Coupled Flight Dynamics and Thermoelasticity Model of Maneuvering Aircraft in an Undeformed Body Frame

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The stress and strain tensors in maneuvering aircraft are affected not only by the aerodynamic and propulsive forces but also by the body translational and angular velocities and high rate temperature distributions. The structural design of flexible aircraft requires the elastic fields to be evaluated at high accuracy. For this, new models which couple the aeroelasticity equations with the heat and translational and rotational equations of flight dynamics are essential. In this paper, the governing equations are derived by describing the motion of a thermoelastic body in an undeformed body frame by means of the variational principle applied to four invariants, the kinetic energy, the gravitational potential energy, the thermoelastic potential, and the dissipation function. Particular cases of longitudinal and lateral flight dynamics of a thermoelastic body including longitudinal flight of a thermoelastic beam are considered. Also, a discrete system for the translational and rotational velocities, the structural modes and nodal temperature is derived. These equations may advance control design of hypersonic platforms, flexible aircraft, and missiles.

Nomenclature

A	=	point in undeformed body
$\mathcal{A}$	=	surface of body $\mathcal{V}$
a_0	=	thermoelastic parameter, m^{-2} s
a_1	=	scalar, J
$\{a_{ij}\}_{i=1}^n$	=	coefficients of quadratic form of thermoelastic potential $\mathcal{W}$
B	=	undeformed body frame
b	=	beam width, m
$\{b_{ij}\}_{i=1}^n$	=	coefficients of quadratic form of dissipation function $\mathcal{D}$
$\mathbf{b} = (b_1, b_2, b_3)$	=	vector, kg m
C_{jq}	=	dimensionless longitudinal stability derivative due to variation of force F_j with q
C_{ju}	=	dimensionless longitudinal stability derivative due to variation of force F_j with u

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C_{jw}	=	dimensionless longitudinal stability derivative due to variation of force F_j with w
$C_{j\dot{w}}$	=	dimensionless longitudinal stability derivative due to variation of force F_j with $\dot{w}$
C_g	=	dimensionless gravity effect parameter
$C_{\mathcal{M}q}$	=	dimensionless longitudinal stability derivative due to variation of moment $\mathcal{M}$ with q
$C_{\mathcal{M}u}$	=	dimensionless longitudinal stability derivative due to variation of moment $\mathcal{M}$ with u
$C_{\mathcal{M}w}$	=	dimensionless longitudinal stability derivative due to variation of moment $\mathcal{M}$ with u
$C_{\mathcal{M}\dot{w}}$	=	dimensionless longitudinal stability derivative due to variation of moment $\mathcal{M}$ with $\dot{w}$
c	=	mean aerodynamic chord, m
c_r	=	wing root chord, m
c_T	=	specific heat of unit volume in absence of deformation, Pa K^{-1}
c_t	=	wing tip chord, m
$\mathbf{c} = (c_1, c_2, c_3)$	=	vector, J s
$D = Eh^2(12\rho)^{-1}$	=	elastic beam parameter, m^{4}s^{-2}
D_0	=	thermoelastic bending rigidity (dimensionless)
$\mathcal{D}$	=	dissipation function, J
E	=	Young modulus, Pa
$\mathbf{E} = (\Phi, \Theta, \Psi)$	=	vector of generalized coordinates (Euler angles), rad
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	=	unit basis vectors of frame B
$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$	=	resultant force applied to body surface due to aerodynamic and propulsive effects, J m^{-1}
f_3	=	force density applied to beam surface x_3 = const, kg m^{-1} s^{-2}
g	=	gravitational constant, m s^{-2}
H	=	direction cosine matrix
h	=	beam height, m
h_w	=	wing half-span, m
I	=	inertial frame
$\tilde{I}$	=	elastic mass moment of inertia, kg m^2
I_j	=	derivative of I with respect to u_j, kg m
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	=	unit basis vectors of frame I
$\mathbf{i}_M, \mathbf{j}_M, \mathbf{k}_M$	=	unit basis vectors of frame M
k_T	=	thermal conductivity, J m^{-1} s^{-1} K^{-1}
l	=	beam length, m
M	=	mean axes body frame

