital subring of $\mathbb{K}$ containing the subgroup

$$\langle \Omega_{\mathbf{r}}(f,g) \mid f,g \in \mathbb{Z}^N \rangle \tag{2.27}$$

of $\mathbb{K}^*$, where $\mathbf{r} = \mathbf{r}(M)$. The ring D will be used as a base ring.

Definition 2.14. Define the *quantum cluster algebra* $\mathcal{A}(M, \widetilde{B}, \mathbf{inv})_D$ associated to the above data to be the unital D-subalgebra of $\mathcal{F}$ generated by all cluster variables $M'(e_k)$, $k \in [1,N]$ and the inverses $M'(e_l)^{-1}$, $l \in \mathbf{inv}$ for all quantum seeds $(M', \widetilde{B}')$ of $\mathcal{F}$ which are mutation-equivalent to $(M, \widetilde{B})$.

Define the *upper quantum cluster algebra* $\mathcal{U}(M, \widetilde{B}, \mathbf{inv})_D$ to be the intersection of all D-subalgebras of $\mathcal{F}$ of the form

$$\mathcal{TA}_{(M',\widetilde{B}'),D} := D\langle M'(e_l)^{\pm1},\, M'(e_k) \mid l \in \mathbf{ex} \sqcup \mathbf{inv},\, k \in [1,N]\backslash(\mathbf{ex}\sqcup\mathbf{inv})\rangle \tag{2.28}$$

taken over all quantum seeds $(M', \widetilde{B}')$ for $\mathcal{F}$ mutation-equivalent to $(M, \widetilde{B})$.

Since $M'(e_l)$ is a frozen variable for $l \in [1,N]\backslash\mathbf{ex}$, we have that $M'(e_l) = M(e_l)$ for all $l \in [1,N]\backslash\mathbf{ex}$ and quantum seeds $(M', \widetilde{B}')$ that are mutation equivalent to $(M, \widetilde{B})$. Because of this, in the definition of quantum cluster algebras it is sufficient to consider the generators $M'(e_j)$, $j \in \mathbf{ex}$, $M(e_l)^{\pm1}$, $l \in \mathbf{inv}$ and $M(e_l)$, $l \in [1,N]\backslash(\mathbf{ex}\sqcup\mathbf{inv})$.

The subalgebra $\mathcal{TA}_{(M',\widetilde{B}'),D}$ is a mixed quantum torus–quantum affine space D-subalgebra of $\mathcal{F}$. The D-subalgebra of $\mathcal{F}$ generated by $M'(e_j)$, $j \in [1,N]$ is isomorphic to a quantum affine space algebra. The algebra $\mathcal{TA}_{(M',\widetilde{B}'),D}$ is equal to the localization of it by the multiplicative subset generated by $M'(e_l)$, $l \in \mathbf{ex}\sqcup\mathbf{inv}$. If one localizes instead by all $M'(e_l)$, $l \in [1,N]$, then the resulting algebra will be the corresponding quantum torus. The mixed nature of $\mathcal{TA}_{(M',\widetilde{B}'),D}$ has to do with this partial localization.

The proof of the Laurent phenomenon of Fomin and Zelevinsky [**11, 12**] carries over to the quantum case. We state this result and provide all needed details for the quantum case in the rest of this section. In the uniparameter case and when all frozen variables are inverted, a quantum Laurent phenomenon was established by Berenstein and Zelevinsky [**3**] following their previous work with Fomin [**1**] using upper bounds.

Theorem 2.15. *For all quantum seeds $(M, \widetilde{B})$ of a division algebra $\mathcal{F}$ over a field $\mathbb{K}$, subrings D of $\mathbb{K}$ containing the subgroup* (2.27) *of $\mathbb{K}^*$, and subsets $\mathbf{inv} \subseteq [1,N]\backslash\mathbf{ex}$, we have the inclusion*

$$\mathcal{A}(M, \widetilde{B}, \mathbf{inv})_D \subseteq \mathcal{U}(M, \widetilde{B}, \mathbf{inv})_D.$$

We will need the following lemma.

LEMMA 2.16. *Let* $\mathbf{r} \in M_N(\mathbb{K}^*)$ *be a multiplicatively skew-symmetric matrix and* D *a subring of* $\mathbb{K}$ *which contains the subgroup* (2.27) *of* $\mathbb{K}^*$*. Consider elements*

$$V := \prod_{i=1}^{m} \big(Y^{(f)} + \xi_i\big) \qquad \text{and} \qquad V' := \prod_{i=1}^{m'} \big(Y^{(f')} + \xi'_i\big)$$

in the D*-quantum torus* $D\langle Y_1^{\pm 1}, \dots, Y_N^{\pm 1}\rangle \subseteq \mathcal{T}_{\mathbf{r}^{\cdot 2}}$*, where* $f, f' \in \mathbb{Z}^N$ *are such that*

$$\Omega_{\mathbf{r}}(f, e_k)^2 = \Omega_{\mathbf{r}}(f', e_k)^2 = 1, \;\; \forall k \in [1, N]$$

and $\xi_1, \dots, \xi_m, \xi'_1, \dots, \xi'_{m'} \in D$.

If f *and* f' *are* $\mathbb{Z}$*-linearly independent, then* V *and* V' *are relatively prime in the center of* $D\langle Y_1^{\pm 1}, \dots, Y_N^{\pm 1}\rangle$.

PROOF. We may assume that $\xi_1, \dots, \xi_m, \xi'_1, \dots, \xi'_{m'}$ are nonzero elements of D, since the factors with vanishing ξ's are central units. Similarly, we may assume that f and f' are nonzero.

The center of $D\langle Y_1^{\pm 1}, \dots, Y_N^{\pm 1}\rangle$ is given by

$$Z = \sum_{g \in L} D Y^{(g)} \tag{2.29}$$

for the sublattice $L \subseteq \mathbb{Z}^N$ defined by

$$L = \{ g \in \mathbb{Z}^N \mid \Omega_{\mathbf{r}}(g, e_k)^2 = 1, \; \forall k \in [1, N]\}.$$

After reducing to the center of $D\langle Y_1^{\pm 1}, \dots, Y_N^{\pm 1}\rangle$, replacing D by the algebraic closure of its quotient field, and changing notation appropriately, we may assume that D is an algebraically closed field and $D\langle Y_1^{\pm 1}, \dots, Y_N^{\pm 1}\rangle$ is a commutative Laurent polynomial ring (hence, a UFD).

Write $f = ng$ and $f' = n'g'$ where $n, n' \in \mathbb{Z}_{>0}$ and the gcd of the entries of g, resp. of g', is 1. By hypothesis, $g \neq \pm g'$. Now the irreducible factors of V, up to associates, are $Y^{(g)} - \zeta$ for n th roots ζ of $-\xi_i$, and similarly for V'. No $Y^{(g)} - \zeta$ is an associate of any $Y^{(g')} - \zeta'$, so V and V' have no irreducible common factors. Thus, they are relatively prime. □

The proof of Theorem 2.15 is analogous to the proof of the Caterpillar Lemma of Fomin and Zelevinsky [**12**, Theorem 2.1]. We will indicate the needed ingredients for the inductive step of the proof in the noncommutative situation.

From now on we will assume that

$$(M_s, \widetilde{B}_s), \;\; s \in [0, 3] \quad \text{are four quantum seeds of } \mathcal{F}$$

and

$$(M_1, \widetilde{B}_1) = \mu_i(M_0, \widetilde{B}_0), \;\; (M_2, \widetilde{B}_2) = \mu_j(M_1, \widetilde{B}_1), \;\; (M_3, \widetilde{B}_3) = \mu_i(M_2, \widetilde{B}_2)$$

for some $i \neq j$ in $\mathbf{ex}$. Denote

$$\begin{aligned} x &:= M_0(e_i), \\ y &:= M_0(e_j) = M_1(e_j), \\ z &:= M_1(e_i) = M_2(e_i), \\ u &:= M_2(e_j) = M_3(e_j), \\ v &:= M_3(e_i). \end{aligned}$$

Set for brevity $\mathcal{TA}_s := \mathcal{TA}_{(M_s, \widetilde{B}_s), D}$. There exist unique elements $P \in \mathcal{TA}_0$, $Q \in \mathcal{TA}_1$ and $R \in \mathcal{TA}_2$ such that

$$z = x^{-1}P, \ \ u = y^{-1}Q \ \text{ and } \ v = z^{-1}R. \tag{2.30}$$

Each of P, Q, R is a sum of two nonzero monomials in the cluster variables for the seeds $(M_s, \widetilde{B}_s)$, $c \in [0,2]$. We will denote by $P(y)$, $Q(z)$, $R(u)$ the elements P, Q, R written as *polynomials* in y, u, z with left hand coefficients that are *polynomials* in the other cluster variables in the seeds $(M_0, \widetilde{B}_0)$, $(M_1, \widetilde{B}_1)$, $(M_2, \widetilde{B}_2)$, respectively. (This uniquely defines $P(y)$, $Q(z)$, $R(u)$.) In this setting one can substitute any element of $\mathcal{F}$ for y, z, or u and the result will be an element of $\mathcal{F}$.

Clearly,

$$z = x^{-1}P, \ \ u = y^{-1}Q = y^{-1}Q(x^{-1}P(y)) \in \mathcal{TA}_0 \quad \text{and} \tag{2.31}$$

$$v = z^{-1}R = z^{-1}R(y^{-1}Q(z)) \in \mathcal{TA}_1. \tag{2.32}$$

Recall that an element p of a (noncommutative) domain A is called prime if it is normal (i.e., $Ap = pA$) and Ap is a completely prime ideal (i.e., A/Ap is a domain). We refer the reader to Section 3.1 for details on this notion. For $c \in A$, we write $p \mid c$ if $c \in Rp$. For $b, c \in A$, we write $b \mid_l c$ if $c \in bA$.

Denote by $\mathcal{TA}_0^\vee$ the D-subalgebra of $\mathcal{TA}_0$ generated by the list of cluster variables and their inverses which appear in (2.28) except for $x^{\pm 1} = M_0(e_i)^{\pm 1}$, i.e.,

$$\mathcal{TA}_0^\vee := D\langle M_0(e_l)^{\pm 1}, \, M_0(e_k) \mid l \in (\mathbf{ex} \sqcup \mathbf{inv}) \backslash \{i\}, \, k \in [1,N] \backslash (\mathbf{ex} \sqcup \mathbf{inv}) \rangle.$$

This is another mixed quantum torus–quantum affine space algebra, but this time of GK dimension $N-1$. Clearly,

$$\mathcal{TA}_0 = \mathcal{TA}_0^\vee[x^{\pm 1}; \sigma], \tag{2.33}$$

where $\sigma \in \mathrm{Aut}(\mathcal{TA}_0^\vee)$ is given by

$$\sigma(M_0(e_l)) = \mathbf{r}(M_0)_{il}^2 M_0(e_l), \ \ \forall l \in [1,N] \backslash \{i\}.$$

We will say that $b \in \mathcal{TA}_0 \backslash \{0\}$ has *leading term* cx^m where $m \in \mathbb{Z}$ and $c \in \mathcal{TA}_0^\vee$ if

$$b - cx^m \in \bigoplus_{m' < m} \mathcal{TA}_0^\vee x^{m'}.$$

The localizations

$$\mathcal{T}_0^\vee := \mathcal{TA}_0^\vee[M_0(e_k)^{-1} \mid k \in [1,N] \backslash (\mathbf{ex} \sqcup \mathbf{inv})]$$
$$\mathcal{T}_0 := \mathcal{TA}_0[M_0(e_k)^{-1} \mid k \in [1,N] \backslash (\mathbf{ex} \sqcup \mathbf{inv})]$$

are quantum tori and

$$\mathcal{T}_0 \cong \mathcal{T}_0^\vee[x^{\pm 1}; \sigma],$$

where by abuse of notation we denote by the same letter the canonical extension of $\sigma \in \mathrm{Aut}(\mathcal{TA}_0^\vee)$ to an automorphism of $\mathcal{T}_0^\vee$.

Lemma 2.17. *The following hold in the above setting:*

(a) *The element*

$$L := z^{-1}R(y^{-1}Q(0))zP^{-1}$$

is a monomial in the elements $M_0(e_k)$, $k \in [1,N] \backslash \{i,j\}$ and $y^{-1} = M_0(e_j)^{-1}$. In particular, it belongs to $\mathcal{TA}_0^\vee$.

(b) *The cluster variables u and v belong to $\mathcal{TA}_0$ and their leading terms are equal to $y^{-1}Q(0)$ and Lx, respectively.*

(c) *If*

$$z^m F = G u^{m_1} v^{m_2}$$

for some $F, G \in \mathcal{TA}_0 \backslash \{0\}$ *and* $m, m_1, m_2 \in \mathbb{Z}_{\geq 0}$*, then* $z^m \mid_l G$ *as elements of* $\mathcal{TA}_0$.

PROOF. We will denote by $b_{s,kl}$ the kl-th entry of the exchange matrix $\widetilde{B}_s$ and by b_s^l the l-th column of $\widetilde{B}_s$.

(a) Set $\epsilon := \operatorname{sign}(b_{1,ij})$. By a direct computation one obtains that

$$L = \xi M_2(E_\epsilon [b_2^i]_\epsilon - [b_1^i]_\epsilon),$$

where ξ is an element of the subgroup (2.27) of $\mathbb{K}^*$ and E_ϵ denotes the first matrix in (2.18) for the mutation $(M_2, \widetilde{B}_2) = \mu_j(M_1, \widetilde{B}_1)$. The i-th coordinate of the vector in the right hand side equals 0, so

$$L = \xi M_1(E_\epsilon [b_2^i]_\epsilon - [b_1^i]_\epsilon).$$

Part (a) now follows by computing the signs of the coordinates of the vector in the right hand side of the last formula.

(b) The statement for u follows from (2.31). By part (a), $z^{-1}R(y^{-1}Q(0)) = LPz^{-1} = Lx$. Analogously to [**12**], write

$$\begin{aligned} v &= z^{-1}\big(R(y^{-1}Q(z)) - R(y^{-1}Q(0))\big) + z^{-1}R(y^{-1}Q(0)) \\ &= z^{-1}\big(R(y^{-1}Q(z)) - R(y^{-1}Q(0))\big) + Lx. \end{aligned}$$

Since

$$R(y^{-1}Q(z)) - R(y^{-1}Q(0)) \in \bigoplus_{m'>0} \mathcal{TA}_0^\vee z^{m'},$$

we have

$$z^{-1}\big(R(y^{-1}Q(z)) - R(y^{-1}Q(0))\big) \in \bigoplus_{m' \geq 0} \mathcal{TA}_0^\vee z^{m'} \subseteq \bigoplus_{m' \geq 0} \mathcal{TA}_0^\vee x^{-m'}.$$

This proves both statements for the cluster variable v.

(c) It is straightforward to see that $M_0(e_k)$, $k \in [1, N] \backslash (\mathbf{ex} \sqcup \mathbf{inv})$ are prime elements of $\mathcal{TA}_0$ and that $M_0(e_k) \nmid z$. Since a prime element p has the property that $p \mid ab \Rightarrow p \mid a$ or $p \mid b$, it is sufficient to prove that

$$z^m \mid_l G \quad \text{in} \quad \mathcal{T}_0.$$

By part (a), L is a unit of $\mathcal{T}_0^\vee$ (in particular, a σ-eigenvector). Noting that $P \in \mathcal{TA}_0^\vee$, using

$$z^m = (\sigma^{-1}(P) \dots \sigma^{-m}(P)) x^{-m},$$

and taking into account part (b) leads to

$$(\sigma^{-1}(P) \dots \sigma^{-m}(P)) F' = G\big((y^{-1}Q(0))^{m_1} x^{m_2} + \text{lower order terms}\big) \tag{2.34}$$

for some $F' \in \mathcal{T}_0$.

Denote $G = \sum_{m' \in \mathbb{Z}} g_{m'} x^{m'}$, where $g_{m'} \in \mathcal{TA}_0^\vee$. We claim that

$$(\sigma^{-1}(P) \dots \sigma^{-m}(P)) \mid_l g_{m'} \quad \text{in} \quad \mathcal{T}_0^\vee, \ \ \forall m' \in \mathbb{Z}.$$

This will imply that $z^m \mid_l G$ in $\mathcal{T}_0$ and will complete the proof of part (c). Assume that this is not the case. Let $\overline{m}$ be the largest integer such that

$$(\sigma^{-1}(P) \dots \sigma^{-m}(P)) \nmid_l g_{\overline{m}} \quad \text{in} \quad \mathcal{T}_0^\vee. \tag{2.35}$$

Comparing the components in $\mathcal{T}_0^\vee x^{\overline{m}+m_2}$ in (2.34) (with respect to the decomposition $\mathcal{T}_0 = \oplus_{m'\in\mathbb{Z}} \mathcal{T}_0^\vee x^{m'}$) we obtain that

$$(\sigma^{-1}(P)\dots\sigma^{-m}(P)) \mid_l g_{\overline{m}} \sigma^{\overline{m}}((y^{-1}Q(0))^{m_1}) \quad \text{in} \quad \mathcal{T}_0^\vee. \tag{2.36}$$

If $b_{1,ij} \neq 0$, then $y^{-1}Q(0)$ is a unit of $\mathcal{T}_0^\vee$ (in particular a σ-eigenvector) and (2.36) implies that $(\sigma^{-1}(P)\dots\sigma^{-m}(P)) \mid_l g_{\overline{m}}$ in $\mathcal{T}_0^\vee$ which contradicts (2.35).

Now consider the case $b_{1,ij} = 0$. Then $Q(0) = Q$ and

$$b_1^j, b_1^i \in \sum_{k\neq i,j} \mathbb{Z}e_k, \tag{2.37}$$

where the second property follows from the fact that the principal part of $\widetilde{B}_1$ is skew-symmetrizable. Thus,

$$\begin{aligned}\sigma^{\overline{m}}((y^{-1}Q(0))^{m_1}) &= (M_1(b_1^j)+\xi_1')\dots(M_1(b_1^j)+\xi_{m_1}')\theta' \\ &= (M_0(b_1^j)+\xi_1)\dots(M_0(b_1^j)+\xi_{m_1})\theta'\end{aligned}$$

for some $\xi_1',\dots,\xi_{m_1}',\xi_1,\dots,\xi_{m_1} \in D$ and $\theta' \in \mathcal{T}_0^\vee$ such that θ' is a Laurent monomial in the generators of $\mathcal{T}_0^\vee$ and its coefficients and $\xi_1,\dots,\xi_{m_1}$ belong to the subgroup (2.27) of the units of D. We also have

$$\begin{aligned}\sigma^{-1}(P)\dots\sigma^{-m}(P) &= \theta(M_0(b_0^i)+\xi_1)\dots(M_0(b_0^i)+\xi_m) \\ &= \theta(M_0(-b_1^i)+\xi_1)\dots(M_0(-b_1^i)+\xi_m),\end{aligned}$$

where $\xi_1,\dots,\xi_m \in D$ and $\theta \in \mathcal{TA}_0^\vee$ have the same properties. In particular, θ and θ' are units of $\mathcal{T}_0^\vee$. Denote the polynomials

$$h(t) := (t+\xi_1)\dots(t+\xi_m), \quad h'(t) := (t+\xi_1')\dots(t+\xi_{m_1}') \in D[t].$$

Substituting the above two expressions in (2.36) and clearing the units gives

$$h(M_0(-b_1^i)) \mid_l g_{\overline{m}} h'(M_0(b_1^j)) \quad \text{in} \quad \mathcal{T}_0^\vee. \tag{2.38}$$

To get to the setting of Lemma 2.16, consider the subtorus

$$\mathcal{T}_0^{\vee\vee} := D\langle M_0(e_k)^{\pm1}, k\neq i,j\rangle \subset \mathcal{T}_0^\vee.$$

It satisfies $\mathcal{T}_0^\vee = \mathcal{T}_0^{\vee\vee}[y^{\pm1};\varphi]$, where $\varphi \in \operatorname{Aut}(\mathcal{T}_0^{\vee\vee})$ is given by $\varphi(M_0(e_k)) = \mathbf{r}(M_0)_{jk}^2 M_0(e_k)$, $k\neq i,j$. By (2.12)–(2.13) and (2.37), $M_0(-b_1^i)$ and $M_0(b_1^j)$ belong to the center of $\mathcal{T}_0^{\vee\vee}$. Denote $q := \varphi(M_0(b_1^j))$ and write

$$g_{\overline{m}} = \sum_{l\in\mathbb{Z}} g_{\overline{m},l} y^l, \quad g_{\overline{m},l} \in \mathcal{T}_0^{\vee\vee}.$$

Eq. (2.38) is equivalent to

$$h(M_0(-b_1^i)) \mid_l g_{\overline{m},l} h'(q^l M_0(b_1^j)) \quad \text{in} \quad \mathcal{T}_0^{\vee\vee}, \quad \forall l\in\mathbb{Z}.$$

Since $\widetilde{B}_1$ has full rank, the vectors $-b_1^i, b_1^j \in \sum_{k\neq i,j}\mathbb{Z}e_k$ are $\mathbb{Z}$-linearly independent. Lemma 2.16 implies that $h(M_0(-b_1^i))$ and $h'(q^l M_0(b_1^j))$ are relatively prime in the center of $\mathcal{T}_0^{\vee\vee}$, $\forall l\in\mathbb{Z}$.

It follows from (2.29) that $\mathcal{T}_0^{\vee\vee}$ is a free module over its center. Applying this and the relative primeness of $h(M_0(-b_1^i))$ and $h'(q^l M_0(b_1^j))$ in the center of $\mathcal{T}_0^{\vee\vee}$, for all $l\in\mathbb{Z}$ gives

$$h(M_0(-b_1^i)) \mid_l g_{\overline{m},l} \quad \text{in} \quad \mathcal{T}_0^{\vee\vee}, \quad \forall l\in\mathbb{Z}.$$

Hence, $h(M_0(-b_1^i)) \mid_l g_{\overline{m}}$ and thus $(\sigma^{-1}(P) \ldots \sigma^{-m}(P)) \mid_l g_{\overline{m}}$ in $\mathcal{T}_0^{\vee\vee}$ which again contradicts (2.35). This completes the proof of part (c). $\square$

Theorem 2.15 follows from Lemma 2.17 (a), (c) by induction using the argument of Fomin and Zelevinsky in the first paragraph of p. 127 in [**12**].

CHAPTER 3

Iterated skew polynomial algebras and noncommutative UFDs

In this chapter, we gather some facts concerning iterated skew polynomial rings (Ore extensions) and noncommutative unique factorization domains which will be used in the paper.

3.1. Equivariant noncommutative unique factorization domains

Recall that a *prime element* of a domain R is any nonzero normal element $p \in R$ (*normality* meaning that $Rp = pR$) such that Rp is a completely prime ideal, i.e., R/Rp is a domain. Assume that in addition R is a $\mathbb{K}$-algebra and $\mathcal{H}$ a group acting on R by $\mathbb{K}$-algebra automorphisms. An *$\mathcal{H}$-prime ideal* of R is any proper $\mathcal{H}$-stable ideal P of R such that ($IJ \subseteq P \implies I \subseteq P$ or $J \subseteq P$) for all $\mathcal{H}$-stable ideals I and J of R. One says that R is an *$\mathcal{H}$-UFD* if each nonzero $\mathcal{H}$-prime ideal of R contains a prime $\mathcal{H}$-eigenvector. This is an equivariant version of Chatters' notion [**6**] of noncommutative unique factorization domain given in [**33**, Definition 2.7].

The following fact is an equivariant version of results of Chatters and Jordan [**6**, Proposition 2.1], [**7**, p. 24], see [**24**, Proposition 2.2] and [**41**, Proposition 6.18 (ii)].

Proposition 3.1. *Let R be a noetherian $\mathcal{H}$-UFD. Every normal $\mathcal{H}$-eigenvector in R is either a unit or a product of prime $\mathcal{H}$-eigenvectors. The factors are unique up to reordering and taking associates.*

3.2. CGL extensions

Convention 3.2. *We use the standard notation $S[x;\sigma,\delta]$ for a* skew polynomial ring*, or* Ore extension*; it denotes a ring generated by a unital subring S and an element x satisfying $xs = \sigma(s)x + \delta(s)$ for all $s \in S$, where σ is a ring endomorphism of S and δ is a (left) σ-derivation of S. The ring $S[x;\sigma,\delta]$ is a free left S-module, with the nonnegative powers of x forming a basis.*

For every Ore extension $S[x;\sigma,\delta]$ appearing in this paper, S is a $\mathbb{K}$-algebra, σ is a $\mathbb{K}$-linear automorphism of S, and δ is a $\mathbb{K}$-linear σ-derivation. As a result, $S[x;\sigma,\delta]$ is a $\mathbb{K}$-algebra.

For convenient reference, we reiterate Definition A:

Definition 3.3. An iterated skew polynomial extension

$$R := \mathbb{K}[x_1][x_2;\sigma_2,\delta_2]\cdots[x_N;\sigma_N,\delta_N] \tag{3.1}$$

is called a *CGL extension* [**33**, Definition 3.1] if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(i) The elements $x_1, \ldots, x_N$ are $\mathcal{H}$-eigenvectors.

(ii) For every $k \in [2, N]$, δ_k is a locally nilpotent σ_k-derivation of the algebra $R_{k-1} = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_{k-1}; \sigma_{k-1}, \delta_{k-1}]$.
(iii) For every $k \in [1, N]$, there exists $h_k \in \mathcal{H}$ such that $\sigma_k = (h_k \cdot)|_{R_{k-1}}$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

Conditions (i) and (iii) imply that

$$\sigma_k(x_j) = \lambda_{kj} x_j \text{ for some } \lambda_{kj} \in \mathbb{K}^*, \quad \forall 1 \leq j < k \leq N.$$

We then set $\lambda_{kk} := 1$ and $\lambda_{jk} := \lambda_{kj}^{-1}$ for $j < k$. This gives rise to a multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} := (\lambda_{kj}) \in M_N(\mathbb{K}^*)$ and a corresponding skew-symmetric bicharacter $\Omega_{\boldsymbol{\lambda}}$, recall (2.1), (2.2).

The CGL extension R is called *torsionfree* if the subgroup $\langle \lambda_{kj} \mid k, j \in [1, N] \rangle$ of $\mathbb{K}^*$ is torsionfree. Define the *length* of R to be N and the *rank* of R by

$$\text{rk}(R) := \{k \in [1, N] \mid \delta_k = 0\} \in \mathbb{Z}_{>0} \tag{3.2}$$

(cf. [**24**, Eq. (4.3)]). Denote the character group of the torus $\mathcal{H}$ by $X(\mathcal{H})$. Up through Chapter 8, we view $X(\mathcal{H})$ as a multiplicative group, with identity 1. The action of $\mathcal{H}$ on R gives rise to an $X(\mathcal{H})$-grading of R. The $\mathcal{H}$-eigenvectors are precisely the nonzero homogeneous elements with respect to this grading. The $\mathcal{H}$-eigenvalue of a nonzero homogeneous element $u \in R$ will be denoted by χ_u. In other words, $\chi_u = X(\mathcal{H})\text{-deg}(u)$ in terms of the $X(\mathcal{H})$-grading.

Example 3.4. For positive integers m and n and a non-root of unity $q \in \mathbb{K}^*$, let $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ be the standard single parameter quantized coordinate ring of the matrix variety $M_{m,n}(\mathbb{K})$. This is a $\mathbb{K}$-algebra with generators t_{ij} for $i \in [1, m]$ and $j \in [1, n]$, and relations

$$\begin{aligned}
t_{ij} t_{kj} &= q t_{kj} t_{ij}, && \text{for } i < k, \\
t_{ij} t_{il} &= q t_{il} t_{ij}, && \text{for } j < l, \\
t_{ij} t_{kl} &= t_{kl} t_{ij}, && \text{for } i < k, \ j > l, \\
t_{ij} t_{kl} - t_{kl} t_{ij} &= (q - q^{-1}) t_{il} t_{kj}, && \text{for } i < k, \ j < l.
\end{aligned}$$

It is well known that $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is an iterated skew polynomial extension

$$\mathcal{O}_q(M_{m,n}(\mathbb{K})) = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N],$$

where $N = mn$ and $x_{(r-1)n+c} = t_{rc}$ for $r \in [1, m]$ and $c \in [1, n]$. It is easy to determine the elements $\sigma_k(x_j)$ and $\delta_k(x_j)$ for $N \geq k > j \geq 1$ and then to see that the skew derivations δ_k are locally nilpotent. For later reference, we note that the scalars λ_{kj} are given by the following powers of q:

$$\lambda_{(r-1)n+c,(r'-1)n+c'} = \begin{cases} q^{\,\text{sign}(r'-r)}, & \text{if } c = c', \\ q^{\,\text{sign}(c'-c)}, & \text{if } r = r', \\ 1, & \text{otherwise}, \end{cases} \quad \begin{matrix} \forall r, r' \in [1, m], \\ c, c' \in [1, n]. \end{matrix} \tag{3.3}$$

There is a rational action of the torus $\mathcal{H} := (\mathbb{K}^*)^{m+n}$ on $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ by $\mathbb{K}$-algebra automorphisms such that

$$(\xi_1, \ldots, \xi_{m+n}) \cdot t_{rc} = \xi_r \xi_{m+c}^{-1} t_{rc}$$

for all $(\xi_1, \ldots, \xi_{m+n}) \in \mathcal{H}$, $r \in [1, m]$, $c \in [1, n]$. Define

$$h_{rc} := (1, \ldots, 1, q^{-1}, 1, \ldots, 1, q, 1, \ldots, 1) \in \mathcal{H},$$

where the entries q^{-1} and q reside in positions r and $m+c$, respectively. Then $\sigma_{(r-1)n+c} = (h_{rc}\cdot)$ and $h_{rc}\cdot t_{rc} = q^{-2}t_{rc}$. Thus, $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is a torsionfree CGL extension of length N. (This is well known; see, e.g., [**33**, Corollary 3.8].)

The algebra $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ and the accompanying $\mathcal{H}$-action are, of course, defined for any nonzero scalar q. The resulting algebra is an iterated skew polynomial ring which satisfies all but one of the conditions for a CGL extension. However, if $m,n\geq 2$ and q is a root of unity, there is no element $h\in\mathcal{H}$ such that $\sigma_{n+2} = (h\cdot)$ and $h\cdot x_{n+2} = \lambda x_{n+2}$ with λ not a root of unity.

The algebras of quantum matrices are special cases of the family of quantum Schubert cell algebras. The latter algebras are treated in detail in Chapters 9 and 10. On the other hand, the quantum Schubert cell algebras are special members in the class of CGL extensions. They have the property that their gradings by the character lattices of the torus $\mathcal{H}$ can be specialized to $\mathbb{Z}_{\geq 0}$-gradings that are connected, meaning that the degree 0 component of the algebra is just $\mathbb{K}\cdot 1$. For example, in the case of quantum matrices we can define a connected $\mathbb{Z}_{\geq 0}$-grading with $\deg t_{ij} = 1$ for all i,j. The next example contains a CGL extension that does not have this property.

Example 3.5. Equip the quantized Weyl algebra

$$R = A_1^q(\mathbb{K}) := \frac{\mathbb{K}\langle x_1, x_2\rangle}{\langle x_2x_1 - qx_1x_2 - 1\rangle}$$

with the action of $\mathcal{H} = \mathbb{K}^*$ by $\mathbb{K}$-algebra automorphisms such that $a\cdot x_1 = ax_1$ and $a\cdot x_2 = a^{-1}x_2$ for $a\in\mathcal{H}$. It is straightforward to verify (and well known) that as long as q is not a root of unity, R is a CGL extension $\mathbb{K}[x_1][x_2;\sigma_2,\delta_2]$ with $h_1 = h_2 := q$.

The grading of R by $X(\mathcal{H})$ cannot be specialized to any nontrivial $\mathbb{Z}_{\geq 0}$-grading. Indeed, if $\deg x_1 = n$, then $\deg x_2 = -n$, hence, $n = 0$.

By [**33**, Proposition 3.2, Theorem 3.7], every CGL extension is an $\mathcal{H}$-UFD. A recursive description of the sets of homogeneous prime elements of the intermediate algebras R_k of a CGL extension R was obtained in [**24**]. It is expressed using the following functions.

For a function $\eta : [1,N]\to\mathbb{Z}$, we denote the predecessor and successor functions for its level sets by

$$p = p_\eta : [1,N]\to[1,N]\sqcup\{-\infty\}, \qquad s = s_\eta : [1,N]\to[1,N]\sqcup\{+\infty\}.$$

They are given by

$$p(k) = \begin{cases} \max\{j<k \mid \eta(j)=\eta(k)\}, & \text{if } \exists j<k \text{ such that } \eta(j)=\eta(k),\\ -\infty, & \text{otherwise}\end{cases} \tag{3.4}$$

and

$$s(k) = \begin{cases} \min\{j>k \mid \eta(j)=\eta(k)\}, & \text{if } \exists j>k \text{ such that } \eta(j)=\eta(k),\\ +\infty, & \text{otherwise.}\end{cases} \tag{3.5}$$

Theorem 3.6. [**24**, Theorem 4.3] *Let R be a CGL extension of length N and rank* $\mathrm{rk}(R)$ *as in* (3.1). *There exist a function $\eta : [1,N]\to\mathbb{Z}$ whose range has cardinality* $\mathrm{rk}(R)$ *and elements*

$$c_k\in R_{k-1} \ \textit{for all}\ k\in[2,N] \ \textit{with}\ p(k)\neq-\infty$$

such that the elements $y_1, \dots, y_N \in R$, recursively defined by

$$y_k := \begin{cases} y_{p(k)} x_k - c_k, & \text{if } p(k) \neq -\infty \\ x_k, & \text{if } p(k) = -\infty, \end{cases} \tag{3.6}$$

are homogeneous and have the property that for every $k \in [1, N]$,

$$\{y_j \mid j \in [1, k],\ s(j) > k\} \tag{3.7}$$

is a list of the homogeneous prime elements of R_k up to scalar multiples.

The elements $y_1, \dots, y_N \in R$ with these properties are unique. The function η satisfying the above conditions is not unique, but the partition of $[1, N]$ into a disjoint union of the level sets of η is uniquely determined by R, as are the predecessor and successor functions p and s. The function p has the property that $p(k) = -\infty$ if and only if $\delta_k = 0$.

The uniqueness of the level sets of η was not stated in [**24**, Theorem 4.3], but it follows at once from [**24**, Theorem 4.2]. This uniqueness immediately implies the uniqueness of p and s. Although one can state the theorem using a partition of $[1, N]$ into $\mathrm{rk}(R)$ subsets, the use of a function will be better suited for combinatorial purposes. In the setting of the theorem, the rank of R is also given by

$$\mathrm{rk}(R) = |\{j \in [1, N] \mid s(j) > N\}| \tag{3.8}$$

[**24**, Eq. (4.3)].

EXAMPLE 3.7. Let $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$ be the CGL extension in Example 3.4. For any subsets $I \subseteq [1, m]$ and $J \subseteq [1, n]$ of the same cardinality d, let $\Delta_{I,J}$ denote the $d \times d$ quantum minor with row index set I and column index set J. Namely, if $I = \{i_1 < \cdots < i_d\}$ and $J = \{j_1 < \cdots < j_d\}$, then

$$\Delta_{I,J} := \sum_{\tau \in S_d} (-q)^{\ell(\tau)} t_{i_1, j_{\tau(1)}} t_{i_2, j_{\tau(2)}} \cdots t_{i_d, j_{\tau(d)}},$$

where $\ell(\tau)$ is the length of τ as a minimal length product of simple transpositions. The homogeneous prime elements y_k from Theorem 3.6 are "solid" quantum minors of the following form:

$$y_{(r-1)n+c} = \Delta_{[r-\min(r,c)+1, r], [c-\min(r,c)+1, c]}, \quad \forall r \in [1, m],\ c \in [1, n].$$

The function $\eta : [1, N] \to \mathbb{Z}$ can be chosen as

$$\eta((r-1)n + c) := c - r, \quad \forall r \in [1, m],\ c \in [1, n].$$

It is easily checked that for the CGL extension presentation of R in Example 3.4, we have $\delta_k = 0$ if and only if $k \in [1, n]$ or $k = (r-1)n + 1$ for some $r \in [2, m]$. Hence, $\mathrm{rk}(R) = m + n - 1$.

There is a universal maximal choice for the torus $\mathcal{H}$ in Definition 3.3 given by the following theorem. Consider the rational action of the torus $(\mathbb{K}^*)^N$ on R by invertible linear transformations given by

$$(\gamma_1, \dots, \gamma_N) \cdot (x_1^{m_1} \cdots x_N^{m_N}) = \gamma_1^{m_1} \cdots \gamma_N^{m_N} x_1^{m_1} \cdots x_N^{m_N}.$$

Denote by $\mathcal{H}_{\max}(R)$ the closed subgroup of $(\mathbb{K}^*)^N$ consisting of those $\psi \in (\mathbb{K}^*)^N$ whose actions give $\mathbb{K}$-algebra automorphisms of R.

THEOREM 3.8. [**24**, Theorems 5.3, 5.5] *For every CGL extension R, $\mathcal{H}_{\max}(R)$ is a $\mathbb{K}$-torus of rank* $\mathrm{rk}(R)$*, and the pair $(R, \mathcal{H}_{\max}(R))$ is a CGL extension for the Ore extension presentation* (3.1).

Theorem 3.8 implies that the torus $\mathcal{H}_{\max}(R)$ is universal (and maximal) in the sense that, if $\mathcal{H}$ is any torus acting rationally on R by algebra automorphisms such that $(R, \mathcal{H})$ is a CGL extension for the presentation (3.1), then the action of $\mathcal{H}$ on R factors through an algebraic group morphism $\mathcal{H} \to \mathcal{H}_{\max}(R)$. The torus $\mathcal{H}_{\max}(R)$ is explicitly described in [**24**, Theorem 5.5] on the basis of the sequence $y_1, \dots, y_N$ from Theorem 3.6.

EXAMPLE 3.9. In the case of $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$ (Example 3.4), $\mathcal{H}_{\max}(R)$ can be computed from [**24**, Theorem 5.5]. It can be expressed as follows:

$$\mathcal{H}_{\max}(R) = \{\psi \in (\mathbb{K}^*)^{mn} \mid \psi_{(r-1)n+c} = \psi_1^{-1}\psi_c\psi_{(r-1)n+1} \ \ \forall r \in [2,m],\ c \in [2,n]\}.$$

This torus is related to the torus $\mathcal{H} = (\mathbb{K}^*)^{m+n}$ of Example 3.4 via a surjective morphism of algebraic groups $\pi : \mathcal{H} \to \mathcal{H}_{\max}(R)$, given by

$$\pi(\xi)_{(r-1)n+c} = \xi_r\xi_{m+c}^{-1}, \ \ \forall r \in [1,m],\ c \in [1,n],$$

and π transports the $\mathcal{H}$-action on R to the $\mathcal{H}_{\max}(R)$-action.

The next result provides a constructive method for finding the sequence of prime elements $y_1, \dots, y_N$ for a given CGL extension R. Note that the elements y'_k are not a priori assumed to be prime, only normal.

PROPOSITION 3.10. *Let R be a CGL extension of length N. Assume that $y'_1, \dots, y'_N$ and $c'_1, \dots, c'_N$ are two sequences of elements of R such that*

(i) *$y'_1, \dots, y'_N$ are homogeneous normal elements of $R_1, \dots, R_N$, respectively.*
(ii) *$c'_k \in R_{k-1}$, $\forall k \in [1,N]$.*
(iii) *For every $k \in [1,N]$, either $y'_k = x_k - c'_k$ or there exists $j \in [1,k-1]$ such that $y'_k = y'_j x_k - c'_k$.*
(iv) *If $p(k) = -\infty$, then the first equality in* (iii) *holds.*

Then $y'_1, \dots, y'_N$ is precisely the sequence of homogeneous prime elements from Theorem 3.6*, and the function p satisfies $p(k) := j$ if the second equality in* (iii) *holds and $p(k) := -\infty$ otherwise.*

PROOF. The given assumptions imply that $c'_1 \in \mathbb{K}$ and $y'_1 = x_1 - c'_1$. Since y'_1 is homogeneous, we must have $y'_1 = x_1 = y_1$.

Now let $k \in [2,N]$. We will prove that, if $y'_i = y_i$, $\forall i \in [1,k-1]$, then $y'_k = y_k$. This implies the first statement of the proposition by induction. By Proposition 3.1 and Theorem 3.6,

$$y'_k = \xi \prod \{y_i^{m_i} \mid i \in [1,k],\ s(i) > k\}$$

for some $\xi \in \mathbb{K}^*$ and $m_i \in \mathbb{Z}_{\geq 0}$, where the factors $y_i^{m_i}$ are taken in ascending order with respect to i. Comparing the coefficients of x_k and using the form of y_k from Theorem 3.6, we obtain that $m_k = 1$. One of the following two situations holds:

(a) $m_i = 0$, $\forall i \in [1,k-1]$ with $s(i) > k$.
(b) $m_{i_0} > 0$ for some $i_0 \in [1,k-1]$ with $s(i_0) > k$.

First we rule out (b). The x_k-coefficient of y'_k is either 1 or y'_j, hence either a unit or a prime element, by induction. If (b) holds, then $y_k = x_k$ and

$$y'_k = \xi y_{i_0} x_k,$$

which contradicts the condition (iv) because $y_k = x_k$ only occurs when $p(k) = -\infty$.

In the situation (a), we have $y'_k = \xi y_k$. So, either

$$x_k - c'_k = \xi x_k, \ \text{ or}$$
$$y'_j x_k - c'_k = \xi(y_{p(k)} x_k - c_k).$$

In the first case, we have $\xi = 1$, $c'_k = 0$ and $y'_k = y_k$. In the second case, using the fact that $y'_i = y_i$, $\forall i \in [1, k-1]$, we obtain $\xi = 1$, $j = p(k)$, $c'_k = c_k$ and $y'_k = y_k$. This argument also proves the second statement of the proposition. $\square$

Although this will not be used later, we also note that the conclusion of the proposition implies

$$c'_k = \begin{cases} c_k, & \text{if } p(k) \neq -\infty \\ 0, & \text{if } p(k) = -\infty. \end{cases}$$

The following fact from [**24**] will be extensively used in the paper to construct quantum clusters.

PROPOSITION 3.11. [**24**, Eq. (4.17) and Theorem 4.6] *For each CGL extension R, the elements y_k of Theorem* 3.6 *quasi-commute: there are scalars $q_{kj} \in \mathbb{K}^*$, given in* (3.23) *below, such that*

$$y_k y_j = q_{kj} y_j y_k, \quad \forall j, k \in [1, N]. \tag{3.9}$$

The quantum torus $\mathcal{T}_{\mathbf{q}}$ embeds in $\mathrm{Fract}(R)$ *via the $\mathbb{K}$-algebra homomorphism $\varphi : \mathcal{T}_{\mathbf{q}} \hookrightarrow \mathrm{Fract}(R)$ given by $\varphi(Y_i) = y_i$, $\forall i \in [1, N]$, and this embedding gives rise to inclusions*

$$\varphi(\mathcal{A}_{\mathbf{q}}) \subseteq R \subset \varphi(\mathcal{T}_{\mathbf{q}}) \subset \mathrm{Fract}(R),$$

recall (2.4).

3.3. Symmetric CGL extensions

The CGL extensions for which we will establish quantum cluster algebra structures are those with suitably many different CGL extension presentations, in which the variables $x_1, \ldots, x_N$ are permuted in various ways. A symmetry condition, which we introduce now, is sufficient to guarantee this.

Given an iterated Ore extension R as in (3.1), for $j, k \in [1, N]$ denote by $R_{[j,k]}$ the unital subalgebra of R generated by $\{x_i \mid j \leq i \leq k\}$. So, $R_{[j,k]} = \mathbb{K}$ if $j \not\leq k$.

DEFINITION 3.12. We call a CGL extension R of length N as in Definition 3.3 *symmetric* if the following two conditions hold:

(i) For all $1 \leq j < k \leq N$,

$$\delta_k(x_j) \in R_{[j+1,k-1]}.$$

(ii) For all $j \in [1, N]$, there exists $h^*_j \in \mathcal{H}$ such that

$$h^*_j \cdot x_k = \lambda_{kj}^{-1} x_k = \lambda_{jk} x_k, \ \ \forall k \in [j+1, N]$$

and $h^*_j \cdot x_j = \lambda^*_j x_j$ for some $\lambda^*_j \in \mathbb{K}^*$ which is not a root of unity.