$\mathbf{M} = \mathcal{L}\mathbf{e}_1 + \mathcal{M}\mathbf{e}_2 + \mathcal{N}\mathbf{e}_3$	=	resultant external moment applied to body, J
m	=	mass of body $\mathcal{V}$, kg
O, O_I, O_M	=	origins of frames B, I, M
p	=	dynamic pressure, Pa
Q_0	=	pitch rate parameter (dimensionless)
$\boldsymbol{Q}$	=	vector of generalized forces associated with vector $\mathbf{q}$
$\boldsymbol{Q}_{\mathbf{R}}$	=	vector of generalized forces associated with vector $\mathbf{R}$
$\boldsymbol{Q}_{\mathbf{E}}$	=	vector of generalized forces associated with vector $\mathbf{E}$
$\mathbf{q}$	=	vector of generalized coordinates
$\mathbf{R}$	=	radius-vector in frame I of origin O of frame B, m
$\mathbf{r}$	=	radius-vector in frame B of point A of undeformed body, m
$\mathbf{r}_1$	=	radius-vector in frame M of point A of undeformed body, m
S	=	wing area, m^2
$\mathbf{S}$	=	entropy flow vector, Pa m K^{-1}
s	=	entropy density, Pa K^{-1}
T	=	increment of temperature above reference absolute temperature T_a, K
T_a	=	reference absolute temperature, K
$\mathcal{T}$	=	kinetic energy, J
t	=	time, s
U_0	=	equilibrium value of the $\mathbf{e}_1$-component U of the translational velocity $\mathbf{V}$, m s^{-1}
$U_0 u, U_0 w, q, \theta$	=	disturbed values of U, W, Q, Θ
$\mathcal{U}$	=	gravitational potential energy, J
$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$	=	displacement vector in frame B, m
$\mathbf{u} = u'_1\mathbf{i}_M + u'_2\mathbf{j}_M + u'_3\mathbf{k}_M$	=	displacement vector in frame M, m
$\mathbf{V} = U\mathbf{e}_1 + V\mathbf{e}_2 + W\mathbf{e}_3$	=	translational velocity of origin O, m s^{-1}
$\mathcal{V}$	=	elastic body
$\mathcal{W}$	=	thermoelastic potential, J
w_T	=	isothermal mechanical energy, Pa
X, Y, Z	=	inertial coordinates, m
$\tilde{X}_i(x_1)$	=	eigenfunctions
x_1, x_2, x_3	=	coordinates in frame B, m
α_T	=	coefficient of thermal expansion, K^{-1}

β	=	thermoelastic diffusivity (dimensionless)
$\beta_T = (3\lambda + 2\mu)\alpha_T$	=	coefficient of thermoelastic expansion, Pa K^{-1}
γ	=	deflection parameter, dimensionless
ϵ	=	divergence of displacement vector $\mathbf{u}$
ϵ_{ij}	=	strain components (dimensionless)
ζ_p	=	damping ratio of phugoid oscillations
ζ_s	=	damping ratio of short-period oscillations
Θ_0	=	equilibrium value of pitch angle Θ, rad
κ_i	=	eigenvalues
λ, μ	=	Lamé constants, Pa
ν	=	Poisson ratio
$\mathbf{\Xi}$	=	heat source, K m^{-1}
ξ_1, ξ_2, ξ_3	=	coordinates in frame M, m
ρ	=	density of body $\mathcal{V}$, kg m^{-3}
ρ_a	=	air density, kg m^{-3}
σ.	=	Laplace transform parameter
Φ, Θ, Ψ	=	Euler angles associated with frames B and I, rad
Φ_1, Θ_1, Ψ_1	=	Euler angles associated with frames M and B, rad
Ω	=	heat capacity, Pa s^{-1}
$\boldsymbol{\omega} = P\mathbf{e}_1 + Q\mathbf{e}_2 + R\mathbf{e}_3$	=	angular velocity of frame B, rad s^{-1}
$\boldsymbol{\omega}_1 = P_1\mathbf{i}_M + Q_1\mathbf{j}_M + R_1\mathbf{k}_M$	=	angular velocity of frame M, rad s^{-1}
ω_{np}	=	frequency of phugoid oscillations
ω_{ns}	=	frequency of short-period oscillations
Superscripts		
$\dot{\mathbf{f}}$	=	time-derivative of vector $\mathbf{f}$ in frame B
$\dot{\mathbf{f}}\|_I$	=	time-derivative of vector $\mathbf{f}$ in frame I
Subscripts		
rb	=	rigid body