For such an algebra R, set

$$\sigma_j^* := (h_j^* \cdot) \in \mathrm{Aut}(R), \quad \forall j \in [1, N-1].$$

Then for all $j \in [1, N-1]$, the inner σ_j^*-derivation on R given by $a \mapsto x_j a - \sigma_j^*(a) x_j$ restricts to a σ_j^*-derivation δ_j^* of $R_{[j+1,N]}$. It is given by

$$\delta_j^*(x_k) := x_j x_k - \lambda_{jk} x_k x_j = -\lambda_{jk} \delta_k(x_j), \quad \forall k \in [j+1, N].$$

For all $1 \leq j < k \leq N$, σ_k and δ_k preserve $R_{[j,k-1]}$ and σ_j^* and δ_j^* preserve $R_{[j+1,k]}$. This gives rise to the skew polynomial extensions

$$R_{[j,k]} = R_{[j,k-1]}[x_k; \sigma_k, \delta_k] \ \text{ and } \ R_{[j,k]} = R_{[j+1,k]}[x_j; \sigma_j^*, \delta_j^*]. \tag{3.10}$$

In particular, it follows that R has a skew polynomial presentation with the variables x_k in descending order:

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \sigma_1^*, \delta_1^*].$$

This is the reason for the name "symmetric".

Example 3.13. For example, the CGL extension $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ of Example 3.4 is symmetric. It is clear that condition (i) of Definition 3.12 holds in this example. Condition (ii) can be verified for the elements

$$h_{(r-1)n+c}^* := (1, \dots, 1, q, 1, \dots, 1, q^{-1}, 1, \dots, 1) \in \mathcal{H},$$

where the entries q and q^{-1} reside in positions r and $m + c$, respectively. Then $h_j^* \cdot x_j = q^2 x_j$ for all j, and so all $\lambda_j^* = q^2$ here.

Denote the following subset of the symmetric group S_N:

$$\begin{aligned} \Xi_N := \{\tau \in S_N \mid \tau(k) = \max \tau([1, k-1]) + 1 \ \text{ or} \\ \tau(k) = \min \tau([1, k-1]) - 1, \ \ \forall k \in [2, N]\}. \end{aligned} \tag{3.11}$$

In other words, Ξ_N consists of those $\tau \in S_N$ such that $\tau([1, k])$ is an interval for all $k \in [2, N]$. For each $\tau \in \Xi_N$, we have the iterated Ore extension presentation

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}] \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}], \tag{3.12}$$

where $\sigma''_{\tau(k)} := \sigma_{\tau(k)}$ and $\delta''_{\tau(k)} := \delta_{\tau(k)}$ if $\tau(k) = \max \tau([1, k-1]) + 1$, while $\sigma''_{\tau(k)} := \sigma^*_{\tau(k)}$ and $\delta''_{\tau(k)} := \delta^*_{\tau(k)}$ if $\tau(k) = \min \tau([1, k-1]) - 1$.

Proposition 3.14. [**24**, Remark 6.5] *For every symmetric CGL extension R and $\tau \in \Xi_N$, the iterated Ore extension presentation* (3.12) *of R is a CGL extension presentation for the same choice of $\mathbb{K}$-torus $\mathcal{H}$, and the associated elements $h''_{\tau(1)}, \dots, h''_{\tau(N)} \in \mathcal{H}$ required by Definition* 3.3(iv) *are given by $h''_{\tau(k)} = h_{\tau(k)}$ if $\tau(k) = \max \tau([1, k-1]) + 1$ and $h''_{\tau(k)} = h^*_{\tau(k)}$ if $\tau(k) = \min \tau([1, k-1]) - 1$.*

Theorem 3.8 implies that the rank of a symmetric CGL extension does not depend on the choice of CGL extension presentation (3.12).

When describing permutations $\tau \in S_N$ as functions, we will use the one-line notation,

$$\tau = [\tau(1), \tau(2), \dots, \tau(N)] := \begin{bmatrix} 1 & 2 & \cdots & N \\ \tau(1) & \tau(2) & \cdots & \tau(N) \end{bmatrix}. \tag{3.13}$$

A special role is played by the longest element of S_N,

$$w_\circ := [N, N-1, \dots, 1]. \tag{3.14}$$

The corresponding CGL extension presentation from Proposition 3.14 is symmetric, while the ones for the other elements of Ξ_N do not possess this property in general.

3.4. Further CGL details

In order to work closely with CGL extensions, we need some further notation and relations, which we give in this section. Throughout, R is a CGL extension as in (3.1); the elements y_k, $k \in [1, N]$ and the functions p and s are as in Theorem 3.6.

As is easily checked (e.g., see the proof of [**33**, Lemma 2.6]),

$$(h\cdot)\delta_k = \chi_{x_k}(h)\delta_k(h\cdot), \ \ \forall h \in \mathcal{H}, \ k \in [2, N]. \tag{3.15}$$

We will use the following convention:

$$\begin{gathered}\text{All products } \prod_{j \in P} y_j^{m_j} \text{ with } P \subseteq [1, N] \text{ and } m_j \in \mathbb{Z}_{\geq 0} \text{ are taken} \\ \text{with the terms in increasing order with respect to } j.\end{gathered} \tag{3.16}$$

Set also

$$y_{-\infty} := 1. \tag{3.17}$$

Define the order functions $O_\pm : [1, N] \to \mathbb{Z}_{\geq 0}$ by

$$\begin{aligned} O_+(k) &:= \max\{m \in \mathbb{Z}_{\geq 0} \mid s^m(k) \neq +\infty\} \\ O_-(k) &:= \max\{m \in \mathbb{Z}_{\geq 0} \mid p^m(k) \neq -\infty\}, \end{aligned} \tag{3.18}$$

where as usual $p^0 = s^0 = \mathrm{id}$. For $k \in [1, N]$, define

$$\overline{e}_k := \sum_{m=0}^{O_-(k)} e_{p^m(k)} \in \mathbb{Z}^N. \tag{3.19}$$

The algebra R has the $\mathbb{K}$-basis

$$\{x^f := x_1^{m_1} \cdots x_N^{m_N} \mid f = (m_1, \dots, m_N)^T \in \mathbb{Z}_{\geq 0}^N\}.$$

Denote by $\prec$ the reverse lexicographic order on $\mathbb{Z}_{\geq 0}^N$:

$$(m_1', \dots, m_N')^T \prec (m_1, \dots, m_N)^T \ \text{ iff } \ \exists j \in [1, N] \ \text{ such that} \\ m_j' < m_j \ \text{ and } \ m_k' = m_k, \ \forall k \in [j+1, N].$$

We say that $b \in R \setminus \{0\}$ has *leading term* ξx^f where $\xi \in \mathbb{K}^*$ and $f \in \mathbb{Z}_{\geq 0}^N$ if

$$b = \xi x^f + \sum_{g \in \mathbb{Z}_{\geq 0}^N, \ g \prec f} \xi_g x^g$$

for some $\xi_g \in \mathbb{K}$. Set $\mathrm{lt}(b) := \xi x^f$. Then

$$\mathrm{lt}(x^f x^{f'}) = \Big(\prod_{k > j} \lambda_{kj}^{m_k m_j'} \Big) x^{f+f'}, \\ \forall \ f = (m_1, \dots, m_N)^T, \ \ f' = (m_1', \dots, m_N')^T \in \mathbb{Z}_{\geq 0}^N. \tag{3.20}$$

We have [**24**, Eq. (4.13)]

$$\mathrm{lt}(y_k) = x^{\overline{e}_k}, \ \ \forall k \in [1, N], \tag{3.21}$$

cf. (3.19).

For $k, j \in [1, N]$, set

$$\alpha_{kj} := \Omega_{\boldsymbol{\lambda}}(e_k, \overline{e}_j) = \prod_{m=0}^{O_-(j)} \lambda_{k,p^m(j)} \in \mathbb{K}^* \tag{3.22}$$

$$q_{kj} := \Omega_{\boldsymbol{\lambda}}(\overline{e}_k, \overline{e}_j) = \prod_{m=0}^{O_-(k)} \prod_{l=0}^{O_-(j)} \lambda_{p^m(k),p^l(j)} = \prod_{m=0}^{O_-(k)} \alpha_{p^m(k),j} \in \mathbb{K}^*, \tag{3.23}$$

recall (2.1) and (3.19). Since $\boldsymbol{\lambda} = (\lambda_{kj})$ is a multiplicatively skew-symmetric matrix, so is $\mathbf{q} := (q_{kj}) \in M_N(\mathbb{K}^*)$. In addition to (3.9), we have [**24**, Corollary 4.8; p. 21; Proposition 4.7(b)]

$$y_j x_k = \alpha_{kj}^{-1} x_k y_j, \;\; \forall j, k \in [1, N] \text{ such that } s(j) > k \tag{3.24}$$

$$\sigma_k(y_j) = \alpha_{kj} y_j, \;\; \text{for } 1 \le j < k \le N \tag{3.25}$$

$$\delta_k(y_{p(k)}) = \alpha_{kp(k)}(\lambda_k - 1)c_k \neq 0, \;\; \forall k \in [2, N] \text{ such that } p(k) \neq -\infty. \tag{3.26}$$

Set also, for $k \in [1, N]$,

$$\alpha_{k,-\infty} = q_{k,-\infty} := 1. \tag{3.27}$$

For $k \in [0, N]$, denote

$$P(k) := \{ j \in [1, k] \mid s(j) > k \}. \tag{3.28}$$

Then $\{y_j \mid j \in P(k)\}$ is a list of the homogeneous prime elements of R_k up to scalar multiples, and $|P(N)| = \mathrm{rk}(R)$ (3.8).

CHAPTER 4

One-step mutations in CGL extensions

In this chapter we obtain a very general way of constructing mutations of toric frames in CGL extensions. The key idea is that, if an algebra R has two different CGL extension presentations obtained by reversing the order in which two adjacent variables x_k and x_{k+1} are adjoined, then the corresponding sequences of prime elements from Theorem 3.6 are obtained by a type of mutation formula. This is realized in Section 4.1. In Section 4.2–4.4 we construct toric frames from Theorem 3.6 and treat their mutations. One problem arises along the way: In the analog of the mutation formula (2.25) for the current situation, the last term has a nonzero coefficient which does not equal one in general. For a one-step mutation such a coefficient can be always made 1 after rescaling, but for the purposes of constructing quantum cluster algebras one needs to be able to synchronize those rescalings to obtain a chain of mutations. This delicate issue is resolved in the next two chapters.

We investigate a CGL extension

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_k; \sigma_k, \delta_k][x_{k+1}; \sigma_{k+1}, \delta_{k+1}] \cdots [x_N; \sigma_N, \delta_N] \tag{4.1}$$

of length N as in Definition 3.3 such that, for some $k \in [1, N-1]$, R has a second CGL extension presentation of the form

$$\begin{aligned} R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_{k-1}; \sigma_{k-1}, \delta_{k-1}][x_{k+1}; \sigma'_k, \delta'_k][x_k; \sigma'_{k+1}, \delta'_{k+1}] \\ [x_{k+2}; \sigma_{k+2}, \delta_{k+2}] \cdots [x_N; \sigma_N, \delta_N]. \end{aligned} \tag{4.2}$$

4.1. A general mutation formula

LEMMA 4.1. *Assume that R is a CGL extension as in* (4.1)*, and that R has a second CGL extension presentation of the form* (4.2)*. Then δ_{k+1} and δ'_{k+1} map R_{k-1} to itself, and*

$$\begin{aligned} \sigma'_k &= \sigma_{k+1}|_{R_{k-1}} & \delta'_k &= \delta_{k+1}|_{R_{k-1}} \\ \sigma'_{k+1}|_{R_{k-1}} &= \sigma_k & \delta'_{k+1}|_{R_{k-1}} &= \delta_k. \end{aligned} \tag{4.3}$$

Moreover, $\delta_{k+1}(x_k) \in \mathbb{K}$ and

$$\sigma'_{k+1}(x_{k+1}) = \lambda_{k,k+1} x_{k+1} \qquad \delta'_{k+1}(x_{k+1}) = -\lambda_{k,k+1}\delta_{k+1}(x_k). \tag{4.4}$$

PROOF. Note first that R_{k-1} is stable under σ_{k+1} and σ'_{k+1}. For $a \in R_{k-1}$, we have $x_{k+1}a = \sigma'_k(a)x_{k+1} + \delta'_k(a)$ with $\sigma'_k(a), \delta'_k(a) \in R_{k-1}$. Comparing this relation with $x_{k+1}a = \sigma_{k+1}(a)x_{k+1} + \delta_{k+1}(a)$, and using the fact that 1, x_{k+1} are left linearly independent over R_k, we conclude that $\sigma'_k(a) = \sigma_{k+1}(a)$ and $\delta'_k(a) = \delta_{k+1}(a)$. Thus, R_{k-1} is stable under δ_{k+1}, and the first line of (4.3) holds. By symmetry (since we may view (4.2) as the initial CGL extension presentation of R and (4.1) as the second one), R_{k-1} is stable under δ'_{k+1}, and the second line of (4.3) holds.

Now $x_k x_{k+1} = \lambda'_{k+1,k} x_{k+1} x_k + \delta'_{k+1}(x_{k+1})$, and so we have

$$\lambda'_{k,k+1} x_k x_{k+1} - \lambda'_{k,k+1} \delta'_{k+1}(x_{k+1}) = x_{k+1} x_k = \lambda_{k+1,k} x_k x_{k+1} + \delta_{k+1}(x_k),$$

with $\delta'_{k+1}(x_{k+1}) \in R'_k = \bigoplus_{l=0}^{\infty} R_{k-1} x_{k+1}^l$ and $\delta_{k+1}(x_k) \in R_k = \bigoplus_{l=0}^{\infty} R_{k-1} x_k^l$. Moreover,

$$R_{k+1} \text{ is a free left } R_{k-1}\text{-module with basis } \{x_k^{l_k} x_{k+1}^{l_{k+1}} \mid l_k, l_{k+1} \in \mathbb{Z}_{\geq 0}\}. \tag{4.5}$$

Hence, we conclude that $\lambda'_{k,k+1} = \lambda_{k+1,k}$ and

$$-\lambda'_{k,k+1} \delta'_{k+1}(x_{k+1}) = \delta_{k+1}(x_k) \in \mathbb{K},$$

from which (4.4) follows. $\square$

THEOREM 4.2. *Assume that R is a CGL extension of length N and rank $\mathrm{rk}(R)$ as in* (4.1)*, and $k \in [1, N-1]$. Denote by $y_1, \ldots, y_N$ and $\eta : [1, N] \to \mathbb{Z}$ the sequence and function from Theorem* 3.6*. Assume that R has a second CGL extension presentation of the form* (4.2)*, and let $y'_1, \ldots, y'_N$ be the corresponding sequence from Theorem* 3.6*.*

(a) *If $\eta(k) \neq \eta(k+1)$, then $y'_j = y_j$ for $j \neq k, k+1$ and $y'_k = y_{k+1}$, $y'_{k+1} = y_k$.*

(b) *If $\eta(k) = \eta(k+1)$, then*

$$y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}$$

is a homogeneous normal element of R_{k-1}, recall (3.17), (3.22), (3.27)*. It normalizes the elements of R_{k-1} in exactly the same way as $y_{p(k)} y_{k+1}$. Furthermore,*

$$y'_j = \begin{cases} \lambda_{k+1,k} y_j, & \text{if } j = s^l(k+1) \text{ for some } l \in \mathbb{Z}_{\geq 0} \\ y_j, & \text{if } j < k, \text{ or } j > k+1 \text{ and } j \neq s^l(k+1)\ \forall l \in \mathbb{Z}_{\geq 0}. \end{cases} \tag{4.6}$$

In both cases (a) *and* (b)*, the function $\eta' : [1, N] \to \mathbb{Z}$ from Theorem* 3.6 *associated to the second presentation can be chosen to be $\eta' = \eta(k, k+1)$, where the last term denotes a transposition in S_N. In particular, the ranges of η and η' coincide and the rank of R is the same for both CGL extension presentations.*

PROOF. Let R'_j be the j-th algebra in the chain (4.2) for $j \in [0, N]$. Obviously $R'_j = R_j$ for $j \neq k, k+1$ and $y'_j = y_j$ for $j \leq k-1$, and we may choose $\eta'(j) = \eta(j)$ for $j \leq k-1$. Since R_{k+1} is a free left R_{k-1}-module with basis (4.5), and R_k and R'_k equal the R_{k-1}-submodules with bases $\{x_k^{l_k} \mid l_k \in \mathbb{Z}_{\geq 0}\}$ and $\{x_{k+1}^{l_{k+1}} \mid l_{k+1} \in \mathbb{Z}_{\geq 0}\}$ respectively, we have

$$R_k \cap R'_k = R_{k-1}. \tag{4.7}$$

In particular, $y'_k \notin R_k$.

Denote by L the number of homogeneous prime elements of R_{k+1} up to taking associates that do not belong to R_{k-1}.

(a) The condition $\eta(k) \neq \eta(k+1)$ implies $L = 2$ and thus $\eta'(k) \neq \eta'(k+1)$. Moreover, y'_k and y'_{k+1} should be scalar multiples of either y_k and y_{k+1} or y_{k+1} and y_k. Since $y'_k \notin R_k$, we must have $y'_k = \xi_{k+1} y_{k+1}$ and $y'_{k+1} = \xi_k y_k$ for some $\xi_k, \xi_{k+1} \in \mathbb{K}^*$. Invoking Theorem 3.6 and looking at leading terms gives $\xi_k = \xi_{k+1} = 1$ and allows us to choose $\eta'(k) = \eta(k+1)$ and $\eta'(k+1) = \eta(k)$. It follows from $R'_j = R_j$ for $j > k+1$ that $y'_j = y_j$ for such j, and that we can choose $\eta'(j) = \eta(j)$ for all $j > k+1$. With these choices, $\eta' = \eta(k, k+1)$.

(b) In this case, $L = 1$ and y'_{k+1} should be a scalar multiple of y_{k+1}. Furthermore, $\eta'(k)$ and $\eta'(k+1)$ must agree, and they need to equal $\eta'(p(k))$ if $p(k) \neq -\infty$, so we can choose $\eta'(k) = \eta'(k+1) = \eta(k) = \eta(k+1)$. By Theorem 3.6,

$$
\begin{aligned}
y_k &= \begin{cases} y_{p(k)}x_k - c_k, & \text{if } p(k) \neq -\infty \\ x_k, & \text{if } p(k) = -\infty \end{cases} & \qquad y_{k+1} &= y_k x_{k+1} - c_{k+1} \\
y'_k &= \begin{cases} y_{p(k)}x_{k+1} - c'_k, & \text{if } p(k) \neq -\infty \\ x_{k+1}, & \text{if } p(k) = -\infty \end{cases} & \qquad y'_{k+1} &= y'_k x_k - c'_{k+1}
\end{aligned} \tag{4.8}
$$

for some $c_k, c'_k \in R_{k-1}$, $c_{k+1} \in R_k$, and $c'_{k+1} \in R'_k$. Write the above elements in terms of the basis (4.5). The coefficients of $x_k x_{k+1}$ in y_{k+1} and y'_{k+1} are $y_{p(k)}$ and $\lambda_{k+1,k} y_{p(k)}$, recall (3.17). Since y'_{k+1} is a scalar multiple of y_{k+1}, we thus see that

$$y'_{k+1} = \lambda_{k+1,k} y_{k+1}. \tag{4.9}$$

Eq. (4.6) and the fact that we may choose $\eta'(j) = \eta(j)$ for $j > k+1$ now follow easily from Theorem 3.6. In particular, $\eta' = \eta = \eta(k, k+1)$.

Next, we verify that

$$y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1} \in R_{k-1}. \tag{4.10}$$

By (3.26), c_{k+1} is a nonzero scalar multiple of $\delta_{k+1}(y_k)$. If $p(k) = -\infty$, then $y_k = x_k$, and so $c_{k+1} \in \mathbb{K}^*$ by Lemma 4.1. From (4.8) and (3.17), (3.27), we then obtain

$$y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1} = x_k x_{k+1} - (x_k x_{k+1} - c_{k+1}) = c_{k+1} \in \mathbb{K}^*. \tag{4.11}$$

This verifies (4.10) (and also shows that $y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}$ is a homogeneous normal element of R_{k-1}) in the present case.

Now assume that $p(k) \neq -\infty$. Invoking (3.9) and the observation that $q_{kp(k)} = \alpha_{kp(k)}$, we have $y_k y_{p(k)} = \alpha_{kp(k)} y_{p(k)} y_k$. First, we obtain

$$
\begin{aligned}
y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1} &= y_k (y_{p(k)} x_{k+1} - c'_k) - \alpha_{kp(k)} y_{p(k)} (y_k x_{k+1} - c_{k+1}) \\
&= -y_k c'_k + \alpha_{kp(k)} y_{p(k)} c_{k+1} \in R_k.
\end{aligned} \tag{4.12}
$$

Using (4.9) together with the fact that $\sigma'_{k+1}(y'_k) = \alpha'_{k+1,k} y'_k$ (3.25) and the observation that $\alpha'_{k+1,k} = \lambda^{-1}_{k+1,k} \alpha_{kp(k)}$, we obtain

$$
\begin{aligned}
y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1} &= y_{p(k)} (\alpha'_{k+1,k} y'_k x_k + \delta'_{k+1}(y'_k)) - c_k y'_k \\
&\qquad - \alpha_{kp(k)} y_{p(k)} \lambda^{-1}_{k+1,k} (y'_k x_k - c'_{k+1}) \\
&= y_{p(k)} \delta'_{k+1}(y'_k) - c_k y'_k + \alpha_{kp(k)} \lambda^{-1}_{k+1,k} y_{p(k)} c'_{k+1} \in R'_k.
\end{aligned}
$$

This equation, combined with (4.12) and (4.7), yields (4.10).

We now use (4.12) to verify that $y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}$ is homogeneous. Note first that $y_k c'_k$ and $y_{p(k)} c_{k+1}$ are homogeneous. By (3.26), c'_k and c_{k+1} are scalar multiples of $\delta'_k(y_{p(k)}) = \delta_{k+1}(y_{p(k)})$ and $\delta_{k+1}(y_k)$, respectively. Hence, it follows from (3.15) that

$$X(\mathcal{H})\text{-deg}(y_k c'_k) = \chi_{y_k} + \chi_{x_{k+1}} + \chi_{y_{p(k)}} = X(\mathcal{H})\text{-deg}(y_{p(k)} c_{k+1}).$$

Thus, $-y_k c'_k + \alpha_{kp(k)} y_{p(k)} c_{k+1}$ is homogeneous, as desired.

Finally, whether $p(k) = -\infty$ or not, it follows from (3.24) that

$$(y_k y'_k) x_j = \beta_j x_j (y_k y'_k) \quad \text{and} \quad (y_{p(k)} y_{k+1}) x_j = \gamma_j x_j (y_{p(k)} y_{k+1}), \quad \forall j \in [1, k-1],$$

where

$$\beta_j = (\alpha'_{jk})^{-1}\alpha_{jk}^{-1} = \Big(\lambda_{j,k+1} \prod_{l=1}^{O_-(k)} \lambda_{j,p^l(k)} \prod_{m=0}^{O_-(k)} \lambda_{j,p^m(k)}\Big)^{-1}$$
$$= \big(\alpha_{j,p(k)}\alpha_{j,k+1}\big)^{-1} = \gamma_j,$$

and so $\big(y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}\big)x_j = \gamma_j x_j \big(y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}\big)$. This shows that $y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}$ is a normal element of R_{k-1} which normalizes the elements of R_{k-1} in exactly the same way as $y_{p(k)} y_{k+1}$, and completes the proof of the theorem. □

Our next result turns the conclusion of Theorem 4.2 (b) into a cluster mutation statement.

THEOREM 4.3. *In the setting of Theorem* 4.2 (b)*, there exist a collection of nonnegative integers* $\{m_i \mid i \in P(k-1),\, i \neq p(k)\}$ *and* $\kappa \in \mathbb{K}^*$ *such that*

$$y'_k = y_k^{-1}\Big(\alpha_{kp(k)} y_{p(k)} y_{k+1} + \kappa \prod_{i \in P(k-1),\, i \neq p(k)} y_i^{m_i}\Big), \tag{4.13}$$

in the conventions from (3.16) *and* (3.17)*, recall also* (3.22)*. If* $p(k) = -\infty$*, then all* $m_i = 0$*.*

PROOF. By Theorem 4.2 (b), $y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}$ is a homogeneous normal element of R_{k-1}. Applying Proposition 3.1 and Theorem 3.6 we obtain

$$y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1} = \kappa \prod_{i \in P(k-1)} y_i^{m_i}$$

for some $\kappa \in \mathbb{K}$ and a collection of nonnegative integers $\{m_i \mid i \in P(k-1)\}$. Recall from (4.11) that if $p(k) = -\infty$, then $y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1}$ is a nonzero scalar. Hence, $m_i = 0$ for all $i \in P(k-1)$ in this case. We need to prove that $\kappa \neq 0$, and that $m_{p(k)} = 0$ if $p(k) \neq -\infty$.

Suppose that $\kappa = 0$. Then

$$y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1} = 0$$

which is a contradiction since y_{k+1} is a prime element of R_{k+1} which does not divide either y_k or y'_k.

Now suppose that $p(k) \neq -\infty$ and $m_{p(k)} \neq 0$. Then $y_{p(k)}$ is a prime element of R_{k-1} and

$$y_k y'_k - \alpha_{kp(k)} y_{p(k)} y_{k+1} \in y_{p(k)} R_{k-1}.$$

Hence,

$$y_k y'_k \in y_{p(k)} R_{k+1}.$$

Furthermore, by Theorem 3.6, $y_k = y_{p(k)} x_k - c_k$ and $y'_k = y_{p(k)} x_{k+1} - c'_k$ for some $c_k, c'_k \in R_{k-1}$ such that $y_{p(k)}$ does not divide c_k or c'_k. Taking into account the normality of $y_{p(k)}$ in R_{k-1} gives

$$c_k c'_k = y_k y'_k - y_{p(k)} x_k y'_k + c_k y_{p(k)} x_{k+1} \in y_{p(k)} R_{k+1} \cap R_{k-1} = y_{p(k)} R_{k-1}.$$

This contradicts the fact that $y_{p(k)}$ is a prime element of R_{k-1} which does not divide c_k or c'_k. □

4.2. A base change and normalization of the elements y_j from Theorem 3.6

If R is a $\mathbb{K}$-algebra with a rational action of a torus $(\mathbb{K}^*)^r$ by algebra automorphisms and $\mathbb{F}/\mathbb{K}$ is a field extension, then there is a canonical rational action of $(\mathbb{F}^*)^r$ on $R \otimes_{\mathbb{K}} \mathbb{F}$ by $\mathbb{F}$-algebra automorphisms. This easily implies the validity of the following lemma.

LEMMA 4.4. *Assume that R is a CGL extension as in* (4.1) *with an action of the torus $\mathcal{H} = (\mathbb{K}^*)^r$, and $\mathbb{F}/\mathbb{K}$ is a field extension. Then, as $\mathbb{F}$-algebras,*

$$R \otimes_{\mathbb{K}} \mathbb{F} \cong \mathbb{F}[x_1][x_2; \sigma_2^\circ, \delta_2^\circ] \dots [x_N; \sigma_N^\circ, \delta_N^\circ]$$

where σ_k° and δ_k° are the automorphisms and skew derivations obtained from the ones in (3.1) *by base change. Furthermore, $R \otimes_{\mathbb{K}} \mathbb{F}$ is a CGL extension with the canonical induced rational action of $(\mathbb{F}^*)^r$ and the same choice of elements h_k.*

REMARK 4.5. After a field extension, every (symmetric) CGL extension R as in Definitions 3.3 and 3.12 can be brought to one that satisfies $\sqrt{\lambda_{lj}} \in \mathbb{K}$ for all $l, j \in [1, N]$ by applying Lemma 4.4 with

$$\mathbb{F} := \mathbb{K}[\sqrt{\lambda_{lj}} \mid 1 \le j < l \le N].$$

Let R be a CGL extension as in (4.1) that satisfies $\sqrt{\lambda_{lj}} \in \mathbb{K}$ for all $l, j \in [1, N]$. Fix

$$\nu_{lj} \in \mathbb{K} \text{ for } 1 \le j < l \le N \quad \text{such that } \nu_{lj}^2 = \lambda_{lj}. \tag{4.14}$$

Set

$$\nu_{ll} := 1 \text{ for } l \in [1, N] \quad \text{and} \quad \nu_{jl} := \nu_{lj}^{-1} \text{ for } 1 \le j < l \le N. \tag{4.15}$$

Then $\boldsymbol{\nu} := (\nu_{lj}) \in M_N(\mathbb{K}^*)$ is a multiplicatively skew-symmetric matrix and $\boldsymbol{\nu}^{\cdot 2} = \boldsymbol{\lambda}$.

Analogously to (3.23), for $l, j \in [1, N]$ set

$$r_{lj} := \Omega_{\boldsymbol{\nu}}(\overline{e}_l, \overline{e}_j) = \prod_{m=0}^{O_-(l)} \prod_{n=0}^{O_-(j)} \nu_{p^m(l), p^n(j)} \in \mathbb{K}^*, \tag{4.16}$$

recall (3.19). The matrix $\mathbf{r} = (r_{lj}) \in M_N(\mathbb{K}^*)$ is multiplicatively skew-symmetric, $\mathbf{r}^{\cdot 2} = \mathbf{q}$, and

$$\Omega_{\mathbf{r}}(e_l, e_j) := \Omega_{\boldsymbol{\nu}}(\overline{e}_l, \overline{e}_j), \quad \forall l, j \in [1, N]. \tag{4.17}$$

We normalize the elements y_j from Theorem 3.6 by

$$\overline{y}_j = \Big(\prod_{0 \le n < m \le O_-(j)} \nu_{p^m(j), p^n(j)}^{-1} \Big) y_j, \quad \forall j \in [1, N].$$

The meaning of this normalization is as follows. The generators x_j of R skew commute up to lower terms of R in the partial order from Section 3.4:

$$x_l x_j - \lambda_{lj} x_j x_l \in \mathrm{Span}\{x^g \mid g \in \mathbb{Z}_{\ge 0}^N,\ g \prec e_l + e_j\}.$$

Ignoring these lower order terms, the above normalization is precisely the normalization from (2.6) applied to the leading term $x^{\overline{e}_j}$ of y_j (cf. (3.21)), thought of as an element of the abstract based quantum torus for the multiplicatively skew-symmetric matrix $\boldsymbol{\nu}$ (recall Section 2.2); that is

$$\overline{y}_j = \mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_j) y_j, \quad \forall j \in [1, N]. \tag{4.18}$$

Proposition 3.11 immediately implies:

PROPOSITION 4.6. *Consider a CGL extension presentation of an algebra R as in (3.1) for which $\sqrt{\lambda_{lj}} \in \mathbb{K}$ for all $l, j \in [1, N]$, and define the matrices $\boldsymbol{\nu}, \mathbf{r} \in M_N(\mathbb{K}^*)$ by (4.14), (4.15), and (4.16). There exists a unique toric frame $M : \mathbb{Z}^N \to \operatorname{Fract}(R)$ having matrix $\mathbf{r}$ and such that $M(e_j) = \overline{y}_j$, $\forall j \in [1, N]$.*

EXAMPLE 4.7. In the case of $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$, the scalars λ_{lj} for $l > j$ are either 1 or q^{-1}, so the only square root needed in $\mathbb{K}$ is $\sqrt{q}$. (We fix a choice of this scalar.) However, $\sqrt{q}$ does not appear in the normalizations $\overline{y}_j$. In fact, $\nu_{p^s(j), p^t(j)} = 1$ for $0 \leq t < s \leq O_-(j)$ (recall the choice of η from Example 3.7 and the description of the λ_{kl} in (3.3)), and so in this example, $\overline{y}_j = y_j$ for all $j \in [1, N]$.

4.3. Almost cluster mutations between CGL extension presentations

Let R be a $\mathbb{K}$-algebra with a CGL extension presentation as in (4.1). We will assume that the torus $\mathcal{H}$ acting on R is the universal maximal torus from Theorem 3.8. It is easy to see that in this setting, the assumption that (4.2) is a second CGL extension presentation of R is equivalent to the following condition:

(i) $\delta_{k+1}(x_j) \in R_{k-1}$, $\forall j \in [1, k]$ and $\exists h'_{k+1} \in \mathcal{H}$ such that $h'_{k+1} \cdot x_k = \lambda'_{k+1} x_k$ for some $\lambda'_{k+1} \in \mathbb{K}^*$ which is not a root of unity, and $h'_{k+1} \cdot x_j = \lambda_{kj} x_j$, $\forall j \in [1, k-1] \cup \{k+1\}$.

The rest of the data for the second CGL extension presentation of R ($\mathbb{K}$-torus $\mathcal{H}'$, scalars $\lambda'_{kj} \in \mathbb{K}^*$, and elements $h'_k \in \mathcal{H}'$) is given by Lemma 4.1 and the following:

(ii) The torus $\mathcal{H}'$ acting on the second CGL extension presentation can be taken as the original universal maximal torus $\mathcal{H}$ from Theorem 3.8. The corresponding elements $h'_j \in \mathcal{H}$ are given by (i) for $j = k+1$ and $h'_j = h_j$ for $j \neq k+1$.

(iii) $\sigma'_{k+1} = (h'_{k+1} \cdot)|_{R'_k}$, where R'_k is the unital $\mathbb{K}$-subalgebra of R generated by $x_1, \ldots, x_{k-1}$, and x_{k+1}.

(iv) $\lambda'_{lj} = \lambda_{(k,k+1)(l),(k,k+1)(j)}$ for $l, j \in [1, N]$.

Here and below, $(k, k+1)$ denotes a transposition in the symmetric group S_N, and S_N is embedded in $GL_N(\mathbb{Z})$ via permutation matrices.

Assume that the condition (i) is satisfied and the base field $\mathbb{K}$ contains all square roots $\sqrt{\lambda_{lj}}$ for $j, l \in [1, N]$, recall Remark 4.5. Fix $\nu_{lj} \in \mathbb{K}^*$ for $1 \leq j < l \leq N$ such that $\nu_{lj}^2 = \lambda_{lj}$ and define a multiplicatively skew-symmetric matrix $\boldsymbol{\nu} := (\nu_{lj}) \in M_N(\mathbb{K}^*)$ by (4.15). Let $\boldsymbol{\nu}' := (\nu'_{lj}) \in M_N(\mathbb{K}^*)$ be the multiplicatively skew-symmetric matrix

$$\boldsymbol{\nu}' := (k, k+1)\boldsymbol{\nu}(k, k+1) \in M_N(\mathbb{K}^*). \tag{4.19}$$

By property (iv), $(\nu'_{lj})^2 = \lambda'_{lj}$, $\forall l, j \in [1, N]$. Denote by $\mathbf{r}' := (r'_{lj}) \in M_N(\mathbb{K}^*)$ the matrix obtained from $\boldsymbol{\nu}'$ by (4.16), where p is replaced by the predecessor function for $\eta' = \eta(k, k+1)$.

Proposition 4.6 implies that $((\overline{y}_1, \ldots, \overline{y}_N), \mathbf{r})$ and $((\overline{y}'_1, \ldots, \overline{y}'_N), \mathbf{r}')$ define two toric frames $M, M' : \mathbb{Z}^N \to \operatorname{Fract}(R)$. Their matrices equal $\mathbf{r}$ and $\mathbf{r}'$, respectively, and $M(e_j) = \overline{y}_j$, $M'(e_j) = \overline{y}'_j$, $\forall j \in [1, N]$. The next theorem describes the relationship between these two toric frames on the basis of Theorem 4.3. Let

$E_+^\circ \in M_N(\mathbb{Z})$ be the matrix with entries

$$(E_+^\circ)_{lj} = \begin{cases} \delta_{lj}, & \text{if } j \neq k \\ -1, & \text{if } l = j = k \\ 1, & \text{if } j = k, \text{ and } l = p(k) \text{ or } k+1, \\ 0, & \text{if } j = k, \text{ and } l \neq p(k),\ k, \text{ or } k+1, \end{cases}$$

cf. (2.18).

For the following theorems, we adopt the convention that

$$e_{-\infty} = \overline{e}_{-\infty} := 0. \tag{4.20}$$

THEOREM 4.8. *Assume the setting of Theorem* 4.2 *and that* $\sqrt{\lambda_{lj}} \in \mathbb{K}$ *for all* $j, l \in [1, N]$.

(a) *If* $\eta(k) \neq \eta(k+1)$, *then* $\mathbf{r}' = (k, k+1)\mathbf{r}(k, k+1)$ *and* $\overline{y}'_j = \overline{y}_{(k,k+1)j}$, *for all* $j \in [1, N]$, *i.e.,* $M' = M(k, k+1)$.

(b) *If* $\eta(k) = \eta(k+1)$, *then* $M'(e_j) = \overline{y}'_j = \overline{y}_j = M(e_j)$, $\forall j \neq k$ *and*

$$M'(e_k) = \overline{y}'_k = M(-e_k + e_{p(k)} + e_{k+1}) + \zeta M\Big(- e_k + \sum_{i \in P(k-1),\, i \neq p(k)} m_i e_i \Big) \tag{4.21}$$

for the collection of nonnegative integers $\{m_i \mid i \in P(k-1),\ i \neq p(k)\}$ *from Theorem* 4.3 *and some* $\zeta \in \mathbb{K}^*$. *Furthermore,*

$$\mathbf{r}' = {}^{E_+^{\circ T}}\mathbf{r}^{E_+^\circ}. \tag{4.22}$$

PROOF. Part (a) of the theorem and the equality $\overline{y}'_j = \overline{y}_j$, $\forall j \neq k$ in (b) follow from Theorem 4.2, once one verifies that

$$\mathcal{S}_{\boldsymbol{\nu}'}(\overline{e}_j) = \begin{cases} \lambda_{k,k+1}\mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_j), & \text{if } \ j = s^l(k+1) \ \text{ for some } \ l \in \mathbb{Z}_{\geq 0} \\ \mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_j), & \text{if } \ j < k, \text{ or } j > k+1 \text{ and } j \neq s^l(k+1)\ \forall l \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Next we verify (4.21). From (3.9) and the observation that $q_{kp(k)} = \alpha_{kp(k)}$, we have

$$\alpha_{kp(k)} y_k^{-1} y_{p(k)} y_{k+1} = y_{p(k)} y_k^{-1} y_{k+1}.$$

In view of (4.18), (2.6), and (2.7), Theorem 4.3 implies that (4.21) is equivalent to the identity

$$r_{p(k),k} r_{k,k+1} r_{p(k),k+1}^{-1} \mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_{p(k)}) \mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_k)^{-1} \mathcal{S}_{\boldsymbol{\nu}}(\overline{e}_{k+1}) = \mathcal{S}_{\boldsymbol{\nu}'}(\overline{e}_k) = \prod \{\nu_{ij}^{-1} \mid i, j = p^{O_-(k)}(k), \ldots, p(k), k+1;\ \ i < j\},$$

where $r_{-\infty,j} := 1$ and $\overline{e}_{-\infty} := 0$. This identity follows from (4.16).

To show (4.22), first note that Eq. (4.17) (applied to the presentation (4.2)) implies $\Omega_{\mathbf{r}'}(e_l, e_j) = \Omega_{\boldsymbol{\nu}}((k, k+1)\overline{e}_l, (k, k+1)\overline{e}_j)$ for all $l, j \in [1, N]$ (recall (4.19)), whence

$$\Omega_{\mathbf{r}'}(e_l, e_j) = \Omega_{\boldsymbol{\nu}}(\overline{e}_l, \overline{e}_j), \quad \forall l, j \neq k,$$
$$\Omega_{\mathbf{r}'}(e_k, e_j) = \Omega_{\boldsymbol{\nu}}(\overline{e}_{p(k)} + e_{k+1}, \overline{e}_j), \quad \forall j \neq k.$$

Since $\overline{e}_{p(k)} + \overline{e}_{k+1} - \overline{e}_k = \overline{e}_{p(k)} + e_{k+1}$, taking into account (4.17) applied to the presentation (4.1), we obtain

$$\Omega_{\mathbf{r}'}(e_l, e_j) = \Omega_{\mathbf{r}}(e_l, e_j), \quad \forall l, j \neq k,$$
$$\Omega_{\mathbf{r}'}(e_k, e_j) = \Omega_{\mathbf{r}}(e_{p(k)} + e_{k+1} - e_k, e_j), \quad \forall j \neq k.$$

These two equalities are equivalent to (4.22). □

4.4. Scalars associated to mutation of prime elements

Next, we derive formulas for certain scalars which play the role of the entries of the matrix $\widetilde{\mathbf{t}}$ from Section 2.3 for the quantum seeds which we construct in Chapter 8. Assume that there exist square roots $\nu_{lj} = \sqrt{\lambda_{lj}} \in \mathbb{K}$ for $1 \le j < l \le N$.

In the next theorem and Chapters 6 and 8, we will consider the following mild condition:

$$\text{The subgroup of } \mathbb{K}^* \text{ generated by } \{\nu_{lj} \mid 1 \le j < l \le N\} \text{ contains no elements of order 2.} \tag{4.23}$$

(This condition automatically holds if $\operatorname{char}\mathbb{K} = 2$; otherwise, it just means that the given subgroup of $\mathbb{K}^*$ does not contain -1.) For all torsionfree CGL extensions R, cf. Section 3.2, one can always choose the ν_{lj} in such a way that (4.23) is satisfied; in fact, it suffices to assume that there are no elements of order 2 in the subgroup of $\mathbb{K}^*$ generated by $\{\lambda_{lj} \mid 1 \le j < l \le N\}$. Condition (4.23) implies that the bicharacter $\Omega_{\boldsymbol{\nu}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^*$ associated to the multiplicatively skew-symmetric matrix $\boldsymbol{\nu}$ does not take the value -1 if $\operatorname{char}\mathbb{K} \ne 2$.

Recall from Section 3.2 that for an $\mathcal{H}$-eigenvector $u \in R$, $\chi_u \in X(\mathcal{H})$ denotes its eigenvalue.

THEOREM 4.9. *Assume the setting of Theorem* 4.3 *and that there exists* $\nu_{lj} = \sqrt{\lambda_{lj}} \in \mathbb{K}$ *for* $1 \le j < l \le N$. *Then*

$$\Omega_{\mathbf{r}}\Big(e_{p(k)} + e_{k+1} - \sum_{i \in P(k-1),\, i \ne p(k)} m_i e_i,\, e_j\Big)^2 = 1, \quad \forall j \ne k \tag{4.24}$$

and

$$\Omega_{\mathbf{r}}\Big(e_{p(k)} + e_{k+1} - \sum_{i \in P(k-1),\, i \ne p(k)} m_i e_i,\, e_k\Big)^2 = \chi_{x_{k+1}}(h_{k+1})^{-1} = \chi_{x_k}(h'_{k+1}) \tag{4.25}$$

for the collection of nonnegative integers $\{m_i \mid i \in P(k-1),\, i \ne p(k)\}$ *from Theorems* 4.3 *and* 4.8 (b). (*Recall also condition* (i) *in Section* 4.3.)

If in addition (4.23) *is satisfied, then*

$$\Omega_{\mathbf{r}}\Big(e_{p(k)} + e_{k+1} - \sum_{i \in P(k-1),\, i \ne p(k)} m_i e_i,\, e_j\Big) = 1, \quad \forall j \ne k. \tag{4.26}$$

PROOF. Denote for brevity the elements

$$g := \sum_{i \in P(k-1),\, i \ne p(k)} m_i e_i \quad \text{and} \quad \overline{g} := \sum_{i \in P(k-1),\, i \ne p(k)} m_i \overline{e}_i$$

in $\mathbb{Z}^N$. Using the fact that $\overline{y}_j$, $j \ne k$ belongs to the images of both toric frames M and M', we obtain

$$\begin{aligned} \xi_{kj}\overline{y}_j\overline{y}'_k = \overline{y}'_k\overline{y}_j &= \big(M(-e_k + e_{p(k)} + e_{k+1}) + \zeta M(-e_k + g)\big)\overline{y}_j \\ &= \overline{y}_j\big(\Omega_{\mathbf{r}}(-e_k + e_{p(k)} + e_{k+1}, e_j)^2 M(-e_k + e_{p(k)} + e_{k+1}) \\ &\qquad + \zeta\Omega_{\mathbf{r}}(-e_k + g, e_j)^2 M(-e_k + g)\big), \end{aligned}$$

for some $\xi_{kj} \in \mathbb{K}^*$. Hence,

$$\Omega_{\mathbf{r}}(-e_k + e_{p(k)} + e_{k+1}, e_j)^2 = \Omega_{\mathbf{r}}(-e_k + g, e_j)^2, \quad \forall j \neq k,$$

which implies (4.24) and (4.26). Applying (4.24) for $j = k+1$ and (4.17) leads to

$$\begin{aligned} \Omega_{\mathbf{r}}(e_{p(k)} + e_{k+1} - g, e_k)^2 &= \Omega_{\mathbf{r}}(e_{p(k)} + e_{k+1} - g, e_k)^2 \Omega_{\mathbf{r}}(e_{p(k)} + e_{k+1} - g, e_{k+1})^{-2} \\ &= \Omega_{\boldsymbol{\nu}}(\overline{e}_{p(k)} + \overline{e}_{k+1} - \overline{g}, \overline{e}_k)^2 \Omega_{\boldsymbol{\nu}}(\overline{e}_{p(k)} + \overline{e}_{k+1} - \overline{g}, \overline{e}_{k+1})^{-2} \\ &= \Omega_{\boldsymbol{\nu}}(\overline{e}_{p(k)} + \overline{e}_{k+1} - \overline{g}, e_{k+1})^{-2} \\ &= \Omega_{\boldsymbol{\nu}}(\overline{e}_{p(k)} + \overline{e}_{k} - \overline{g}, e_{k+1})^{-2}. \end{aligned}$$

Since $\overline{y}'_k$ is an $\mathcal{H}$-eigenvector, it follows from (4.21) that $\chi_{M(e_{p(k)}+e_{k+1}-g)} = 1$. Using that $\chi_{M(e_{p(k)}+e_{k+1}-g)} = \chi_{M(e_{p(k)}+e_k-g)} + \chi_{x_{k+1}}$, we obtain

$$\begin{aligned} \Omega_{\boldsymbol{\nu}}(\overline{e}_{p(k)} + \overline{e}_k - \overline{g}, e_{k+1})^{-2} &= \chi_{M(e_{p(k)}+e_k-g)}(h_{k+1}) \\ &= \chi_{M(e_{p(k)}+e_{k+1}-g)}(h_{k+1}) \chi_{x_{k+1}}(h_{k+1})^{-1} \\ &= \chi_{x_{k+1}}(h_{k+1})^{-1}, \end{aligned}$$

which proves the first equality in (4.25).

It follows from (4.24) that $M(e_{p(k)} + e_{k+1} - g)$ commutes with $M(g)$, and hence also with $M(e_{p(k)} + e_{k+1})$. By Theorem 4.8 (b), $\overline{y}_k \overline{y}'_k$ is a linear combination of $M(e_{p(k)} + e_{k+1})$ and $M(g)$, so $M(e_{p(k)} + e_{k+1} - g)$ also commutes with $\overline{y}_k \overline{y}'_k$. By the first equality in (4.25),

$$M(e_{p(k)} + e_{k+1} - g)\overline{y}_k = \chi_{x_{k+1}}(h_{k+1})^{-1} \overline{y}_k M(e_{p(k)} + e_{k+1} - g).$$

Interchanging the roles of x_k and x_{k+1} and using the symmetric nature of the assumption of Theorem 4.2 (b) shows that

$$M'(e_{p(k)} + e_{k+1} - g)\overline{y}'_k = \chi_{x_k}(h'_{k+1})^{-1} \overline{y}'_k M'(e_{p(k)} + e_{k+1} - g).$$

In view of Theorem 4.8 (b), the element $M'(e_{p(k)} + e_{k+1} - g)$ is a scalar multiple of $M(e_{p(k)} + e_{k+1} - g)$, so we conclude that

$$M(e_{p(k)} + e_{k+1} - g)\overline{y}'_k = \chi_{x_k}(h'_{k+1})^{-1} \overline{y}'_k M(e_{p(k)} + e_{k+1} - g).$$

Therefore, $\chi_{x_{k+1}}(h_{k+1})\chi_{x_k}(h'_{k+1}) = 1$, which proves the second equality in (4.24). $\square$

CHAPTER 5

Homogeneous prime elements for subalgebras of symmetric CGL extensions

Each symmetric CGL extension R of length N has many different CGL extension presentations given by (3.12). They are parametrized by the elements of the subset Ξ_N of S_N, cf. (3.11). In order to phrase Theorem 4.8 into a mutation statement between toric frames for $\mathrm{Fract}(R)$ associated to the elements of Ξ_N and to make the scalars ζ from Theorem 4.8 equal to one, we need a good picture of the sequences of homogeneous prime elements $y_1, \dots, y_N$ from Theorem 3.6 associated to each presentation (3.12). This is obtained in Theorem 5.3. Theorem 5.1 contains a description of the homogeneous prime elements that enter into this result. Those prime elements (up to rescaling) comprise the cluster variables that will be used in Chapter 8 to construct quantum cluster algebra structures on symmetric CGL extensions. Along the way, we explicitly describe the elements of Ξ_N and prove an invariance property of the scalars λ_l and λ_l^* from Definitions 3.3 and 3.12. Theorem 5.13, which appears at the end of the chapter, contains a key result used in the next chapter to normalize the generators x_j of symmetric CGL extensions so that all scalars ζ in Theorem 4.8 become equal to one.

Throughout the chapter, η will denote a function $[1, N] \to \mathbb{Z}$ satisfying the conditions of Theorem 3.6, with respect to the original CGL extension presentation (3.1) of R, and p and s will denote the corresponding predecessor and successor functions. We will repeatedly use the one-line notation (3.13) for permutations.

5.1. The elements $y_{[i,s^m(i)]}$

Recall from Section 3.3 that for a symmetric CGL extension R of rank N and $1 \le j \le k \le N$, $R_{[j,k]}$ denotes the unital subalgebra of R generated by $x_j, \dots, x_k$. It is an Ore extension of both $R_{[j,k-1]}$ and $R_{[j+1,k]}$. All such subalgebras are (symmetric) CGL extensions and Theorem 3.6 applies to them.

For $i \in [1, N]$ and $0 \le m \le O_+(i)$, recall (3.18) (i.e., $s^m(i) \in [1, N]$), set

$$e_{[i,s^m(i)]} = e_i + e_{s(i)} + \cdots + e_{s^m(i)} \in \mathbb{Z}^N. \tag{5.1}$$

The vectors (3.19) are special cases of those:

$$\overline{e}_k = e_{[p^{O_-(k)}(k),k]}, \quad \forall k \in [1, N].$$

We also set $e_\varnothing = 0$. The next theorem treats the prime elements that will appear as cluster variables for symmetric CGL extensions. It will be proved in Section 5.4.

THEOREM 5.1. *Assume that R is a symmetric CGL extension of length N, and $i \in [1, N]$ and $m \in \mathbb{Z}_{\ge 0}$ are such that $s^m(i) \in [1, N]$, i.e., $s^m(i) \ne +\infty$. Then the following hold:*

(a) *All homogeneous prime elements of* $R_{[i,s^m(i)]}$ *that do not lie in* $R_{[i,s^m(i)-1]}$ *are associates of each other.*

(b) *All homogeneous prime elements of* $R_{[i,s^m(i)]}$ *that do not lie in* $R_{[i+1,s^m(i)]}$ *are associates of each other. In addition, the set of these homogeneous prime elements coincides with the set of homogeneous prime elements in part* (a).

(c) *The homogeneous prime elements in* (a) *and* (b) *have leading terms of the form*

$$\xi x^{e_{[i,s^m(i)]}} = \xi x_i x_{s(i)} \dots x_{s^m(i)}$$

for some $\xi \in \mathbb{K}^*$, *see* §3.4. *For each* $\xi \in \mathbb{K}^*$, *there is a unique homogeneous prime element of* $R_{[i,s^m(i)]}$ *with such a leading term. Denote by* $y_{[i,s^m(i)]}$ *the prime element with leading term* $x_i x_{s(i)} \dots x_{s^m(i)}$. *Let* $y_{\varnothing} := 1$.

(d) *We have*

$$y_{[i,s^m(i)]} = y_{[i,s^{m-1}(i)]} x_{s^m(i)} - c_{[i,s^m(i)-1]} = x_i y_{[s(i),s^m(i)]} - c'_{[i+1,s^m(i)]}$$

for some $c_{[i,s^m(i)-1]} \in R_{[i,s^m(i)-1]}$ *and* $c'_{[i+1,s^m(i)]} \in R_{[i+1,s^m(i)]}$.

(e) *For all* $k \in [1, N]$ *such that* $p(i) < k < s^{m+1}(i)$, *we have*

$$y_{[i,s^m(i)]} x_k = \Omega_{\boldsymbol{\lambda}}(e_{[i,s^m(i)]}, e_k) x_k y_{[i,s^m(i)]}.$$

The case $m = 0$ of this theorem is easy to verify. In that case $y_{[i,i]} = x_i$, $\forall i \in [1, N]$.

Example 5.2. In the case of $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$, the elements $y_{[i,s^l(i)]}$ of Theorem 5.1 are solid quantum minors, just as in Example 3.7. More precisely, if $i \in [1, N]$ and $l \in \mathbb{Z}_{\geq 0}$ with $s^l(i) \neq +\infty$, then $i = (r-1)n + c$ with $r, r+l \in [1, m]$ and $c, c+l \in [1, n]$, and $y_{[i,s^l(i)]} = \Delta_{[r,r+l],[c,c+l]}$.

The following theorem describes the y-sequences from Theorem 3.6 associated to the CGL extension presentations (3.12) in terms of the prime elements from Theorem 5.1. It will be proved in Section 5.4. Recall that for every $\tau \in \Xi_N$ and $k \in [1, N]$, $\tau([1, k])$ is an interval.

Theorem 5.3. *Assume that* R *is a symmetric CGL extension of length* N *and* τ *an element of the subset* Ξ_N *of* S_N, *cf.* (3.11). *Let* $y_{\tau,1}, \dots, y_{\tau,N}$ *be the sequence in* R *from Theorem* 3.6 *applied to the CGL extension presentation* (3.12) *of* R *corresponding to* τ. *Let* $k \in [1, N]$.

If $\tau(k) \geq \tau(1)$, *then* $y_{\tau,k}$ *is a scalar multiple of* $y_{[p^m(\tau(k)),\tau(k)]}$, *where*

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\tau(k)) \in \tau([1, k])\}. \tag{5.2}$$

If $\tau(k) \leq \tau(1)$, *then* $y_{\tau,k}$ *is a scalar multiple of* $y_{[\tau(k),s^m(\tau(k))]}$, *where*

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid s^n(\tau(k)) \in \tau([1, k])\}. \tag{5.3}$$

In both cases, the predecessor and successor functions are with respect to the original CGL extension presentation (3.1) *of* R.

5.2. The elements of Ξ_N

In this and the next section, we investigate the elements of the subset Ξ_N of S_N defined in (3.11). It follows from (3.11) that every element $\tau \in \Xi_N$ has the property that either $\tau(N) = 1$ or $\tau(N) = N$. This implies the following recursive description of Ξ_N.

Lemma 5.4. *For each $\tau \in \Xi_N$, there exists $\tau' \in \Xi_{N-1}$ such that either*

$$\tau(i) = \tau'(i), \ \ \forall i \in [1, N-1] \quad \text{and} \quad \tau(N) = N$$

or

$$\tau(i) = \tau'(i) + 1, \ \ \forall i \in [1, N-1] \quad \text{and} \quad \tau(N) = 1.$$

For all $\tau' \in \Xi_{N-1}$, the above define elements of Ξ_N.

Given $k \in [1, N]$ and a sequence $k \le j_k \le \cdots \le j_1 \le N$, define

$$\tau_{(j_k, \ldots, j_1)} := (k(k+1) \ldots j_k) \ldots (23 \ldots j_2)(12 \ldots j_1) \in S_N, \tag{5.4}$$

where in the right hand side we use the standard notation for cycles in S_N. Lemma 5.4 implies by induction the following characterization of the elements of Ξ_N. We leave its proof to the reader.

Lemma 5.5. *The subset $\Xi_N \subset S_N$ consists of the elements of the form $\tau_{(j_k, \ldots, j_1)}$, where $k \in [1, N]$ and $k \le j_k \le \cdots \le j_1 \le N$.*

The representation of an element of Ξ_N in the form (5.4) is not unique. One way to visualize $\tau_{(j_k, \ldots, j_1)}$ is that the sequence $\tau(1), \ldots, \tau(N)$ is obtained from the sequence $1, \ldots, N$ by the following procedure:

($*$) *The number 1 is pulled to the right to position j_1 (preserving the order of the other numbers), then the number 2 is pulled to the right to position $j_2 - 1$, ..., at the end the number k is pulled to the right to position $j_k - k + 1$.*

For example, for $k = 2$ the following illustrates how $\tau_{(j_2, j_1)}$ is obtained from the identity permutation:

$$\begin{aligned} &[1, 2, 3, 4, \ldots, j_2, j_2+1, \ldots, j_1, j_1+1, \ldots, N] \mapsto \\ &[2, 3, 4, \ldots, j_2, j_2+1, \ldots, j_1, 1, j_1+1, \ldots, N] \mapsto \\ &[3, 4, \ldots, j_2, 2, j_2+1, \ldots, j_1, 1, j_1+1, \ldots, N], \end{aligned}$$

where the numbers that are pulled (1 and 2) are circled.

If we perform the above procedure one step at a time, so in each step we only interchange the positions of two adjacent numbers, then the elements of S_N from all intermediate steps will belong to Ξ_N. For example, this requires factoring the cycle $(1, 2, \ldots, j_1)$ as $(1, j_1)(1, j_1 - 1) \cdots (1, 2)$ rather than as $(1, 2)(2, 3) \cdots (j_1 - 1, j_1)$. This implies at once the first part of the next corollary.

Corollary 5.6. *Let R be a symmetric CGL extension of length N and $\tau \in \Xi_N$.*

(a) *There exists a sequence $\tau_0 = \mathrm{id}, \tau_1, \ldots, \tau_n = \tau$ in Ξ_N such that for all $l \in [1, n]$,*

$$\tau_l = (\tau_{l-1}(k_l), \tau_{l-1}(k_l + 1))\tau_{l-1} = \tau_{l-1}(k_l, k_l + 1)$$

for some $k_l \in [1, N-1]$ such that $\tau_{l-1}(k_l) < \tau_{l-1}(k_l + 1)$.

(b) *If $\eta : [1, N] \to \mathbb{Z}$ is a function satisfying the conditions of Theorem 3.6 for the original CGL presentation of R, then*

$$\eta_\tau := \eta\tau : [1, N] \to \mathbb{Z} \tag{5.5}$$

satisfies the conditions of Theorem 3.6 for the η-function of the CGL extension presentation (3.12) of R corresponding to τ.

The above described sequence for the first part of the corollary for the element $\tau_{(j_k,\ldots,j_1)} \in \Xi_N$ has length $j_1+\cdots+j_k-k(k+1)/2$. The second part of the corollary follows by recursively applying Theorem 4.2 to the CGL extension presentations (3.12) for the elements τ_{l-1} and τ_l. Corollary 5.6 (b) gives a second proof of the fact that the rank of a symmetric CGL extension does not depend on the choice of CGL extension presentation of the form (3.12), see Section 3.3.

Corollary 5.7. *Assume that R is a symmetric CGL extension of length N and $\eta : [1,N] \to \mathbb{Z}$ is a function satisfying the conditions of Theorem* 3.6*. Then for all $1 \le j \le k \le N$, the function $\eta_{[j,k]} \colon [1, k-j+1] \to \mathbb{Z}$ given by*

$$\eta_{[j,k]}(l) = \eta(j+l-1), \ \ \forall l \in [1, k-j+1]$$

satisfies the conditions of Theorem 3.6 *for the symmetric CGL extension $R_{[j,k]}$.*

The meaning of Corollary 5.7 is that the η-function for "interval subalgebras" of symmetric CGL extensions can be chosen to be the restriction of the original η-function up to a shift. This fact makes it possible to run induction on the rank of R in various situations. Corollary 5.7 follows by applying Corollary 5.6 (b) to

$$\tau = [j, \ldots, k, j-1, \ldots, 1, k+1, \ldots, N] \in \Xi_N$$

and considering the $(k-j+1)$-st subalgebra of the corresponding Ore extension, which equals $R_{[j,k]}$.

5.3. A subset of Ξ_N

We investigate a subset of Ξ_N which will play an important role in Chapter 8. Write elements τ of the symmetric group S_N in the form $[\tau(1), \tau(2), \ldots, \tau(N)]$. Recall (3.14) that $w_\circ = [N, N-1, \ldots, 1]$ denotes the longest element of S_N. For $1 \le i \le j \le N$, define the following elements of Ξ_N:

$$\tau_{i,j} := [i+1, \ldots, j, i, j+1, \ldots, N, i-1, i-2, \ldots, 1] \in \Xi_N. \tag{5.6}$$

They satisfy

$$\tau_{1,1} = \mathrm{id}, \quad \tau_{i,N} = \tau_{i+1,i+1}, \ \forall i \in [1, N-1], \quad \tau_{N,N} = w_\circ.$$

Denote by Γ_N the subset of Ξ_N consisting of all $\tau_{i,j}$'s and consider the following linear ordering on it

$$\begin{aligned} \Gamma_N := \{ \mathrm{id} = \tau_{1,1} \prec \ldots \prec \tau_{1,N} = \tau_{2,2} \prec \ldots \prec \tau_{2,N} = \tau_{3,3} \prec \ldots \prec \\ \prec \tau_{3,N} = \tau_{4,4} \prec \ldots \prec \tau_{N-2,N} = \tau_{N-1,N-1} \prec \tau_{N-1,N} = \tau_{N,N} = w_\circ \}. \end{aligned} \tag{5.7}$$

In the notation of (5.4), the elements of Γ_N are given by

$$\tau_{i,j} := \tau_{(j,N,\ldots,N)}, \text{ where } N \text{ is repeated } i-1 \text{ times}, \ \forall 1 \le i \le j \le N.$$

The sequence of elements (5.7) is nothing but the sequence of the intermediate steps of the procedure $(*)$ from Section 5.2 applied to the longest element $w_\circ \in S_N$ (in which case $k = N-1$ and $j_1 = j_2 = \cdots = j_{N-1} = N$).

Assume that R is a symmetric CGL extension. To each element of Γ_N, Proposition 3.14 associates a CGL extension presentation of R. Each two consecutive presentations are associated to a pair $\tau_{i,j}, \tau_{i,j+1} \in \Xi_N$ for some $1 \le i \le j < N$. They have the forms

$$R = \mathbb{K}[x_{i+1}] \dots [x_j; \sigma_j, \delta_j][x_i; \sigma_i^*, \delta_i^*][x_{j+1}; \sigma_{j+1}, \delta_{j+1}] \dots [x_N; \sigma_N, \delta_N][x_{i-1}; \sigma_{i-1}^*, \delta_{i-1}^*] \dots [x_1; \sigma_1^*, \delta_1^*] \tag{5.8}$$

and

$$R = \mathbb{K}[x_{i+1}] \dots [x_j; \sigma_j, \delta_j][x_{j+1}; \sigma_{j+1}, \delta_{j+1}][x_i; \sigma_i^*, \delta_i^*][x_{j+2}; \sigma_{j+2}, \delta_{j+2}] \dots [x_N; \sigma_N, \delta_N][x_{i-1}; \sigma_{i-1}^*, \delta_{i-1}^*] \dots [x_1; \sigma_1^*, \delta_1^*], \tag{5.9}$$

respectively. These two presentations satisfy the assumptions of Theorem 4.2. Recall from Definitions 3.3 and 3.12 that $\sigma_l = (h_l \cdot)$, $\sigma_l^* = (h_l^* \cdot)$ and $h_l \cdot x_l = \lambda_l x_l$, $h_l^* \cdot x_l = \lambda_l^* x_l$ with $h_l, h_l^* \in \mathcal{H}$, $\lambda_l, \lambda_l^* \in \mathbb{K}^*$. By Corollary 5.6(b), the η-function for the CGL extension presentation (5.8) can be taken to be $\eta\tau_{i,j}$. In particular, the values of this function on $j+1-i$ and $j+2-i$ are $\eta(i)$ and $\eta(j+1)$, respectively. If $\eta(i) = \eta(j+1)$, then Eq. (4.25) of Theorem 4.9 applied to the presentations (5.8) and (5.9) implies

$$\lambda_i^* \lambda_{j+1} = \chi_{x_i}(h_i^*) \chi_{x_{j+1}}(h_{j+1}) = 1. \tag{5.10}$$

PROPOSITION 5.8. *Let R be a symmetric CGL extension of length N and $a \in \mathbb{Z}$ be such that $|\eta^{-1}(a)| > 1$. Denote*

$$\eta^{-1}(a) = \{l, s(l), \dots, s^m(l)\}$$

where $l \in [1, N]$ and $m = O_+(l) \in \mathbb{Z}_{>0}$. Then

$$\lambda_l^* = \lambda_{s(l)}^* = \dots = \lambda_{s^{m-1}(l)}^* = \lambda_{s(l)}^{-1} = \lambda_{s^2(l)}^{-1} = \dots = \lambda_{s^m(l)}^{-1}. \tag{5.11}$$

PROOF. It follows from (5.10) that

$$\lambda_{s^{m_1}(l)}^* = \lambda_{s^{m_2}(l)}^{-1}, \quad \forall 0 \le m_1 < m_2 \le m$$

which is equivalent to the statement of the proposition. $\square$

5.4. Sequences of homogeneous prime elements

We proceed with the proofs of the two theorems formulated in Section 5.1.

PROOF OF THEOREM 5.1. As already noted, the case $m = 0$ is easily verified. Assume now that $m > 0$.

(a) Consider the CGL extension presentation of R associated to

$$\tau_{(s^m(i)-1, s^m(i), \dots, s^m(i))} = [i+1, \dots, s^m(i)-1, i, s^m(i), i-1, \dots, 1, s^m(i)+1, \dots, N] \in \Xi_N, \tag{5.12}$$

where $s^m(i)$ is repeated $i-1$ times. The $(s^m(i)-i)$-th and $(s^m(i)-i+1)$-st algebras in the chain are precisely $R_{[i,s^m(i)-1]}$ and $R_{[i,s^m(i)]}$. Theorem 3.6 implies that the homogeneous prime elements of $R_{[i,s^m(i)]}$ that do not belong to $R_{[i,s^m(i)-1]}$ are associates of each other. Denote by $z_{[i,s^m(i)]}$ one such element. Set $z_\varnothing = 1$. (One can prove part (a) using the simpler presentation of R associated to the permutation

$$[i, i+1, \dots, s^m(i), i-1, \dots, 1, s^m(i)+1, \dots, N] \in \Xi_N$$

but the first presentation will also play a role in the proof of part (b).) The equivariance property of the η-function from Corollary 5.6 (b) and Theorem 3.6 imply that

$$z_{[i,s^m(i)]} = \xi_{i;m}(z_{[i,s^{m-1}(i)]}x_{s^m(i)} - c_{[i,s^m(i)-1]}) \tag{5.13}$$

for some $c_{[i,s^m(i)-1]} \in R_{[i,s^m(i)-1]}$ and $\xi_{i;m} \in \mathbb{K}^*$.

(b) Now consider the CGL extension presentation of R associated to

$$\tau_{(s^m(i),\dots,s^m(i))} = [i+1,\dots,s^m(i), i,\dots,1, s^m(i)+1,\dots,N] \in \Xi_N, \tag{5.14}$$

where $s^m(i)$ is repeated i times. The $(s^m(i)-i)$-th and $(s^m(i)-i+1)$-st algebras in the chain are $R_{[i+1,s^m(i)]}$ and $R_{[i,s^m(i)]}$. Theorem 3.6 implies that the homogeneous prime elements of $R_{[i,s^m(i)]}$ that do not belong to $R_{[i+1,s^m(i)]}$ are associates of each other. They are associates of $z_{[i,s^m(i)]}$. This follows from Theorem 4.2 (b) applied to the CGL extension presentations of R associated to the elements (5.12) and (5.14). The fact that these two CGL extension presentations satisfy the assumptions of Theorem 4.2 (b) follows from the equivariance of the η-function from Corollary 5.6 (b). This equivariance and Theorem 3.6 applied to the presentation for (5.14) also imply that

$$z_{[i,s^m(i)]} = \xi'_{i;m}(x_i z_{[s(i),s^m(i)]} - c'_{[i+1,s^m(i)]}) \tag{5.15}$$

for some $c'_{[i+1,s^m(i)]} \in R_{[i+1,s^m(i)]}$ and $\xi'_{i;m} \in \mathbb{K}^*$.

The rest of part (b) and parts (c)–(d) follow at once by comparing the leading terms in Eqs. (5.13) and (5.15), and the fact that the group of units of an Ore extension is reduced to scalars.

For part (e), we apply (3.24) in $R_{[\min\{k,i\},\max\{k,s^m(i)\}]}$ with $j = s^m(i)$. $\square$

PROOF OF THEOREM 5.3. For $k \in [0,N]$, denote by $R_{\tau,k}$ the k-th algebra in the chain (3.12). We consider the case when $\tau(k) \geq \tau(1)$, leaving the analogous case $\tau(k) \leq \tau(1)$ to the reader. Since $\tau([1,j])$ is an interval for all $j \leq k$,

$$\tau([1,k]) = [\tau(i), \tau(k)] \quad \text{for some } \ i \in [1,k].$$

Therefore $R_{\tau,k} = R_{[\tau(i),\tau(k)]}$ and $R_{\tau,k-1} = R_{[\tau(i),\tau(k)-1]}$. For $m \in \mathbb{Z}_{\geq 0}$ given by (5.2) we have

$$\tau(i) \leq p^m(\tau(k)) \leq \tau(k).$$

Theorem 3.6 implies that $y_{[p^m(\tau(k)),\tau(k)]}$ is a homogeneous prime element of the algebra $R_{[\tau(i),\tau(k)]} = R_{\tau,k}$. It does not belong to $R_{\tau,k-1} = R_{[\tau(i),\tau(k)-1]}$ because of Theorem 5.1 (c). It follows from Theorem 3.6 that the homogeneous prime elements of $R_{\tau,k}$ that do not belong to $R_{\tau,k-1}$ are associates of $y_{\tau,k}$. Hence, $y_{\tau,k}$ is a scalar multiple of $y_{[p^m(\tau(k)),\tau(k)]}$. $\square$

COROLLARY 5.9. *If R is a symmetric CGL extension of length N and $i, j \in [1,N]$, $m, n \in \mathbb{Z}_{\geq 0}$ are such that*

$$i \leq j \leq s^n(j) \leq s^m(i) \leq N,$$

then

$$y_{[i,s^m(i)]}y_{[j,s^n(j)]} = \Omega_{\boldsymbol{\lambda}}(e_{[i,s^m(i)]}, e_{[j,s^n(j)]})y_{[j,s^n(j)]}y_{[i,s^m(i)]},$$

recall (2.1).

The corollary follows by applying Theorem 5.3 to the CGL extension presentation of R associated to

$$\tau' := [j, \dots, s^m(i), j-1, \dots, 1, s^m(i)+1, \dots, N] \in \Xi_N$$

and then using (3.9), (3.23). For this permutation, Theorem 5.3 implies that $y_{\tau', s^n(j)-j+1}$ and $y_{\tau', s^m(i)-i+1}$ are scalar multiples of $y_{[j,s^n(j)]}$ and $y_{[i,s^m(i)]}$, respectively.

5.5. An identity for normal elements

For $1 \leq j \leq l \leq N$, set

$$O^j_-(l) = \max\{m \in \mathbb{Z}_{\geq 0} \mid p^m(l) \geq j\}. \tag{5.16}$$

The following fact follows directly from Theorems 3.6 and 5.3.

COROLLARY 5.10. *Let R be a symmetric CGL extension of length N and $1 \leq j \leq k \leq N$. Then*

$$\Big\{ y_{[p^{O^j_-(i)}(i), i]} \;\Big|\; i \in [j,k],\ s(i) > k \Big\}$$

is a list of the homogeneous prime elements of $R_{[j,k]}$ up to scalar multiples.

Applying Theorems 4.2, 4.3 to the CGL extension presentations (3.12) of a symmetric CGL extension of length N associated to the elements

$$\tau_{i,s^m(i)-1} = [i+1, \dots, s^m(i)-1, i, s^m(i), \dots, N, i-1, \dots, 1] \in \Xi_N$$

and

$$\tau_{i,s^m(i)} = [i+1, \dots, s^m(i)-1, s^m(i), i, s^m(i)+1, \dots, N, i-1, \dots, 1] \in \Xi_N$$

for $i \in [1,N]$ and $m \in \mathbb{Z}_{>0}$ such that $s^m(i) \in [1,N]$ and using Theorem 5.3 yields the following result.

COROLLARY 5.11. *Assume that R is a symmetric CGL extension of length N, and $i \in [1,N]$ and $m \in \mathbb{Z}_{>0}$ are such that $s^m(i) \in [1,N]$. Then*

$$u_{[i,s^m(i)]} := y_{[i,s^{m-1}(i)]} y_{[s(i),s^m(i)]} - \Omega_{\boldsymbol{\lambda}}(e_i, e_{[s(i),s^{m-1}(i)]}) y_{[s(i),s^{m-1}(i)]} y_{[i,s^m(i)]} \tag{5.17}$$

$$= y_{[i,s^{m-1}(i)]} y_{[s(i),s^m(i)]} - \Omega_{\boldsymbol{\lambda}}(e_{s^m(i)}, e_{[s(i),s^{m-1}(i)]})^{-1} y_{[i,s^m(i)]} y_{[s(i),s^{m-1}(i)]} \tag{5.18}$$

is a nonzero homogeneous normal element of $R_{[i+1,s^m(i)-1]}$ which is not a multiple of $y_{[s(i),s^{m-1}(i)]}$ if $m \geq 2$, recall (2.1)*. It normalizes the elements of $R_{[i+1,s^m(i)-1]}$ in exactly the same way as $y_{[s(i),s^{m-1}(i)]} y_{[i,s^m(i)]}$ does. Moreover,*

$$u_{[i,s^m(i)]} = \psi \prod_{k \in P} y^{m_k}_{[p^{O^{i+1}_-(k)}(k), k]} \tag{5.19}$$

where $\psi \in \mathbb{K}^*$,

$$P = P_{[i,s^m(i)]} := \{k \in [i, s^m(i)] \setminus \{i, s(i), \dots, s^m(i)\} \mid s(k) > s^m(i)\}, \tag{5.20}$$

and the integers m_k are those from Theorems 4.3, 4.8*. Consequently, the leading term of $u_{[i,s^m(i)]}$ has the form*

$$\xi x^{n_{i+1}e_{i+1} + \cdots + n_{s^m(i)-1} e_{s^m(i)-1}}$$

for some $\xi \in \mathbb{K}^*$ and $n_{i+1}, \dots, n_{s^m(i)-1} \in \mathbb{Z}_{\geq 0}$ such that $n_{s(i)} = \ldots = n_{s^{m-1}(i)} = 0$ and $n_j = n_l$ for all $i+1 \leq j \leq l \leq s^m(i)-1$ with $\eta(j) = \eta(l)$.

The equality (5.18) follows from Corollary 5.9. The scalar in the right hand side of Eq. (5.17) is the one that cancels out the leading terms of the two products, cf. Theorem 5.1 (c). The last statement of Corollary 5.11 follows from Eqs. (5.19) and (3.20).

Since $y_{\varnothing} = 1$ and $y_{[i,i]} = x_i$, the $m = 1$ case of the corollary states that

$$u_{[i,s(i)]} := x_i x_{s(i)} - y_{[i,s(i)]} = c_{[i,s(i)-1]} = c'_{[i+1,s(i)]} \tag{5.21}$$

is a nonzero normal element of $R_{[i+1,s(i)-1]}$, cf. Theorem 5.1 (d). We set

$$u_{[i,i]} := 1.$$

EXAMPLE 5.12. In the case of $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$, Example 3.7 shows how to express each $u_{[i,s^l(i)]}$ as a difference of products of solid quantum minors. In particular, if $i = (r-1)n + c$ with $s(i) \neq +\infty$, then $r < m$, $c < n$, and

$$\begin{aligned} u_{[i,s(i)]} &:= x_i x_{s(i)} - y_{[i,s(i)]} = t_{rc} t_{r+1,c+1} - \Delta_{[r,r+1],[c,c+1]} \\ &= q t_{r,c+1} t_{r+1,c} = q x_{i+1} x_{i+n}. \end{aligned} \tag{5.22}$$

More generally, if $i = (r-1)n + c$, $l \in \mathbb{Z}_{\geq 0}$, and $s^l(i) \neq +\infty$, then

$$u_{[i,s^l(i)]} = q\Delta_{[r,r+l-1],[c+1,c+l]} \Delta_{[r+1,r+l],[c,c+l-1]} = q y_{[i+1,s^{l-1}(i+1)]} y_{[i+n,s^{l-1}(i+n]}.$$

This follows from an identity for quantum minors:

$$\begin{aligned} \Delta_{[r,r+l-1],[c,c+l-1]} \Delta_{[r+1,r+l],[c+1,c+l]} - q\Delta_{[r,r+l-1],[c+1,c+l]} \Delta_{[r+1,r+l],[c,c+l-1]} \\ = \Delta_{[r,r+l],[c,c+l]} \Delta_{[r+1,r+l-1],[c+1,c+l-1]}, \end{aligned}$$

for all $r \in [1, m-l]$, $c \in [1, n-l]$, $l \in [2, \min(m,n)]$. This identity can be obtained from the equation

$$t_{rc} t_{r+l,c+l} - q t_{r,c+l} t_{r+l,c} = \Delta_{\{r,r+l\},\{c,c+l\}}$$

by applying the antipode in a copy of $\mathcal{O}_q(GL_{l+1}(\mathbb{K}))$. (See [**32**, Lemma 4.1] for how the antipode applies to quantum minors.)

The next result, which is the main result in this section, describes the relationship between the normal elements in Corollary 5.11. It will play a key role in normalizing the scalars ζ in Theorem 4.8.

THEOREM 5.13. *Let R be a symmetric CGL extension of length N. For all $i \in [1, N]$ and $m \in \mathbb{Z}_{>0}$ such that $s^{m+1}(i) \in [1, N]$,*

$$\operatorname{lt}(u_{[s(i),s^m(i)]} u_{[i,s^{m+1}(i)]}) = \theta_m \operatorname{lt}(u_{[i,s^m(i)]} u_{[s(i),s^{m+1}(i)]}) \tag{5.23}$$

where $\theta_m = (\lambda_i^*)^{-1} \Omega_{\boldsymbol{\lambda}}(e_{[s^2(i),s^m(i)]}, 2e_i + 2e_{s(i)}) \Omega_{\boldsymbol{\lambda}}(e_{s^m(i)}, e_i)^{-1} \Omega_{\boldsymbol{\lambda}}(e_{s(i)}, e_i)$, recall Section 3.4 for the definition of leading terms $\operatorname{lt}(-)$.

For $m = 1$, Eq. (5.23) simplifies to

$$\operatorname{lt}(u_{[i,s^2(i)]}) = (\lambda_i^*)^{-1} \operatorname{lt}(u_{[i,s(i)]} u_{[s(i),s^2(i)]}). \tag{5.24}$$

PROOF. For a subspace W of a $\mathbb{K}$-vector space V and $v_1, v_2 \in V$, we will write

$$v_1 \equiv v_2 \mod W \quad \text{iff} \quad v_1 - v_2 \in W.$$

Denote

$$W_m := \big(\oplus_{n_1,n_2 \in \mathbb{Z}_{\geq 0}} x_i^{n_1} y_{[s(i),s^m(i)]} R_{[i+1,s^{m+1}(i)-1]} x_{s^{m+1}(i)}^{n_2} \big) \subset R_{[i,s^{m+1}(i)]}.$$

By [**24**, Proposition 4.7(b)] applied to $R_{[i+1,s^{m+1}(i)]}$,

$$\delta_{s^{m+1}(i)}(y_{[s(i),s^m(i)]}) = \Omega_\nu(e_{s^{m+1}(i)}, e_{[s(i),s^m(i)]})(\lambda_{s^{m+1}(i)} - 1)c_{[s(i),s^{m+1}(i)-1]},$$

cf. also Theorem 5.1 (d). Thus,

$$\begin{aligned} y_{[i,s^{m+1}(i)]}y_{[s(i),s^m(i)]} &\equiv y_{[i,s^m(i)]}x_{s^{m+1}(i)}y_{[s(i),s^m(i)]} \equiv y_{[i,s^m(i)]}\delta_{s^{m+1}(i)}(y_{[s(i),s^m(i)]}) \\ &= \Omega_\nu(e_{s^{m+1}(i)}, e_{[s(i),s^m(i)]})(\lambda_{s^{m+1}(i)} - 1)y_{[i,s^m(i)]}c_{[s(i),s^{m+1}(i)-1]} \\ &\equiv \Omega_\nu(e_{s^{m+1}(i)}, e_{[s(i),s^m(i)]})(1 - \lambda_{s^{m+1}(i)})y_{[i,s^m(i)]}y_{[s(i),s^{m+1}(i)]} \qquad \text{mod } W_m. \end{aligned}$$

It follows from Corollary 5.11 and Proposition 5.8 that

$$\begin{aligned} u_{[i,s^{m+1}(i)]} &= y_{[i,s^m(i)]}y_{[s(i),s^{m+1}(i)]} - \\ &\qquad \Omega_{\boldsymbol\lambda}(e_{s^{m+1}(i)}, e_{[s(i),s^m(i)]})^{-1}y_{[i,s^{m+1}(i)]}y_{[s(i),s^m(i)]} \\ &\equiv (\lambda_i^*)^{-1}y_{[i,s^m(i)]}y_{[s(i),s^{m+1}(i)]} \qquad \text{mod } W_m. \end{aligned} \tag{5.25}$$

First, we consider the case $m > 1$. Using Eq. (5.25), Corollary 5.9 and the fact that $y_{[s(i),s^m(i)]}$ is a prime element of $R_{[i+1,s^{m+1}(i)-1]}$, which follows from Corollary 5.10, we obtain

$$\begin{aligned} &u_{[s(i),s^m(i)]}u_{[i,s^{m+1}(i)]} \\ &\quad\equiv \big(y_{[s(i),s^{m-1}(i)]}y_{[s^2(i),s^m(i)]} - \Omega_{\boldsymbol\lambda}(e_{s(i)}, e_{[s^2(i),s^{m-1}(i)]})y_{[s^2(i),s^{m-1}(i)]}y_{[s(i),s^m(i)]}\big) \\ &\qquad\quad \times (\lambda_i^*)^{-1}y_{[i,s^m(i)]}y_{[s(i),s^{m+1}(i)]} \\ &\quad\equiv (\lambda_i^*)^{-1}y_{[s(i),s^{m-1}(i)]}y_{[s^2(i),s^m(i)]}y_{[i,s^m(i)]}y_{[s(i),s^{m+1}(i)]} \\ &\quad= (\lambda_i^*)^{-1}\Omega_{\boldsymbol\lambda}(e_{[s^2(i),s^m(i)]}, e_{[i,s^m(i)]})y_{[s(i),s^{m-1}(i)]}y_{[i,s^m(i)]}y_{[s^2(i),s^m(i)]}y_{[s(i),s^{m+1}(i)]} \\ &\quad\equiv \theta_m u_{[i,s^m(i)]}u_{[s(i),s^{m+1}(i)]} \qquad \text{mod } W_m. \end{aligned}$$

Since the first and last products above belong to $R_{[i+1,s^{m+1}(i)-1]}$,

$$u_{[s(i),s^m(i)]}u_{[i,s^{m+1}(i)]} = \theta_m u_{[i,s^m(i)]}u_{[s(i),s^{m+1}(i)]} + y_{[s(i),s^m(i)]}r \tag{5.26}$$

for some $r \in R_{[i+1,s^{m+1}(i)-1]}$. The leading term of $y_{[s(i),s^m(i)]}r$ has the form ξx^f where $\xi \in \mathbb{K}^*$ and $f = n_{i+1}e_{i+1} + \cdots + n_{s^{m+1}(i)-1}e_{s^{m+1}(i)-1}$ are such that $n_{s(i)}, \ldots, n_{s^m(i)} \geq 1$. Combining this with the last statement of Corollary 5.11 implies that the exponent of the leading term of the third product in (5.26) is different from the exponents of the leading terms of the first two products. Now the the validity of (5.23) for $m > 1$ follows from (5.26).

To verify (5.24), we use Eq. (5.25) and the fact that $x_{s(i)}$ is a prime element of $R_{[i+1,s^2(i)-1]}$ to obtain

$$u_{[i,s^2(i)]} \equiv (\lambda_i^*)^{-1}y_{[i,s(i)]}y_{[s(i),s^2(i)]} \equiv (\lambda_i^*)^{-1}u_{[i,s(i)]}u_{[s(i),s^2(i)]} \qquad \text{mod } W_1.$$

Hence,

$$u_{[i,s^2(i)]} = (\lambda_i^*)^{-1}u_{[i,s(i)]}u_{[s(i),s^2(i)]} + x_{s(i)}r$$

for some $r \in R_{[i+1,s^2(i)-1]}$. Now (5.24) can be deduced analogously to the case of $m > 1$. □

CHAPTER 6

Chains of mutations in symmetric CGL extensions

For a given CGL extension R one has the freedom of rescaling the generators x_j by elements of the base field $\mathbb{K}$. The prime elements y_j and $\overline{y}_j$ from Theorem 3.6 and Eq. (4.18) obviously depend (again up to rescaling) on the choice of x_j. In this chapter we prove that for each symmetric CGL extension R its generators can be rescaled in such a way that all scalars ζ in Theorem 4.8 become equal to one. This implies a mutation theorem, proved in Chapter 8, for toric frames for $\mathrm{Fract}(R)$ associated to the elements of Ξ_N via the sequences of prime elements from Theorem 5.3.

6.1. The leading coefficients of $u_{[i,s^m(i)]}$

Throughout this chapter we will assume that R is a symmetric CGL extension of length N and that there exist square roots $\nu_{lj} = \sqrt{\lambda_{lj}} \in \mathbb{K}$ such that the condition (4.23) is satisfied. Recall (5.17). For $i \in [1,N]$ and $m \in \mathbb{Z}_{\geq 0}$ such that $s^m(i) \in [1,N]$, let

$$\pi_{[i,s^m(i)]} \in \mathbb{K}^* \quad \text{and} \quad f_{[i,s^m(i)]} \in \sum_{j=i+1}^{s^m(i)-1} \mathbb{Z}_{\geq 0}\, e_j \subset \mathbb{Z}^N$$

be given by

$$\mathrm{lt}(u_{[i,s^m(i)]}) = \pi_{[i,s^m(i)]} x^{f_{[i,s^m(i)]}}. \tag{6.1}$$

Note that $\pi_{[i,i]} = 1$ and $f_{[i,i]} = 0$, because $u_{[i,i]} = 1$.

A quick induction using (5.23) shows that

$$f_{[i,s^m(i)]} = f_{[i,s(i)]} + \cdots + f_{[s^{m-1}(i),s^m(i)]} \tag{6.2}$$

for $m > 0$, and thus

$$x^{f_{[i,s^m(i)]}} = x^{f_{[i,s(i)]}} x^{f_{[s(i),s^m(i)]}} = x^{f_{[i,s^{m-1}(i)]}} x^{f_{[s^{m-1}(i),s^m(i)]}}. \tag{6.3}$$

Since all terms in (5.17) are homogeneous of the same $X(\mathcal{H})$-degree,

(6.4)
$$\begin{aligned} X(\mathcal{H})\text{-deg}(x^{f_{[i,s^m(i)]}}) &= X(\mathcal{H})\text{-deg}(u_{[i,s^m(i)]}) = X(\mathcal{H})\text{-deg}(y_{[s(i),s^{m-1}(i)]} y_{[i,s^m(i)]}) \\ &= X(\mathcal{H})\text{-deg}(x^{e_i + 2e_{[s(i),s^{m-1}(i)]} + e_{s^m(i)}}) \end{aligned}$$

for $m > 0$. The next result describes recursively the leading coefficients $\pi_{[i,s^m(i)]}$. Recall (2.7).