I. Introduction

A variety of aerospace structures can be modeled as unrestrained elastic and thermoelastic bodies. Examples of these include helicopters, aircraft, and missiles. Building a flight dynamics model for elastic aircraft is particularly important for frequency domain analysis and control design of such structures as NASA Helios type aircraft which are highly flexible and fly at low speeds and high altitude. To advance the control systems for flexible aircraft or hybrid structures having flexible wings, it is required to accurately derive the governing dynamical systems for the body translational and angular velocities whose coefficients are affected by the aircraft deformations. On the other hand, the structural design of a maneuvering vehicle needs the strain and stress fields to be evaluated to a high degree of accuracy. Therefore, the impact of the translational and rotational velocities on these fields has to be taken into account. In addition to elastic deformations caused by the aerodynamic and propulsive forces, accurate modeling of missiles flight at hypersonic regimes and during the atmospheric reentry necessitates to capture the effects of high temperatures in the shock layer, large aerodynamic heating of the vehicle [1], [2], [3], and the chemically reacting gas surrounding the vehicle [4].

Simultaneous consideration of both rigid-body flight dynamics and elastic deformation has received substantial attention since the sixties. By using a Newtonian approach, the principal axes as the body axes and assuming that the elastic deformations have a negligible effect on the mass moment of inertia Bisplinghoff and Ashley [5] decoupled the system of governing equations into the rigid body flight dynamics equations and equations for natural modes associated with the elastic deformations. Applying essentially the same approach Blakelock [6] derived uncoupled rigid body and vibrations equations, where corrections to the aerodynamic forces due to structural flexibility were made.

Another way of modeling flight of flexible aircraft was proposed by Milne [7]. Its main feature is the use of mean axes. The mean axes system of coordinates is chosen such that the relative linear and angular momenta, due to elastic deformation, are zero at every instant. The mean axes were systematically applied for decoupling the left-hand side of the system into the rigid body dynamics and natural modes equations by Waszak and Schmidt [8] and Schmidt [9]. In their derivations they assumed that the elastic displacements were small and the origin of the mean-axes frame was located at the instantaneous center of mass. These and Milne's requirements resulted in decoupling of the left-hand sides of the flight dynamics and the structural dynamics equations. The coupling was preserved in the right-hand sides of the equations at the level of aerodynamical forces. Recently, Schmidt [10] modified the equations of motion under the mean axes formulation. The translational equations still coincide with those of the rigid-body dynamics, while the rotational and elastic equations for forced vibration modes have some new terms which represent inertial coupling.

An approach that is also based on the Euler-Lagrange equation but avoids the use of the mean axes concept was advanced by Meirovitch and Tuzcu [11], [12] [13]. To find an approximate solution that would be acceptable for time simulations of maneuvering flexible aircraft, they discretized the elastic displacements and expressed them in terms of matrices of shape functions and finite-dimensional vectors of generalized coordinates. In their derivations they considered the coupling of the flight dynamics equations with the discretized elastic displacements and did not analyze the impact of the translational and rotational velocities on the structural modes themselves. Bolender and Doman [14] used an approach similar to that of Meirovitch and Tuzcu [11] and derived nonlinear longitudinal dynamics equations for an elastic vehicle modeled as a beam with free ends and clamped at the center of mass. Their model captures the interactions between the first elastic mode and the rigid body modes. General Lagrangian equations for dynamics of a flexible satellite which couple rigid-body dynamics and flexible modes were presented by Mazzini [15].

An excellent survey of reference frames used in flight dynamics and aeroelasticity including the mean axes frame and general body axes was presented in [16], [17]. Under the assumption of small elastic displacements a structural-dynamic finite element formulation of the flight dynamics of flexible aircraft was analyzed by Neto et al [17] in three body-axes frames, the dually-constrained body axes, the structural axes, and the mean axes. The assumption of small deformations in linear structural dynamic models was validated and applied to different configurations of X-HALE aircraft in [18]. For modeling, attached dually-constrained body axes were selected. The origin of this frame is rigidly connected to a material point that remains fixed when structural deformation occurs. By employing the projective control technique an output-feedback-based stability augmentation system was designed [19] to improve the handling qualities of flexible aircraft and increase the structural damping of some of its aeroelastic modes. In the work cited, the aerodynamic model of aircraft is based on the doublet-lattice and vortex-lattice methods, the aerodynamic forces are approximated by rational functions, a beam-finite-element method is applied to model the structural dynamics of wings, and the technique is tested on X-HALE flexible aircraft. To pursue the goal to develop a simple model of slightly flexible aircraft, Silvestre and Luckner [20] used the linearized mean-axes formulation [8]. This enabled them to formulate the governing equations as a system of classical rigid-body dynamic equations and the aeroelastic dynamic equations in modal coordinates coupled by the aerodynamic forces. Unsteady aerodynamics effects caused by elastic deformations were accounted for by applying Yates' modified strip analysis method. The methodology was tested in [20] on a Stemme S15 prototype by comparing the simulation and flight-test data. A fluid-structure interaction model for a slender high-speed deformable vehicle was considered by Kitson and Cesnik [21]. They applied a reduced modeling method based on the high-fidelity steady-state computational fluid dynamics solutions. This technique enables vehicle flight simulation with coupled aeroelastic and flight mechanics equations.