PROPOSITION 6.1. *Let R be a symmetric CGL extension for which there exist $\nu_{kj} = \sqrt{\lambda_{kj}} \in \mathbb{K}^*$, $0 \leq j < k \leq N$ satisfying condition (4.23). Then the following hold:*

(a) *The scalars* $\pi_{[i,s^m(i)]}$ *satisfy the recursive relation*

$$\pi_{[s(i),s^m(i)]}\pi_{[i,s^{m+1}(i)]} = (\lambda_i^*)^{-\delta_{m,1}}\Omega_{\boldsymbol{\lambda}}(e_{s^m(i)}, e_{s(i)})\pi_{[i,s^m(i)]}\pi_{[s(i),s^{m+1}(i)]}$$

for all $i \in [1,N]$ *and* $m \in \mathbb{Z}_{>0}$ *such that* $s^{m+1}(i) \in [1,N]$.

(b) *If*

$$\pi_{[i,s(i)]} = \mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s(i)]})$$

for all $i \in [1,N]$ *such that* $s(i) \neq \infty$, *then*

$$\pi_{[i,s^m(i)]} = \mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})^{-2}\mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s^m(i)]})$$

for $i \in [1,N]$ *and* $m \in \mathbb{Z}_{\geq 0}$ *with* $s^m(i) \in [1,N]$.

Before we proceed with the proof of Proposition 6.1 we derive several important properties of the form $\Omega_{\boldsymbol{\lambda}}$.

LEMMA 6.2. *Let* R *be a symmetric CGL extension of rank* N *and* $i \in [1,N]$, $m \in \mathbb{Z}_{>0}$ *be such that* $s^m(i) \in [1,N]$. *Then the following hold:*

(a) *For all* $0 \leq k < n \leq m$,

$$\Omega_{\boldsymbol{\lambda}}(e_{[i,s^m(i)]}, f_{[s^k(i),s^n(i)]}) = \Omega_{\boldsymbol{\lambda}}(e_{[i,s^m(i)]}, e_{s^k(i)} + 2e_{[s^{k+1}(i),s^{n-1}(i)]} + e_{s^n(i)}).$$

(b) *If* $j \in [i,s^m(i)]\backslash\{i, s(i), \ldots, s^m(i)\}$ *and* $k \in \mathbb{Z}_{\geq 0}$ *are such that* $p(j) < i$ *and* $s^{k+1}(j) > s^m(i) > s^k(j)$, *then*

$$\Omega_{\boldsymbol{\lambda}}(f_{[i,s^m(i)]}, e_{[j,s^k(j)]}) = \Omega_{\boldsymbol{\lambda}}(e_i + 2e_{[s(i),s^{m-1}(i)]} + e_{s^m(i)}, e_{[j,s^k(j)]}).$$

(c) *For all* $0 \leq k < n \leq m$ *and* $j \in \{i, s(i), \ldots, s^m(i)\} \cup [1, s^k(i)] \cup [s^n(i), N]$,

$$\Omega_{\boldsymbol{\lambda}}(e_j, f_{[s^k(i),s^n(i)]}) = (\lambda_i^*)^{\delta_{j,s^k(i)}-\delta_{j,s^n(i)}}\Omega_{\boldsymbol{\lambda}}(e_j, e_{s^k(i)} + 2e_{[s^{k+1}(i),s^{n-1}(i)]} + e_{s^n(i)}).$$

(d) *For all* $0 \leq k < n \leq m$,

$$\begin{aligned}&\Omega_{\boldsymbol{\lambda}}(f_{[i,s^m(i)]}, f_{[s^k(i),s^n(i)]})\\ &= (\lambda_i^*)^{-\delta_{k,0}+\delta_{n,m}}\Omega_{\boldsymbol{\lambda}}(e_i + 2e_{[s(i),s^{m-1}(i)]} + e_{s^m(i)}, e_{s^k(i)} + 2e_{[s^{k+1}(i),s^{n-1}(i)]} + e_{s^n(i)}).\end{aligned}$$

PROOF. (a) By [**24**, Theorem 5.3] $y_{[i,s^m(i)]}$ is an $\mathcal{H}$-normal element of $R_{[i,s^m(i)]}$ when $\mathcal{H}$ is replaced with the universal maximal torus from Theorem 3.8. This means that there exists $h \in \mathcal{H}$ such that $y_{[i,s^m(i)]}u = (h.u)y_{[i,s^m(i)]}$ for all $u \in R_{[i,s^m(i)]}$. In particular, $y_{[i,s^m(i)]}u = \chi_u(h)uy_{[i,s^m(i)]}$ when u is homogeneous. Taking account of Theorem 5.1(e) and the result of (6.4) applied to the interval $[s^k(i), s^n(i)]$, we obtain the statement of (a).

Part (b) is obtained just as in (a), since under the given conditions, $y_{[j,s^k(j)]}$ is an $\mathcal{H}_{\max}$-normal element of $R_{[i,s^m(i)]}$.

(c) In view of (6.2), it suffices to prove that

$$\Omega_{\boldsymbol{\lambda}}(e_j, f_{[s^{l-1}(i),s^l(i)]}) = (\lambda_i^*)^{\delta_{j,s^{l-1}(i)}-\delta_{j,s^l(i)}}\Omega_{\boldsymbol{\lambda}}(e_j, e_{s^{l-1}(i)} + e_{s^l(i)}) \tag{6.5}$$

for all $l \in [1,m]$ and $j \in [1, s^{l-1}(i)] \cup [s^l(i), N]$. Fix such l and j, let χ denote the $\mathcal{H}$-eigenvalue of $x^{f_{[s^{l-1}(i),s^l(i)]}}$, and recall from (6.4) that χ equals the $\mathcal{H}$-eigenvalue of $x_{s^{l-1}(i)}x_{s^l(i)}$.

Assume first that $j \geq s^l(i)$, and recall that

$$\chi_{x_{i'}}(h_j) = \begin{cases}\lambda_{ji'} & (1 \leq i' < j)\\ \lambda_j & (i' = j).\end{cases}$$

Since $x^{f_{[s^{l-1}(i),s^l(i)]}} \in R_{[1,j-1]}$, it follows that

$$\chi(h_j) = \Omega_{\boldsymbol{\lambda}}(e_j, f_{[s^{l-1}(i),s^l(i)]}). \tag{6.6}$$

Since χ is also the $\mathcal{H}$-eigenvalue of $x_{s^{l-1}(i)}x_{s^l(i)}$, we have

$$\chi(h_j) = \begin{cases} \Omega_{\boldsymbol{\lambda}}(e_j, e_{s^{l-1}(i)} + e_{s^l(i)}) & (j > s^l(i)) \\ \Omega_{\boldsymbol{\lambda}}(e_j, e_{s^{l-1}(i)})\lambda_{s^l(i)} & (j = s^l(i)) \end{cases} \tag{6.7}$$

as well. In the second case of (6.7), we invoke (5.11) and the skewsymmetry of $\Omega_{\boldsymbol{\lambda}}$ to obtain

$$\chi(h_j) = (\lambda_i^*)^{-1}\Omega_{\boldsymbol{\lambda}}(e_j, e_{s^{l-1}(i)} + e_{s^l(i)}) \qquad (j = s^l(i)). \tag{6.8}$$

Combining (6.6)–(6.8) yields (6.5) for $j \in [s^l(i), N]$.

We obtain (6.5) for $j \in [1, s^{l-1}(i)]$ in the same manner, by working with $\chi(h_j^*)$.

(d) The case $k = 0$, $n = m$ is obvious since $\Omega_{\boldsymbol{\lambda}}$ is multiplicatively skew-symmetric.

In general, since $\Omega_{\boldsymbol{\lambda}}(f_{[s^k(i),s^n(i)]}, f_{[s^k(i),s^n(i)]}) = 1$ and $f_{[i,s^k(i)]} + f_{[s^n(i),s^m(i)]}$ is a $\mathbb{Z}_{\geq 0}$-linear combination of e_j s with $j < s^k(i)$ or $j > s^n(i)$, we see from (6.2) and part (c) that

$$\begin{aligned} \Omega_{\boldsymbol{\lambda}}(f_{[i,s^m(i)]},\, f_{[s^k(i),s^n(i)]}) &= \Omega_{\boldsymbol{\lambda}}(f_{[i,s^k(i)]} + f_{[s^n(i),s^m(i)]}, f_{s^k(i),s^n(i)]}) \\ &= \Omega_{\boldsymbol{\lambda}}(f_{[i,s^k(i)]} + f_{[s^n(i),s^m(i)]}, e_{s^k(i)} + 2e_{[s^{k+1}(i),s^{n-1}(i)]} + e_{s^n(i)}). \end{aligned} \tag{6.9}$$

We now apply part (c) again, with $j = s^k(i), \dots, s^n(i)$. If $0 < k < n < m$, then (6.9) yields

$$\begin{aligned} &\Omega_{\boldsymbol{\lambda}}(f_{[i,s^m(i)]}, f_{[s^k(i),s^n(i)]}) \\ &\quad = \Omega_{\boldsymbol{\lambda}}(e_i + 2e_{[s(i),s^{k-1}(i)]} + e_{s^k(i)} + e_{s^n(i)} + 2e_{[s^{n+1}(i),s^{m-1}(i)]} + e_{s^m(i)}, \\ &\qquad\qquad e_{s^k(i)} + 2e_{[s^{k+1}(i),s^{n-1}(i)]} + e_{s^n(i)}), \end{aligned}$$

and the desired result follows because

$$\Omega_{\boldsymbol{\lambda}}(e_{s^k(i)} + 2e_{[s^{k+1}(i),s^{n-1}(i)]} + e_{s^n(i)}, e_{s^k(i)} + 2e_{[s^{k+1}(i),s^{n-1}(i)]} + e_{s^n(i)}) = 1.$$

If $k = 0$ and $n < m$, then $f_{[i,s^k(i)]} = 0$ and (6.9) yields

$$\begin{aligned} &\Omega_{\boldsymbol{\lambda}}(f_{[i,s^m(i)]}, f_{[s^k(i),s^n(i)]}) \\ &\quad = (\lambda_i^*)^{-1}\Omega_{\boldsymbol{\lambda}}(e_{s^n(i)} + 2e_{[s^{n+1}(i),s^{m-1}(i)]} + e_{s^m(i)}, e_i + 2e_{[s(i),s^{n-1}(i)]} + e_{s^n(i)}), \end{aligned}$$

from which the desired result follows. The case $k > 0$, $n = m$ is covered in the same fashion. $\square$

Note from Lemma 6.2(c) that

$$\Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s(i)]} - e_{s(i)})^2 = \Omega_{\boldsymbol{\lambda}}(e_i, f_{[i,s(i)]} - e_{s(i)}) = \lambda_i^*, \quad \text{when } i, s(i) \in [1, N],$$

and thus λ_i^* has a square root in the subgroup $\langle \nu_{kj} \mid k, j \in [1, N] \rangle$ of $\mathbb{K}^*$. All values of $\mathcal{S}_{\boldsymbol{\nu}}$, $\Omega_{\boldsymbol{\nu}}$, and $\Omega_{\boldsymbol{\lambda}}$ lie in this group.

Proof of Proposition 6.1. (a) Theorem 5.13 and Eq. (6.3) imply

$$\begin{aligned} &\pi_{[s(i),s^m(i)]}\pi_{[i,s^{m+1}(i)]} \operatorname{lt}(x^{f_{[s(i),s^m(i)]}} x^{f_{[i,s(i)]}} x^{f_{[s(i),s^{m+1}(i)]}}) \\ &\qquad = \theta_m \pi_{[i,s^m(i)]}\pi_{[s(i),s^{m+1}(i)]} \operatorname{lt}(x^{f_{[i,s(i)]}} x^{f_{[s(i),s^m(i)]}} x^{f_{[s(i),s^{m+1}(i)]}}) \end{aligned}$$

for the scalar $\theta_m \in \mathbb{K}^*$ from Theorem 5.13. It follows from Eq. (3.20), Lemma 6.2 (d) and Eq. (6.2) that

$$\frac{\pi_{[s(i),s^m(i)]}\pi_{[i,s^{m+1}(i)]}}{\pi_{[i,s^m(i)]}\pi_{[s(i),s^{m+1}(i)]}} = \theta_m \Omega_{\boldsymbol{\lambda}}(f_{[i,s(i)]}, f_{[s(i),s^m(i)]}) = \theta_m \Omega_{\boldsymbol{\lambda}}(f_{[i,s(i)]}, f_{[i,s^m(i)]})$$
$$= \theta_m (\lambda_i^*)^{1-\delta_{m,1}} \Omega_{\boldsymbol{\lambda}}(e_i + e_{s(i)}, e_i + 2e_{[s(i),s^{m-1}(i)]} + e_{s^m(i)}).$$

Simplifying the last expression using the definition of θ_m from Theorem 5.13 proves part (a).

(b) Denote

$$\varphi_{[i,s^m(i)]} := \mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})^{-2} \mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s^m(i)]}).$$

We will prove that

$$\varphi_{[s(i),s^m(i)]}\varphi_{[i,s^{m+1}(i)]} = (\lambda_i^*)^{-\delta_{m,1}} \Omega_{\boldsymbol{\lambda}}(e_{s^m(i)}, e_{s(i)}) \varphi_{[i,s^m(i)]}\varphi_{[s(i),s^{m+1}(i)]} \tag{6.10}$$

for all $i \in [1, N]$ and $m \in \mathbb{Z}_{>0}$ such that $s^{m+1}(i) \in [1, N]$. Part (b) trivially holds for $m = 0, 1$ and follows from the identity (6.10) and part (a) by induction on m.

For all $j \in [2, N]$ and $f \in \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_j$, $g \in \mathbb{Z}e_{j+1} + \cdots + \mathbb{Z}e_N$,

$$\mathcal{S}_{\boldsymbol{\nu}}(f + g) = \Omega_{\boldsymbol{\nu}}(f, g)^{-1} \mathcal{S}_{\boldsymbol{\nu}}(f) \mathcal{S}_{\boldsymbol{\nu}}(g). \tag{6.11}$$

Hence, taking also (6.2) into account,

$$\frac{\varphi_{[i,s^{m+1}(i)]}}{\varphi_{[i,s^m(i)]}} = \frac{\Omega_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]}, e_{s^{m+1}(i)})^2 \mathcal{S}_{\boldsymbol{\nu}}(f_{[s^m(i),s^{m+1}(i)]})}{\Omega_{\boldsymbol{\nu}}(-e_i + f_{[i,s^{m+1}(i)]}, f_{[s^m(i),s^{m+1}(i)]})}.$$

Similarly,

$$\frac{\varphi_{[s(i),s^m(i)]}}{\varphi_{[s(i),s^{m+1}(i)]}} = \frac{\Omega_{\boldsymbol{\nu}}(-e_{s(i)} + f_{[s(i),s^{m+1}(i)]}, f_{[s^m(i),s^{m+1}(i)]})}{\Omega_{\boldsymbol{\nu}}(e_{[s^2(i),s^m(i)]}, e_{s^{m+1}(i)})^2 \mathcal{S}_{\boldsymbol{\nu}}(f_{[s^m(i),s^{m+1}(i)]})}.$$

Multiplying these two identities and simplifying leads to

$$\frac{\varphi_{[s(i),s^m(i)]}\varphi_{[i,s^{m+1}(i)]}}{\varphi_{[i,s^m(i)]}\varphi_{[s(i),s^{m+1}(i)]}} = \frac{\Omega_{\boldsymbol{\nu}}(e_{s(i)}, e_{s^{m+1}(i)})^2 \Omega_{\boldsymbol{\nu}}(e_i - e_{s(i)}, f_{[s^m(i),s^{m+1}(i)]})}{\Omega_{\boldsymbol{\nu}}(f_{[i,s^{m+1}(i)]} - f_{[s(i),s^{m+1}(i)]}, f_{[s^m(i),s^{m+1}(i)]})}.$$

The square of the last term in the numerator simplifies, using Lemma 6.2(c), to

$$(\lambda_i^*)^{-\delta_{m,1}} \Omega_{\boldsymbol{\lambda}}(e_i - e_{s(i)}, e_{s^m(i)} + e_{s^{m+1}(i)}).$$

Using Lemma 6.2(d), the square of the denominator simplifies to

$$(\lambda_i^*)^{\delta_{m,1}} \Omega_{\boldsymbol{\lambda}}(e_i + e_{s(i)}, e_{s^m(i)} + e_{s^{m+1}(i)}).$$

Consequently, after some further simplifications we find that

$$\left(\frac{\varphi_{[s(i),s^m(i)]}\varphi_{[i,s^{m+1}(i)]}}{\varphi_{[i,s^m(i)]}\varphi_{[s(i),s^{m+1}(i)]}} \right)^2 = (\lambda_i^*)^{-2\delta_{m,1}} \Omega_{\boldsymbol{\lambda}}(e_{s^m(i)}, e_{s(i)})^2.$$

Invoking condition (4.23), we obtain (6.10), as desired. $\square$

6.2. Rescaling of the generators of a symmetric CGL extension

For $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_N) \in (\mathbb{K}^*)^N$ and $f := (n_1, \dots, n_N) \in \mathbb{Z}^N$ set

$$\boldsymbol{\gamma}^f := \gamma_1^{n_1} \dots \gamma_N^{n_N} \in \mathbb{K}^*.$$

Given a symmetric CGL extension R of length N and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_N) \in (\mathbb{K}^*)^N$, one can rescale the generators x_j of R,

$$x_j \mapsto \gamma_j x_j, \quad j \in [1, N], \tag{6.12}$$

by which we mean that one can use $\gamma_1 x_1, \dots, \gamma_N x_N$ as a new sequence of generators of R. This obviously does not effect the $\mathcal{H}$-action and the matrix $\boldsymbol{\lambda}$, but one obtains a new set of elements y_k, $y_{[i,s^m(i)]}$, $u_{[i,s^m(i)]}$ by applying Theorems 3.6, 5.1 and Corollary 5.11 for the new set of generators. (Note that this is not the same as substituting (6.12) in the formulas for y_k and $y_{[i,s^m(i)]}$; those other kind of transformed elements will not be even prime because (6.12) does not determine an algebra automorphism.) The uniqueness part of Theorem 5.1 implies that the effect of (6.12) on the elements $y_{[i,s^m(i)]}$ is that they are rescaled by the rule

$$y_{[i,s^m(i)]} \mapsto \boldsymbol{\gamma}^{e_{[i,s^m(i)]}} y_{[i,s^m(i)]} = (\gamma_i \dots \gamma_{s^m(i)}) y_{[i,s^m(i)]}$$

for all $i \in [1, N]$, $m \in \mathbb{Z}_{\geq 0}$ such that $s^m(i) \in [1, N]$. Hence, the effect of (6.12) on the elements $u_{[i,s^m(i)]}$ is that they are rescaled by

$$u_{[i,s^m(i)]} \mapsto \boldsymbol{\gamma}^{e_{[i,s^m(i)]} + e_{[s(i),s^{m-1}(i)]}} u_{[i,s^m(i)]} = (\gamma_i \gamma_{s(i)}^2 \dots \gamma_{s^{m-1}(i)}^2 \gamma_{s^m(i)}) u_{[i,s^m(i)]}$$

for all $i \in [1, N]$, $m \in \mathbb{Z}_{>0}$ such that $s^m(i) \in [1, N]$. It follows from (6.1) that the effect of (6.12) on the scalars $\pi_{[i,s^m(i)]}$ is that they are rescaled by

$$\pi_{[i,s^m(i)]} \mapsto \boldsymbol{\gamma}^{e_{[i,s^m(i)]} + e_{[s(i),s^{m-1}(i)]} - f_{[i,s^m(i)]}} \pi_{[i,s^m(i)]} = \\ \gamma_i \gamma_{s^m(i)} \boldsymbol{\gamma}^{2e_{[s(i),s^{m-1}(i)]} - f_{[i,s^m(i)]}} \pi_{[i,s^m(i)]}.$$

(Note that the rescaling (6.12) has no effect on the integer vector $f_{[i,s^m(i)]}$.) This implies at once the following fact.

PROPOSITION 6.3. *Let R be a symmetric CGL extension of length N and rank $\mathrm{rk}(R)$ for which there exist $\nu_{kj} = \sqrt{\lambda_{kj}} \in \mathbb{K}^*$, $0 \leq j < k \leq N$. Then there exist N-tuples $\boldsymbol{\gamma} \in (\mathbb{K}^*)^N$ such that after the rescaling (6.12) we have*

$$\pi_{[i,s(i)]} = \mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s(i)]}), \quad \forall i \in [1, N] \text{ such that } s(i) \neq +\infty. \tag{6.13}$$

The set of those N-tuples is parametrized by $(\mathbb{K}^*)^{\mathrm{rk}(R)}$ and the entries $\gamma_1, \dots, \gamma_N$ of all such N-tuples are recursively determined by

$$\gamma_i \text{ is arbitrary if } p(i) = -\infty$$

and

$$\gamma_i = \gamma_{p(i)}^{-1} \boldsymbol{\gamma}^{f_{[p(i),i]}} \pi_{[p(i),i]}^{-1} \mathcal{S}_\nu(-e_{p(i)} + f_{[p(i),i]}), \quad \text{if } p(i) \neq -\infty, \tag{6.14}$$

where in the right hand side the π-scalars are the ones for the original generators $x_1, \dots, x_N$ of R.

Note that the product of the first two terms of the right hand side of (6.14) is a product of powers of $\gamma_{p(i)}, \dots, \gamma_{i-1}$ since $f_{[p(i),i]} \in \mathbb{Z}_{\geq 0} e_{p(i)+1} + \dots + \mathbb{Z}_{\geq 0} e_{i-1}$.

We refer the reader to Chapter 10 for an explicit example on how the rescaling in Proposition 6.3 works for the canonical generators of the quantum Schubert cell algebras. As for quantum matrices, we have the following

EXAMPLE 6.4. In the case of $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$, condition (6.13) is already satisfied, and so no rescaling of the generators is necessary. To see this, let $i \in [1, N]$ with $s(i) \neq +\infty$. Then $i = (r-1)n+c$ for some $r \in [1, m-1]$ and $c \in [1, n-1]$, and $s(i) = rn + c + 1$. From (5.22), we see that $\pi_{[i,s(i)]} = q$ and $f_{[i,s(i)]} = e_{i+1} + e_{i+n}$. Recalling the choice of $\boldsymbol{\nu}$ from Example 4.7, we find that

$$\mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s(i)]}) = \nu_{i,i+1}\nu_{i,i+n}\nu_{i+1,i+n}^{-1} = (\sqrt{q})(\sqrt{q})(1) = q = \pi_{[i,s(i)]}.$$

6.3. Toric frames and mutations in symmetric CGL extensions

Let R be a symmetric CGL extension of length N for which $\sqrt{\lambda_{kj}} \in \mathbb{K}^*$ for all $k, j \in [1, N]$ and let $\boldsymbol{\nu} = (\nu_{kj}) \in M_N(\mathbb{K}^*)$ be a multiplicatively skew-symmetric matrix such that $\boldsymbol{\nu}^{\cdot 2} = \boldsymbol{\lambda}$.

The procedure from Section 4.3 defines a toric frame for $\mathrm{Fract}(R_{[i,s^m(i)]})$ for all $i \in [1, N]$ and $m \in \mathbb{Z}_{>0}$ such that $s^m(i) \in [1, N]$. We apply the construction from Section 4.3 to the CGL extension presentation of $R_{[i,s^m(i)]}$ associated to the following order of adjoining the generators:

$$x_{i+1}, \dots, x_{s^m(i)-1}, x_i, x_{s^m(i)}. \tag{6.15}$$

Extending (4.18), set

$$\overline{y}_{[p^n(l),l]} := \mathcal{S}_{\boldsymbol{\nu}}(e_{[p^n(l),l]})y_{[p^n(l),l]}, \ \ \forall j \in [1, N], \ n \in \mathbb{Z}_{\geq 0} \ \text{ with } \ p^n(j) \in [1, N]. \tag{6.16}$$

Let $1 \leq j \leq k \leq N$, and recall the definition (5.16) of the function O_-^j. By Corollary 5.10,

$$\left\{ \overline{y}_{[p^{O_-^j(l)}(l),l]} \;\middle|\; l \in [j, k], \ s(l) > k \right\}$$

is a list of the the homogeneous prime elements of $R_{[j,k]}$ up to scalar multiples. The intermediate subalgebras for the CGL extension presentation of $R_{[i,s^m(i)]}$ associated to (6.15) are

$$R_{[i+1,i+1]}, \dots, R_{[i+1,s^m(i)-1]}, R_{[i,s^m(i)-1]}, R_{[i,s^m(i)]}.$$

We identify $\mathbb{Z}^{s^m(i)-i+1} \cong \mathbb{Z}e_i + \cdots + \mathbb{Z}e_{s^m(i)} \subseteq \mathbb{Z}^N$ and enumerate the rows and columns of square matrices of size $(s^m(i) - i + 1)$ by $i, \dots, s^m(i)$. Define the multiplicatively skew-symmetric matrix $\mathbf{r}_{[i,s^m(i)]} \in M_{s^m(i)-i+1}(\mathbb{K}^*)$ by

$$(\mathbf{r}_{[i,s^m(i)]})_{k,j} := \Omega_{\boldsymbol{\nu}}(e'_k, e'_j), \ \ j, k \in [i, s^m(i)], \tag{6.17}$$

where

$$e'_k = \begin{cases} e_{[p^{O_-^{i+1}(k)}(k),k]}, & \text{if } \ k \in [i+1, s^m(i) - 1] \\ e_{[i,s^{m-1}(i)]}, & \text{if } \ k = i \\ e_{[i,s^m(i)]}, & \text{if } \ k = s^m(i). \end{cases} \tag{6.18}$$

Proposition 4.6 implies that there is a unique toric frame $M_{[i,s^m(i)]} : \mathbb{Z}^{s^m(i)-i+1} \to \mathrm{Fract}(R_{[i,s^m(i)]})$ with matrix $\mathbf{r}_{[i,s^m(i)]}$ satisfying

$$M_{[i,s^m(i)]}(e_k) = \begin{cases} \overline{y}_{[p^{O_-^{i+1}(k)}(k),k]}, & \text{if } \ k \in [i+1, s^m(i) - 1] \\ \overline{y}_{[i,s^{m-1}(i)]}, & \text{if } \ k = i \\ \overline{y}_{[i,s^m(i)]}, & \text{if } \ k = s^m(i). \end{cases}$$

Recall the definition (6.1) of the vectors $f_{[i,s^m(i)]}$. It follows from Corollary 5.11 that there exists a unique vector

$$g_{[i,s^m(i)]} = \sum \{m_k e_k \mid k \in P_{[i,s^m(i)]}\}$$

such that

$$f_{[i,s^m(i)]} = \sum \Big\{ m_k e_{[p^{O_-^{i+1}(k)}(k),k]} \;\Big|\; k \in P_{[i,s^m(i)]} \Big\},$$

where the integers m_k are those from Theorems 4.3, 4.8.

Set $t := s^m(i) - i + 1$. As above, we identify $\mathbb{Z}^t$ with $\mathbb{Z}e_i \oplus \cdots \oplus \mathbb{Z}e_{s^m(i)}$, that is, with the sublattice of $\mathbb{Z}^N$ with basis e_j, $j \in [i, s^m(i)]$. Define $\sigma \in M_t(\mathbb{Z})$ by $\sigma(e_k) = e'_k$ for all $k \in [i, s^m(i)]$, where the e'_k are as in (6.18). It is easy to check that

$$\sigma \text{ is invertible and } \sigma(\mathbb{Z}^t_{\geq 0}) \subseteq \mathbb{Z}^t_{\geq 0}.$$

This choice of σ ensures that

$$\sigma(g_{[i,s^m(i)]}) = f_{[i,s^m(i)]} \tag{6.19}$$

and

$$\mathbf{r}_{[i,s^m(i)]} = \big(\boldsymbol{\nu}|_{[i,s^m(i)]\times[i,s^m(i)]}\big)_\sigma. \tag{6.20}$$

The definition of the toric frame $M_{[i,s^m(i)]}$ also implies that

$$\operatorname{lt}\big(M_{[i,s^m(i)]}(e_k)\big) = \mathcal{S}_{\boldsymbol{\nu}}(\sigma(e_k))x^{\sigma(e_k)}, \quad \forall k \in [i, s^m(i)]. \tag{6.21}$$

Lemma 6.5. *For all $g \in \mathbb{Z}^t_{\geq 0}$,*

$$\operatorname{lt}\big(M_{[i,s^m(i)]}(g)\big) = \mathcal{S}_{\boldsymbol{\nu}}(\sigma(g))x^{\sigma(g)}.$$

Proof. We prove the lemma by using the inductive argument from the proof of Proposition 2.1. By Eq. (6.21), the lemma is valid for $g = e_k$. The validity of the lemma for g and $g' \in \mathbb{Z}^t_{\geq 0}$ implies its validity for $g + g' \in \mathbb{Z}^t_{\geq 0}$ because of Eq. (2.8) (applied to the toric frame $M_{[i,s^m(i)]}$), Eq. (6.20) above, and the fact that

$$\operatorname{lt}\Big(\mathcal{S}_{\boldsymbol{\nu}}(f)x^f \mathcal{S}_{\boldsymbol{\nu}}(f')x^{f'}\Big) = \Omega_{\boldsymbol{\nu}}(f,f')\mathcal{S}_{\boldsymbol{\nu}}(f+f')x^{f+f'}, \quad \forall f, f' \in \mathbb{Z}^N_{\geq 0}.$$

The last identity is an immediate consequence of Eqs. (3.20) and (2.8). $\square$

Theorem 6.6. *Let R be a symmetric CGL extension of length N. Assume that the base field $\mathbb{K}$ contains square roots ν_{kj} of all scalars λ_{kj}, such that the subgroup of $\mathbb{K}^*$ generated by ν_{kj}, $1 \leq j < k \leq N$ does not contain elements of order 2. Assume also that the generators $x_1, \ldots, x_N$ of R are normalized so that the condition (6.13) is satisfied. Then for all $i \in [1, N]$ and $m \in \mathbb{Z}_{>0}$ such that $s^m(i) \in [1, N]$,*

$$\overline{y}_{[s(i),s^m(i)]} = \begin{cases} M_{[i,s(i)]}(e_{s(i)} - e_i) + M_{[i,s(i)]}(g_{[i,s(i)]} - e_i), & \textit{if } m = 1 \\ M_{[i,s^m(i)]}(e_{s^{m-1}(i)} + e_{s^m(i)} - e_i) + \\ \qquad\qquad M_{[i,s^m(i)]}(g_{[i,s^m(i)]} - e_i), & \textit{if } m > 1. \end{cases} \tag{6.22}$$

Proof. Lemma 6.5 and Eq. (6.19) imply

$$\operatorname{lt}\big(M_{[i,s^m(i)]}(g_{[i,s^m(i)]})\big) = \mathcal{S}_{\boldsymbol{\nu}}(f_{[i,s^m(i)]})x^{f_{[i,s^m(i)]}}.$$

By Eq. (5.19), $u_{[i,s^m(i)]}$ is a scalar multiple of $M_{[i,s^m(i)]}(g_{[i,s^m(i)]})$, and so we find that

$$u_{[i,s^m(i)]} = \pi_{[i,s^m(i)]}\mathcal{S}_{\boldsymbol{\nu}}(f_{[i,s^m(i)]})^{-1}M_{[i,s^m(i)]}(g_{[i,s^m(i)]}). \tag{6.23}$$

In case $m > 1$, we observe that

$$\overline{y}_{[s(i),s^m(i)]} - M_{[i,s^m(i)]}(e_{s^{m-1}(i)} + e_{s^m(i)} - e_i) = \\ \mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})y_{[s(i),s^m(i)]} - \xi y^{-1}_{[i,s^{m-1}(i)]} y_{[s(i),s^{m-1}(i)]} y_{[i,s^m(i)]}$$

where, taking account of (6.11) and (6.17) and simplifying,

$$\begin{aligned}\xi &= \mathcal{S}_{\mathbf{r}_{[i,s^m(i)]}}(e_{s^{m-1}(i)} + e_{s^m(i)} - e_i)\mathcal{S}_{\boldsymbol{\nu}}(e_{[i,s^{m-1}(i)]})^{-1}\mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^{m-1}(i)]})\mathcal{S}_{\boldsymbol{\nu}}(e_{[i,s^m(i)]}) \\ &= \mathcal{S}_{\mathbf{r}_{[i,s^m(i)]}}(e_{s^{m-1}(i)} + e_{s^m(i)} - e_i)\times \\ &\qquad \times \Omega_{\boldsymbol{\nu}}(e_i, e_{[s(i),s^{m-1}(i)]})\Omega_{\boldsymbol{\nu}}(e_i, e_{[s(i),s^m(i)]})^{-1}\mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]}) \\ &= \Omega_{\boldsymbol{\lambda}}(e_i, e_{[s(i),s^{m-1}(i)]})\mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]}).\end{aligned}$$

Thus, we have

$$\overline{y}_{[s(i),s^m(i)]} - M_{[i,s^m(i)]}(e_{s^{m-1}(i)} + e_{s^m(i)} - e_i) = \\ \mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})y^{-1}_{[i,s^{m-1}(i)]}u_{[i,s^m(i)]}, \quad \text{if } m > 1. \tag{6.24}$$

A similar argument shows that

$$\overline{y}_{[s(i),s(i)]} - M_{[i,s(i)]}(e_{s(i)} - e_i) = x_i^{-1}u_{[i,s(i)]}, \quad \text{if } m = 1. \tag{6.25}$$

From now until the last sentence of the proof, we combine the cases $m = 1$ and $m > 1$. Using the definition of $M_{[i,s^m(i)]}$ together with (6.11) and simplifying, we obtain

$$\begin{aligned}&M_{[i,s^m(i)]}(g_{[i,s^m(i)]} - e_i) \\ &\qquad = \mathcal{S}_{\mathbf{r}_{[i,s^m(i)]}}(g_{[i,s^m(i)]} - e_i)M_{[i,s^m(i)]}(e_i)^{-1}\times \\ &\qquad\qquad \mathcal{S}_{\mathbf{r}_{[i,s^m(i)]}}(g_{[i,s^m(i)]})^{-1}M_{[i,s^m(i)]}(g_{[i,s^m(i)]}) \\ &\qquad = \Omega_{\mathbf{r}_{[i,s^m(i)]}}(e_i, g_{[i,s^m(i)]})\mathcal{S}_{\boldsymbol{\nu}}(e_{[i,s^{m-1}(i)]})^{-1}y^{-1}_{[i,s^{m-1}(i)]}M_{[i,s^m(i)]}(g_{[i,s^m(i)]}).\end{aligned}$$

We also have

$$\begin{aligned}\Omega_{\mathbf{r}_{[i,s^m(i)]}}(e_i, g_{[i,s^m(i)]}) &= \Omega_{\boldsymbol{\nu}}(e_{[i,s^{m-1}(i)]}, f_{[i,s^m(i)]}) \\ &= \Omega_{\boldsymbol{\nu}}(e_i + e_{[s(i),s^{m-1}(i)]}, f_{[i,s^m(i)]}) \\ &= \Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s^m(i)]})\times \\ &\qquad \Omega_{\boldsymbol{\nu}}(e_{[s(i),s^{m-1}(i)]}, e_i + 2e_{[s(i),s^{m-1}(i)]} + e_{s^m(i)}) \\ &= \Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s^m(i)]})\Omega_{\boldsymbol{\nu}}(e_{[s(i),s^{m-1}(i)]}, e_i + e_{s^m(i)}),\end{aligned}$$

where the first equality follows from the definition of $\mathbf{r}_{[i,s^m(i)]}$ and the third from Lemma 6.2 (c). It follows from Eq. (6.11) that

$$\begin{aligned}\mathcal{S}_{\boldsymbol{\nu}}(e_{[i,s^{m-1}(i)]}) &= \Omega_{\boldsymbol{\nu}}(e_i, e_{[s(i),s^{m-1}(i)]})^{-1}\mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^{m-1}(i)]}) \\ &= \Omega_{\boldsymbol{\nu}}(e_i, e_{[s(i),s^{m-1}(i)]})^{-1}\Omega_{\boldsymbol{\nu}}(e_{[s(i),s^{m-1}(i)]}, e_{s^m(i)})\mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})\end{aligned}$$

and

$$\mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s^m(i)]}) = \Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s^m(i)]})\mathcal{S}_{\boldsymbol{\nu}}(f_{[i,s^m(i)]}). \tag{6.26}$$

Combining the first three of the previous four equations yields

$$M_{[i,s^m(i)]}(g_{[i,s^m(i)]} - e_i) = \\ \Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s^m(i)]})\mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})^{-1}y^{-1}_{[i,s^{m-1}(i)]}M_{[i,s^m(i)]}(g_{[i,s^m(i)]}). \tag{6.27}$$

Finally, applying Proposition 6.1 (b) and (6.11) to (6.23) yields

$$u_{[i,s^m(i)]} = \mathcal{S}_{\boldsymbol{\nu}}(e_{[s(i),s^m(i)]})^{-2}\Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s^m(i)]})M_{[i,s^m(i)]}(g_{[i,s^m(i)]}).$$

Substituting this identity into (6.24), (6.25) and taking account of (6.27) yields (6.22) and completes the proof of the theorem. □

Remark 6.7. Because of Theorem 5.1, all instances when Theorem 4.8 is applicable to a symmetric CGL extension with respect to the CGL extension presentations associated to the elements of the set Ξ_N are covered by Theorem 6.6. We refer the reader to Proposition 8.13 for details.

Excerpting several steps from the proof of Theorem 6.6 yields the following:

Corollary 6.8. *Let R be a symmetric CGL extension of length N. Assume that the base field $\mathbb{K}$ contains square roots ν_{kj} of all scalars λ_{kj}, such that the subgroup of $\mathbb{K}^*$ generated by ν_{kj}, $1 \le j < k \le N$ does not contain elements of order 2.*

(a) *For all $i \in [1, N]$ and $m \in \mathbb{Z}_{>0}$ such that $s^m(i) \in [1, N]$, eq.* (6.23) *holds:*

$$u_{[i,s^m(i)]} = \pi_{[i,s^m(i)]}\mathcal{S}_{\boldsymbol{\nu}}(f_{[i,s^m(i)]})^{-1}M_{[i,s^m(i)]}(g_{[i,s^m(i)]}).$$

(b) *Condition* (6.13) *is satisfied if and only if*

$$M_{[i,s(i)]}(g_{[i,s(i)]} - e_i) = x_i^{-1}u_{[i,s(i)]}, \;\; \forall i \in [1, N], \; s(i) \neq +\infty. \tag{6.28}$$

Proof. Observe that (6.13) is not used in the proof of Theorem 6.6 until the very last paragraph, where Proposition 6.1 (b) is applied. Thus, all equations in the proof up through (6.27) hold without assuming (6.13). This gives part (a).

(b) In case (6.13) holds, we obtain (6.28) by substituting (6.25) into the case $m = 1$ of (6.22).

Conversely, if (6.28) holds for some $i \in [1, N]$ with $s(i) \neq +\infty$, then (6.27) implies

$$u_{[i,s(i)]} = \Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s(i)]})M_{[i,s(i)]}(g_{[i,s(i)]}).$$

Comparing this with (6.23) yields

$$\pi_{[i,s(i)]} = \Omega_{\boldsymbol{\nu}}(e_i, f_{[i,s(i)]})\mathcal{S}_{\boldsymbol{\nu}}(f_{[i,s(i)]}) = \mathcal{S}_{\boldsymbol{\nu}}(-e_i + f_{[i,s(i)]}),$$

recall (6.26), and verifies (6.13). □

CHAPTER 7

Division properties of mutations between CGL extension presentations

7.1. Main result

In this chapter we describe the intersection of the localizations of a CGL extension R by the two sets of cluster variables (y-elements) in the setting of Theorem 4.2 (b). This plays a key role in the next chapter where we prove that every symmetric CGL extension is a quantum cluster algebra which equals the corresponding upper quantum cluster algebra.

Versions of the following two lemmas are well known. We give a proof of the first for completeness.

Lemma 7.1. *Assume that R is a $\mathbb{K}$-algebra domain and E a left or right Ore set in R. Let $p \in R$ be a prime element such that $\mathbb{K}^* pE = \mathbb{K}^* Ep$ and $Rp \cap E = \varnothing$. Then p is a prime element of $R[E^{-1}]$. If $s \in R$ and $p \nmid s$ as elements of R, then $p \nmid s$ as elements of $R[E^{-1}]$.*

Proof. If $e \in E$, then $pe = \alpha f p$ for some $\alpha \in \mathbb{K}^*$ and $f \in E$, whence $pe^{-1} = \alpha^{-1} f^{-1} p$ in $R[E^{-1}]$. Similarly, $e^{-1}p = \beta^{-1} p g^{-1}$ for some $\beta \in \mathbb{K}^*$ and $g \in E$. These observations, together with the normality of p in R, imply that p is normal in $R[E^{-1}]$. In particular, $R[E^{-1}]p$ is an ideal of $R[E^{-1}]$.

It follows from the assumption $Rp \cap E = \varnothing$ and the complete primeness of Rp that $R[E^{-1}]p \cap R = Rp$. Consequently, $R[E^{-1}]/R[E^{-1}]p$ is an Ore localization of the domain R/Rp, so it too is a domain, and thus p is a prime element of $R[E^{-1}]$.

The last statement of the lemma follows from the equality $R[E^{-1}]p \cap R = Rp$. □

Lemma 7.2. *Let $B[x; \sigma, \delta]$ be an Ore extension (recall Convention 3.2) and E a (left or right) Ore set of regular elements in B. If $\mathbb{K}^* E$ is σ-stable, then E is a (left or right) Ore set of regular elements in $B[x; \sigma, \delta]$.*

Proof. Replace E by $\mathbb{K}^* E$ and apply [**20**, Lemma 1.4]. □

Recall that given a (noncommutative) domain A and $a, b \in A$, we write $a \mid_l b$ to denote that $b \in aR$.

Assume now that R is a CGL extension of length N as in (3.1). Let $y_1, \ldots, y_N \in R$ be the sequence of elements from Theorem 3.6. Then y_j is a normal element of R_j and by Lemma 7.2, $\{y_j^m \mid m \in \mathbb{Z}_{\geq 0}\}$ is an Ore set in R, $\forall j \in [1, N]$. It follows from (3.9) that for all $I \subseteq [1, N]$,

$$E_I := \mathbb{K}^* \prod_{j \in I} \{y_j^m \mid m \in \mathbb{Z}_{\geq 0}\} \tag{7.1}$$

is a multiplicative set and thus is an Ore set in R. Set $E := E_{[1,N]}$. We will say that $y_1^{m_1} \dots y_N^{m_N}$ is a *minimal denominator* of a nonzero element $v \in R[E^{-1}]$ if

$$v = y_1^{-m_1} \dots y_N^{-m_N} s$$

for some $s \in R$ such that $y_j \nmid_l s$ for all $j \in [1, N]$ with $m_j > 0$.

For the rest of this chapter, we assume the setting of Theorem 4.2 (b). Then for all $I' \subseteq [1, N]$,

$$E'_{I'} := \mathbb{K}^* \prod_{j \in I'} \{(y'_j)^m \mid m \in \mathbb{Z}_{\geq 0}\} \tag{7.2}$$

is another Ore set in R. Recall from Theorem 4.2 (b) that y'_j is a scalar multiple of y_j for $j \neq k$. Hence, $E'_{I'} = E_{I'}$ if $k \notin I'$. Denote $E' := E'_{[1,N]}$.

LEMMA 7.3. *For all $I, I' \subseteq [1, N]$,*

$$E_I \cap E'_{I'} = E_{(I \cap I') \setminus \{k\}} = E'_{(I \cap I') \setminus \{k\}}. \tag{7.3}$$

PROOF. Let $e \in E_I \cap E'_{I'}$, and write

$$e = \beta \prod_{j=1}^{N} y_j^{m_j} = \beta' \prod_{j=1}^{N} (y'_j)^{m'_j}$$

for some $\beta, \beta' \in \mathbb{K}^*$ and $m_j, m'_j \in \mathbb{Z}_{\geq 0}$ such that $m_j = 0$ for $j \notin I$ and $m'_j = 0$ for $j \notin I'$. We will show that

$$m_k = m'_k = 0 \quad \text{and} \quad m_j = m'_j, \ \ \forall j \in [1, N], \ j \neq k, \tag{7.4}$$

which then implies $e \in E_{(I \cap I') \setminus \{k\}} = E'_{(I \cap I') \setminus \{k\}}$.

Since y_N and y'_N are prime elements and associates in R, and they do not divide $y_1, \dots, y_{N-1}$, unique factorization implies that $m_N = m'_N$. Then, after replacing e by $e y_N^{-m_N}$ and adjusting β', we may reduce to the case in which $m_N = m'_N = 0$. Continuing in this manner, we reduce to the case in which $m_j = m'_j = 0$ for all $j > k$.

If $m_k > 0$, we observe using Theorem 4.2 (b) that

$$e y'_k = \beta \Big(\prod_{j<k} y_j^{m_j} \Big) y_k^{m_k - 1} (u + \alpha_{kp(k)} y_{p(k)} y_{k+1})$$

for some $u \in R_{k-1}$. This implies that $e y'_k$ is an element of R_{k+1} which has degree 1 with respect to x_{k+1}. On the other hand, the equation

$$e y'_k = \beta' \Big(\prod_{j<k} (y'_j)^{m'_j} \Big) (y'_k)^{m'_k + 1}$$

shows that $e y'_k$ has degree $m'_k + 1$ with respect to x_{k+1}, in view of (4.8). Hence, $m'_k = 0$ and $e \in R_{k-1}$, contradicting our assumption that $m_k > 0$.

Thus $m_k = 0$, and similarly $m'_k = 0$. Proceeding as above, we conclude that $m_j = m'_j$ for all $j < k$, verifying (7.4). □

The next theorem contains the main result of this chapter.

THEOREM 7.4. *Assume the setting of Theorem 4.2 (b). Let*

$$y_1^{m_1} \dots y_N^{m_N} \quad \textit{and} \quad y_1^{m'_1} \dots y_{k-1}^{m'_{k-1}} (y'_k)^{m'_k} y_{k+1}^{m'_{k+1}} \dots y_N^{m'_N}$$

be two minimal denominators of a nonzero element $v \in R[E^{-1}] \cap R[(E')^{-1}]$ (*with respect to the two different localizations*). *Then*

$$m_k = m'_k = 0 \quad \textit{and} \quad m_j = m'_j, \ \ \forall j \in [1, N], \ j \neq k.$$

In particular,

$$R[(E_I)^{-1}] \cap R[(E'_{I'})^{-1}] = R[(E_I \cap E'_{I'})^{-1}] = R[(E_{(I \cap I') \setminus \{k\}})^{-1}]$$

for all $I, I' \subseteq [1, N]$.

Theorem 7.4 implies that each nonzero element of $R[E^{-1}]$ has a unique minimal denominator. The theorem was inspired by [**17**, Theorem 4.1].

7.2. Proof of the main result

For $s \in R$ and $j \in [1, N]$, denote

$$s = \sum_{l_j, \ldots, l_N \in \mathbb{Z}_{\geq 0}} s_{l_j, \ldots, l_N} x_j^{l_j} \ldots x_N^{l_N}, \quad s_{l_j, \ldots, l_N} \in R_{j-1}. \tag{7.5}$$

For the proof of Theorem 7.4 we will need the following two lemmas.

Lemma 7.5. *Let* R *be a CGL extension of length* N *and* $y_1, \ldots, y_N$ *the sequence of elements from Theorem* 3.6. *If*

$$y_j \mid_l y_1^{n_1} \ldots y_{j-1}^{n_{j-1}} s$$

for some $j \in [1, N]$, $s \in R$, *and* $n_1, \ldots, n_{j-1} \in \mathbb{Z}_{\geq 0}$, *then* $y_j \mid_l s$.

Proof. There exists $s' \in R$ such that $y_j s' = y_1^{n_1} \ldots y_{j-1}^{n_{j-1}} s$. Comparing the coefficients of $x_{j+1}^{l_{j+1}} \ldots x_N^{l_N}$ leads to

$$y_j \mid_l y_1^{n_1} \ldots y_{j-1}^{n_{j-1}} s_{l_{j+1}, \ldots, l_N}, \quad \forall l_{j+1}, \ldots, l_N \in \mathbb{Z}_{\geq 0}.$$

By Theorem 3.6, y_j is a prime element of R_j and $y_j \nmid y_1, \ldots, y_{j-1}$. So $y_j \mid s_{l_{j+1}, \ldots, l_N}$, for all $l_{j+1}, \ldots, l_N \in \mathbb{Z}_{\geq 0}$, and thus $y_j \mid_l s$. □

Lemma 7.6. *If, in the setting of Theorem* 4.2 (b), $y_k \mid_l s(y'_k)^n$ *for some* $s \in R_{k+1}$ *and* $n \in \mathbb{Z}_{\geq 0}$, *then* $y_k \mid_l s$.

Proof. Write $s = \sum_{l \in \mathbb{Z}_{\geq 0}} s_l x_{k+1}^l$ for some $s_l \in R_k$. Similarly to the proof of Lemma 7.5, the conclusion of this lemma is equivalent to $y_k \mid s_l$, $\forall l \in \mathbb{Z}_{\geq 0}$. Assuming this is not the case, we replace s with

$$s - \sum \{ s_l x_{k+1}^l \mid l \in \mathbb{Z}_{\geq 0}, y_k \nmid s_l \}.$$

Note that this new version of s satisfies the assumption of the lemma. Moreover, it has the form

$$s = s_L x_{k+1}^L + \sum_{l=0}^{L-1} s_l x_{k+1}^l$$

for some $L \in \mathbb{Z}_{\geq 0}$ such that $y_k \nmid s_L$. Then

$$s(y'_k)^n - \xi s_L y_{p(k)}^n x_{k+1}^{L+n} \in \oplus_{l=0}^{L+n-1} R_k x_{k+1}^l$$

for some $\xi \in \mathbb{K}^*$. The assumption $y_k \mid_l s(y'_k)^n$ implies that $y_k \mid s_L y_{p(k)}^n$ which is a contradiction since y_k is a prime element of R_k which does not divide s_L or $y_{p(k)}$. □

Proof of Theorem 7.4. After multiplying v by $\prod_{j \neq k} y_j^{\min(m_j, m'_j)}$ and using (3.9) for both sets of y-elements, we can assume that

$$\min(m_j, m'_j) = 0, \quad \forall j \neq k. \tag{7.6}$$

(Under this condition we have to prove that $m_1 = m'_1 = \ldots = m_N = m'_N = 0$.) Write

$$v = y_1^{-m_1} \ldots y_N^{-m_N} s = y_1^{-m'_1} \ldots y_{k-1}^{-m'_{k-1}} (y'_k)^{-m'_k} y_{k+1}^{-m'_{k+1}} \ldots y_N^{-m'_N} s'$$

for some nonzero elements $s, s' \in R$.

First, we prove that

$$m_{k+1} = m'_{k+1} = \ldots = m_N = m'_N = 0. \tag{7.7}$$

Assume that this is not the case. Then

$$i = \max\{j \mid \max(m_j, m'_j) > 0\} \geq k+1.$$

Without loss of generality, we can assume that $m'_i > 0$. Applying (3.9) to both sets of y-elements we obtain

$$y_i^{m'_i} \Big(\prod_{j \in [1,i-1] \backslash \{k\}} y_j^{m'_j} \Big) s = \xi \Big(\prod_{j \in [1,i-1]} y_j^{m_j} \Big) (y'_k)^{-m'_k} s'$$

for some $\xi \in \mathbb{K}^*$. Comparing the coefficients of $x_{i+1}^{l_{i+1}} \ldots x_N^{l_N}$ gives

$$y_i^{m'_i} \Big(\prod_{j \in [1,i-1] \backslash \{k\}} y_j^{m'_j} \Big) s_{l_{i+1}, \ldots, l_N} = \xi \Big(\prod_{j \in [1,i-1]} y_j^{m_j} \Big) (y'_k)^{-m'_k} s'_{l_{i+1}, \ldots, l_N} \in R_i. \tag{7.8}$$

Since y_i is a prime element of R_i, by Lemma 7.1, it is a prime element of $R_i[(y'_k)^{-1}]$. Furthermore, by Theorem 3.6 and Lemma 7.1, $y_i \nmid y_j$ in $R_i[(y'_k)^{-1}]$ for $j \in [1, i-1]$. Therefore for all $l_{i+1}, \ldots, l_N \in \mathbb{Z}_{\geq 0}$, $y_i \mid s'_{l_{i+1}, \ldots, l_N}$ in $R_i[(y'_k)^{-1}]$ and thus in R_i (by Lemma 7.1). Hence, $y_i \mid_l s'$ which contradicts the minimality assumption on the denominators.

Next, we prove that

$$m_k = m'_k = 0. \tag{7.9}$$

Assume the opposite, e.g., $m_k > 0$. (The other case is analogous because of the symmetric nature of y_k and y'_k.) Similarly to (7.8), one obtains that there exists $\xi \in \mathbb{K}^*$ such that

$$\Big(\prod_{j=1}^{k-1} y_j^{m'_j} \Big) s_{l_{k+2}, \ldots, l_N} = \xi (y_k)^{m_k} \Big(\prod_{j=1}^{k-1} y_j^{m_j} \Big) (y'_k)^{-m'_k} s'_{l_{k+2}, \ldots, l_N} \in R_{k+1}$$

for all $l_{k+2}, \ldots, l_N \in \mathbb{Z}_{\geq 0}$. Since $\{(y'_k)^m \mid m \in \mathbb{Z}_{\geq 0}\}$ is an Ore set in R_{k+1},

$$(y'_k)^{-m'_k} s'_{l_{k+2}, \ldots, l_N} = s''_{l_{k+2}, \ldots, l_N} (y'_k)^{-m''_k}$$

for some $s''_{l_{k+2}, \ldots, l_N} \in R_{k+1}$ and $m''_k \in \mathbb{Z}_{\geq 0}$ (depending on $l_{k+2}, \ldots, l_N$). Therefore

$$y_k \mid_l \Big(\prod_{j=1}^{k-1} y_j^{m'_j} \Big) s_{l_{k+2}, \ldots, l_N} (y'_k)^{m''_k}.$$

It follows from Lemmas 7.5 and 7.6 that $y_k \mid s_{l_{k+2}, \ldots, l_N}$, $\forall l_{k+2}, \ldots, l_N \in \mathbb{Z}_{\geq 0}$ and thus $y_k \mid_l s$. This contradicts the minimality assumption on denominators and proves (7.9).

Finally, analogously to (7.7) one shows that

$$m_1 = m_1' = \ldots = m_{k-1} = m_{k-1}' = 0.$$

This completes the proof of Theorem 7.4. $\square$

CHAPTER 8

Symmetric CGL extensions and quantum cluster algebras

In this chapter we prove that every symmetric CGL extension possesses a quantum cluster algebra structure under a couple of very mild additional assumptions. Those assumptions are satisfied by all known examples of symmetric CGL extensions. The quantum cluster algebra structure is constructed in an explicit fashion. We furthermore prove that each of these quantum cluster algebras equals the corresponding upper quantum cluster algebra. The proofs work for base fields of arbitrary characteristic.

8.1. General setting

Fix a symmetric CGL extension R of length N and rank $\mathrm{rk}(R)$ as in Section 3.3 over a base field $\mathbb{K}$ which contains the square roots of all scalars λ_{lj}, $1 \leq j < l \leq N$. Choose $\nu_{lj} = \sqrt{\lambda_{lj}} \in \mathbb{K}^*$ and extend (as in (4.15)) to a multiplicatively skew-symmetric matrix $\boldsymbol{\nu} = (\nu_{lj}) \in M_N(\mathbb{K}^*)$ such that $\boldsymbol{\nu}^{\cdot 2} = \boldsymbol{\lambda}$. Define a multiplicatively skew-symmetric matrix $\mathbf{r} \in M_N(\mathbb{K}^*)$ by (4.16) and a sequence of normalized homogeneous prime elements $\overline{y}_1, \ldots, \overline{y}_N$ by (4.18). (We recall that generally each of those is a prime element of some of the subalgebras R_l, not of the full algebra $R = R_N$.) By Proposition 4.6 we obtain a toric frame $M : \mathbb{Z}^N \to \mathrm{Fract}(R)$ whose matrix is $\mathbf{r}(M) := \mathbf{r}$ and such that $M(e_k) := \overline{y}_k$, for all $k \in [1, N]$.

Next, consider an arbitrary element $\tau \in \Xi_N \subset S_N$, recall (3.11). We will associate to it a toric frame $M_\tau : \mathbb{Z}^N \to \mathrm{Fract}(R)$ such that $M_{\mathrm{id}} = M$. By Proposition 3.14 we have the CGL extension presentation

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}] \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}] \tag{8.1}$$

with $\sigma''_{\tau(k)} = (h''_{\tau(k)} \cdot)$ where $h''_{\tau(k)} \in \mathcal{H}$, $\forall k \in [1, N]$ and

$$\begin{aligned} h''_{\tau(k)} &= h_{\tau(k)}, \ \ \delta''_{\tau(k)} = \delta_{\tau(k)}, \quad \text{if } \tau(k) = \max \tau([1, k-1]) + 1 \\ h''_{\tau(k)} &= h^*_{\tau(k)}, \ \ \delta''_{\tau(k)} = \delta^*_{\tau(k)}, \quad \text{if } \tau(k) = \min \tau([1, k-1]) - 1. \end{aligned}$$

The $\boldsymbol{\lambda}$-matrix of the presentation (8.1) is $\boldsymbol{\lambda}_\tau := \tau^{-1} \boldsymbol{\lambda} \tau$ (where as before we use the canonical embedding $S_N \hookrightarrow GL_N(\mathbb{Z})$ via permutation matrices). In other words, the entries of $\boldsymbol{\lambda}_\tau$ are given by $(\boldsymbol{\lambda}_\tau)_{lj} := \lambda_{\tau(l)\tau(j)}$. For this presentation we choose the multiplicatively skew-symmetric matrix of square roots $\boldsymbol{\nu}_\tau := \tau^{-1} \boldsymbol{\nu} \tau \in M_N(\mathbb{K}^*)$ (with entries $(\boldsymbol{\nu}_\tau)_{lj} := \nu_{\tau(l)\tau(j)}$). Denote by $\widehat{\mathbf{r}}_\tau$ the corresponding multiplicatively skew-symmetric matrix derived from $\boldsymbol{\nu}_\tau$ by applying (4.16). Let $\overline{y}_{\tau,1}, \ldots, \overline{y}_{\tau,N}$ be the sequence of normalized prime elements given by (4.18) applied for the presentation (8.1). By Proposition 4.6 there exists a unique toric frame $\widehat{M}_\tau : \mathbb{Z}^N \to \mathrm{Fract}(R)$ whose matrix is $\mathbf{r}(\widehat{M}_\tau) := \widehat{\mathbf{r}}_\tau$ and such that $\widehat{M}_\tau(e_k) := \overline{y}_{\tau,k}$, $\forall k \in [1, N]$.

The toric frames $\widehat{M}_\tau$ will be the main toric frames that will be used to construct a quantum cluster algebra structure on the symmetric CGL extension R. In order to connect those frames with mutations (without the need of additional permutations) we need to permute the order of the cluster variables in each frame $\widehat{M}_\tau$. This is done by applying a composition of τ and a permutation $\tau_\bullet \in \prod_{a \in \mathbb{Z}} S_{\eta^{-1}(a)}$, where $\eta \colon [1, N] \to \mathbb{Z}$ is a function satisfying the conditions of Theorem 3.6 for the original CGL extension presentation of R. Note that all terms in the above product are trivial except for the terms coming from the range of η. For a in the range of η, denote for brevity $|a| := |\eta^{-1}(a)|$. Consider the set

$$\eta^{-1}(a) = \{\tau(k_1), \ldots, \tau(k_{|a|}) \mid k_1 < \cdots < k_{|a|}\} \tag{8.2}$$

and order its elements in an increasing order. Define $\tau_\bullet \in S_N$ by setting $\tau_\bullet(\tau(k_i))$ to be equal to the i-th element in the list (for all choices of a and i).

EXAMPLE 8.1. Let $N = 6$ and $\eta : [1, 6] \to \mathbb{Z}$ be given by

$$\eta(1) = \eta(4) = \eta(6) = 1, \quad \eta(2) = \eta(5) = 2, \quad \eta(3) = 3.$$

(This is the η-function for the standard CGL extension presentation of the quantum Schubert cell algebra $\mathcal{U}^-[w_\circ] \subset \mathcal{U}_q(\mathfrak{sl}_4)$ from Chapter 9, where $w_\circ$ is the longest element of S_4.) Let

$$\tau := \tau_{(4,5)} = [3, 4, 2, 5, 1, 6]$$

in the notation (5.4) from Section 5.2. Then

$$\eta^{-1}(1) = \{\tau(5) = 1 < \tau(2) = 4 < \tau(6) = 6\}$$

and

$$\tau_\bullet(4) = 1, \quad \tau_\bullet(1) = 4, \quad \tau_\bullet(6) = 6.$$

Similarly, one computes that $\tau_\bullet(j) = j$ for $j = 2$, 3 and 5.

Define the toric frame

$$M_\tau := \widehat{M}_\tau(\tau_\bullet \tau)^{-1} : \mathbb{Z}^N \to \mathrm{Fract}(R). \tag{8.3}$$

It satisfies $M_\tau(e_k) = \overline{y}_{\tau, (\tau_\bullet \tau)^{-1}(k)}$, $\forall k \in [1, N]$ and its matrix equals

$$\mathbf{r}_\tau := \mathbf{r}(M_\tau) = (\tau_\bullet \tau) \widehat{\mathbf{r}}_\tau (\tau_\bullet \tau)^{-1}.$$

The point of applying τ in this normalization is to match the indexing of the $\overline{y}$-elements with the one of the x-elements in (8.1). (Note that the order of the x-elements in (8.1) is $x_{\tau(1)}, \ldots, x_{\tau(N)}$.) The application of $\tau_\bullet$ then rearranges the η-preimages $\tau(k_1), \ldots, \tau(k_{|a|})$ from (8.2) in increasing order. This is needed because in the setting of Theorem 4.2 (b) the element y_k (not y_{k+1}) gets mutated. Clearly, $\tau_\bullet$ preserves the level sets of η.

Recall that $P(N) = \{k \in [1, N] \mid s(k) = +\infty\}$ parametrizes the set of homogeneous prime elements of R, i.e.,

$$\{y_k \mid k \in P(N)\} \text{ is a list of the homogeneous prime elements of } R \tag{8.4}$$

up to associates. Define

$$\mathbf{ex} := [1, N] \setminus P(N) = \{k \in [1, N] \mid s(k) \neq +\infty\}. \tag{8.5}$$

Since $|P(N)| = \mathrm{rk}(R)$, the cardinality of this set is $|\mathbf{ex}| = N - \mathrm{rk}(R)$. Finally, recall that for a homogeneous element $u \in R$, $\chi_u \in X(\mathcal{H})$ denotes its $\mathcal{H}$-eigenvalue.

8.2. Statement of the main result

The next theorem contains the main result of the paper. Keep M, $\tau_\bullet$, M_τ, $\mathbf{r}_\tau$, $\mathbf{ex}$, etc. as in Section 8.1.

THEOREM 8.2. *Let R be a symmetric CGL extension of length N and rank* $\mathrm{rk}(R)$ *as in Definition* 3.12. *Define* $\mathbf{ex} \subset [1, N]$ *by* (8.5). *Assume that the base field* $\mathbb{K}$ *contains square roots ν_{lj} of all scalars λ_{lj}, such that $\boldsymbol{\nu} := (\nu_{lj})$ is a multiplicatively skew-symmetric matrix and the subgroup of $\mathbb{K}^*$ generated by $\{\nu_{lj} \mid 1 \leq j < l \leq N\}$ does not contain elements of order* 2. *Assume also that there exist positive integers d_i, $i \in \mathrm{range}(\eta)$ such that*

$$(\lambda_l^*)^{d_{\eta(j)}} = (\lambda_j^*)^{d_{\eta(l)}}, \quad \forall j, l \in \mathbf{ex}, \tag{8.6}$$

recall the equality (5.11). *Let the sequence of generators $x_1, \ldots, x_N$ of R be normalized* (*rescaled*) *so that* (6.13) *is satisfied* (*recall Proposition* 6.3).

Then the following hold:

(a) *For all $\tau \in \Xi_N$* (*see* (3.11)) *and $l \in \mathbf{ex}$, there exists a unique vector $b_\tau^l \in \mathbb{Z}^N$ such that $\chi_{M_\tau(b_\tau^l)} = 1$ and*

$$\Omega_{\mathbf{r}_\tau}(b_\tau^l, e_j) = 1, \ \ \forall j \in [1, N], \ j \neq l \quad \textit{and} \quad \Omega_{\mathbf{r}_\tau}(b_\tau^l, e_l)^2 = \lambda_l^*. \tag{8.7}$$

Denote by $\widetilde{B}_\tau \in M_{N \times |\mathbf{ex}|}(\mathbb{Z})$ the matrix with columns b_τ^l, $l \in \mathbf{ex}$. Let $\widetilde{B} := \widetilde{B}_{\mathrm{id}}$.

(b) *For all $\tau \in \Xi_N$, the pair $(M_\tau, \widetilde{B}_\tau)$ is a quantum seed for* $\mathrm{Fract}(R)$. *The principal part of $\widetilde{B}_\tau$ is skew-symmetrizable via the integers $d_{\eta(k)}$, $k \in \mathbf{ex}$.*

(c) *All such quantum seeds are mutation-equivalent to each other. More precisely, they are linked by the following one-step mutations. Let $\tau, \tau' \in \Xi_N$ be such that*

$$\tau' = (\tau(k), \tau(k+1))\tau = \tau(k, k+1)$$

for some $k \in [1, N-1]$. If $\eta(\tau(k)) \neq \eta(\tau(k+1))$, then $M_{\tau'} = M_\tau$. If $\eta(\tau(k)) = \eta(\tau(k+1))$, then $M_{\tau'} = \mu_{k_\bullet}(M_\tau)$, where $k_\bullet = \tau_\bullet \tau(k)$.

(d) *We have the following equality between the CGL extension R and the quantum cluster and upper cluster algebras associated to M, $\widetilde{B}$, $\varnothing$:*

$$R = \mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} = \mathcal{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}.$$

In particular, $\mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$ and $\mathcal{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$ are affine and noetherian, and more precisely $\mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$ is generated by the cluster variables in the seeds parametrized by the subset Γ_N of Ξ_N, recall (5.7).

(e) *Let $\mathbf{inv}$ be any subset of the set $P(N)$ of frozen variables, cf.* (8.4). *Then*

$$R[y_k^{-1} \mid k \in \mathbf{inv}] = \mathcal{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}} = \mathcal{U}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}}.$$

Theorem 8.2 is proved in Section 8.6–8.9. The strategy of the proof is summarized in Section 8.5. In Section 8.3 we derive an explicit formula for the cluster variables of the quantum seeds that appear in the statement of Theorem 8.2. Proposition 2.6 (b) and Lemma 2.13 imply at once the following properties of all quantum seeds of the quantum cluster algebras in Theorem 8.2.

PROPOSITION 8.3. *All quantum seeds $(M_\star, \widetilde{B}_\star)$ of the quantum cluster algebras in Theorem* 8.2 *have the properties that*

$$\chi_{M_\star(b^l)} = 1, \quad \Omega_{\mathbf{r}(M_\star)}(b_\star^l, e_j) = 1, \ \ \forall j \in [1, N], \ j \neq l, \quad \textit{and} \quad \Omega_{\mathbf{r}(M_\star)}(b_\star^l, e_l)^2 = \lambda_l^*$$

for all $l \in \mathbf{ex}$, where the $b_\star^l$, $l \in \mathbf{ex}$ are the columns of $\widetilde{B}_\star$.

In Section 8.10, we give a ring theoretic interpretation of some columns of the initial exchange matrix $\widetilde{B}$, in terms of the exponent vectors $f_{[i,s(i)]}$ for certain of the elements $u_{[i,s(i)]}$.

EXAMPLE 8.4. Let $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$ with the CGL extension presentation given in Example 3.4, and assume there exists a square root of q in $\mathbb{K}^*$. Fix a choice of $\sqrt{q}$. Define η for R as in Example 3.7 and $\mathbf{ex}$ as in (8.5). Then

$$\mathbf{ex} = \{(r-1)n + c \mid r \in [1, m-1],\ c \in [1, n-1]\}.$$

Next, define ν_{lj} for $l, j \in [1, N]$ by replacing q with $\sqrt{q}$ in (3.3). Then the matrix (ν_{lj}) is multiplicatively skew-symmetric, and the subgroup $\langle \nu_{lj} \mid 1 \le j < l \le N \rangle$ of $\mathbb{K}^*$ contains no elements of order 2. We saw in Example 3.13 that the CGL extension presentation of R is symmetric and that all $\lambda_j^* = q^2$. Hence, (8.6) is trivially satisfied with $d_i := 1$ for all $i \in \operatorname{range}(\eta)$. Finally, Example 6.4 shows that condition (6.13) is satisfied.

Therefore all hypotheses of Theorem 8.2 are satisfied, and the theorem provides a quantum cluster algebra structure on R. The mutation matrix $\widetilde{B} = (b_{ij})$ for R is easily computed:

$$b_{(r-1)n+c,\,(r'-1)n+c'} = \begin{cases} \pm 1, & \text{if } r = r',\ c' = c \pm 1 \\ & \text{or } c = c',\ r' = r \pm 1 \\ & \text{or } r = r' \pm 1,\ c = c' \pm 1, \\ 0, & \text{otherwise} \end{cases} \qquad \begin{array}{c} \forall r, r' \in [1, m], \\ c, c' \in [1, n]. \end{array}$$

REMARK 8.5. As noted in Remark 4.5, after a field extension every symmetric CGL extension can be brought to one that satisfies $\sqrt{\lambda_{lj}} \in \mathbb{K}^*$ for all $1 \le j < l \le N$. Recall that a CGL extension R is called torsionfree if the subgroup of $\mathbb{K}^*$ generated by $\{\lambda_{lj} \mid 1 \le j < l \le N\}$ is torsionfree. The torsionfree symmetric CGL extensions form the largest and most interesting class of known symmetric CGL extensions. All CGL extensions coming from the theory of quantum groups are in this class. For every torsionfree symmetric CGL extension R, one can always make a choice of the square roots $\nu_{lj} = \sqrt{\lambda_{lj}} \in \mathbb{K}$, $1 \le j < k \le N$ such that the subgroup of $\mathbb{K}^*$ generated by all of them is torsionfree and thus does not contain -1 (i.e., condition (4.23) is satisfied for all of them).

REMARK 8.6. All of the symmetric CGL extensions that we are aware of satisfy

$$\lambda_k^* = q^{m_k},\ \forall k \in [1, N]$$

for some non-root of unity $q \in \mathbb{K}^*$ and $m_1, \ldots, m_N \in \mathbb{Z}_{>0}$. It follows from Proposition 5.8 that $m_j = m_k$ for all $j, k \in \mathbf{ex}$ with $\eta(j) = \eta(k)$. This implies that all such CGL extensions have the property (8.6) where the integers d_i are simply chosen as $d_{\eta(k)} := m_k$, $\forall k \in \mathbf{ex}$.

REMARK 8.7. For applications, it is useful to determine the exchange matrix $\widetilde{B}$ *before* the generators x_k have been rescaled to satisfy (6.13). This is possible because the rescaling does not change $\widetilde{B}$, as we next note.

Assume R is a symmetric CGL extension satisfying all the hypotheses of Theorem 8.2 *except* (6.13). Let the elements y_k, $\overline{y}_k$, the skew-symmetric matrix $\mathbf{r}$, and the toric frame M be as in Section 8.1. Now suppose that we rescale the x_k according to Proposition 6.3, say with new generators $x'_1, \ldots, x'_N$, to make (6.13) hold. Build the toric frame and its matrix for the new setting as in Section 8.1.

Since the scalars λ_{kj} and ν_{kj} do not change, the matrix $\mathbf{r}$ does not change. The new versions of the $\overline{y}_k$, call them $\overline{y}'_k$, are scalar multiples of the old ones, and the new toric frame, call it M', satisfies $M'(e_k) = \overline{y}'_k$ for all k. Hence, for any vector $b \in \mathbb{Z}^N$, the element $M'(b)$ is a scalar multiple of $M(b)$, and thus $\chi_{M'(b)} = \chi_{M(b)}$. Finally, the same elements $h_k, h_k^* \in \mathcal{H}$ which enter into the symmetric CGL conditions for the original generators are used with respect to the new generators, which means that the scalars λ_k and λ_k^* do not change under the rescaling.

Thus, the conditions in Theorem 8.2 (a) which uniquely determine the columns of $\widetilde{B} = \widetilde{B}_{\mathrm{id}}$ are the same before and after rescaling. Therefore $\widetilde{B}$ does not change under the rescaling.

Remark 8.8. Theorem 8.2 raises several questions and problems.

(a) For a symmetric quantum nilpotent algebra R, the elements of Ξ_N give rise to seeds for the quantum cluster algebras structure on R whose cluster variables are prime elements in some subalgebras of R. Do the remaining seeds of R have any similar ring-theoretic meaning?

(b) In general Theorem 8.2 constructs quantum cluster algebras whose quantum tori are multiparameter in the sense of Chapter 2 rather than one-parameter in the sense of [**3**]. However, all examples of such multiparameter quantum cluster algebras that we are aware of come from 2-cocycle twists of one-parameter CGL extensions (e.g., quantum Schubert cell algebras). Are there any families of symmetric CGL extensions that are genuinely multiparameter, in the sense that they are not obtained from each other by 2-cocycle twists?

8.3. Cluster variables

The next result gives an explicit formula for the cluster variables that appear in Theorem 8.2.

Proposition 8.9. *Assume the setting of Theorem* 8.2. *Let* $\tau \in \Xi_N$ *and* $k \in [1, N]$.

If $\tau(k) \geq \tau(1)$, *then* $\overline{y}_{\tau,k} = \overline{y}_{[p^m(\tau(k)),\tau(k)]}$, *where*

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\tau(k)) \in \tau([1, k])\}.$$

If $\tau(k) \leq \tau(1)$, *then* $\overline{y}_{\tau,k} = \overline{y}_{[\tau(k),s^m(\tau(k))]}$, *where*

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid s^n(\tau(k)) \in \tau([1, k])\}.$$

Here the predecessor and successor functions are computed with respect to the original CGL extension presentation (3.1) *of* R.

Remark 8.10. Theorem 8.2 (d) and Proposition 8.9 imply that the quantum cluster algebra $\mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} = R$ coincides with the subalgebra of $\mathrm{Fract}(R)$ generated by the cluster variables in the (finite!) set of toric frames $\{M_\tau \mid \tau \in \Gamma_N\}$.

Proof of Proposition 8.9. We prove the two cases of the proposition in parallel, by induction on k. If $k = 1$, both cases are clear because $\overline{y}_{\tau,1} = x_{\tau(1)} = \overline{y}_{[\tau(1),\tau(1)]}$. Assume now that $k > 1$. For the induction step, we will restrict to the second of the two cases, i.e., the case where $\tau(k) < \tau(1)$. The first case is similar; it is left to the reader.

By Theorem 5.3, $\overline{y}_{\tau,k} = \xi\, \overline{y}_{[\tau(k),s^m(\tau(k))]}$ for some $\xi \in \mathbb{K}^*$, so all we need to show is that the leading terms of the two prime elements are equal. From the fact that $\tau([1,j])$ is an interval for all $j \le k$ and the assumption that $\tau(k) \le \tau(1)$ it follows that

$$\tau([1,k]) = [\tau(k), \tau(i)] \quad \text{for some} \quad i \in [1,k].$$

Hence, $\tau([1,k-1]) = \tau([1,k]) \backslash \{\tau(k)\} = [\tau(k)+1, \tau(i)]$ and thus

$$R_{\tau,k} = R_{[\tau(k),\tau(i)]} \quad \text{and} \quad R_{\tau,k-1} = R_{[\tau(k)+1,\tau(i)]}.$$

All leading terms in what follows will be computed with respect to the standard CGL extension presentation of $R_{[\tau(k),\tau(i)]}$ obtained by adjoining $x_{\tau(k)}, x_{\tau(k)+1}, \ldots, x_{\tau(i)-1}, x_{\tau(i)}$ in this order. By Theorem 3.6 and Corollary 5.6 (b),

$$y_{\tau,k} = y_{\tau,p_\tau(k)} x_{\tau(k)} - d_{k-1}$$

for some $d_{k-1} \in R_{\tau,k-1}$, where p_τ is the predecessor function for the level sets of $\eta_\tau = \eta\tau : [1,N] \to \mathbb{Z}$.

Suppose first that $\tau(p_\tau(k)) \le \tau(1)$, and note that $\tau([1,p_\tau(k)]) = [\tau(p_\tau(k)), \tau(i')]$ for some $i' \in [1, p_\tau(k)]$. By definition of $p_\tau(k)$, we have $\eta_\tau(j) \ne \eta_\tau(k)$, $\forall j \in [p_\tau(k)+1, k-1]$, and so $\tau(p_\tau(k)) = s(\tau(k))$. Hence,

$$\max\{n \in \mathbb{Z}_{\ge 0} \mid s^n(\tau(p_\tau(k))) \in \tau([1, p_\tau(k)])\} = m-1.$$

The induction hypothesis thus implies that $\overline{y}_{\tau,p_\tau(k)} = \overline{y}_{[s(\tau(k)),s^m(\tau(k))]}$. On the other hand, if $\tau(p_\tau(k)) \ge \tau(1)$, we find that $\tau(p_\tau(k)) = s^m(\tau(k))$, and the induction hypothesis again implies $\overline{y}_{\tau,p_\tau(k)} = \overline{y}_{[s(\tau(k)),s^m(\tau(k))]}$.

Applying Eq. (6.11) for $\mathcal{S}_{\boldsymbol{\nu}_\tau}$ and $\mathcal{S}_{\boldsymbol{\nu}}$ gives

$$\begin{aligned}
\operatorname{lt}(\overline{y}_{\tau,k}) &= \mathcal{S}_{\boldsymbol{\nu}_\tau}(e^\tau_{[p^m_\tau(k),k]}) \operatorname{lt}(y_{\tau,p_\tau(k)} x_{\tau(k)}) \\
&= \mathcal{S}_{\boldsymbol{\nu}_\tau}(e^\tau_{[p^m_\tau(k),k]}) \Omega_{\boldsymbol{\nu}}(e_{\tau(k)}, e_{[s(\tau(k)),s^m(\tau(k))]})^{-2} \mathcal{S}_{\boldsymbol{\nu}_\tau}(e^\tau_{[p^m_\tau(k),p_\tau(k)]})^{-1} \times \\
&\qquad\qquad x_{\tau(k)} \operatorname{lt}(\overline{y}_{\tau,p_\tau(k)}) \\
&= \Omega_{\boldsymbol{\nu}}(e_{\tau(k)}, e_{[s(\tau(k)),s^m(\tau(k))]})^{-1} x_{\tau(k)} \operatorname{lt}(\overline{y}_{[s(\tau(k)),s^m(\tau(k))]}) \\
&= \operatorname{lt}(\overline{y}_{[\tau(k),s^m(\tau(k))]}),
\end{aligned}$$

where $e^\tau_{[p^m_\tau(k),p^l_\tau(k)]} = e_{p^m_\tau(k)} + e_{p^{m-1}_\tau(k)} + \cdots + e_{p^l_\tau(k)}$ for $m \ge l \ge 0$. This completes the proof of the proposition. $\square$

Corollary 8.11. *Assume the setting of Theorem 8.2. The cluster variables of the seeds $(M_\tau, \widetilde{B}_\tau)$ for τ in Ξ_N or in Γ_N are exactly the homogeneous prime elements $\overline{y}_{[i,j]}$. More precisely,*

$$\begin{aligned}
\{M_\tau(e_k) \mid \tau \in \Xi_N,\ k \in [1,N]\} &= \{M_\tau(e_k) \mid \tau \in \Gamma_N,\ k \in [1,N]\} \\
&= \{\overline{y}_{[i,j]} \mid 1 \le i \le j \le N,\ \eta(i) = \eta(j)\}.
\end{aligned}$$

Proof. The second of the displayed sets is contained in the first a priori, and the first is contained in the third by Proposition 8.9 and the definition of the M_τ. It remains to show that the third set is contained in the second. Thus, let $1 \le i \le j \le N$ with $\eta(i) = \eta(j)$. Then $i = p^m(j)$ where

$$m = \max\{n \in \mathbb{Z}_{\ge 0} \mid p^n(j) \in [i,j]\}.$$

Set $k' := j - i + 1$, and choose $\tau \in \Gamma_N$ as follows:

$$\tau = \begin{cases} \mathrm{id} = \tau_{1,1} & (\text{if } i = 1) \\ \tau_{i-1,j} = [i, \ldots, j, i-1, j+1, \ldots, N, i-1, \ldots, 1] & (\text{if } i > 1). \end{cases}$$

Then $\tau(1) = i \leq j = \tau(k')$ and $\tau([1,k']) = [i,j]$. Consequently,

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n(\tau(k')) \in \tau([1,k'])\},$$

and Proposition 8.9 shows that

$$\overline{y}_{\tau,k'} = \overline{y}_{[p^m(\tau(k')),\tau(k')]} = \overline{y}_{[i,j]}.$$

Therefore $\overline{y}_{[i,j]} = \overline{y}_{\tau,(\tau_\bullet\tau)^{-1}(k)} = M_\tau(e_k)$ where $k := \tau_\bullet\tau(k')$. □

EXAMPLE 8.12. Let $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$, assume there exists $\sqrt{q} \in \mathbb{K}^*$, and let R have the quantum cluster algebra structure coming from Theorem 8.2 as in Example 8.4. The cluster variables of the seeds $(M_\tau, \widetilde{B}_\tau)$ in Theorem 8.2 are exactly the solid quantum minors within $[1,m]\times[1,n]$. To see this, first recall from Example 4.7 that $\nu_{ij} = 1$ for all $i,j \in [1,N]$ with $\eta(i) = \eta(j)$. Consequently, (6.16) yields $\overline{y}_{[i,s^l(i)]} = y_{[i,s^l(i)]}$ for all $i \in [1,N]$ and $l \in \mathbb{Z}_{\geq 0}$ with $s^l(i) \neq +\infty$. These elements, by Corollary 8.11, are exactly the cluster variables $M_\tau(e_k)$ for $\tau \in \Xi_N$ and $k \in [1,N]$. On the other hand, each $y_{[i,s^l(i)]}$ is a solid quantum minor by Example 5.2. Conversely, for any $l \in \mathbb{Z}_{\geq 0}$ and $r, r+l \in [1,m]$, $c, c+l \in [1,n]$, we have

$$\Delta_{[r,r+l],[c,c+l]} = y_{[(r-1)n+c,\, s^l((r-1)n+c)]}.$$

8.4. Auxiliary results

In this section we establish two results that will be needed for the proof of Theorem 8.2. The first one uses Theorem 6.6 and Proposition 8.9 to construct mutations between pairs of the toric frames M_τ for $\tau \in \Xi_N$. The corresponding mutations of quantum seeds (Theorem 8.2 (c)) are constructed in Section 8.6–8.7.

For $g = \sum_j g_j e_j \in \mathbb{Z}^N$ set

$$\operatorname{supp}(g) := \{j \in [1,N] \mid g_j \neq 0\}.$$

PROPOSITION 8.13. *Let R be a symmetric CGL extension of length N and $\nu_{kj} = \sqrt{\lambda_{kj}} \in \mathbb{K}^*$ for $1 \leq j < k \leq N$ such that (4.23) is satisfied. Assume that the generators of R are rescaled so that the condition (6.13) is satisfied.*

Let $\tau, \tau' \in \Xi_N$ be such that

$$\tau' = (\tau(k), \tau(k+1))\tau = \tau(k, k+1)$$

for some $k \in [1, N-1]$ such that $\tau(k) < \tau(k+1)$.

(a) *If $\eta(\tau(k)) \neq \eta(\tau(k+1))$, then $M_{\tau'} = M_\tau$.*

(b) *Let $\eta(\tau(k)) = \eta(\tau(k+1))$. Set $k_\bullet := \tau_\bullet\tau(k)$. Then $k_\bullet = (\tau')_\bullet\tau'(k)$ and*

$$M_{\tau'}(e_j) = \begin{cases} M_\tau(e_j), & \text{if } j \neq k_\bullet \\ M_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - e_{k_\bullet}) + M_\tau(g - e_{k_\bullet}), & \text{if } j = k_\bullet \end{cases} \tag{8.8}$$

for some $g \in \mathbb{Z}^N_{\geq 0}$ such that $\operatorname{supp}(g) \cap \eta^{-1}\eta(k_\bullet) = \varnothing$ and $|\operatorname{supp}(g) \cap \eta^{-1}(a)| \leq 1$ for all $a \in \mathbb{Z}$. Furthermore, the vector $e_{p(k_\bullet)} + e_{s(k_\bullet)} - g \in \mathbb{Z}^N$ satisfies the identities

$$\Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_j) = 1, \quad \forall j \neq k_\bullet, \tag{8.9}$$

$$\Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_{k_\bullet})^2 = \lambda^*_{k_\bullet}, \tag{8.10}$$

and

$$\chi_{M_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g)} = 1. \tag{8.11}$$

Note that the condition $\tau(k) < \tau(k+1)$ is not essential since, if $\tau(k) > \tau(k+1)$ and all other conditions are satisfied, then one can interchange the roles of τ and τ'.

PROOF. Part (a) follows from Theorem 4.2 (a) applied to the CGL extension presentation (8.1) of R associated to τ.

(b) Since $\tau([1,j])$ is an interval for all $j \leq k+1$ and $\tau(k) < \tau(k+1)$, we have $\tau'([1,k+1]) = \tau([1,k+1]) = [\tau(i), \tau(k+1)]$ and $\tau([1,k]) = [\tau(i), \tau(k+1)-1]$ for some $i \in [1,k]$. On the other hand, the set

$$\tau'([1,k]) = \tau'([1,k+1]) \setminus \{\tau'(k+1)\} = \tau([1,k+1]) \setminus \{\tau(k)\}$$

must be also an interval, so $\tau(k) = \tau(i)$ and $i = k$. Therefore,

$$\begin{aligned} \tau'([1,k+1]) = \tau([1,k+1]) &= [\tau(k), \tau(k+1)], \\ \tau([1,k]) = [\tau(k), \tau(k+1)-1], \quad &\text{and} \quad \tau'([1,k]) = [\tau(k)+1, \tau(k+1)]. \end{aligned} \tag{8.12}$$

This implies that

$$\tau(k+1) = s^m(\tau(k)) \ \text{ for some } \ m \in \mathbb{Z}_{>0}$$

and

$$\eta^{-1}(\eta\tau(k)) \cap \tau([1,k+1]) = \{\tau(k), s(\tau(k)), \ldots, \tau(k+1) = s^m(\tau(k))\}. \tag{8.13}$$

From the last identity and the fact that $\tau(j) = \tau'(j)$ for $j \neq k, k+1$ we infer

$$\tau^{-1}\eta^{-1}(\eta\tau(k)) = (\tau')^{-1}\eta^{-1}(\eta\tau(k)).$$

By the definition of the permutations $\tau_\bullet$ and $(\tau')_\bullet$,

$$k_\bullet = \tau_\bullet\tau(k) = (\tau')_\bullet\tau'(k) = s^{m-1}(\tau(k)).$$

From Theorem 4.8 (b) we have $\widehat{M}_{\tau'}(e_l) = \widehat{M}_\tau(e_l)$ for $l \neq k$. Eq. (8.8) for $j \neq k_\bullet$ follows from this.