To study thermoeffects on vibration of elastic bodies, Biot [22], [23] introduced two invariants, the thermoelastic potential and the dissipation function, used a variational formulation and derived the governing differential equations of coupled thermoelasticity. In [24], [25] Biot studied nonlinear effects of temperature dependence of the heat equation parameters on heating, cooling, and torsional and anticlastic stability of flight structures. Note that in his analysis [22] - [25] Biot focused on the thermoelastic effects and did not analyze the body flight dynamics. A one-dimensional model of uncoupled thermoelasticity for an Euler-Bernoulli beam was explored and applied for analysis of the hypersonic cruise of aircraft by Culler et al [26]. An average temperature distribution along the beam was used to obtain the frequencies and mode shapes. In the framework of this model they found that the frequency of the first mode remained unchanged, and the effects of the aerodynamic heating on aircraft's rigid-body poles and zeros were negligible. This was due [26] to the uncoupling of the heat and elasticity equations and to the fact that in this model, the rigid-body dynamics did not excite the flexible states, and the flexible-body dynamics did not excite the rigid-body states. Experimental data, approximate methods including Van Dyke's formula for unsteady aerodynamic forces were employed in [27], [28] to discover high temperatures and thermal gradients effects on thermal stresses and flutter boundaries. Aerothermoelastic analysis in the conditions of high temperatures was implemented in [29], [30] by taking into account the effects of thermal stresses and variation of material properties with the temperature growth. There are also numerical studies, for example [31], [32], [33] [34], where fluid-structure thermal problems are analyzed by developing finite elements method codes or combining and applying the existing solvers. Linear and nonlinear aerothermoelastic reduced-order structural dynamics models for hypersonic vehicles were proposed by Falkiewicz and Cesnik [35], [36] and Klock and Cesnik [37]. Finite element analysis of the impact of high-speed thermal and structural loads on metal-ceramic spatially graded composite high-speed panels was presented in [38].

In this paper, we aim to develop a model that couples flight dynamics modes with elastic displacements in both flight dynamics equations and the system of dynamic elasticity equations. In contrast to the classical rigid body flight dynamics six motion equations, this model comprises ten equations. Six of them are generalized translational and rotational equations which couple the six rigid body modes with integral characteristics of the elastic displacements. The others are four thermoelastic equations which are different from the classical dynamic equations of couple thermoelasticity [39]. In the model to be proposed, the generalized Navier elastic equations couple the elastic displacements not only with temperature but also with the translational and rotational velocities of the body.

This paper is organized as follows. In Section II, we derive and analyze the kinetic energy expressions in two reference frames, a system of coordinates fixed in the undeformed body and the mean axes frame. We employ the undeformed body frame in the rest of the paper. In Section III, based on four invariants, the kinetic energy, the gravitational potential energy, the thermoelastic potential, and a dissipation function, by means of the variational principle we describe motion of a maneuvering thermoelastic body by modeling it by a system of ten nonlinear differential equations. Six equations of this system are flight dynamics ordinary differential equations which couple the translational and rotational velocities with integral characteristics of the elastic displacements and their time derivatives. The rest four equations represent a generalization of the classical thermoelasticity partial differential equations in which the elastic displacements are coupled with not only temperature but also the translational and rotational velocities. Particular cases of longitudinal and lateral dynamics including linearized longitudinal flight dynamics equations of thermoelastic maneuvering aircraft are considered in Section IV. The longitudinal equations are linearized, and a thermoelastic analog of Blakelock's analysis [6] of short-period and phugoid oscillations of a transport is presented. Longitudinal flight of thermoelastic and purely elastic beams are analyzed in Section V. A numerically-analytical method of integral transformations and successive approximations for a dynamic model of coupled thermoeasticity for a beam was proposed in [40]. In model [40], the effects of translational and rotational velocities were not considered. We advance this method to the longitudinal flight regime. Numerical results for the deflection and temperature and their dependence on the dimensionless pitch rate and thermoelastic diffusivity are reported. In Section VI, discrete flight dynamics equations with respect to the translational and rotational velocities and structural modes and nodal temperature are derived.