The identities in (8.12) and Proposition 8.9 give

$$\begin{aligned} \overline{y}_{\tau,k} &= \overline{y}_{[\tau(k), s^{m-1}(\tau(k))]}, \qquad & \overline{y}_{\tau,k+1} = \overline{y}_{[\tau(k), s^m(\tau(k))]}, \\ \overline{y}_{\tau',k} &= \overline{y}_{[s(\tau(k)), s^m(\tau(k))]}. \end{aligned} \tag{8.14}$$

Recall the definition of the toric frames $M_{[i,s^m(i)]}$ from Section 6.3. We will construct an isomorphism $\dot{w}$ from $\mathbb{Z}^{k+1}$ to $\mathbb{Z}e_{\tau(k)} + \cdots + \mathbb{Z}e_{s^m(\tau(k))}$ such that the frames $M_{[\tau(k),s^m(\tau(k))]}\dot{w}$ and $\widehat{M}_\tau$ agree on suitable e_j.

Recall the definition of the set $P_{[\tau(k),s^m(\tau(k))]}$ from (5.20), set

$$\begin{aligned} A &:= \eta(P_{[\tau(k),s^m(\tau(k))]}) = \eta(\tau(1,k+1]) \setminus \{\eta(\tau(k))\} \\ Q &:= \{j \in [1,k-1] \mid \eta(\tau(j)) \neq \eta(\tau(l)), \ \ \forall l \in [j+1,k+1]\}, \end{aligned}$$

and note that $\eta\tau$ restricts to a bijection of Q onto A. Thus,

$$|Q| = |A| = |P_{[\tau(k),s^m(\tau(k))]}|. \tag{8.15}$$

The definition of Q also ensures that

$$\{t \in [\tau(k), s^m(\tau(k))] \mid \eta(t) = \eta(\tau(j))\} \subseteq \tau([1,j]), \ \ \forall j \in Q. \tag{8.16}$$

If $j \in Q$ and $\tau(j) \geq \tau(1)$, then $\tau([1,j]) = [\tau(i_j), \tau(j)]$ for some $i_j \in [1,j]$, and we observe that $\tau(j) \in P_{[\tau(k),s^m(\tau(k))]}$. Moreover, the integer m_j corresponding to j in Proposition 8.9 equals $O_-^{\tau(k)+1}(\tau(j))$, and hence we obtain

$$\overline{y}_{\tau,j} = \overline{y}_{[p^{m_j}(\tau(j)),\tau(j)]} = M_{[\tau(k),s^m(\tau(k))]}(e_{\tau(j)}), \ \ \forall j \in Q \ \text{ with } \ \tau(j) \geq \tau(1). \tag{8.17}$$

On the other hand, if $\tau(j) < \tau(1)$, then $\tau([1,j]) = [\tau(j), \tau(i_j)]$ for some $i_j \in [1, j-1]$. Let m_j denote the integer corresponding to j in Proposition 8.9, and observe that $s^{m_j}(\tau(j)) = \tau(j^-) \in P_{[\tau(k), s^m(\tau(k))]}$ for some $j^- \in [1,j]$. Moreover, $m_j = O_-^{\tau(k)+1}(\tau(j^-))$, and so

$$(8.18)\quad \overline{y}_{\tau,j} = \overline{y}_{[p^{m_j}(\tau(j^-)), \tau(j^-)]} = M_{[\tau(k), s^m(\tau(k))]}(e_{\tau(j^-)}), \qquad \forall j \in Q \text{ with } \tau(j) < \tau(1).$$

In case $m > 1$, we set

$$t := \max(\tau^{-1}\{s(\tau(k)), \dots, s^{m-1}(\tau(k))\}) = \max\{j \in [1, k-1] \mid \eta(\tau(j)) = \eta(\tau(k))\}$$

(see (8.13)). Then either $\tau(t) = s^{m-1}(\tau(k))$ or $\tau(t) = s(\tau(k))$, and Proposition 8.9 yields

$$(8.19)\qquad \overline{y}_{\tau,t} = \overline{y}_{[s(\tau(k)), s^{m-1}(\tau(k))]} = M_{[\tau(k), s^m(\tau(k))]}(e_{s^{m-1}(\tau(k))}), \text{ if } m > 1.$$

Now choose a bijection $w : [1, k+1] \to [\tau(k), s^m(\tau(k))]$ such that

$$\begin{aligned} w(k+1) &= s^m(\tau(k)) \\ w(k) &= \tau(k) \\ w(t) &= s^{m-1}(\tau(k)), \text{ if } m > 1 \\ w(j) &= \tau(j), \ \forall j \in Q \text{ with } \tau(j) \geq \tau(1) \\ w(j) &= \tau(j^-), \ \forall j \in Q \text{ with } \tau(j) < \tau(1), \end{aligned}$$

and let $\dot{w}$ denote the isomorphism $\mathbb{Z}^{k+1} \to \mathbb{Z}e_{\tau(k)} + \dots + \mathbb{Z}e_{s^m(\tau(k))}$ such that $\dot{w}(e_j) = e_{w(j)}$ for $j \in [1, k+1]$. In particular, $\eta w(j) = \eta(\tau(j))$, $\forall j \in Q$, so $\eta w|_Q$ is injective. By construction, $w(Q) \subseteq P_{[\tau(k), s^m(\tau(k))]}$, and so we conclude from (8.15) that

$$(8.20)\qquad w|_Q : Q \longrightarrow P_{[\tau(k), s^m(\tau(k))]} \quad \text{is a bijection.}$$

Combining Eqs. (8.14) and (8.17)–(8.19) with the definition of w, we see that

$$(8.21)\quad \widehat{M}_\tau(e_j) = \overline{y}_{\tau,j} = M_{[\tau(k), s^m(\tau(k))]}\dot{w}(e_j), \quad \begin{cases} \forall j \in Q \cup \{t, k, k+1\} & (m > 1) \\ \forall j \in Q \cup \{k, k+1\} & (m = 1). \end{cases}$$

Comparing the matrices of the toric frames $\widehat{M}_\tau|_{\mathbb{Z}^{k+1}}$ and $M_{[\tau(k), s^m(\tau(k))]}\dot{w}$ (recall (4.16), (6.17), and (2.11)), we find that

$$(8.22)\quad \big(\mathbf{r}(\widehat{M}_\tau|_{\mathbb{Z}^{k+1}})\big)_{lj} = \big((\mathbf{r}_{[\tau(k), s^m(\tau(k))]})_{\dot{w}}\big)_{lj}, \qquad \begin{cases} \forall l, j \in Q \cup \{t, k, k+1\} & (m > 1) \\ \forall l, j \in Q \cup \{k, k+1\} & (m = 1). \end{cases}$$

The next step is to apply Theorem 6.6. We do the case $m > 1$ and leave the case $m = 1$ to the reader. (In the latter case, $p(k_\bullet) = -\infty$ and $e_{p(k_\bullet)} = 0$.) Observe that (8.21) and (8.22) together imply that

$$(8.23)\quad \widehat{M}_\tau(f) = M_{[\tau(k), s^m(\tau(k))]}\dot{w}(f), \ \forall f \in \mathbb{Z}^N \text{ with } \operatorname{supp}(f) \subseteq Q \cup \{t, k, k+1\}.$$

Thus, taking account of (8.14) and (8.20), Theorem 6.6 implies that

$$\overline{y}_{\tau',k} = \widehat{M}_\tau(e_t + e_{k+1} - e_k) + \widehat{M}_\tau(g' - e_k),$$

where $g' \in \mathbb{Z}_{\geq 0}e_1 + \cdots + \mathbb{Z}_{\geq 0}e_{k-1}$ is such that $\operatorname{supp}(g') \subseteq Q$. By the definition of $\tau_\bullet$ and Eq. (8.13),

$$\tau_\bullet \tau(t) = p(k_\bullet) \text{ and } \tau_\bullet \tau(k+1) = s(k_\bullet).$$

Therefore,

$$M_{\tau'}(e_{k_\bullet}) = \widehat{M}_{\tau'}(e_k) = \overline{y}_{\tau',k} = M_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - e_{k_\bullet}) + M_\tau(\tau_\bullet \tau(g') - e_{k_\bullet}),$$

which implies the validity of (8.8) for $j = k_\bullet$.

Finally, the identities (8.9)–(8.10) follow from Theorem 4.9, Eq. (8.23) and the fact that $R_{\tau,k+1} = R_{[\tau(k),s^m(\tau(k))]}$, see (8.12). We note that $\eta(k_\bullet) = \eta(\tau(k))$ and $s(\tau(k)) \neq +\infty$, $s(k_\bullet) \neq +\infty$, which follow from the definition of $\tau_\bullet$ and Eq. (8.13). Because of this and Proposition 5.8, $\chi_{x_{\tau(k)}}(h^*_{\tau(k)}) = \lambda^*_{\tau(k)} = \lambda^*_{k_\bullet}$. The identity (8.11) follows from the fact that $\overline{y}_{\tau,j}$ and $\overline{y}_{\tau',j}$ are $\mathcal{H}$-eigenvectors for all $j \in [1, N]$ and Eq. (8.8). □

The next lemma proves uniqueness of integral vectors satisfying bilinear identities of the form (8.9)–(8.10) from strong $\mathcal{H}$-rationality of CGL extensions.

LEMMA 8.14. *Assume that R is a symmetric CGL extension of length N and $(\nu_{kj}) \in M_N(\mathbb{K}^*)$ is a multiplicatively skew-symmetric matrix with $\nu_{kj}^2 = \lambda_{kj}$, $\forall j, k \in [1, N]$. Then for all $\tau \in \Xi_N$, $\theta \in X(\mathcal{H})$, and $\xi_1, \dots, \xi_N \in \mathbb{K}^*$, there exists at most one vector $b \in \mathbb{Z}^N$ such that $\chi_{M_\tau(b)} = \theta$ and $\Omega_{\mathbf{r}_\tau}(b, e_j)^2 = \xi_j$, $\forall j \in [1, N]$.*

PROOF. Let $b_1, b_2 \in \mathbb{Z}^N$ be such that $\chi_{M_\tau(b_1)} = \chi_{M_\tau}(b_2) = \theta$ and $\Omega_{\mathbf{r}_\tau}(b_1, e_j)^2 = \Omega_{\mathbf{r}_\tau}(b_2, e_j)^2 = \xi_j$, $\forall j \in [1, N]$. Then $M_\tau(b_1)M_\tau(b_2)^{-1}$ commutes with $M_\tau(e_j)$ for all $j \in [1, N]$. This implies that $M_\tau(b_1)M_\tau(b_2)^{-1}$ belongs to the center of $\operatorname{Fract}(R)$, because by Proposition 3.11 $\operatorname{Fract}(R)$ is generated (as a division algebra) by $M_\tau(e_1)^{\pm 1}, \dots, M_\tau(e_N)^{\pm 1}$. Furthermore,

$$\chi_{M_\tau(b_1)M_\tau(b_2)^{-1}} = 1.$$

By the strong $\mathcal{H}$-rationality of the 0 ideal of a CGL extension [**4**, Theorem II.6.4],

$$Z(\operatorname{Fract}(R))^{\mathcal{H}} = \mathbb{K},$$

where $Z(.)$ stands for the center of an algebra and $(.)^{\mathcal{H}}$ for the subalgebra fixed by $\mathcal{H}$. Hence, $M_\tau(b_1)M_\tau(b_2)^{-1} \in \mathbb{K}$, which is only possible if $b_1 = b_2$. □

8.5. An overview of the proof of Theorem 8.2

In this section we give a summary of the strategy of our proof of Theorem 8.2.

In Section 8.1 we constructed quantum frames $M_\tau : \mathbb{Z}^N \to \operatorname{Fract}(R)$ associated to the elements of the set Ξ_N. In order to extend them to quantum seeds of $\operatorname{Fract}(R)$, one needs to construct a compatible matrix $\widetilde{B}_\tau \in M_{N \times |\mathbf{ex}|}(\mathbb{Z})$ for each of them. This will be first done for the subset Γ_N of Ξ_N in an iterative fashion with respect to the linear ordering (5.7). If τ and τ' are two consecutive elements of Γ_N in that linear ordering, then $\tau' = \tau(k, k+1)$ for some $k \in [1, N]$ such that $\tau(k) < \tau(k+1)$. If $\eta(\tau(k)) \neq \eta(\tau(k+1))$ then $M_{\tau'} = M_\tau$ by Proposition 8.13 (a) and nothing happens at that step. If $\eta(\tau(k)) = \eta(\tau(k+1))$, then we use Proposition 8.13 (b) to construct $b^{k_\bullet}_\tau$ and $b^{k_\bullet}_{\tau'}$ where $k_\bullet := (\tau_\bullet \tau)(k)$. Up to $\pm$ sign these vectors are equal to $e_{p(k_\bullet)} + e_{s(k_\bullet)} - g$, where $g \in \mathbb{Z}^N_{\geq 0}$ is the vector from Proposition 8.13 (b). Then we use "reverse" mutation to construct $b^{k_\bullet}_\sigma$ for $\sigma \in \Gamma_N$, $\sigma \prec \tau$ in the linear order (5.7). Effectively this amounts to starting with a quantum cluster algebra in which all variables are frozen and then recursively adding more exchangeable variables.

There are two things that could go wrong with this. Firstly, the reverse mutations from different stages might not be synchronized. Secondly, there are many pairs of consecutive elements τ, τ' for which $k_\bullet$ is the same. So we need to prove that $b_\sigma^{k_\bullet}$ is not *overdetermined*. We use strong rationality of CGL extensions to handle both via Lemma 8.14. This part of the proof (of parts (a) and (b) of Theorem 8.2) is carried out in Section 8.6.

Once $\widetilde{B}_{\mathrm{id}}$ is (fully) constructed then the $\widetilde{B}_\tau$ are constructed inductively by applying Proposition 8.13 and using the sequences of elements of Ξ_N from Corollary 5.6 (a). At each step Lemma 8.14 is applied to match columns of mutation matrices. This proves parts (a) and (b) of Theorem 8.2. Part (c) of the theorem is obtained in a somewhat similar manner from Proposition 8.13. This is done in Section 8.7.

The last two parts (d)–(e) of Theorem 8.2 are proved in Section 8.9. For each $\tau \in \Xi_N$ we denote by E_τ the multiplicative subset of R generated by $\mathbb{K}^*$ and $M_\tau(e_j)$ for $j \in \mathbf{ex}$. An application of Section 7.1 gives that it is an Ore subset of R. The idea of the proof of Theorem 8.2 (d) is to obtain the following chain of embeddings

$$R \subseteq \mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} \subseteq \mathcal{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} \subseteq \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}] = R,$$

which clearly implies the desired result. The first inclusion follows from the fact that each generator x_j of R is a cluster variable for the toric frame M_τ associated to some $\tau \in \Gamma_N$. The hardest is the last equality. It is derived from Theorem 7.4 by using that the consecutive toric frames associated to the sequence of elements (5.7) are obtained from each other by one-step mutations and each $j \in \mathbf{ex}$ gets mutated at least one time along the way. Part (e) of Theorem 8.2 is proved in a similar fashion.

Both parts (d) and (e) follow from results about intersections of localizations of symmetric CGL extensions. These intersection results, which do not require any rescaling of variables, are proved in Section 8.8.

8.6. Recursive construction of quantum seeds for $\tau \in \Gamma_N$

Recall the linear ordering (5.7) on $\Gamma_N \subset \Xi_N$. We start by constructing a chain of subsets $\mathbf{ex}_\tau \subseteq \mathbf{ex}$ indexed by the elements of Γ_N such that

$$\mathbf{ex}_{\mathrm{id}} = \varnothing, \quad \mathbf{ex}_{w_\circ} = \mathbf{ex} \quad \text{and} \quad \mathbf{ex}_\sigma \subseteq \mathbf{ex}_\tau, \ \ \forall \sigma, \tau \in \Gamma_N, \ \sigma \prec \tau.$$

This is constructed inductively by starting with $\mathbf{ex}_{\mathrm{id}} = \varnothing$. If $\tau \prec \tau'$ are two consecutive elements in the linear ordering, then for some $1 \le i < j \le N$,

$$\tau = \tau_{i,j-1} \ \text{ and } \ \tau' = \tau_{i,j}$$

(recall (5.6) and the equalities in (5.7)). Thus,

$$\tau' = (ij)\tau = \tau(j-i, j-i+1).$$

Assuming that $\mathbf{ex}_\tau$ has been constructed, we define

$$\mathbf{ex}_{\tau'} := \begin{cases} \mathbf{ex}_\tau \cup \{p(j)\}, & \text{if } p(i) = -\infty, \ \eta(i) = \eta(j) \\ \mathbf{ex}_\tau, & \text{otherwise.} \end{cases}$$

It is clear that this process ends with $\mathbf{ex}_{w_\circ} = \mathbf{ex}$.

For a subset $X \subseteq \mathbf{ex}$, by an $N \times X$ matrix we will mean a matrix of size $N \times |X|$ whose columns are indexed by the set X. The set of such matrices with integral entries will be denoted by $M_{N\times X}(\mathbb{Z})$. (Recall from Section 2.3 that the columns of all of our $N \times |\mathbf{ex}|$ matrices are indexed by $\mathbf{ex} \subset [1, N]$.) The following lemma provides an inductive procedure for establishing Theorem 8.2 (a), (b) for $\tau \in \Gamma_N$.

Lemma 8.15. *Assume that R is a symmetric CGL extension of length N satisfying* (8.6) *and $\{\nu_{kj} \mid 1 \le j < k \le N\} \subset \mathbb{K}^*$ is a set of square roots of λ_{kj} which satisfies* (4.23). *Assume also that the generators of R are rescaled so that the condition* (6.13) *is satisfied.*

Let $\tau \in \Gamma_N$. For all $\sigma \in \Gamma_N$, $\sigma \preceq \tau$, there exists a unique matrix $\widetilde{B}_{\sigma,\tau} \in M_{N\times \mathbf{ex}_\tau}(\mathbb{Z})$ whose columns $b^l_{\sigma,\tau} \in \mathbb{Z}^N$, $l \in \mathbf{ex}_\tau$ satisfy

$$\begin{aligned} \chi_{M_\sigma(b^l_{\sigma,\tau})} = 1, \qquad \Omega_{\mathbf{r}_\sigma}(b^l_{\sigma,\tau}, e_j) = 1, \ \ \forall j \in [1, N], \ j \neq l \\ \Omega_{\mathbf{r}_\sigma}(b^l_{\sigma,\tau}, e_l)^2 = \lambda^*_l \end{aligned} \tag{8.24}$$

for all $l \in \mathbf{ex}_\tau$. The principal part of the matrix $\widetilde{B}_{\sigma,\tau}$ is skew-symmetrizable via the integers $\{d_{\eta(k)} \mid k \in \mathbf{ex}_\tau\}$.

Proof. The uniqueness statement follows at once from Lemma 8.14. If a matrix $\widetilde{B}_{\sigma,\tau}$ with such properties exists, then its principal part is skew-symmetrizable by Lemma 2.4 and the condition (8.6), since the conditions (8.24) imply that $(\mathbf{r}_\sigma, \widetilde{B}_{\sigma,\tau})$ is a compatible pair.

What remains to be proved is the existence statement in the lemma. It trivially holds for $\tau = \mathrm{id}$ since $\mathbf{ex}_{\mathrm{id}} = \varnothing$.

Let $\tau \prec \tau'$ be two consecutive elements of Γ_N in the linear ordering (5.7). Assuming that the existence statement in the lemma holds for τ, we will show that it holds for τ'. The lemma will then follow by induction.

From our inductive assumption, $(\mathbf{r}_\tau, \widetilde{B}_{\tau,\tau})$ is a compatible pair, and consequently $(M_\tau, \widetilde{B}_{\tau,\tau})$ is a quantum seed.

As noted above, for some $1 \le i < j \le N$ we have $\tau = \tau_{i,j-1}$ and $\tau' = \tau_{i,j}$. In particular, $\tau' = (ij)\tau = \tau(j-i, j-i+1)$ and $\tau(j-i) = i < j = \tau(j-i+1)$, so Proposition 8.13 is applicable to the pair (τ, τ'), with $k := j - i$.

If $\eta(i) \neq \eta(j)$, then $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau$ and $M_{\tau'} = M_\tau$ (by Proposition 8.13 (a)). So, $\Omega_{\mathbf{r}_{\tau'}} = \Omega_{\mathbf{r}_\tau}$. These identities imply that the following matrices have the properties (8.24) for the element $\tau' \in \Gamma_N$: $\widetilde{B}_{\sigma,\tau'} := \widetilde{B}_{\sigma,\tau}$ for $\sigma \preceq \tau$ and $\widetilde{B}_{\tau',\tau'} := \widetilde{B}_{\tau,\tau}$.

Next, we consider the case $\eta(i) = \eta(j)$. This implies that $j = s^m(i)$ for some $m \in \mathbb{Z}_{>0}$. This fact and the definition of $\tau_\bullet$ give that the element $\tau_\bullet\tau(j-i)$ equals the m-th element of $\eta^{-1}(\eta(i))$ when the elements in the preimage are ordered from least to greatest. Therefore this element is explicitly given by

$$\tau_\bullet\tau(j-i) = s^{m-1}p^{O_-(i)}(i). \tag{8.25}$$

Now set

$$k_\bullet := \tau_\bullet\tau(j-i)$$

as in Proposition 8.13 (b). There are two subcases: (1) $p(i) \neq -\infty$ and (2) $p(i) = -\infty$.

Subcase (1). In this situation $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau$, so we do not need to generate an "extra column" for each matrix. Set $\widetilde{B}_{\sigma,\tau'} := \widetilde{B}_{\sigma,\tau}$ for $\sigma \in \Gamma_N$, $\sigma \preceq \tau$. Eq. (8.24) for the pairs (σ, τ') with $\sigma \preceq \tau$ follows from the equality $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau$.

Next we deal with the pair $(\sigma = \tau', \tau')$. Applying the inductive assumption (8.24) for $\widetilde{B}_{\tau,\tau}$ and Proposition 8.13 (b) shows that the vector $g \in \mathbb{Z}^N_{\geq 0}$ has the properties

$$\chi_{M_\tau(b^{k_\bullet}_{\tau,\tau})} = \chi_{M_\tau(e_{p(k_\bullet)}+e_{s(k_\bullet)}-g)}$$

and

$$\Omega_{\mathbf{r}_\tau}(b^{k_\bullet}_{\tau,\tau}, e_t)^2 = \Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_t)^2, \ \ \forall t \in [1, N].$$

Lemma 8.14 implies that $e_{p(k_\bullet)} + e_{s(k_\bullet)} - g = b^{k_\bullet}_{\tau,\tau}$. It follows from this, Corollary 2.11 (a), and Eq. (8.8) that $\mu_{k_\bullet}(M_\tau)(e_l) = M_{\tau'}(e_l)$ for all $l \in [1, N]$. Consequently,

$$\Omega_{\mathbf{r}_{\tau'}}(e_t, e_l)^2 M_{\tau'}(e_l) M_{\tau'}(e_t) = M_{\tau'}(e_t) M_{\tau'}(e_l) = \Omega_{\mu_{k_\bullet}(\mathbf{r}_\tau)}(e_t, e_l)^2 M_{\tau'}(e_l) M_{\tau'}(e_t)$$

for all $t, l \in [1, N]$, so $\Omega_{\mathbf{r}_{\tau'}}(e_t, e_l)^2 = \Omega_{\mu_{k_\bullet}(\mathbf{r}_\tau)}(e_t, e_l)^2$, for all $j, l \in [1, N]$. This is an equality in the subgroup of $\mathbb{K}^*$ generated by $\{\nu_{lt} \mid 1 \leq t < l \leq N\}$ and the condition (4.23) implies that

$$\Omega_{\mathbf{r}_{\tau'}}(e_t, e_l) = \Omega_{\mu_{k_\bullet}(\mathbf{r}_\tau)}(e_t, e_l), \ \ \forall t, l \in [1, N]. \tag{8.26}$$

Therefore $\mathbf{r}_{\tau'} = \mu_{k_\bullet}(\mathbf{r}_\tau)$ and $M_{\tau'} = \mu_{k_\bullet}(M_\tau)$. We set $\widetilde{B}_{\tau',\tau'} := \mu_{k_\bullet}(\widetilde{B}_{\tau,\tau})$. Lemma 2.13 implies that the columns of $\widetilde{B}_{\tau',\tau'}$ have the property $\chi_{M_{\tau'}(b^l_{\tau',\tau'})} = 1$ for all $l \in \mathbf{ex}_\tau = \mathbf{ex}_{\tau'}$. Proposition 2.6 (b) implies that $(\mathbf{r}_{\tau'}, \widetilde{B}_{\tau',\tau'})$ is a compatible pair and

$$\Omega_{\mathbf{r}_{\tau'}}(b^l_{\tau',\tau'}, e_t) = 1, \ \ \forall t \in [1, N], \ t \neq l \quad \text{and} \quad \Omega_{\mathbf{r}_{\tau'}}(b^l_{\tau',\tau'}, e_l)^2 = \lambda^*_l$$

for all $l \in \mathbf{ex}_{\tau'}$, which completes the proof of the inductive step of the lemma in this subcase.

Subcase (2). In this case, $\mathbf{ex}_{\tau'} = \mathbf{ex}_\tau \sqcup \{k_\bullet\}$ and $k_\bullet = s^{m-1}(i) = p(j)$. Define the matrix $\widetilde{B}_{\tau,\tau'}$ by

$$b^l_{\tau,\tau'} = \begin{cases} b^l_{\tau,\tau}, & \text{if } \ l \neq k_\bullet \\ e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, & \text{if } \ l = k_\bullet, \end{cases}$$

where $g \in \mathbb{Z}^N_{\geq 0}$ is the vector from Proposition 8.13 (b). Applying the assumption (8.24) for $\widetilde{B}_{\tau,\tau}$ and Proposition 8.13 (b), we obtain that the matrix $\widetilde{B}_{\tau,\tau'}$ has the properties (8.24). We set $\widetilde{B}_{\tau',\tau'} := \mu_{k_\bullet}(\widetilde{B}_{\tau,\tau'})$. As in subcase (1), using Lemma 2.13 and Proposition 2.6 (b), one derives that $\widetilde{B}_{\tau',\tau'}$ satisfies the properties (8.24).

We are left with constructing $\widetilde{B}_{\sigma,\tau'} \in M_{N\times \mathbf{ex}_{\tau'}}(\mathbb{Z})$ for $\sigma \in \Gamma_N$, $\sigma \prec \tau$. We do this by a downward induction on the linear ordering (5.7) in a fashion that is similar to the proof of the lemma in the subcase (1). Assume that $\sigma \prec \sigma'$ is a pair of consecutive elements of Γ_N such that $\sigma' \preceq \tau$. As in the beginning of the section, we have that for some $1 \leq i^\vee < j^\vee \leq N$,

$$\sigma = \tau_{i^\vee, j^\vee - 1} \ \text{ and } \ \sigma' = \tau_{i^\vee, j^\vee},$$

so

$$\tau' = (i^\vee j^\vee)\tau = \tau(j^\vee - i^\vee, j^\vee - i^\vee + 1).$$

Assume that there exists a matrix $\widetilde{B}_{\sigma',\tau'} \in M_{N\times \mathbf{ex}_{\tau'}}(\mathbb{Z})$ that satisfies (8.24). We define the matrix $\widetilde{B}_{\sigma,\tau'} \in M_{N\times \mathbf{ex}_{\tau'}}(\mathbb{Z})$ by

$$\widetilde{B}_{\sigma,\tau'} := \begin{cases} \widetilde{B}_{\sigma',\tau'}, & \text{if } \ \eta(i^\vee) \neq \eta(j^\vee) \\ \mu_{k^\vee_\bullet}(\widetilde{B}_{\sigma,\tau'}), & \text{if } \ \eta(i^\vee) = \eta(j^\vee), \end{cases}$$

where

$$k_\bullet^\vee := \sigma_\bullet \sigma(j^\vee - i^\vee).$$

Analogously to the proof of the lemma in the subcase (1), using Propositions 2.6 (b) and 8.13 and Lemmas 2.13 and 8.14, one proves that the matrix $\widetilde{B}_{\sigma,\tau'}$ has the properties (8.24). This completes the proof of the lemma. □

REMARK 8.16. It follows from Lemma 8.14 that the matrices $\widetilde{B}_{\sigma,\tau}$ in Lemma 8.15 have the following restriction property:

For all triples $\sigma \preceq \tau \prec \tau'$ of elements of Γ_N,

$$b^l_{\sigma,\tau} = b^l_{\sigma,\tau'}, \quad \forall l \in \mathbf{ex}_\tau.$$

In other words, $\widetilde{B}_{\sigma,\tau}$ is obtained from $\widetilde{B}_{\sigma,\tau'}$ by removing all columns indexed by the set $\mathbf{ex}_{\tau'} \backslash \mathbf{ex}_\tau$.

This justifies that Lemma 8.15 gradually enlarges a matrix $\widetilde{B}_{\sigma,\sigma} \in M_{N\times \mathbf{ex}_\sigma}(\mathbb{Z})$ to a matrix $\widetilde{B}_{\sigma,w_0} \in M_{N\times\mathbf{ex}}(\mathbb{Z})$, for all $\sigma \in \Gamma_N$. In the case of $\sigma = \mathrm{id}$, we start with an empty matrix ($\mathbf{ex}_{\mathrm{id}} = \varnothing$) and obtain a matrix $\widetilde{B}_{\mathrm{id},w_\circ} \in M_{N\times\mathbf{ex}}(\mathbb{Z})$ which will be the needed mutation matrix for the initial toric frame M_{id}.

PROOF OF THEOREM 8.2 (a), (b) FOR $\tau \in \Gamma_N$. Change τ to σ in these statements. These parts of the theorem for the elements of Γ_N follow from Lemma 8.15 applied to $(\sigma, \tau) = (\sigma, w_\circ)$. For all $\sigma \in \Gamma_N$ we set $\widetilde{B}_\sigma := \widetilde{B}_{\sigma,w_\circ}$ and use that $\mathbf{ex}_{w_\circ} = \mathbf{ex}$. □

8.7. Proofs of parts (a), (b) and (c) of Theorem 8.2

Next we establish Theorem 8.2 (a)–(b) in full generality. This will be done by using the result of Section 8.6 for $\tau = \mathrm{id}$ and iteratively applying the following proposition.

PROPOSITION 8.17. *Let R be a symmetric CGL extension of length N satisfying (8.6) and $\{\nu_{kj} \mid 1 \le j < k \le N\} \subset \mathbb{K}^*$ a set of square roots of λ_{kj} which satisfies (4.23). Assume also that the generators of R are rescaled so that the condition (6.13) is satisfied. Let $\tau, \tau' \in \Xi_N$ be such that*

$$\tau' = (\tau(k), \tau(k+1))\tau = \tau(k, k+1)$$

for some $k \in [1, N-1]$ and $\tau'(k) < \tau'(k+1)$, $\eta(\tau(k)) = \eta(\tau(k+1))$. Set $k_\bullet := \tau_\bullet \tau(k)$.

Assume that there exists an $N \times |\mathbf{ex}|$ matrix $\widetilde{B}_\tau$ with integral entries whose columns $b^l_\tau \in \mathbb{Z}^N$, $l \in \mathbf{ex}$ satisfy

$$\begin{aligned} &\chi_{M_\tau(b^l_\tau)} = 1, \qquad \Omega_{\mathbf{r}_\tau}(b^l_\tau, e_j) = 1, \ \ \forall j \in [1,N], \ j \neq l \\ &\Omega_{\mathbf{r}_\tau}(b^l_\tau, e_l)^2 = \lambda^*_l \end{aligned} \tag{8.27}$$

for all $l \in \mathbf{ex}$. Then its principal part is skew-symmetrizable and the columns $b^j_{\tau'} \in \mathbb{Z}^N$, $j \in \mathbf{ex}$ of the matrix $\mu_{k_\bullet}(\widetilde{B_\tau})$ satisfy

$$\begin{aligned} &\chi_{M_{\tau'}(b^l_{\tau'})} = 1, \qquad \Omega_{\mathbf{r}_{\tau'}}(b^l_{\tau'}, e_j) = 1, \ \ \forall j \in [1,N], \ j \neq l \\ &\Omega_{\mathbf{r}_{\tau'}}(b^l_{\tau'}, e_l)^2 = \lambda^*_l \end{aligned} \tag{8.28}$$

for all $l \in \mathbf{ex}$. Furthermore,

$$\mathbf{r}_{\tau'} = \mu_k(\mathbf{r}_\tau) \tag{8.29}$$

and

$$b_\tau^{k_\bullet} = -b_{\tau'}^{k_\bullet} = e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, \tag{8.30}$$

where $g \in \mathbb{Z}_{\geq 0}^N$ *is the vector from Proposition* 8.13.

The conditions (8.27)–(8.28) are stronger than saying that the pairs $(\mathbf{r}_\tau, \widetilde{B}_\tau)$ and $(\mathbf{r}_{\tau'}, \mu_{k_\bullet}(\widetilde{B_\tau}))$ are compatible. The point is to recursively establish a stronger condition which matches the setting of Lemma 8.14 in order to use the uniqueness conclusion of the lemma.

Remark 8.18. The statements of Lemma 8.15 and Proposition 8.17 have many similarities and their proofs use similar ideas. However, we note that there is a principal difference between the two results. In the former case we have no mutation matrices to start with and we use Proposition 8.13 (b) to gradually add columns. In the latter case we already have a mutation matrix for one toric frame and construct a mutation matrix for another toric frame.

Proof of Proposition 8.17. The fact that the principal part of $\widetilde{B}_\tau$ is skew-symmetrizable follows from Lemma 2.4 and the condition (8.6). The assumptions on $\widetilde{B}_\tau$ and Proposition 8.13 (b) imply

$$\chi_{M_\tau(b_\tau^{k_\bullet})} = \chi_{M_\tau(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g)}$$

and

$$\Omega_{\mathbf{r}_\tau}(b_\tau^{k_\bullet}, e_j)^2 = \Omega_{\mathbf{r}_\tau}(e_{p(k_\bullet)} + e_{s(k_\bullet)} - g, e_j)^2, \ \forall j \in [1, N].$$

By Lemma 8.14, $b_\tau^{k_\bullet} = e_{p(k_\bullet)} + e_{s(k_\bullet)} - g$. The mutation formula for $\mu_{k_\bullet}(\widetilde{B}_\tau)$ also gives that $b_{\tau'}^{k_\bullet} = -b_\tau^{k_\bullet}$, so we obtain (8.30). Analogously to the proof of (8.26), one shows that

$$\Omega_{\mathbf{r}_{\tau'}}(e_j, e_l) = \Omega_{\mu_{k_\bullet}(\mathbf{r}_\tau)}(e_j, e_l), \ \ \forall j, l \in [1, N],$$

which is equivalent to (8.29). Finally, all identities in (8.28) follow from the general mutation facts in Proposition 2.6 (b) and Lemma 2.13. $\square$

Proof of Theorem 8.2 (a), (b) for all $\tau \in \Xi_N$. Similarly to the proof of Lemma 8.15, the uniqueness statement in part (a) follows from Lemma 8.14. We will prove the existence statement in part (a) by an inductive argument on τ. Once the existence of the matrix $\widetilde{B}_\tau$ with the stated properties is established, the fact that the principal part of $\widetilde{B}_\tau$ is skew-symmetrizable follows from Lemma 2.4 and the condition (8.6). Hence, $(M_\tau, \widetilde{B}_\tau)$ is a quantum seed and this yields part (b) of the theorem for the given $\tau \in \Xi_N$.

For the existence statement in part (a) we fix $\tau \in \Xi_N$. By Corollary 5.6 (a), there exists a sequence $\tau_0 = \mathrm{id}, \tau_1, \ldots, \tau_n = \tau$ in Ξ_N with the property that for all $l \in [1, n]$,

$$\tau_l = (\tau_{l-1}(k_l), \tau_{l-1}(k_l + 1))\tau_{l-1} = \tau_{l-1}(k_l, k_l + 1)$$

for some $k_l \in [1, N-1]$ such that $\tau_{l-1}(k_l) < \tau_{l-1}(k_l + 1)$. In Section 8.6 we established the validity of Theorem 8.2 (a) for the identity element of S_N. By induction on l we prove the validity of Theorem 8.2 (a) for τ_l. If $\eta(\tau_{l-1}(k_l)) \neq \eta(\tau_l(k_l))$, then Proposition 8.13 (a) implies that $M_{\tau_l} = M_{\tau_{l-1}}$ and we can choose $\widetilde{B}_{\tau_l} = \widetilde{B}_{\tau_{l-1}}$. If $\eta(\tau_{l-1}(k_l)) = \eta(\tau_l(k_l))$, then Proposition 8.17 proves that the validity of Theorem 8.2 (a) for τ_{l-1} implies the validity of Theorem 8.2 (a) for τ_l.

In this case $\widetilde{B}_{\tau_l} := \mu_{(k_l)_\bullet}(\widetilde{B}_{\tau_{l-1}})$, where $(k_l)_\bullet = ((\tau_{k_l})_\bullet \tau_{k_l})(k_l)$. This completes the proof of Theorem 8.2 (a) and (b). $\square$

Proof of Theorem 8.2 (c). The one-step mutation statement in part (c) of Theorem 8.2 and Corollary 5.6 (a) imply that all quantum seeds associated to the elements of Ξ_N are mutation-equivalent to each other.

In the rest we prove the one-step mutation statement in part (c) of the theorem. If $\eta(\tau(k)) \neq \eta(\tau(k+1))$, then the statement follows from Proposition 8.13 (a).

Now let $\eta(\tau(k)) = \eta(\tau(k+1))$. We have that either $\tau(k) < \tau(k+1)$ or $\tau'(k) = \tau(k+1) < \tau'(k+1) = \tau(k)$. In the first case we apply Proposition 8.17 to the pair (τ, τ') and in the second case to the pair (τ', τ). The one-step mutation statement in Theorem 8.2 (c) follows from this, the uniqueness statement in part (a) of the theorem and the involutivity of mutations of quantum seeds (Corollary 2.11 (b)). $\square$

8.8. Intersections of localizations

Parts (d) and (e) of Theorem 8.2 involve showing that R and $R[y_k^{-1} \mid k \in \mathbf{inv}]$ are equal to intersections of appropriate localizations. Proving this does not require either the existence of square roots of the λ_{kj} in $\mathbb{K}^*$ or the condition (6.13), and leads to a general result of independent interest, showing that all symmetric CGL extensions are intersections of partially localized quantum affine space algebras.

Throughout this section, we will assume that R is a symmetric CGL extension of rank N as in Definition 3.3, and we fix a function $\eta : [1, N] \to \mathbb{Z}$ satisfying the conditions of Theorem 3.6. For each $\tau \in \Xi_N$, there is a CGL presentation

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}] \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}]$$

as in (3.12). Let p_τ and s_τ denote the predecessor and successor functions for the level sets of $\eta_\tau := \eta\tau$, which by Corollary 5.6 (b) can be chosen as the η-function for the presentation (3.12). Let $y_{\tau,1}, \ldots, y_{\tau,N}$ be the corresponding sequence of homogeneous prime elements of R from Theorem 3.6, and denote

$$\begin{aligned}
\mathcal{A}_\tau &:= \text{the } \mathbb{K}\text{-subalgebra of } R \text{ generated by } \{y_{\tau,k} \mid k \in [1, N]\} \\
\mathcal{T}_\tau &:= \text{the } \mathbb{K}\text{-subalgebra of Fract}(R) \text{ generated by } \{y_{\tau,k}^{\pm 1} \mid k \in [1, N]\} \\
E_\tau &:= \text{the multiplicative subset of } \mathcal{A}_\tau \text{ generated by} \\
&\qquad\qquad \mathbb{K}^* \sqcup \{y_{\tau,k} \mid k \in [1, N],\ s_\tau(k) \neq +\infty\}.
\end{aligned}$$

By Proposition 3.11, $\mathcal{T}_\tau$ is a quantum torus with corresponding quantum affine space algebra $\mathcal{A}_\tau$. In particular, E_τ consists of normal elements of $\mathcal{A}_\tau$, so it is an Ore set in $\mathcal{A}_\tau$.

Theorem 8.19. *Let R be a symmetric CGL extension of length N.*
(a) $\mathcal{A}_\tau \subseteq R \subseteq \mathcal{A}_\tau[E_\tau^{-1}] \subseteq \mathcal{T}_\tau \subseteq \text{Fract}(R)$, *for all* $\tau \in \Xi_N$.
(b) *R is generated as a $\mathbb{K}$-algebra by* $\{y_{\tau,k} \mid \tau \in \Gamma_N,\ k \in [1, N]\}$.
(c) *Each E_τ is an Ore set in R, and* $R[E_\tau^{-1}] = \mathcal{A}_\tau[E_\tau^{-1}]$.
(d) $R = \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}] = \bigcap_{\tau \in \Gamma_N} \mathcal{A}_\tau[E_\tau^{-1}]$.
(e) *Let* $\mathbf{inv}$ *be any subset of* $\{k \in [1, N] \mid s(k) = +\infty\}$. *Then*

$$R[y_k^{-1} \mid k \in \mathbf{inv}] = \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}][y_k^{-1} \mid k \in \mathbf{inv}].$$

PROOF OF PARTS (a), (b), (c), (e). For part (a), the first, third, and fourth inclusions are clear. For the second, it suffices to show that $x_{\tau(k)} \in \mathcal{A}_\tau[E_\tau^{-1}]$ for all $k \in [1, N]$, which we do by induction on k. The case $k = 1$ is immediate from the fact that $x_{\tau(1)} = y_{\tau,1}$.

Now let $k \in [2, N]$. If $p_\tau(k) = -\infty$, then $x_{\tau(k)} = y_{\tau,k} \in \mathcal{A}_\tau[E_\tau^{-1}]$. If $p_\tau(k) = l \neq -\infty$, then $l < k$ and $y_{\tau,k} = y_{\tau,l}x_{\tau(k)} - c_{\tau,k}$ for some element $c_{\tau,k}$ in the $\mathbb{K}$-subalgebra of R generated by $x_{\tau(1)}, \dots, x_{\tau(k-1)}$. By induction, $c_{\tau,k} \in \mathcal{A}_\tau[E_\tau^{-1}]$. Further, $y_{\tau,l} \in E_\tau$ because $s_\tau(l) = k \neq +\infty$, and thus

$$x_{\tau(k)} = y_{\tau,l}^{-1}\big(y_{\tau,k} + c_{\tau,k}\big) \in \mathcal{A}_\tau[E_\tau^{-1}]$$

in this case also.

(b) For each $j \in [1, N]$, we have $\tau_{j,j} \in \Gamma_N$ with $\tau_{j,j}(1) = j$. Theorem 5.3 implies that $y_{\tau,1}$ is a scalar multiple of $y_{[j,j]} = x_j$. Thus, in fact, R is generated by $\{y_{\tau,1} \mid \tau \in \Gamma_N\}$.

(c) By (7.1), for each $\tau \in \Xi_N$ the set E_τ is an Ore set in R. It is clear from part (a) that $R[E_\tau^{-1}] = \mathcal{A}_\tau[E_\tau^{-1}]$.

(e) Let $v \in \bigcap_{\tau\in\Gamma_N} R[E_\tau^{-1}][y_k^{-1} \mid k \in \mathbf{inv}]$. For each $\tau \in \Gamma_N$, we can write v as a fraction with numerator from $R[E_\tau^{-1}]$ and right-hand denominator from the multiplicative set Y generated by $\{y_k \mid k \in \mathbf{inv}\}$. (Note that Y is generated by normal elements of R, so it is an Ore set in R and in $R[E_\tau^{-1}]$.) Hence, choosing a common denominator, we obtain $y \in Y$ such that $vy \in R[E_\tau^{-1}]$ for all $\tau \in \Gamma_N$. Once part (d) is proved, we can conclude that $vy \in R$, and thus $v \in R[y_k^{-1} \mid k \in \mathbf{inv}]$, as required. $\square$

In proving Theorem 8.19 (d), we need to compare fraction expressions for an element v of $\bigcap_{\tau\in\Gamma_N} R[E_\tau^{-1}]$ across the given localizations, which is done via the reenumerations $\tau_\bullet\tau$ from Section 8.1. Some properties of these permutations are given next. We keep the notation $\mathbf{ex} := \{j \in [1, N] \mid s(j) \neq +\infty\}$.

LEMMA 8.20. (a) *For any $\tau \in \Xi_N$, the permutation $(\tau_\bullet\tau)^{-1}$ maps* $\mathbf{ex}$ *bijectively onto the set $\{k \in [1, N] \mid s_\tau(k) \neq +\infty\}$.*

(b) *Suppose that $\tau, \tau' \in \Xi_N$ and $\tau' = \tau(k, k+1)$ for some $k \in [1, N-1]$. If $\eta\tau(k) = \eta\tau(k+1)$, then $\tau'_\bullet\tau' = \tau_\bullet\tau$.*

(c) *Suppose that $\tau, \tau' \in \Xi_N$ and $\tau' = \tau(k, k+1)$ for some $k \in [1, N-1]$. If $\eta\tau(k) \neq \eta\tau(k+1)$, then*

$$\tau'_\bullet\tau'(k) = \tau_\bullet\tau(k+1) \qquad \text{and} \qquad \tau'_\bullet\tau'(k+1) = \tau_\bullet\tau(k),$$

while $\tau'_\bullet\tau'(l) = \tau_\bullet\tau(l)$ for all $l \in [1, N] \setminus \{k, k+1\}$.

(d) *Let $j \in$ **ex**. Then there exist $\tau, \tau' \in \Gamma_N$ such that $\tau' = \tau(k, k+1)$ for some $k \in [1, N-1]$ with $\eta\tau(k) = \eta\tau(k+1)$ and $\tau_\bullet\tau(k) = j$.*

PROOF. (a) Let $j \in [1, N]$ and $k = (\tau_\bullet\tau)^{-1}(j)$, and note that $\eta(j) = \eta\tau(k)$. Then $j \in \mathbf{ex}$ if and only if j is not the largest element of the set $\eta^{-1}(\eta(j))$, if and only if k is not the largest element of $\tau^{-1}\eta^{-1}(\eta(j)) = \eta_\tau^{-1}(\eta_\tau(k))$, if and only if $s_\tau(k) \neq +\infty$.

(b) This follows from the observation that $\tau^{-1}(L) = (\tau')^{-1}(L)$ for all level sets L of η.

(c) If L is a level set of η not containing k or $k+1$, then $\tau^{-1}(L) = (\tau')^{-1}(L)$ and so $\tau'_\bullet\tau'(l) = \tau_\bullet\tau(l)$ for all $l \in L$. For the case $L = \eta^{-1}(\eta\tau(k))$, we have

$$L = \{\tau(i_1) = \tau'(i_1), \dots, \tau(i_{r-1}) = \tau'(i_{r-1}),\, \tau(k) = \tau'(k+1),$$
$$\tau(i_{r+1}) = \tau'(i_{r+1}), \dots, \tau(i_t) = \tau'(i_t)\}$$

for some $1 \le i_1 < \dots < i_{r_1} < k < k+1 < i_{r+1} < \dots < i_t \le N$, from which it is clear that $\tau'_\bullet\tau'(k+1) = \tau_\bullet\tau(k)$ and $\tau'_\bullet\tau'(l) = \tau_\bullet\tau(l)$ for all $l \in \{i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_t\}$. Similarly, we see that $\tau'_\bullet\tau'(k) = \tau_\bullet\tau(k+1)$ and $\tau'_\bullet\tau'(l) = \tau_\bullet\tau(l)$ for all $l \in \tau^{-1}\eta^{-1}(\tau(k+1)) \setminus \{k+1\}$.

(d) Set $m = O_-(j) \in \mathbb{Z}_{\ge 0}$ and $i = p^m(j) \in [1, N]$, so that $p(i) = -\infty$ and $s^m(i) = j$. Since $s(j) \in [i+1, N]$, we may take $\tau := \tau_{i,s(j)-1}$ and $\tau' := \tau_{i,s(j)}$ in Γ_N. Then $\tau' = \tau(k, k+1)$ where $k = s(j) - i$, and $\eta\tau(k) = \eta(i) = \eta(s(j)) = \eta\tau(k+1)$. Moreover, the level set $L := \eta^{-1}(\eta\tau(k)) = \eta^{-1}(\eta(i))$ equals $\eta^{-1}(\eta(j))$ and τ maps the elements of $[1, k] \cap \tau^{-1}(L)$ to $s(i), \dots, s^m(i), i$ in that order. Hence, $\tau_\bullet\tau(k)$ equals the $(m+1)$-st element of L when L is written in ascending order. That element is j. $\square$

Proof of Theorem 8.19 (d). The second equality follows from part (c), and one inclusion of the first equality is obvious.

Let $v \in \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}]$ be a nonzero element. For each $\tau \in \Gamma_N$, let

$$\prod_{l \in [1,N]} y_{\tau,l}^{m_{\tau,l}}$$

be a minimal denominator of v with respect to the localization $R[E_\tau^{-1}]$, where all $m_{\tau,l} \in \mathbb{Z}_{\ge 0}$ and $m_{\tau,l} = 0$ when $s_\tau(l) = +\infty$. We first verify the following

Claim. Let $\tau, \tau' \in \Gamma_N$ such that $\tau' = \tau(k, k+1)$ for some $k \in [1, N-1]$.

(1) $m_{\tau,(\tau_\bullet\tau)^{-1}(j)} = m_{\tau',(\tau'_\bullet\tau')^{-1}(j)}$, for all $j \in [1, N]$.
(2) If $\eta\tau(k) = \eta\tau(k+1)$, then $m_{\tau,k} = 0$.

If $\eta\tau(k) = \eta\tau(k+1)$, then (1) and (2) follow from Lemma 8.20 (b), Theorem 7.4, and Theorem 4.2 (b). If $\eta\tau(k) \ne \eta\tau(k+1)$, we obtain from Theorem 4.2 (a) that $y_{\tau',k} = y_{\tau,k+1}$ and $y_{\tau',k+1} = y_{\tau,k}$, while $y_{\tau',l} = y_{\tau,l}$ for all $l \ne k, k+1$. As a result, $E_{\tau'} = E_\tau$ and we see that $m_{\tau',k} = m_{\tau,k+1}$ and $m_{\tau',k+1} = m_{\tau,k}$, while $m_{\tau',l} = m_{\tau,l}$ for all $l \ne k, k+1$. In this case, (1) follows from Lemma 8.20 (c).

Since all the permutations in Γ_N appear in the chain (5.7), part (1) of the claim implies that

$$m_{\sigma,(\sigma_\bullet\sigma)^{-1}(j)} = m_{\tau,(\tau_\bullet\tau)^{-1}(j)}, \ \ \forall \sigma, \tau \in \Gamma_N,\ j \in [1, N]. \tag{8.31}$$

For any $j \in \mathbf{ex}$, Lemma 8.20 (d) shows that there exist $\tau, \tau' \in \Gamma_N$ such that $\tau' = \tau(k, k+1)$ for some $k \in [1, N-1]$ with $\eta\tau(k) = \eta\tau(k+1)$ and $\tau_\bullet\tau(k) = j$. Part (2) of the claim above then implies that $m_{\tau,(\tau_\bullet\tau)^{-1}}(j) = 0$. From (8.31), we thus get

$$m_{\sigma,(\sigma_\bullet\sigma)^{-1}(j)} = 0, \ \ \forall \sigma \in \Gamma_N,\ j \in \mathbf{ex}.$$

In particular, $m_{\mathrm{id},j} = 0$ for all $j \in \mathbf{ex}$, whence $m_{\mathrm{id},j} = 0$ for all $j \in [1, N]$ and therefore $v \in R$, which completes the proof of Theorem 8.19 (d). $\square$

8.9. Completion of the proof of Theorem 8.2

We finally prove the last and most important part of Theorem 8.2 which establishes an equality between the CGL extension R, the quantum cluster algebra $\mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$ and the corresponding upper quantum cluster algebra $\mathcal{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}$.

Recall the setting of Section 7.1. For all $\tau \in \Xi_N$, define the multiplicative subsets

$$E_\tau := \mathbb{K}^* \Big\{ M_\tau(f) \mid f \in \sum_{j \in \mathbf{ex}} \mathbb{Z}_{\geq 0} e_j \Big\}.$$

By (7.1), E_τ is an Ore subset of R for all $\tau \in \Xi_N$. In view of Lemma 8.20 (a) and the definition of M_τ, we see that E_τ is generated (as a multiplicative set) by

$$\mathbb{K}^* \sqcup \{ y_{\tau,k} \mid k \in [1, N],\ s_\tau(k) \neq +\infty \},$$

matching the definition used in Section 8.8.

Proof of Theorem 8.2 (d)–(e). By Theorem 8.2 (c), for all $\tau \in \Xi_N$ the quantum seeds $(M_\tau, \widetilde{B}_\tau)$ of $\operatorname{Fract}(R)$ are mutation-equivalent to each other. For each $j \in [1, N]$, we have $\tau_{j,j} \in \Gamma_N$ with $\tau_{j,j}(1) = j$, and so $\overline{y}_{\tau_{j,j},1} = \overline{y}_{[j,j]} = x_j$ by Proposition 8.9. Thus, each generator x_j of R is a cluster variable for a quantum seed associated to some $\tau \in \Gamma_N$. Hence,

$$R \subseteq \mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}. \tag{8.32}$$

The quantum Laurent phenomenon (Theorem 2.15) implies that

$$\mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} \subseteq \mathcal{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}}. \tag{8.33}$$

Since $M_\tau(e_j) \in R$ for all $\tau \in \Xi_N$, $j \in [1, N]$,

$$\mathcal{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} \subseteq \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}]. \tag{8.34}$$

Combining the embeddings (8.32), (8.33), (8.34) and Theorem 8.19 (d) leads to

$$R \subseteq \mathcal{A}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} \subseteq \mathcal{U}(M, \widetilde{B}, \varnothing)_{\mathbb{K}} \subseteq \bigcap_{\tau \in \Gamma_N} R[E_\tau^{-1}] = R,$$

which establishes all equalities in Theorem 8.2 (d).

For part (e) we have the embeddings

$$\begin{aligned} R[y_k^{-1} \mid k \in \mathbf{inv}] \subseteq \mathcal{A}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}} \subseteq \\ \subseteq \mathcal{U}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}} \subseteq \bigcap_{\tau \in \Gamma_N} R[y_k^{-1} \mid k \in \mathbf{inv}][E_\tau^{-1}], \end{aligned} \tag{8.35}$$

which follow from the quantum Laurent phenomenon and the fact that each generator of R is a cluster variable in one of the seeds indexed by Γ_N. Theorem 8.19 (e) and the chain of embeddings (8.35) imply the validity of part (e) of the theorem. □

The chain of embeddings (8.35) and Theorem 8.19 (e) also imply the following description of the upper quantum cluster algebra in Theorem 8.2 as a finite intersection of mixed quantum tori–quantum affine space algebras of the form (2.28).

Corollary 8.21. *In the setting of Theorem* 8.2*,*

$$\mathcal{U}(M, \widetilde{B}, \mathbf{inv})_{\mathbb{K}} = \bigcap_{\tau \in \Delta} \mathcal{T}\mathcal{A}_\tau(\mathbf{inv})$$

for every subset Δ of Γ_N which is an interval with respect to the linear ordering (5.7) *and has the property that for each $k \in \mathbf{ex}$ there exist two consecutive elements $\tau = \tau_{i,j-1} \prec \tau' = \tau_{i,j}$ of Δ such that $\eta(i) = \eta(j)$ and $k = \tau_\bullet \tau(j-i)$, recall Section* 8.6. *Here $\mathcal{TA}_\tau(\mathbf{inv})$ denotes the mixed quantum torus–quantum affine space algebra from* (2.28) *associated to the permutation τ, $D = \mathbb{K}$, and the set of inverted frozen variables* **inv**.

In particular, this property holds for $\Delta = \Gamma_N$.

8.10. Some vectors $f_{[i,s(i)]}$

The normalization condition (6.13) in Theorem 8.2 depends on the leading terms of the elements $u_{[i,s(i)]}$ for $i \in [1, N]$ with $s(i) \neq +\infty$, and thus on the exponent vectors $f_{[i,s(i)]}$. The entries of these vectors actually arise from the exchange matrix $\widetilde{B} = (b_{lk})$, as we now show in the case when $[i, s(i)] = [1, N]$. This will be used in Chapter 10.

Proposition 8.22. *Let R be a symmetric CGL extension of length N which satisfies the hypotheses of Theorem* 8.2. *Assume that $N \geq 3$ and $s(1) = N$, and recall from* (5.20) *that $P_{[1,N]} = \{l \in [2, N-1] \mid s(l) = +\infty\}$ in this case. Then*

$$f_{[1,N]} = -\sum_{l \in P_{[1,N]}} b_{l1} e_{[p^{O_-(l)}(l),l]}. \tag{8.36}$$

Proof. We will apply Proposition 8.13 using the permutations

$$\tau := \tau_{1,N-1} = [2, \dots, N-1, 1, N] \qquad \text{and} \qquad \tau' := \tau_{1,N} = [2, \dots, N, 1]$$

from the subset Γ_N of Ξ_N. In this case, $\tau' = \tau(k, k+1)$ for $k = N-1$ and $\tau_\bullet \tau = \tau$, so that $k_\bullet = 1$. We first show that

$$M_\tau = M_{\mathrm{id}} = M \qquad \text{and} \qquad \widetilde{B}_\tau = \widetilde{B}_{\mathrm{id}} = \widetilde{B}. \tag{8.37}$$

We have $\tau_{1,1} = \mathrm{id} \prec \tau_{1,2} \prec \cdots \prec \tau_{1,N-2} \prec \tau$ in the linear ordering (5.7). Note that $\tau_{1,j+1} = \tau_{1,j}(j, j+1)$ for all $j \in [1, N-2]$, and that $\eta(\tau_{1,j}(j)) = \eta(1) \neq \eta(j+1) = \eta(\tau_{1,j}(j+1))$ for such j. Theorem 8.2 (c) implies that $M_{\tau_{1,j+1}} = M_{\tau_{1,j}}$, and hence also $\mathbf{r}_{\tau_{1,j+1}} = \mathbf{r}_{\tau_{1,j}}$, for all $j \in [1, N-2]$. It follows that the conditions determining the columns of $\widetilde{B}_{\tau_{1,j}}$ and $\widetilde{B}_{\tau_{1,j+1}}$ in Theorem 8.2 (a) coincide, and thus $\widetilde{B}_{\tau_{1,j+1}} = \widetilde{B}_{\tau_{1,j}}$ by uniqueness. This verifies (8.37).

Recall the toric frame $M_{[1,N]}$ from Section 6.3, and apply Proposition 8.9 to see that

$$M_{[1,N]}(e_l) = \overline{y}_{\tau,\tau^{-1}(l)} = \widehat{M}_\tau \tau^{-1}(e_l) = M_\tau(e_l), \ \ \forall l \in [1, N].$$

Thus, $M_{[1,N]} = M_\tau = M$. From Section 6.3 we also recall the vector $g_{[1,N]} = \sum_{l \in P_{[1,N]}} m_l e_l$ and the relation

$$f_{[1,N]} = \sum_{l \in P_{[1,N]}} m_l e_{[p^{O_-^2(l)}(l),l]} = \sum_{l \in P_{[1,N]}} m_l e_{[p^{O_-(l)}(l),l]},$$

with the last equality following from the fact that $\eta(l) \neq \eta(1)$ for all $l \in P_{[1,N]}$. Corollary 6.8 (a) implies that

$$u_{[1,N]} \ \text{ is a scalar multiple of } M_{[1,N]}(g_{[1,N]}) = M(g_{[1,N]}). \tag{8.38}$$

Set $g' := \sum_{l \in P_{[1,N]}} m_l e_{l-1}$, and observe from the proof of Proposition 8.13 that the vector g appearing in the statement of the proposition is $\tau_\bullet \tau(g') = g_{[1,N]}$. Thus, equations (8.9)–(8.11) imply, recalling (8.37), that

$$\Omega_{\mathbf{r}}(e_N - g_{[1,N]}, e_j) = 1, \quad \forall j \neq 1,$$
$$\Omega_{\mathbf{r}}(e_N - g_{[1,N]}, e_1)^2 = \lambda_1^*,$$

and

$$\chi_{M(e_N - g_{[1,N]})} = 1.$$

Theorem 8.2 (a) now implies that $e_N - g_{[1,N]} = b^1$. From this we obtain $b_{N1} = 1$ and

$$b_{l1} = \begin{cases} -m_l, & l \in P_{[1,N]} \\ 0 & l \in [1, N-1] \setminus P_{[1,N]}. \end{cases}$$

Consequently,

$$M(g_{[1,N]}) = \prod_{l \in P_{[1,N]}} \overline{y}_l^{\,-b_{l1}}.$$

Eq. (8.36) follows from this and (8.38). □

CHAPTER 9

Quantum groups and quantum Schubert cell algebras

In this chapter we first set up notation and review material on quantum groups and quantum Schubert cell algebras. Then we derive an explicit description of the sequences of prime elements of the latter algebras constructed in the previous chapters.

9.1. Quantized universal enveloping algebras

Fix a finite dimensional complex simple Lie algebra $\mathfrak{g}$ of rank r with Weyl group W and set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Let $\langle ., . \rangle$ be the invariant bilinear form on $\mathbb{R}\Pi$ normalized by $\langle \alpha_i, \alpha_i \rangle = 2$ for short roots α_i. For $\gamma \in \mathbb{R}\Pi$, set

$$\|\gamma\|^2 = \langle \gamma, \gamma \rangle.$$

Denote by $\{s_i\}$, $\{\alpha_i^\vee\}$ and $\{\varpi_i\}$ the corresponding sets of simple reflections, coroots and fundamental weights. Let $\mathcal{Q}$ and $\mathcal{P}$ be the root and weight lattices of $\mathfrak{g}$, and $\mathcal{P}^+ = \sum_i \mathbb{Z}_{\geq 0} \varpi_i$ the set of dominant integral weights of $\mathfrak{g}$. Denote the Cartan matrix of $\mathfrak{g}$ by

$$(c_{ij}) := (\langle \alpha_i^\vee, \alpha_j \rangle) \in M_r(\mathbb{Z}). \tag{9.1}$$

As in the previous chapters, we will work over a base field $\mathbb{K}$ of arbitrary characteristic. Choose a non-root of unity $q \in \mathbb{K}^*$ and denote by $\mathcal{U}_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{K}$ with deformation parameter q. We will mostly follow the notation of Jantzen [**28**], except for denoting the standard generators of $\mathcal{U}_q(\mathfrak{g})$ by $K_i^{\pm 1}$, E_i, F_i instead of $K_{\pm\alpha_i}$, E_{α_i}, F_{α_i} which better fits with the combinatorial notation from Chapters 3–8. We will use the form of the Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ with relations listed in [**28**, §4.3], and comultiplication, counit, and antipode given by

$$\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, && \epsilon(K_i) = 1, && S(K_i) = K_i^{-1}, \\
&\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, && \epsilon(E_i) = 0, && S(E_i) = -K_i^{-1} E_i, \\
&\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, && \epsilon(F_i) = 0, && S(F_i) = -F_i K_i,
\end{aligned}$$

for all $i \in [1, r]$ [**28**, Proposition 4.11].

The algebra $\mathcal{U}_q(\mathfrak{g})$ is $\mathcal{Q}$-graded with $\deg K_i = 0$, $\deg E_i = \alpha_i$, and $\deg F_i = -\alpha_i$ for all $i \in [1, r]$ [**28**, §4.7]. The corresponding graded components will be denoted by $\mathcal{U}_q(\mathfrak{g})_\gamma$, $\gamma \in \mathcal{Q}$. Define the torus

$$\mathcal{H} := (\mathbb{K}^*)^r. \tag{9.2}$$

Its rational character lattice is isomorphic to $\mathcal{Q}$, where the simple root α_j is mapped to the character

$$h \mapsto h^{\alpha_j} := t_j, \ \ \forall h = (t_1, \dots, t_r) \in \mathcal{H}. \tag{9.3}$$

The $\mathcal{Q}$-grading of $\mathcal{U}_q(\mathfrak{g})$ gives rise to the rational $\mathcal{H}$-action on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms such that

$$h \cdot u = h^{\gamma} u, \ \ \forall \gamma \in \mathcal{Q}, \ u \in \mathcal{U}_q(\mathfrak{g})_{\gamma}. \tag{9.4}$$

9.2. Quantum Schubert cell algebras

Let $\mathcal{B}_{\mathfrak{g}}$ denote the braid group of $\mathfrak{g}$ and $\{T_i\}_{i=1}^r$ its standard generating set. We will use Lusztig's action of $\mathcal{B}_{\mathfrak{g}}$ on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms in the version given in [**28**, §8.14, Eqs. 8.14 (2), (3), (7), (8)]. The canonical section $W \to \mathcal{B}_{\mathfrak{g}}$ will be denoted by $w \mapsto T_w$. It follows from [**28**, §4.7, Eq. (1) and §8.14, Eq. (2)] that

$$T_i\big(\mathcal{U}_q(\mathfrak{g})_{\gamma}\big) = \mathcal{U}_q(\mathfrak{g})_{s_i\gamma}, \ \ \forall i \in [1, r], \ \gamma \in \mathcal{Q}. \tag{9.5}$$

The quantum Schubert cell algebras $\mathcal{U}^{\pm}[w]$, $w \in W$ were defined by De Concini, Kac, and Procesi [**8**], and Lusztig [**34**, §40.2] as follows. Fix a reduced expression

$$w = s_{i_1} \dots s_{i_N} \tag{9.6}$$

and denote

$$w_{\leq k} := s_{i_1} \dots s_{i_k}, \qquad w_{\leq 0} := 1,$$
$$w_{[j,k]} := \begin{cases} s_{i_j} \dots s_{i_k}, & \text{if } \ j \leq k \\ 1, & \text{if } \ j > k \end{cases}$$

for all $j, k \in [1, N]$. (The above notation depends on the choice of reduced expression, but this dependence will not be displayed explicitly for simplicity of the notation.) Define the roots

$$\beta_k := w_{\leq (k-1)} \alpha_{i_k}, \ \ \forall k \in [1, N] \tag{9.7}$$

of $\mathfrak{g}$ and the Lusztig root vectors

$$\begin{aligned} E_{\beta_k} &:= T_{w_{\leq k-1}}(E_{i_k}) = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}), \\ F_{\beta_k} &:= T_{w_{\leq k-1}}(F_{i_k}) = T_{i_1} \dots T_{i_{k-1}}(F_{i_k}) \in \mathcal{U}_q(\mathfrak{g}), \ \ \forall k \in [1, N], \end{aligned} \tag{9.8}$$

see [**34**, §39.3]. Note from (9.5) that $E_{\beta_k} \in \mathcal{U}_q(\mathfrak{g})_{\beta_k}$ and $F_{\beta_k} \in \mathcal{U}_q(\mathfrak{g})_{-\beta_k}$.

By [**8**, Proposition 2.2] and [**34**, Proposition 40.2.1], the subalgebras $\mathcal{U}^{\pm}[w]$ of $\mathcal{U}_q(\mathfrak{g})$ generated by E_{β_k}, $k \in [1, N]$ and F_{β_k}, $k \in [1, N]$, respectively, do not depend on the choice of a reduced expression for w and have the $\mathbb{K}$-bases

$$\begin{aligned} &\{(E_{\beta_N})^{m_N} \dots (E_{\beta_1})^{m_1} \mid m_N, \dots, m_1 \in \mathbb{Z}_{\geq 0}\} \ \text{ and} \\ &\{(F_{\beta_N})^{m_N} \dots (F_{\beta_1})^{m_1} \mid m_1, \dots, m_N \in \mathbb{Z}_{\geq 0}\}. \end{aligned} \tag{9.9}$$

The algebras $\mathcal{U}^{\pm}[w]$ are $\mathcal{Q}$-graded subalgebras of $\mathcal{U}_q(\mathfrak{g})$ and are thus stable under the action (9.4).

There is a unique algebra automorphism ω of $\mathcal{U}_q(\mathfrak{g})$ such that

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_i) = K_i^{-1}, \quad \forall i \in [1, r].$$

By [**28**, eq. 8.14(9)], $\omega(T_i(u)) = (-1)^{\langle \alpha_i^\vee, \gamma \rangle} q^{\langle \alpha_i, \gamma \rangle} T_i(\omega(u))$ for all $i \in [1, r]$, $\gamma \in \mathcal{Q}$, and $u \in \mathcal{U}_q(\mathfrak{g})_{\gamma}$. This implies that the restrictions of ω induce isomorphisms

$$\omega : \mathcal{U}^+[w] \xrightarrow{\cong} \mathcal{U}^-[w], \quad \omega(E_{\beta_k}) = (-1)^{\langle \beta_k - \alpha_{i_k}, \rho^\vee \rangle} q^{-\langle \beta_k - \alpha_{i_k}, \rho \rangle} F_{\beta_k}, \ \ \forall k \in [1, N],$$

where ρ and $\rho^\vee$ are the half sums of the positive roots and coroots of $\mathfrak{g}$, respectively.

We will restrict ourselves to $\mathcal{U}^-[w]$ since these algebras are naturally realized in terms of highest weight vectors for $\mathcal{U}_q(\mathfrak{g})$-modules (see Section 9.3 for details), while the $\mathcal{U}^+[w]$ are realized in terms of lowest weight vectors [**41**, Theorem 2.6]. The above isomorphisms can be used to translate all results to $\mathcal{U}^+[w]$.

The Levendorskii–Soibelman straightening law is the following commutation relation in $\mathcal{U}^-[w]$:

$$
\begin{aligned}
(9.10)\quad & F_{\beta_k}F_{\beta_j} - q^{-\langle \beta_k, \beta_j\rangle} F_{\beta_j}F_{\beta_k} \\
&= \sum_{\mathbf{m}=(m_{j+1},\dots,m_{k-1})\,\in\,\mathbb{Z}_{\geq 0}^{k-j-1}} \xi_{\mathbf{m}} (F_{\beta_{k-1}})^{m_{k-1}} \dots (F_{\beta_{j+1}})^{m_{j+1}}, \ \ \xi_{\mathbf{m}} \in \mathbb{K},
\end{aligned}
$$

for all $1 \leq j < k \leq N$ (see e.g. [**4**, Proposition I.6.10] and apply ω). This identity and (9.9) easily imply, as recorded in the following lemma, that $\mathcal{U}^-[w]$ is a symmetric CGL extension for the obvious order of its canonical generators:

$$
x_1 := F_{\beta_1}, \dots, x_N := F_{\beta_N}. \tag{9.11}
$$

Note that the unital subalgebra of $\mathcal{U}^-[w]$ generated by $F_{\beta_1}, \dots, F_{\beta_k}$ is equal to $\mathcal{U}^-[w_{\leq k}]$. It follows from (9.3) that for all $k \in [1,N]$ there exist $h_k, h_k^* \in \mathcal{H}$ such that

$$
h_k^{\beta_j} = q^{\langle \beta_k, \beta_j \rangle}, \ \ \forall j \in [1,k] \quad \text{and} \quad (h_k^*)^{\beta_l} = q^{-\langle \beta_k, \beta_l\rangle}, \ \ \forall l \in [k,N]. \tag{9.12}
$$

For $i \in [1,r]$, set $q_i := q^{\|\alpha_i\|^2/2} \in \mathbb{K}^*$. Recall that $\|\alpha_i\|^2/2 \in \{1,2,3\}$ for all $i \in [1,r]$.

Lemma 9.1. *For all Weyl group elements $w \in W$ of length N and reduced expressions* (9.6) *of w we have:*

(a) *For all $k \in [1,N]$, the algebra $\mathcal{U}^-[w_{\leq k}]$ is an Ore extension of the form $\mathcal{U}^-[w_{\leq (k-1)}][F_{\beta_k}; \sigma_k, \delta_k]$, where $\sigma_k = (h_k \cdot) \in \mathrm{Aut}(\mathcal{U}^-[w_{\leq(k-1)}])$ and δ_k is the locally nilpotent σ_k-derivation of $\mathcal{U}^-[w_{\leq (k-1)}]$ given by*

$$
\delta_k(u) := F_{\beta_k} u - q^{\langle \beta_k, \gamma\rangle} u F_{\beta_k}, \ \ \forall u \in (\mathcal{U}^-[w_{\leq(k-1)}])_\gamma, \ \gamma \in \mathcal{Q}.
$$

The h_k-eigenvalue of F_{β_k} is $q_{i_k}^{-2}$ and is not a root of unity.

(b) *The algebra*

$$
\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_1}][F_{\beta_2}; \sigma_2, \delta_2] \cdots [F_{\beta_N}; \sigma_N, \delta_N] \tag{9.13}
$$

is a symmetric CGL extension for the choice of generators (9.11) *and the choice of elements $h_k, h_k^* \in \mathcal{H}$.*

(c) *The corresponding "interval" subalgebras are given by*

$$
\mathcal{U}^-[w]_{[j,k]} = T_{w_{\leq j-1}}\big(\mathcal{U}^-[w_{[j,k]}]\big).
$$

For parts (a)-(b) of the lemma see e.g. [**13**, Lemma 2.1]. Part (c) of the lemma follows at once from the definition of the root vectors F_{β_k}. The multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} \in M_N(\mathbb{K}^*)$ associated to the above CGL extension presentation of $\mathcal{U}^-[w]$ is given by

$$
\lambda_{kj} = q^{-\langle \beta_k, \beta_j\rangle}, \ \ \forall 1 \leq j < k \leq N. \tag{9.14}
$$

Furthermore,

$$
\lambda_k = q_{i_k}^{-2} \ \text{ and } \ \lambda_k^* = q_{i_k}^2, \ \ \forall k \in [1,N], \tag{9.15}
$$

because $\|\beta_k\|^2 = \|\alpha_{i_k}\|^2$.

9.3. Quantum function algebras and homomorphisms

Recall that a $\mathcal{U}_q(\mathfrak{g})$-module V is a type one module if it is a direct sum of its weight spaces defined by

$$V_\mu := \{v \in V \mid K_i v = q^{\langle \mu, \alpha_i \rangle} v, \ \ \forall i \in [1,r]\}, \ \mu \in \mathcal{P}.$$

The category of finite dimensional type one (left) $\mathcal{U}_q(\mathfrak{g})$-modules is semisimple (see [**28**, Theorem 5.17] and the remark on p. 85 of [**28**]), and is closed under taking tensor products and duals. The irreducible modules in this category are classified by the dominant integral weights of $\mathfrak{g}$ ([**28**, Theorem 5.10]). For each $\mu \in \mathcal{P}^+$, denote by $V(\mu)$ the corresponding irreducible module. Let v_μ be a highest weight vector of $V(\mu)$.

Let G denote the connected, simply connected, complex simple algebraic group with Lie algebra $\mathfrak{g}$. The quantum function algebra $R_q[G]$ is the Hopf subalgebra of the restricted dual $\mathcal{U}_q(\mathfrak{g})^\circ$ spanned by the matrix coefficients $c^\mu_{\xi,v}$ of the modules $V(\mu)$, $\mu \in \mathcal{P}^+$, defined by

$$c^\mu_{\xi,v}(u) := \xi(uv), \ \ \forall v \in V(\mu), \ \xi \in V(\mu)^*, \ u \in \mathcal{U}_q(\mathfrak{g}). \tag{9.16}$$

Set for brevity

$$c^\mu_\xi := c^\mu_{\xi,v_\mu}, \ \ \forall \mu \in \mathcal{P}^+, \ \xi \in V(\mu)^*.$$

The space

$$R^+ := \operatorname{Span}\{c^\mu_\xi \mid \mu \in \mathcal{P}^+, \ \xi \in V(\mu)^*\}$$

is a subalgebra of $R_q[G]$ [**30**, §9.1.6]. The braid group $\mathcal{B}_\mathfrak{g}$ acts on the finite dimensional type one $\mathcal{U}_q(\mathfrak{g})$-modules V (see [**28**, §8.6]) in a compatible way with its action on $\mathcal{U}_q(\mathfrak{g})$:

$$T_w(u.v) := (T_w u).(T_w v), \ \ \forall w \in W, \ u \in \mathcal{U}_q(\mathfrak{g}), \ v \in V$$

(cf. [**28**, eq. 8.14 (1)]). Given $\mu \in \mathcal{P}^+$ and $w \in W$, there exists a unique vector $\xi_{w,\mu} \in (V(\mu)^*)_{-w\mu}$ such that

$$\langle \xi_{w,\mu}, T^{-1}_{w^{-1}} v_\mu \rangle = 1. \tag{9.17}$$

For $y, w \in W$ and $\mu \in \mathcal{P}^+$ define the (generalized) quantum minors

$$\Delta_{y\mu,w\mu} := c^\mu_{\xi_{y,\mu}, T^{-1}_{w^{-1}} v_\mu} \in R_q[G] \ \text{ and } \ \Delta_{w\mu} := \Delta_{w\mu,\mu} = c^\mu_{\xi_{w,\mu}} \in R^+. \tag{9.18}$$

For all $\mu \in \mathcal{P}^+$ and $y, w \in W$, the quantum minor $\Delta_{y\mu,w\mu}$ above equals the quantum minor $\Delta_{y\mu,w\mu}$ of Berenstein–Zelevinsky from [**3**, Eq. (9.10)]. One shows this by a repetitive application of the formulas

$$T_i^{-1} v_{m\varpi_i} = \frac{1}{[m]_{q_i}!} F_i^m v_{m\varpi_i} \quad \text{and} \quad E_i^m F_i^m v_{m\varpi_i} = ([m]_{q_i}!)^2 v_{m\varpi_i}$$

for $m \in \mathbb{Z}_{\geq 0}$, $i \in [1,r]$. (Here and below we use the notation of [**4, 28**] for q-integers and factorials.) This is why everywhere in this and the next chapter we use $T^{-1}_{w^{-1}}$ instead of T_w. Using the latter would bring some unwanted extra scalars coming from the formula

$$T_i v_{m\varpi_i} = \frac{(-q_i)^m}{[m]_{q_i}!} F_i^m v_{m\varpi_i}.$$

For $w \in W$, the set $E_w := \{\Delta_{w\mu} \mid \mu \in \mathcal{P}^+\} \subset R^+$ is a multiplicative subset of R^+ because $\Delta_{w\mu_1}\Delta_{w\mu_2} = \Delta_{w\mu_1 + w\mu_2}$ for all $\mu_1, \mu_2 \in \mathcal{P}^+$, $w \in W$. Joseph proved that it is an Ore set [**30**, Lemma 9.1.10] for $\operatorname{char} \mathbb{K} = 0$ and q transcendental. His proof works for all base fields $\mathbb{K}$, $q \in \mathbb{K}^*$ not a root of unity, [**40**, §2.2]. Following [**30**, §10.4.8], define the algebras

$$R^w := R^+[E_w^{-1}] \text{ and } R_0^w := \{c_\xi^\mu \Delta_{w\mu}^{-1} \mid \mu \in \mathcal{P}^+,\ \xi \in V(\mu)^*\}.$$

One does not need to take a span in the definition of the second algebra, but note that the elements of R_0^w do not have unique presentations in the form $c_\xi^\mu \Delta_{w\mu}^{-1}$. The algebra R_0^w is $\mathcal{Q}$-graded by

$$(R_0^w)_\gamma := \{c_\xi^\mu \Delta_{w\mu}^{-1} \mid \mu \in \mathcal{P}^+,\ \xi \in (V(\mu)^*)_{\gamma - w\mu}\},\ \ \forall\, \gamma \in \mathcal{Q}.$$

In particular,

$$\Delta_{y\mu}\Delta_{w\mu}^{-1} = c^\mu_{\xi_{y,\mu}} \Delta_{w\mu}^{-1} \in (R_0^w)_{(w-y)\mu},\ \ \forall \mu \in \mathcal{P}^+,\ y, w \in W.$$

The algebra $\mathcal{U}^-[w]$ is realized as a quotient of R_0^w as follows. Denote $\mathcal{Q}^+ := \sum_i \mathbb{Z}_{\geq 0}\alpha_i$. For $\gamma \in \mathcal{Q}^+$, set $m_w(\gamma) := \dim(\mathcal{U}^+[w])_\gamma = \dim(\mathcal{U}^-[w])_{-\gamma}$. Let $\{u_{\gamma,n}\}_{n=1}^{m_w(\gamma)}$ and $\{u_{-\gamma,n}\}_{n=1}^{m_w(\gamma)}$ be dual bases of $(\mathcal{U}^+[w])_\gamma$ and $(\mathcal{U}^-[w])_{-\gamma}$ with respect to the Rosso–Tanisaki form, see [**28**, Ch. 6]. The quantum R-matrix corresponding to w is the element

$$\mathcal{R}^w := 1 \otimes 1 + \sum_{\gamma \in \mathcal{Q}^+,\, \gamma \neq 0} \sum_{n=1}^{m_w(\gamma)} u_{\gamma,n} \otimes u_{-\gamma,n}$$

of the completion $\mathcal{U}^+ \widehat{\otimes} \mathcal{U}^-$ of $\mathcal{U}^+ \otimes \mathcal{U}^-$ with respect to the descending filtration [**34**, §4.1.1].

There is a unique graded algebra antiautomorphism τ of $\mathcal{U}_q(\mathfrak{g})$ given by

$$\tau(E_i) = E_i,\ \ \tau(F_i) = F_i,\ \ \tau(K_i) = K_i^{-1},\ \ \forall i \in [1, r], \tag{9.19}$$

see [**28**, Lemma 4.6(b)] for details. It is compatible with the braid group action:

$$\tau(T_w u) = T_{w^{-1}}^{-1}(\tau(u)),\ \ \forall u \in \mathcal{U}_q(\mathfrak{g}),\ w \in W, \tag{9.20}$$

see [**28**, Eq. 8.18(6)].

Theorem 9.2. [**41**, Theorem 2.6] *For all Weyl group elements $w \in W$, the map $\varphi_w : R_0^w \to \mathcal{U}^-[w]$ given by*

$$\varphi_w\big(c_\xi^\mu \Delta_{w\mu}^{-1}\big) := \big(c^\mu_{\xi, T^{-1}_{w^{-1}} v_\mu} \otimes \mathrm{id}\big)(\tau \otimes \mathrm{id})\mathcal{R}^w,\ \ \forall \mu \in \mathcal{P}^+,\ \xi \in V(\mu)^*$$

is a well defined surjective $\mathcal{Q}$-graded algebra antihomomorphism.

The kernel of φ_w has an explicit description in terms of Demazure modules, [**41**, Theorem 2.6]. Later we will need the following explicit formula for φ_w which follows at once from the standard formula for the inner product of pairs of monomials with respect to the Rosso–Tanisaki form [**28**, Eqs. 8.30 (1) and (2)]:

$$\begin{aligned}\varphi_w(c_\xi^\mu \Delta_{w\mu}^{-1}) = \sum_{m_1, \ldots, m_N \in \mathbb{Z}_{\geq 0}} &\left(\prod_{j=1}^N \frac{(q_{i_j}^{-1} - q_{i_j})^{m_j}}{q_{i_j}^{m_j(m_j-1)/2} [m_j]_{q_{i_j}}!} \right) \\ &\times \langle \xi, (\tau E_{\beta_1})^{m_1} \ldots (\tau E_{\beta_N})^{m_N} T^{-1}_{w^{-1}} v_\mu \rangle F_{\beta_N}^{m_N} \ldots F_{\beta_1}^{m_1},\end{aligned} \tag{9.21}$$

for all $\mu \in \mathcal{P}^+$, $\xi \in V(\mu)^*$.

9.4. Quantum minors and sequences of prime elements

For $y, w \in W$ and $i \in [1, r]$, set

$$\widetilde{\Delta}_{y\varpi_i, w\varpi_i} := \varphi_w(\Delta_{y\varpi_i}\Delta^{-1}_{w\varpi_i}) = (\Delta_{y\varpi_i, w\varpi_i}\tau \otimes \mathrm{id})\mathcal{R}^w \in \mathcal{U}^-[w]_{(w-y)\varpi_i}. \tag{9.22}$$

The elements $\Delta_\mu \Delta^{-1}_{w\mu} \in R^w_0$ are normal modulo $\ker \varphi_w$ for all $\mu \in \mathcal{P}^+$, which implies that $\widetilde{\Delta}_{\varpi_i, w\varpi_i}$ are (nonzero) normal elements of $\mathcal{U}^-[w]$ for all $i \in [1, r]$. More precisely,

$$\widetilde{\Delta}_{\varpi_i, w\varpi_i} u = q^{-\langle (1+w)\varpi_i, \gamma\rangle} u \widetilde{\Delta}_{\varpi_i, w\varpi_i}, \ \ \forall i \in [1, r], \ u \in \mathcal{U}^-[w]_\gamma, \ \gamma \in \mathcal{Q}, \tag{9.23}$$

see [**41**, Eq. (3.30)].

Denote the support of $w \in W$ by

$$\mathcal{S}(w) := \{i \in [1, r] \mid s_i \leq w\} = \{i_1, \dots, i_N\},$$

where $\leq$ refers to the Bruhat order on W. The homogeneous prime elements of the algebras $\mathcal{U}^-[w]$ are given by the following theorem.

Theorem 9.3. [**41**, Theorem 6.2] *For all $w \in W$,*

$$\{\widetilde{\Delta}_{\varpi_i, w\varpi_i} \mid i \in \mathcal{S}(w)\}$$

is a list of the homogeneous prime elements of $\mathcal{U}^-[w]$ up to scalar multiples.

Corollary 9.4. *In the above setting,*

$$\mathrm{rk}(\mathcal{U}^-[w]) = |\mathcal{S}(w)|.$$

Lemma 9.1 (c) and Theorem 9.3 imply that all prime elements $y_{[j, s^m(j)]}$ are scalar multiples of elements of the form $T_{w'}(\widetilde{\Delta}_{\varpi_i, w''\varpi_i})$ where w' and w'' are subwords of (9.6), $i \in \mathcal{S}(w)$. In the remaining part of this chapter we obtain a more explicit form of this fact.

From now on we fix a reduced expression (9.6) of $w \in W$. Consider the function

$$\eta : [1, N] \to [1, r], \ \ \eta(k) = i_k. \tag{9.24}$$

The associated functions $p : [1, N] \to [1, N] \sqcup \{-\infty\}$ and $s : [1, N] \to [1, N] \sqcup \{+\infty\}$ are the functions $k \mapsto k^-$ and $k \mapsto k^+$ from [**1**]:

$$p(k) = \begin{cases} \max\{j < k \mid i_j = i_k\}, & \text{if such } j \text{ exists} \\ -\infty, & \text{otherwise} \end{cases} \tag{9.25}$$

and

$$s(k) = \begin{cases} \min\{j > k \mid i_j = i_k\}, & \text{if such } j \text{ exists} \\ +\infty, & \text{otherwise.} \end{cases} \tag{9.26}$$

(One should note that [**11**] uses integers instead of $\pm\infty$.) Recall from (3.14) that $w_\circ$ denotes the longest element of S_N. The CGL extension presentation of $\mathcal{U}^-[w]$ corresponding to this permutation has the form

$$\mathcal{U}^-[w] = \mathbb{K}[F_{\beta_N}][F_{\beta_{N-1}}; \sigma^*_{N-1}, \delta^*_{N-1}] \cdots [F_{\beta_1}; \sigma^*_1, \delta^*_1]. \tag{9.27}$$

We write $y_{w_\circ,1}, \dots, y_{w_\circ,N}$ for the sequence of y-elements given by Theorem 3.6 for the above presentation.

THEOREM 9.5. *For all Weyl group elements $w \in W$ of length N, reduced expressions* (9.6) *of w, and $k \in [1, N]$, we have*

$$y_{w_\circ,k} = \big(q_{i_{N-k+1}}^{-1} - q_{i_{N-k+1}}\big)^{-O_+(N-k+1)-1}\, \widetilde{\Delta}_{w_{\leq N-k}\varpi_{i_{N-k+1}}, w\varpi_{i_{N-k+1}}}. \tag{9.28}$$

The function η from (9.24) *satisfies the conditions of Theorem* 3.6 *for the CGL extension presentation* (9.13) *of $\mathcal{U}^-[w]$, and the corresponding functions p and s are given by the formulas* (9.25) *and* (9.26).