II. Kinetic energy in an undeformed body frame and the mean-axes

A. Inertial and undeformed body frames

Let X, Y, Z be inertial right-hand Cartesian coordinates with the origin at a point O_I (frame I) fixed in the surface of Earth whose motion and rotation are disregarded. Denote by $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ the unit basis vectors of the frame I. Assume that an elastic body $\mathcal{V}$ of constant mass m is subjected to aerodynamic and propulsive effects and moves in the inertial space. Due to the body maneuvers and the forces and moments applied the body experiences time-dependent deformation. At time $t = 0$, select a system of orthogonal coordinates with the body center of mass as its origin O, the x_1-axis along the undeformed fuselage centerline pointing forward, the x_2-axis pointing to the starboard that is to the right of the forward pointing body, and the x_3-axis pointing downward (a right-handed Cartesian system). The position of this body frame, B, is uniquely described by the radius-vector $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ of its origin O (Figure 1) and the three Euler angles Φ, Θ, and Ψ, the bank, pitch, and heading angles, associated with the frames I and B. Select a point A of the undeformed body with Lagrangian coordinates $\mathbf{r} = (x_1, x_2, x_3)$. With respect to the reference frame B we define the Eulerian coordinates $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ is the elastic displacement, while $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are the unit basis vectors of the frame B. Thus the actual position of the point A at time $t = 0$ in the inertial frame is characterized by the vector $\mathbf{R}' = \mathbf{R} + \mathbf{r} + \mathbf{u}$. Denote the body translational and rotational velocities as $\mathbf{V} = U\mathbf{e}_1 + V\mathbf{e}_2 + W\mathbf{e}_3$ and $\omega = P\mathbf{e}_1 + Q\mathbf{e}_2 + R\mathbf{e}_3$, respectively. Here, P, Q, and R are the roll, pitch, and yaw rates. At any instant, as the body maneuvers, the frame B does not move with respect to the maneuvering undeformed body. This implies that the derivative with respect to time of the vector $\mathbf{r}$ in this frame equals zero at any time.

The kinetic energy of the body is given by

$$\mathcal{T} = \frac{1}{2}\int_{\mathcal{V}} \dot{\mathbf{R}}'|_I \cdot \dot{\mathbf{R}}'|_I \rho d\mathcal{V}, \tag{1}$$

where $\dot{\mathbf{R}}'|_I$ is the derivative of the vector $\mathbf{R}'$ with respect to time t in the inertial frame, ρ is the body density, and $d\mathcal{V}$ is the elementary volume. Express now the total time-derivative of the vector $\mathbf{r}' = \mathbf{r} + \mathbf{u}$ with respect to the inertial frame through its time-derivative with respect to the moving body frame

$$\dot{\mathbf{r}}'|_I = \dot{\mathbf{r}}' + \omega \times \mathbf{r}'. \tag{2}$$

Here and further, to distinguish the time-derivative of a vector $\boldsymbol{\gamma}$ in the undeformed body frame from the time-derivative

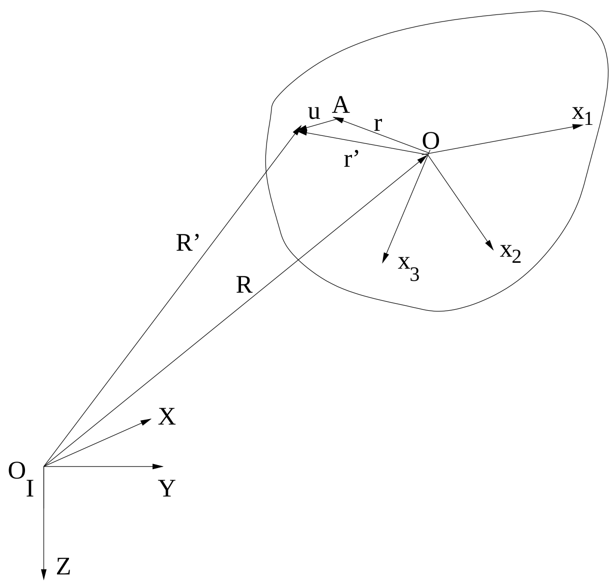

Fig. 1 Inertial system I and the undeformed body frame B.

in the inertial frame we write $\dot{\boldsymbol{\gamma}}$ and $\dot{\boldsymbol{\gamma}}|_I$, respectively. We next substitute the derivative (2) into (1) to obtain

$$\mathcal{T} = \frac{1}{2}\int_{\mathcal{V}} (\dot{\mathbf{R}}|_I + \dot{\mathbf{r}}' + \omega \times \mathbf{r}') \cdot (\dot{\mathbf{R}}|_I + \dot{\mathbf{r}}' + \omega \times \mathbf{r}')\rho d\mathcal{V} \tag{3}$$

and use the relations

$$\dot{\mathbf{r}} = 0, \quad \int_{\mathcal{V}} \mathbf{r}\rho dV = 0. \tag{4}$$

The last formula is due to the choice of the center of mass of the body as the origin of the frame B. Due to this and the vector identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}), \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2, \tag{5}$$

we deduce

$$\mathcal{T} = \frac{m}{2}\dot{\mathbf{R}}|_I\cdot\dot{\mathbf{R}}|_I + \frac{1}{2}\int_{\mathcal{V}}\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\rho d\mathcal{V} + \dot{\mathbf{R}}|_I\cdot\left(\int_{\mathcal{V}}\dot{\mathbf{u}}\rho d\mathcal{V} + \omega\times\int_{\mathcal{V}}\mathbf{u}\rho d\mathcal{V}\right) + \omega\cdot\int_{\mathcal{V}}(\mathbf{r}+\mathbf{u})\times\dot{\mathbf{u}}\rho d\mathcal{V} + \mathcal{J}. \tag{6}$$

Here, m is the body mass independent of time,

$$\dot{\mathbf{R}}|_I = \dot{X}|_I\mathbf{i} + \dot{Y}|_I\mathbf{j} + \dot{Z}|_I\mathbf{k} = U\mathbf{e}_1 + V\mathbf{e}_2 + W\mathbf{e}_3 \tag{7}$$

is the velocity vector $\mathbf{V}$ in the inertial and the body frames, and

$$\mathcal{J} = \frac{1}{2}\int_{\mathcal{V}}(\omega\times\mathbf{r}')\cdot(\omega\times\mathbf{r}')\rho d\mathcal{V}. \tag{8}$$

Since $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, it is a matter of simple algebra to represent the integral (8) in the classical form

$$\mathcal{J} = \frac{1}{2}\omega\tilde{I}\omega^t, \tag{9}$$

where $\boldsymbol{\omega}^t$ is the column-vector associated with the rotational velocity row-vector $\omega = (P, Q, R)$, and $\tilde{I}$ is the elastic mass moment of inertia

$$\tilde{I} = \begin{pmatrix} \tilde{I}_{11} & -\tilde{I}_{12} & -\tilde{I}_{13} \\ -\tilde{I}_{12} & \tilde{I}_{22} & -\tilde{I}_{23} \\ -\tilde{I}_{13} & -\tilde{I}_{23} & \tilde{I}_{33} \end{pmatrix} \tag{10}$$

obtained from the classical mass moment of inertia by replacing the rigid body vector $\mathbf{r}$ by the vector $\mathbf{r}' = \mathbf{r} + \mathbf{u}$ or, equivalently, x_i by $x_i + u_i$, $i = 1, 2, 3$,

$$\tilde{I}_{ii} = \int_{\mathcal{V}} \sum_{j=1,\, j\neq i}^{3} (x_j + u_j)^2 \rho d\mathcal{V}, \quad \tilde{I}_{ij} = \int_{\mathcal{V}} (x_i + u_i)(x_j + u_j)\rho d\mathcal{V}, \quad i \neq j. \tag{11}$$

In contrast to the mass moment of inertia of a rigid body, the elastic moment is a function of the three displacements $u_i(t)$, $\tilde{I} = \tilde{I}(\mathbf{u}(t))$.