Furthermore, for all $1 \leq j < k \leq N$,

$$\begin{aligned}\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}} \widetilde{\Delta}_{w_{\leq k-1}\varpi_{i_k}, w\varpi_{i_k}} &= \\ &= q^{\langle (w_{\leq j-1}+w)\varpi_{i_j}, (w_{\leq k-1}-w)\varpi_{i_k}\rangle} \widetilde{\Delta}_{w_{\leq k-1}\varpi_{i_k}, w\varpi_{i_k}} \widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}}.\end{aligned} \tag{9.29}$$

Using the definition of the braid group action, one easily sees that

$$\widetilde{\Delta}_{\varpi_i, w\varpi_i} = \widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}}, \quad \text{for } j = \min\{k \in [1, N] \mid i_k = i\}.$$

PROOF. First we prove (9.29). Applying [**3**, Eq. (10.2)], we obtain

$$\Delta_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}} \Delta_{w_{\leq j-1}\varpi_{i_j}, \varpi_{i_j}} = \\ q^{\langle w_{\leq k-1}\varpi_{i_k}, w_{\leq j-1}\varpi_{i_j}\rangle - \langle \varpi_{i_k}, \varpi_{i_j}\rangle} \Delta_{w_{\leq j-1}\varpi_{i_j}, \varpi_{i_j}} \Delta_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}}$$

for all $1 \leq j < k \leq N$. Eq. (10.2) in [**3**] was stated for $\mathbb{K} = \mathbb{Q}(q)$, but its proof works for all fields $\mathbb{K}$ since it only uses the standard left and right actions of $\mathcal{U}_q(\mathfrak{g})$ on $R_q[G]$. In addition, we have

$$\Delta_{w\mu,\mu} u = q^{-\langle w\mu, \gamma\rangle} u \Delta_{w\mu,\mu} \mod R^+[E_w^{-1}] \ker \varphi_w, \ \ \forall u \in (R_0^w)_\gamma, \ \gamma \in \mathcal{Q}, \ \mu \in \mathcal{P}^+.$$

see [**41**, Eq. (2.22) and Theorem 2.6]. Eq. (9.29) follows from these two identities using that $\varphi_w : R_0^w \to \mathcal{U}^-[w]$ is a graded antihomomorphism by Theorem 9.2.

Proposition 3.3 from [**13**] implies that $\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}} \in \mathcal{U}^-[w]_{[j,N]}$ for all $j \in [1, N]$ and that $\mathcal{U}^-[w]_{[j,N]}$ sits inside the subalgebra of $\mathrm{Fract}(\mathcal{U}^-[w])$ generated by the elements $\widetilde{\Delta}^{\pm 1}_{w_{\leq l-1}\varpi_{i_l}, w\varpi_{i_l}}$, $l \in [j, N]$. Since $\widetilde{\Delta}_{w_{\leq k-1}\varpi_{i_k}, w\varpi_{i_k}}$ belongs to $(\mathcal{U}^-[w])_{-(w_{\leq k-1}-w)\varpi_{i_k}}$, it follows from (9.29) that $\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}}$ is a normal element of $\mathcal{U}^-[w]_{[j,N]}$:

$$\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}} u = q^{-\langle (w_{\leq j-1}+w)\varpi_{i_j}, \gamma\rangle} u \widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}}$$

for all $u \in \big(\mathcal{U}^-[w]_{[j,N]}\big)_\gamma$, $\gamma \in \mathcal{Q}$.

Invoking again [**13**, Proposition 3.3], we have

$$\begin{aligned}\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}} &\equiv (q_{i_j}^{-1} - q_{i_j}) \widetilde{\Delta}_{w_{\leq s(j)-1}\varpi_{i_{s(j)}}, w\varpi_{i_{s(j)}}} F_{\beta_j} \\ &\mod \mathcal{U}^-[w]_{[j+1,N]}, \ \text{ if } \ s(j) \neq +\infty\end{aligned} \tag{9.30}$$

and

$$\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}} \equiv (q_{i_j}^{-1} - q_{i_j}) F_{\beta_j} \mod \mathcal{U}^-[w]_{[j+1,N]}, \ \text{ if } \ s(j) = +\infty. \tag{9.31}$$

We apply Proposition 3.10 for the presentation (9.27) of $\mathcal{U}^-[w]$. The previous arguments verify conditions (i)–(iii) of the proposition, where y'_k is the element on the right hand side of (9.28) for $k \in [1, N]$ and $c'_1, \ldots, c'_N$ are obtained from (9.30) and (9.31). We are left with showing that condition (iv) is satisfied. Denote by $\widetilde{s}$: $[1, N] \to [1, N] \sqcup \{+\infty\}$ the successor function from Theorem 3.6 for the presentation

(9.27) of $\mathcal{U}^-[w]$. We need to prove that, if $\widetilde{s}(j) = +\infty$, then $s(j) = +\infty$, i.e we are in the case of Eq. (9.31). If $\widetilde{s}(j) = +\infty$, then $\mathrm{rk}\,\mathcal{U}^-[w_{[j,N]}] = \mathrm{rk}\,\mathcal{U}^-[w_{[j+1,N]}] + 1$. Applying Corollary 9.4, we obtain that $|\mathcal{S}(w_{[j,N]})| = |\mathcal{S}(w_{[j+1,N]})| + 1$, and thus $s(j) = +\infty$. Eq. (9.28) and the statements for η, p and s now follow from Proposition 3.10. $\square$

Combining Theorem 9.5 (applied to $\mathcal{U}^-[w_{\leq s^m(j)}]$) and Proposition 8.9 (applied to the case when τ equals the longest element of $S_{s^m(j)}$) leads to the first part of the following:

COROLLARY 9.6. *In the setting of Theorem 9.5, assume there exists $\sqrt{q} \in \mathbb{K}^*$. Then for all $j \in [1, N]$ and $m \in \mathbb{Z}_{\geq 0}$ such that $s^m(j) \in [1, N]$,*

$$y_{[j,s^m(j)]} = \left(q_{i_j}^{-1} - q_{i_j}\right)^{-m-1} \left(\prod_{0 \leq l < n \leq m} q^{\langle \beta_{s^l(j)}, \beta_{s^n(j)} \rangle} \right) \widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w_{\leq s^m(j)}\varpi_{i_j}}. \tag{9.32}$$

and

$$\beta_j + \beta_{s(j)} + \cdots + \beta_{s^m(j)} = (w_{\leq j-1} - w_{\leq s^m(j)})\varpi_{i_j}. \tag{9.33}$$

Eq. (9.33) follows from (9.32) by inspecting degrees with respect to the $\mathcal{Q}$-grading. On one hand, $\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w_{\leq s^m(j)}\varpi_{i_j}}$ has degree $(w_{s^m(j)} - w_{\leq j-1})\varpi_{i_j}$ by (9.22). On the other hand,

$$\deg(y_{[j,s^m(j)]}) = \sum_{l=0}^{m} \deg(F_{\beta_{s^l(j)}}) = -\sum_{l=0}^{m} \beta_{s^l(j)}.$$

CHAPTER 10

Quantum cluster algebra structures on quantum Schubert cell algebras

As an application of the results of Chapter 8, we prove that for all finite dimensional simple Lie algebras $\mathfrak{g}$, the quantum Schubert cell algebras $\mathcal{U}^-[w]$ have canonical structures of quantum cluster algebras. Previously this was known for symmetric Kac–Moody algebras $\mathfrak{g}$ due to Geiß, Leclerc and Schröer [**15**]. Our proof works under very general assumptions on the base field $\mathbb{K}$ and the deformation parameter $q \in \mathbb{K}^*$. The field can have arbitrary characteristic, it does not need to be algebraically closed, and q is only assumed to be a non-root of unity.

The existence of *some* quantum cluster algebra structure on $U^-[w]$ is guaranteed by Theorem 8.2, once we enlarge the base field to include a square root of q and rescale the generators $F_{\beta_1}, \dots, F_{\beta_N}$ to satisfy condition (6.13). We actually put these generators in reverse order, $F_{\beta_N}, \dots, F_{\beta_1}$, before arranging to apply Theorem 8.2. What then remains is to identify the initial quantum seed and the cluster variables of this structure.

10.1. Statement of the main result

Fix a finite dimensional complex simple Lie algebra $\mathfrak{g}$, a Weyl group element $w \in W$, and a reduced expression (9.6) of w. Throughout this chapter, the predecessor and successor functions p and s will refer to the ones given by (9.25)–(9.26). Recall that the multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} \in M_N(\mathbb{K}^*)$ associated to the CGL extension presentation of $\mathcal{U}^-[w]$ for the sequence of generators (9.11) is given by

$$\lambda_{jk} = q^{\langle \beta_j, \beta_k \rangle}, \ \ \forall 1 \le j < k \le N.$$

From now on we will assume that the base field $\mathbb{K}$ contains a square root of q and fix such a root $\sqrt{q} \in \mathbb{K}^*$. Let $\boldsymbol{\nu} \in M_N(\mathbb{K}^*)$ be the unique multiplicatively skew-symmetric matrix given by

$$\nu_{jk} = \sqrt{q}^{\,\langle \beta_j, \beta_k \rangle}, \ \ \forall 1 \le j < k \le N.$$

The results of Chapters 4–8 will be applied to the algebra $\mathcal{U}^-[w]$ for this choice of the matrix $\boldsymbol{\nu}$.

Theorem 9.5 implies that there is a unique toric frame $M^w \colon \mathbb{Z}^N \to \mathrm{Fract}(\mathcal{U}^-[w])$ such that

$$M^w(e_k) = \sqrt{q}^{\,\|(w - w_{\le k-1})\varpi_{i_k}\|^2/2}\, \widetilde{\Delta}_{w_{\le k-1}\varpi_{i_k}, w\varpi_{i_k}}, \ \ \forall k \in [1, N]$$

whose matrix is given by

$$\mathbf{r}(M^w)_{jk} = \sqrt{q}^{\,\langle (w_{\le j-1} + w)\varpi_{i_j}, (w_{\le k-1} - w)\varpi_{i_k} \rangle}, \ \ \forall 1 \le j < k \le N. \tag{10.1}$$

The cluster variables $M^w(e_k)$ are also given by the expressions

$$\begin{aligned} M^w(e_k) &= \sqrt{q}^{\,\|(w_{\leq O_+(k)} - w_{\leq k-1})\varpi_{i_k}\|^2/2}\, \widetilde{\Delta}_{w_{\leq k-1}\varpi_{i_k}, w\varpi_{i_k}} \\ &= \sqrt{q}^{\,\|(w_{[k,N]} - 1)\varpi_{i_k}\|^2/2}\, \widetilde{\Delta}_{w_{\leq k-1}\varpi_{i_k}, w\varpi_{i_k}}. \end{aligned}$$

The frozen variables of the quantum cluster algebra structure that we will define will be indexed by

$$\{k \in [1,N] \mid p(k) = -\infty\}. \tag{10.2}$$

Note that this set has the same cardinality as $\mathcal{S}(w)$. We will use the convention that

(10.3) *the columns of all* $n \times (N - |\mathcal{S}(w)|)$ *matrices are indexed by* $\{k \in [1,N] \mid p(k) \neq -\infty\}$.

Under this convention, define the $N \times (N - |\mathcal{S}(w)|)$-matrix $\widetilde{B}^w$ with entries

$$b_{jk} = \begin{cases} 1, & \text{if } j = p(k) \\ -1, & \text{if } j = s(k) \\ c_{i_j i_k}, & \text{if } p(j) < p(k) < j < k \\ -c_{i_j i_k}, & \text{if } p(k) < p(j) < k < j \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 10.1. *Consider an arbitrary finite dimensional complex simple Lie algebra* $\mathfrak{g}$*, a Weyl group element* $w \in W$*, a reduced expression* (9.6) *of* w*, an arbitrary base field* $\mathbb{K}$ *and a non-root of unity* $q \in \mathbb{K}^*$ *such that* $\sqrt{q} \in \mathbb{K}$*. Then* $(M^w, \widetilde{B}^w)$ *is a quantum seed and the quantum Schubert cell algebra* $\mathcal{U}^-[w]$ *equals the quantum cluster algebra* $\mathcal{A}(M^w, \widetilde{B}^w, \varnothing)_{\mathbb{K}}$ *with set of frozen variables indexed by* (10.2)*. Furthermore, this quantum cluster algebra equals the upper quantum cluster algebra* $\mathcal{U}(M^w, \widetilde{B}^w, \varnothing)_{\mathbb{K}}$*. For all* $j \in [1,N]$ *and* $m \in \mathbb{Z}_{\geq 0}$ *such that* $s^m(j) \in [1,N]$*, the elements*

$$\overline{y}_{[j,s^m(j)]} = \sqrt{q}^{\,\|(w_{\leq s^m(j)} - w_{\leq j-1})\varpi_{i_j}\|^2/2}\, \widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w_{\leq s^m(j)}\varpi_{i_j}} \tag{10.4}$$

are cluster variables of $\mathcal{U}^-[w]$.

As mentioned, Geiß, Leclerc and Schröer [**15**] constructed quantum cluster algebra structures on the algebras $\mathcal{U}^-[w]$ in the case of symmetric Kac–Moody algebras $\mathfrak{g}$. We note that the normalization scalars for the cluster variables in Theorem 10.1 match the ones used in [**15**]. In the case when w is the square of a Coxeter element and $\mathfrak{g}$ is an arbitrary Kac–Moody algebra, a quantum cluster algebra structure on $\mathcal{U}^-[w]$ was constructed by Berenstein and Rupel [**2**] simultaneously to our work.

Because of the invariance of the bilinear form $\langle ., . \rangle$ with respect to the Weyl group W, the scalar in (10.4) equals

$$\sqrt{q}^{\,\|(w_{[j,s^m(j)]} - 1)\varpi_{i_j}\|^2/2}.$$

Theorem 10.1 is proved in Section 10.3. In Section 10.2 we prove that the toric frame M^w and the matrix $\widetilde{B}^w$ are compatible. Most of the conditions needed to apply Theorem 8.2 are straightforward to verify for the algebra $\mathcal{U}^-[w]$, except for the condition (6.13). Set for brevity

$$\sqrt{q_i} := \sqrt{q}^{\,\|\alpha_i\|^2/2}, \quad \forall i \in [1,r].$$

We prove in Section 10.3 that the condition (6.13) is satisfied after the rescaling

$$x_k \mapsto \sqrt{q_{i_k}}(q_{i_k}^{-1} - q_{i_k})F_{\beta_k}, \quad \forall k \in [1, N]$$

of the standard generators (9.11) of $\mathcal{U}^-[w]$ and the reversal of the order of these generators.

By setting $m = 0$ in the last statement of Theorem 10.1, one obtains that the rescaled generators

$$\sqrt{q_{i_k}}(q_{i_k}^{-1} - q_{i_k})F_{\beta_k}$$

of $\mathcal{U}^-[w]$ are cluster variables for all $k \in [1, N]$.

Remark 10.2. The quantum seed of $\mathcal{U}^-[w]$ in Theorem 10.1 comes from the one in Theorem 8.2 associated to the longest element $w_\circ \in \Xi_N$. More precisely, $M^w = M_{w_\circ}(w_\circ)_\bullet$ if the generators are rescaled appropriately. Using (10.4) and Proposition 8.9 it is straightforward to write down explicitly the toric frames of the quantum seeds of $\mathcal{U}^-[w]$ associated to all elements of Ξ_N via the construction of Theorem 8.2. To compute the matrices for all those other seeds one needs to solve explicitly the system of linear equations (8.7).

There is one additional difference between Theorems 8.2 and 10.1. In the first case the cluster variables for the seed corresponding to $w_\circ \in \Xi_N$ (and as a matter of fact for all clusters corresponding to elements of Ξ_N) are reenumerated according to the rule of Section 8.1. This is needed in order to match the combinatorics for the different seeds. In the case of Theorem 10.1, we do not perform this reenumeration, in order to match our results to the conventions in the existing literature. Because of this difference, the roles of the predecessor and successor functions in Theorems 8.2 and 10.1 are interchanged. Details are given next.

For the remainder of Chapter 10, we fix the following choice of generators of $\mathcal{U}^-[w]$:

$$x_k = \sqrt{q_{i_k}}(q_{i_k}^{-1} - q_{i_k})F_{\beta_k}, \quad \forall k \in [1, N], \tag{10.5}$$

which as previously noted are rescalings of the canonical generators (9.11). Since $\mathcal{U}^-[w]$ is a symmetric CGL extension with respect to these generators, it is also a symmetric CGL extension with respect to the presentation with the order of generators reversed:

$$\mathcal{U}^-[w] = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \cdots [x_1; \sigma^*_1, \delta^*_1]. \tag{10.6}$$

We will apply Theorem 8.2 to this presentation as the *starting* presentation of $\mathcal{U}^-[w]$. To do so, we must verify that conditions (8.6) and (6.13) hold for the presentation (10.6).

We will need to identify the data involved when Theorem 8.2 is applied as above. (These data are not the same as those appearing at the stage $\tau = w_\circ$ when the starting presentation is the one with the generators in the usual order $x_1, \ldots, x_N$.) First, we label the generators of (10.6) in ascending order in the form $x_{w_\circ, k} = x_{w_\circ(k)}$. By Corollary 5.6, we can use $\eta_{w_\circ} := \eta w_\circ$ for the η-function going with this indexing. The corresponding predecessor and successor functions are given by $p_{w_\circ} = w_\circ s w_\circ$ and $s_{w_\circ} = w_\circ p w_\circ$. The multiplicatively skew-symmetric $\boldsymbol{\lambda}$-matrix associated to (10.6) is $w_\circ \boldsymbol{\lambda} w_\circ$, where $\boldsymbol{\lambda}$ is the one associated to the presentation (9.13), recall (9.14). Given the choice of $\boldsymbol{\nu}$ above, we choose $w_\circ \boldsymbol{\nu} w_\circ$ as the $\boldsymbol{\nu}$-matrix associated to (10.6).

We have already labelled the sequence of normalized prime elements for the presentation (10.6) coming from Theorem 3.6 as $\overline{y}_{w_\circ,k}$, $k \in [1,N]$, and we will show later that $M^w(e_k) = \overline{y}_{w_\circ,w_\circ(k)}$ for all $k \in [1,N]$.

Proposition 3.14 shows that the elements of $\mathcal{H}$ needed for the CGL conditions on (10.6) can be chosen to be $h_N^*, \dots, h_1^*$. Since $h_j^*.x_j = \lambda_j^* x_j$ for all $j \in [1,N]$, the singly-indexed λ-elements for (10.6), which we denote $\lambda_{w_\circ,k}$, have the form

$$\lambda_{w_\circ,k} = \lambda^*_{w_\circ(k)} = q^2_{i_{w_\circ(k)}}, \quad \forall k \in [1,N],$$

recall (9.15). The set of exchangeable indices for the presentation (10.6) is

$$\begin{aligned}\mathbf{ex}_\circ &:= \{k \in [1,N] \mid s_{w_\circ}(k) \neq +\infty\} = \{k \in [1,N] \mid pw_\circ(k) \neq -\infty\}\\ &= \{k \in [1,N] \mid s(w_\circ)_\bullet w_\circ(k) \neq +\infty\} = ((w_\circ)_\bullet w_\circ)^{-1}(\mathbf{ex}).\end{aligned}$$

In the third equality, we have used the observation that $(w_\circ)_\bullet$ acts by reversing the order of the elements in each level set of η. Taking account of Proposition 5.8, we see that

$$\lambda^*_{w_\circ,k} := (\lambda_{w_\circ,k})^* = \lambda^{-1}_{w_\circ,w_\circ pw_\circ(k)} = q^{-2}_{i_{pw_\circ(k)}} = q^{-2}_{i_{w_\circ(k)}}, \quad \forall k \in \mathbf{ex}_\circ. \tag{10.7}$$

In particular, it follows that

$$\big(\lambda^*_{w_\circ,l}\big)^{\|\alpha_{i_{w_\circ(j)}}\|^2} = q^{-\|\alpha_{i_{w_\circ(j)}}\|^2\|\alpha_{i_{w_\circ(l)}}\|^2} = \big(\lambda^*_{w_\circ,j}\big)^{\|\alpha_{i_{w_\circ(l)}}\|^2}, \quad \forall j,l \in \mathbf{ex}_\circ,$$

which verifies (8.6) for the presentation (10.6).

Lemma 10.3. *Let $j \in [1,N]$.*
(a) *If $m \in \mathbb{Z}_{\geq 0}$ such that $s^m(j) \in [1,N]$, then* (10.4) *holds:*

$$\overline{y}_{[j,s^m(j)]} = \sqrt{q}^{\,\|(w_{\leq s^m(j)} - w_{\leq j-1})\varpi_{i_j}\|^2/2}\, \widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w_{\leq s^m(j)}\varpi_{i_j}}.$$

(b) *If $k = w_\circ(j)$, then*

$$\overline{y}_{w_\circ,k} = \sqrt{q}^{\,\|(w - w_{\leq j-1})\varpi_{i_j}\|^2/2}\, \widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w\varpi_{i_j}}.$$

(c) *If the presentation* (10.6) *satisfies condition* (6.13)*, then*

$$M^w = M_{w_\circ}(w_\circ)_\bullet.$$

Proof. (a) Using Corollary 9.6, the form of the rescaling, and the equality $\sqrt{q_{i_k}} = \sqrt{q}^{\,\|\alpha_{i_k}\|^2/2} = \sqrt{q}^{\,\|\beta_k\|^2/2}$, we obtain $\overline{y}_{[j,s^m(j)]} = \Theta\widetilde{\Delta}_{w_{\leq j-1}\varpi_{i_j}, w_{\leq s^m(j)}\varpi_{i_j}}$ where

$$\begin{aligned}\Theta &= \Bigg(\prod_{0\leq l<n\leq m} \sqrt{q}^{\,-\langle\beta_{s^l(j)},\beta_{s^n(j)}\rangle}\Bigg)\sqrt{q_{i_j}}^{\,m+1}\Bigg(\prod_{0\leq l<n\leq m} q^{\langle\beta_{s^l(j)},\beta_{s^n(j)}\rangle}\Bigg)\\ &= \sqrt{q}^{\,(m+1)\|\beta_j\|^2/2}\Bigg(\prod_{0\leq l<n\leq m}\sqrt{q}^{\,\langle\beta_{s^l(j)},\beta_{s^n(j)}\rangle}\Bigg) = \sqrt{q}^{\,\|\beta_j+\beta_{s(j)}+\dots+\beta_{s^m(j)}\|^2/2}\\ &= \sqrt{q}^{\,\|(w_{\leq s^m(j)} - w_{\leq j-1})\varpi_{i_j}\|^2/2}.\end{aligned}$$

(b) Proposition 8.9 shows that $\overline{y}_{w_\circ,k} = \overline{y}_{[j,s^m(j)]}$ where

$$m = \max\{n \in \mathbb{Z}_{\geq 0} \mid s^n(j) \in [j,N]\} = O_+(j).$$

Thus $w_{\leq s^m(j)}\varpi_{i_j} = w\varpi_{i_j}$, so the desired equation follows from part (a).

(c) Under the given assumptions, it follows from part (b) and the definition of M^w that

$$M^w(e_k) = \overline{y}_{w_\circ, w_\circ(k)} = \widehat{M}_{w_\circ}(e_{w_\circ(k)}) = M_{w_\circ}(w_\circ)_\bullet(e_k), \quad \forall k \in [1, N]. \tag{10.8}$$ □

10.2. Compatibility of the toric frame M^w and the matrix $\widetilde{B}^w$

Recall that the rational character lattice $X(\mathcal{H})$ of the torus $\mathcal{H} = (\mathbb{K}^*)^r$ can be identified with $\mathcal{Q}$ via (9.3). Thus, we now view $X(\mathcal{H})$ as an additive group with identity 0.

PROPOSITION 10.4. *In the setting of Theorem* 9.5*, the multiplicatively skew-symmetric matrix* $\mathbf{r}(M^w)$ *and the matrix* $\widetilde{B}^w$ *are compatible, and more precisely*

$$\Omega_{\mathbf{r}(M^w)}(b^k, e_l) = q_{i_k}^{-\delta_{kl}}, \quad \forall k, l \in [1, N], \ p(k) \neq -\infty. \tag{10.9}$$

Moreover, the columns b^k *of the matrix* $\widetilde{B}^w$ *satisfy*

$$\chi_{M^w(b^k)} = 0, \quad \forall k \in [1, N], \ p(k) \neq -\infty. \tag{10.10}$$

PROOF. We derive the proposition from results of [**1, 3**]. This appears to be the shortest way to prove the proposition, although some constructions that match our setting to the setting of [**1, 3**] might appear to be a bit artificial. To avoid this, one could prove the proposition directly following the idea of [**1, 3**]. We leave this to the interested reader.

Define the function $\eta^\vee : [1, N + r] \to [1, r]$ by

$$\eta^\vee(j) = i_j, \ \ \forall j \in [1, N] \quad \text{and} \quad \eta^\vee(N + i) = i, \ \ \forall i \in [1, r].$$

The predecessor map $p^\vee : [1, N + r] \to [1, N + r] \sqcup \{-\infty\}$ for the level sets of $\eta^\vee$ coincides with p on $[1, N]$ and satisfies

$$p^\vee(N + i) := \begin{cases} \max\{j \in [1, N] \mid i_j = i\}, & \text{if such exists} \\ -\infty, & \text{otherwise} \end{cases}$$

for $i \in [1, r]$. Similarly, the successor map $s^\vee$ for the level sets of $\eta^\vee$ is given by

$$s^\vee(j) = \begin{cases} s(j), & \text{if } \ s(j) \neq +\infty \\ N + i_j, & \text{if } \ s(j) = +\infty \end{cases}$$

for $j \in [1, N]$ and $s^\vee(N + i) = +\infty$ for $i \in [1, r]$. Our setting differs from [**1, 3**] in that we place additional indices at the end of a word for a Weyl group element, not at the beginning. The reason for this is that the results of [**1, 3**] will be applied to the inverse of a certain reduced expression.

We extend $\widetilde{B}^w$ to an $(N + r) \times (N - |\mathcal{S}(w)|)$-matrix by setting

$$b_{N+i,k} := \begin{cases} -1, & \text{if } \ s^\vee(k) = N + i \\ -c_{i,i_k}, & \text{if } \ p(k) < p^\vee(N + i) < k \\ 0, & \text{otherwise} \end{cases}$$

for all $i \in [1, r]$ and $k \in [1, N]$, $p(k) \neq -\infty$, using the convention (10.3). Applying [**3**, Theorem 8.3] for the Weyl group elements $u := 1$, $v := w^{-1}$ and the double word corresponding to the reduced expression $s_N \dots s_1$ which is reverse to (9.6) leads to

$$\sum_{j=1}^{N} b_{jk} \mathrm{sign}(j-l)\big(\langle \varpi_{i_j}, \varpi_{i_l}\rangle - \langle w_{\leq j-1}\varpi_{i_j}, w_{\leq l-1}\varpi_{i_l}\rangle\big) \\ - \sum_{i=1}^{r} b_{N+i,k}\langle w\varpi_i, (w_{\leq l-1} - w)\varpi_{i_l}\rangle = -\delta_{kl}\langle \alpha_{i_l}, \alpha_{i_l}\rangle \tag{10.11}$$

for $k, l \in [1, N]$, $p(k) \neq -\infty$.

Using the fact that the one-step mutations in [**3**] are eigenfunctions of both the left and right regular actions of the maximal torus of the connected, simply connected complex algebraic group with Lie algebra $\mathfrak{g}$, we obtain that

$$\sum_{j=1}^{N} b_{jk}\varpi_{i_j} + \sum_{i=1}^{r} b_{N+i,k}\varpi_i = 0 \tag{10.12}$$

$$\sum_{j=1}^{N} b_{jk} w_{\leq j-1}\varpi_{i_j} + \sum_{i=1}^{r} b_{N+i,k} w\varpi_i = 0 \tag{10.13}$$

for all $k \in [1, N]$, $p(k) \neq -\infty$.

Acting by w on (10.12) and subtracting (10.13) gives

$$\chi_{M(b^k)} = \sum_{j=1}^{N} b_{jk}(w - w_{\leq j-1})\varpi_{i_j} = 0, \ \ \forall k \in [1, N], \ p(k) \neq -\infty,$$

which proves (10.10). Similarly, (10.12) and (10.13) imply

$$\sum_{j=1}^{N} b_{jk}\langle (w_{\leq j-1} - w)\varpi_{i_j}, w\varpi_{i_l}\rangle = 0 \quad \text{and}$$
$$\sum_{j=1}^{N} b_{jk}\langle w\varpi_{i_j}, (w_{\leq l-1} - w)\varpi_{i_l}\rangle + \sum_{i=1}^{r} b_{N+i,k}\langle w\varpi_i, (w_{\leq l-1} - w)\varpi_{i_l}\rangle = 0$$

for the same values of k and all $l \in [1, N]$. Combining these two identities, (10.11), and the fact that $\Omega_{\mathbf{r}(M^w)}(e_j, e_l) = \sqrt{q}^{\,m_{jl}}$ where

$$m_{jl} := \mathrm{sign}(j-l)\big(\langle \varpi_{i_j}, \varpi_{i_l}\rangle - \langle w_{\leq j-1}\varpi_{i_j}, w_{\leq l-1}\varpi_{i_l}\rangle\big) \\ - \langle (w_{\leq j-1} - w)\varpi_{i_j}, w\varpi_{i_l}\rangle + \langle w\varpi_{i_j}, (w_{\leq l-1} - w)\varpi_{i_l}\rangle, \ \ \forall j, l \in [1, N]$$

(which follows from (10.1)), we obtain $\Omega_{\mathbf{r}(M^w)}(b^k, e_l) = \sqrt{q}^{\,-\delta_{kl}\|\alpha_{i_k}\|^2}$ and thus (10.9). $\square$

Up to a possible additional rescaling, Theorem 8.2 applied to (10.6) produces an initial quantum seed that we will label $(M_\circ, \widetilde{B}_\circ)$. For $k \in [1, N]$, the toric frame $M_\circ$ sends e_k to $\overline{y}_{w_\circ,k}$ (possibly rescaled), so we see from Lemma 10.3 (b) and the definition of M^w that up to rescaling, $M_\circ$ agrees with $M^w w_\circ$. Since $\widetilde{B}_\circ$ is not affected by any rescaling, recall Remark 8.7, we may use $M^w w_\circ$ in place of $M_\circ$ to determine $\widetilde{B}_\circ$, as follows.

Corollary 10.5. $\widetilde{B}_\circ = w_\circ \widetilde{B}^w \big(w_\circ|_{\mathbf{ex}_\circ}\big)$.

PROOF. Recalling (2.11), we have

$$\mathbf{r}(M^w w_\circ) = {}^{w_\circ^T}\mathbf{r}(M^w)^{w_\circ} = w_\circ \mathbf{r}(M^w) w_\circ.$$

Hence, Proposition 10.4 implies that

$$\Omega_{\mathbf{r}(M^w w_\circ)}(w_\circ b^{w_\circ(k)}, e_l) = \Omega_{\mathbf{r}(M^w)}(b^{w_\circ(k)}, e_{w_\circ(l)}) = q_{i_{w_\circ(k)}}^{-\delta_{kl}} = \sqrt{\lambda^*_{w_\circ,k}}^{\,\delta_{kl}},$$
$$\forall k \in \mathbf{ex}_\circ,\ l \in [1, N],$$

and also

$$\chi_{(M^w w_\circ)(w_\circ b^{w_\circ(k)})} = 0, \quad \forall k \in \mathbf{ex}_\circ.$$

Recall from Remark 8.7 that neither $\mathbf{r}(M_\circ)$ nor $\chi_{M_\circ(b)}$, for $b \in \mathbb{Z}^N$, changes under a rescaling of the generators $x_{w_\circ,k}$. Consequently, the two previous equations also hold when $M^w w_\circ$ is replaced by $M_\circ$. Theorem 8.2 (a) thus implies that $w_\circ b^{w_\circ(k)}$ equals the k-th column of $\widetilde{B}_\circ$, for all $k \in \mathbf{ex}_\circ$, as desired. $\square$

10.3. Proof of Theorem 10.1

Let us write $y_{w_\circ,[i,s^m(i)]}$ and $u_{w_\circ,[i,s^m(i)]}$ for the "interval prime elements" and corresponding u-elements obtained from Theorem 5.1 and Corollary 5.11 applied to the CGL extension (10.6). In particular,

$$u_{w_\circ,[i,s(i)]} = x_{w_\circ,i} x_{w_\circ,s(i)} - y_{w_\circ,[i,s(i)]}, \quad \forall i \in [1, N],\ s(i) \neq +\infty.$$

As in eq. (6.1), write

$$\mathrm{lt}(u_{w_\circ,[i,s(i)]}) = \pi_{w_\circ,[i,s(i)]}(x_{w_\circ})^{f_{w_\circ,[i,s(i)]}}, \quad \forall i \in [1, N],\ s(i) \neq +\infty,$$

for some $\pi_{w_\circ,[i,s(i)]} \in \mathbb{K}^*$ and $f_{w_\circ,[i,s(i)]} \in \sum_{j=i+1}^{s(i)-1} \mathbb{Z}_{\geq 0} e_j$, where the leading term and the monomial $(x_{w_\circ})^{f_{w_\circ,[i,s(i)]}}$ are computed with respect to the order of generators $x_N, \dots, x_1$.

LEMMA 10.6. *Assume that in the setting of Theorem* 10.1*,*

$$w = s_{i_1} \dots s_{i_N}$$

is a reduced expression such that $i_1 = i_N = i$ and $i_k \neq i$, $\forall k \in [2, N-1]$, for some $i \in [1, r]$. Then

$$f_{w_\circ,[1,N]} = -\sum_{j=2}^{N-1} c_{i_{w_\circ(j)} i} e_j,$$

so that

$$\mathrm{lt}(u_{w_\circ,[1,N]}) = \pi_{w_\circ,[1,N]} x_{N-1}^{-c_{i_{N-1} i}} \cdots x_2^{-c_{i_2 i}}. \tag{10.14}$$

PROOF. In view of Corollary 10.5, the entries in the first column of $\widetilde{B}_\circ$ satisfy

$$\big(\widetilde{B}_\circ\big)_{j1} = \big(\widetilde{B}^w\big)_{w_\circ(j),N} = c_{i_{w_\circ(j)} i}, \quad \forall j \in [2, N-1].$$

Since the vector $f_{w_\circ,[1,N]}$ does not change under a rescaling of the generators $x_{w_\circ,k}$, we obtain the desired conclusion by applying Proposition 8.22 to the (possibly rescaled) presentation (10.6). $\square$

Note that in the setting of Lemma 10.6, we have $\beta_1 = \alpha_i$, $\beta_N = -w\alpha_i$ and

$$y_{w_\circ,N} = q_i \widetilde{\Delta}_{\varpi_i, w\varpi_i}$$

because of Theorem 9.5 and the effect of the rescaling of the generators (10.5). Hence,

$$u_{w_\circ,[1,N]} = x_N x_1 - q_i \widetilde{\Delta}_{\varpi_i, w\varpi_i}. \tag{10.15}$$

For use in the next proof, we note the following identities for the braid group action in the case of $\mathfrak{g} = \mathfrak{sl}_2$:

$$E_1^m T_1^{-1} v_{m\varpi_1} = [m]_q! \, v_{m\varpi_1} \quad \text{and} \quad T_1^2 v_{m\varpi_1} = \frac{(-q)^m}{([m]_q!)^2} v_{m\varpi_1}, \quad \forall m \in \mathbb{Z}_{\geq 0}. \tag{10.16}$$

They are easily deduced from the standard facts ([**28**, §8.6 and Lemma 1.7]) for the $\mathcal{U}_q(\mathfrak{sl}_2)$-braid group action. We leave this to the reader.

Lemma 10.7. *Under the assumptions of Lemma* 10.6*,*

$$\pi_{w_\circ,[1,N]} = \mathcal{S}_{w_\circ \boldsymbol{\nu} w_\circ}(-e_1 + f_{w_\circ,[1,N]}).$$

Proof. Using (6.11), we see that

$$\begin{aligned}
\mathcal{S}_{w_\circ \boldsymbol{\nu} w_\circ}(-e_1 + f_{w_\circ,[1,N]}) &= \Omega_{w_\circ \boldsymbol{\nu} w_\circ}(e_1, f_{w_\circ,[1,N]}) \mathcal{S}_{w_\circ \boldsymbol{\nu} w_\circ}(f_{w_\circ,[1,N]}) \\
&= \Omega_{\boldsymbol{\nu}}(e_N, f) \mathcal{S}_{\boldsymbol{\nu}}(f)^{-1} = \Omega_{\boldsymbol{\nu}}(f, -e_N) \mathcal{S}_{\boldsymbol{\nu}}(f)^{-1} \\
&= \mathcal{S}_{\boldsymbol{\nu}}(-e_N + f)^{-1},
\end{aligned}$$

where

$$f := w_\circ f_{w_\circ,[1,N]} = -\sum_{j=2}^{N-1} c_{i_j i} e_j.$$

Hence, what must be proved is that

$$\pi_{w_\circ,[1,N]} = \mathcal{S}_{\boldsymbol{\nu}}(-e_N + f)^{-1}. \tag{10.17}$$

Since $x_N x_1$, $y_{w_\circ,N}$, and $u_{w_\circ,[1,N]}$ are homogeneous of the same degree with respect to the $\mathcal{Q}$-grading of $\mathcal{U}^-[w]$,

$$-\beta_N - \beta_1 = (w-1)\varpi_i = (-c_{i_{N-1} i})(-\beta_{N-1}) + \cdots + (-c_{i_2 i})(-\beta_2), \tag{10.18}$$

recall (9.22). Therefore,

$$\begin{aligned}
\mathcal{S}_{\boldsymbol{\nu}}(-e_N + f)^{-1} &= \sqrt{q}^{\,\langle \beta_N, c_{i_2 i}\beta_2 + \cdots + c_{i_{N-1} i}\beta_{N-1} \rangle} \prod_{1<j<k<N} \sqrt{q}^{\,c_{i_j i} c_{i_k i} \langle \beta_j, \beta_k \rangle} \\
&= \sqrt{q}^{\,\langle \alpha_i, (w-1)\alpha_i \rangle} \prod_{1<j<k<N} \sqrt{q}^{\,c_{i_j i} c_{i_k i} \langle \beta_j, \beta_k \rangle},
\end{aligned}$$

because $\langle -w\alpha_i, (w-1)\alpha_i \rangle = \langle \alpha_i, (w-1)\alpha_i \rangle$. Taking the square length of the vector in (10.18) and using that $\|\beta_k\|^2 = \|\alpha_{i_k}\|^2$ leads to

$$\begin{aligned}
\sum_{k=2}^{N-1} c_{i_k i}^2 \|\alpha_{i_k}\|^2 + 2 \sum_{1<j<k<N} c_{i_j i} c_{i_k i} \langle \beta_j, \beta_k \rangle &= \langle (1-w)\alpha_i, (1-w)\alpha_i \rangle \\
&= 2 \langle \alpha_i, (1-w)\alpha_i \rangle.
\end{aligned}$$

Hence,

$$\mathcal{S}_{\boldsymbol{\nu}}(-e_N + f)^{-1} = \prod_{k=2}^{N-1} \sqrt{q_{i_k}}^{\,-c_{i_k i}^2} \tag{10.19}$$

because $\sqrt{q_{i_k}} = \sqrt{q}^{\,\|\alpha_{i_k}\|^2/2}$ for all $k \in [1, N]$.

Denote for brevity $\xi_\mu := \xi_{1,\mu}$ for $\mu \in \mathcal{P}^+$ (i.e., $\xi_\mu \in (V(\mu)^*)_{-\mu}$ is the unique vector such that $\langle \xi_\mu, v_\mu \rangle = 1$). The explicit form (9.21) of the antihomomorphism φ_w and the rescaling of the generators (10.5) imply

$$\pi_{w_\circ,[1,N]} = -q_i \frac{\langle \xi_{\varpi_i}, (\tau E_{\beta_2})^{-c_{i_2 i}} \dots (\tau E_{\beta_{N-1}})^{-c_{i_{N-1} i}} T_{w^{-1}}^{-1} v_{\varpi_i} \rangle}{\prod_{j=2}^{N-1} \sqrt{q_{i_j}}^{\,c_{i_j i}^2} [-c_{i_j i}]_{q_{i_j}}!}. \tag{10.20}$$

For all $i' \in [1, N]$ with $i' \neq i$, the element $T_i^{-1} v_{\varpi_i}$ is a highest weight vector for the $\mathcal{U}_{q_{i'}}(\mathfrak{sl}_2)$-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{E_{i'}, F_{i'}, K_{i'}^{-1}\}$ with highest weight

$$\langle s_i \varpi_i, \alpha_{i'}^\vee \rangle \varpi_{i'} = \langle \varpi_i - \alpha_i, \alpha_{i'}^\vee \rangle \varpi_{i'} = -c_{i' i} \varpi_{i'},$$

because $E_{i'} F_i^m v_{\varpi_i} = F_i^m E_{i'} v_{\varpi_i} = 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Applying (10.16), we obtain

$$E_{i_j}^{-c_{i_j i}} T_{i_j}^{-1} (T_i^{-1} v_{\varpi_i}) = [-c_{i_j i}]_{q_{i_j}}! \, T_i^{-1} v_{\varpi_i}, \ \ \forall j \in [2, N-1].$$

Using that $\tau(E_{\beta_j}) = T_{i_1}^{-1} \dots T_{i_{j-1}}^{-1} (E_{i_j})$ and repeatedly applying the above identity gives

$$\langle \xi_{\varpi_i}, (\tau E_{\beta_2})^{-c_{i_2 i}} \dots (\tau E_{\beta_{N-1}})^{-c_{i_{N-1} i}} T_{w^{-1}}^{-1} v_{\varpi_i} \rangle = \\ = \langle \xi_{\varpi_i}, T_i^{-2} v_{\varpi_i} \rangle \prod_{j=2}^{N-1} [-c_{i_j i}]_{q_{i_j}}! = -q_i^{-1} \prod_{j=2}^{N-1} [-c_{i_j i}]_{q_{i_j}}!.$$

For the last equality we use the second identity in (10.16) for $m = 1$. Hence,

$$\pi_{w_\circ,[1,N]} = \prod_{j=2}^{N-1} \sqrt{q_{i_j}}^{\,-c_{i_j i}^2} = \mathcal{S}_{\boldsymbol{\nu}}(-e_N + f)^{-1},$$

which verifies (10.17) and completes the proof of the proposition. $\square$

Proof of Theorem 10.1. Applying Lemma 10.7 to each of the interval subalgebras

$$\mathbb{K}[x_{w_\circ,i}][x_{w_\circ,i+1}; \sigma^*_{w_\circ(i+1)}, \delta^*_{w_\circ(i+1)}] \cdots [x_{w_\circ,s_{w_\circ}(i)}; \sigma^*_{w_\circ s_{w_\circ}(i)}, \delta^*_{w_\circ s_{w_\circ}(i)}],$$

for $i \in [1, N]$ such that $s_{w_\circ}(i) \neq +\infty$, we conclude that the symmetric CGL extension (10.6) satisfies condition (6.13). Thus, the hypotheses of Theorem 8.2 hold for (10.6). Eq. (10.4) has been verified in Lemma 10.3 (b).

By Theorem 8.2, $(M_\circ, \widetilde{B}_\circ)$ is a quantum seed and

$$\mathcal{U}^-[w] = \mathcal{A}(M_\circ, \widetilde{B}_\circ, \varnothing)_{\mathbb{K}} = \mathcal{U}(M_\circ, \widetilde{B}_\circ, \varnothing)_{\mathbb{K}},$$

with set of frozen variables indexed by $\mathbf{ex}_\circ$. Applying this result to $\mathcal{U}^-[w_{\leq s^m(j)}]$, for $j \in [1, N]$ and $m \in \mathbb{Z}_{\geq 0}$ such that $s^m(j) \in [1, N]$, we see that $\overline{y}_{[j, s^m(j)]}$ is a cluster variable of $\mathcal{U}^-[w_{\leq s^m(j)}]$ and hence a cluster variable of $\mathcal{U}^-[w]$.

Since $M_\circ(e_k) = \overline{y}_{w_\circ,k} = M^w(e_{w_\circ(k)})$ for all $k \in [1, N]$, we have $M_\circ = M^w w_\circ$. Taking Corollary 10.5 into account, we therefore obtain Theorem 10.1. $\square$

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