Now, upon substituting the expression (9) into (6) we derive the kinetic energy in the undeformed body frame $B = Ox_1x_2x_3$ in the form

$$\mathcal{T} = \frac{m}{2}\dot{\mathbf{R}}|_I \cdot \dot{\mathbf{R}}|_I + \frac{1}{2}\omega\tilde{I}\omega^t + \dot{\mathbf{R}}|_I \cdot (\dot{\mathbf{b}} + \omega \times \mathbf{b}) + \omega \cdot \mathbf{c} + a_1. \tag{12}$$

Here, $\mathbf{b}$ and $\mathbf{c}$ are vectors and a_1 is a scalar given by

$$\mathbf{b} = \int_{\mathcal{V}} \mathbf{u}\rho d\mathcal{V}, \quad \mathbf{c} = \int_{\mathcal{V}} (\mathbf{r} + \mathbf{u}) \times \dot{\mathbf{u}}\rho d\mathcal{V}, \quad a_1 = \frac{1}{2}\int_{\mathcal{V}} \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\rho d\mathcal{V}. \tag{13}$$

On comparing the kinetic energy of the undeformed body $\mathcal{V}$

$$\mathcal{T}_{rb} = \frac{m}{2}\dot{\mathbf{R}}|_I \cdot \dot{\mathbf{R}}|_I + \frac{1}{2}\omega I_{rb}\omega^t \tag{14}$$

where I_{rb} is the mass moment of inertia, with formula (12) we see that in the elastic case the kinetic energy has three additional terms, and the mass moment of inertia $\tilde{I}$ depends on the elastic displacements.

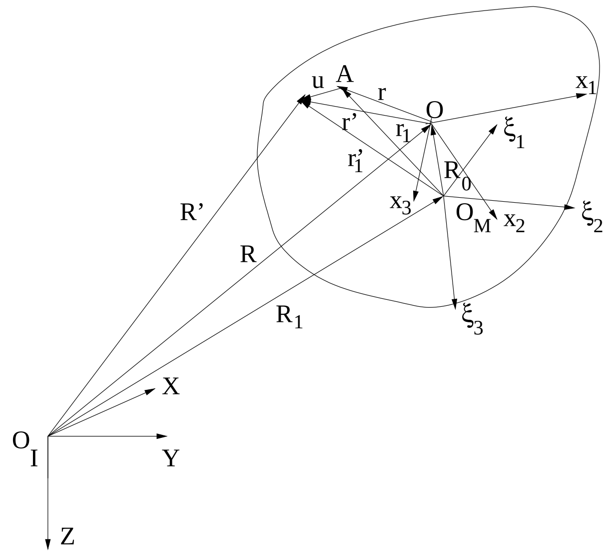

Fig. 2 Mean axes and the undeformed body frames M and B.

B. Mean-axes body frame

The mean-axes body frame, $M = O_M\xi_1\xi_2\xi_3$, is a right-hand Cartesian system of coordinates (Figure 2) with the axes being selected such that the following two vector (six scalar) conditions are satisfied:

$$\int_{\mathcal{V}} \dot{\mathbf{r}}_1'|_M \rho d\mathcal{V} = 0, \quad \int_{\mathcal{V}} \mathbf{r}_1' \times \dot{\mathbf{r}}_1'|_M \rho d\mathcal{V} = 0. \tag{15}$$

Here, $\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}$, $\mathbf{r}_1$ is the radius-vector of a point A of the undeformed body $\mathcal{V}$, $\mathbf{u}$ is the vector of elastic displacement in the mean-axes frame, and $\dot{\mathbf{r}}_1'|_M$ is the time-derivative of the vector $\mathbf{r}_1'$ with respect to the frame M. Denote by $\mathbf{i}_M$, $\mathbf{j}_M$, and $\mathbf{k}_M$ the unit basis vectors of the frame M. Then the displacement vector $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ in the frame B can be represented in the frame M as $\mathbf{u} = u_1'\mathbf{i}_M + u_2'\mathbf{j}_M + u_3'\mathbf{k}_M$. If the body is rigid, the conditions (15) are automatically satisfied, and the mean-axes frame coincides with the undeformed body frame. For an elastic body, the undeformed body frame B and the mean axes frame M do not coincide, and in general, the origins and the directions of the axes are different. Assume that the frame M exists, and it has been found. Since $\mathbf{R}' = \mathbf{R}_1 + \mathbf{r}_1'$ and $\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}$, similarly to (6), first we have

$$\mathcal{T} = \frac{m}{2}\dot{\mathbf{R}}_1|_I \cdot \dot{\mathbf{R}}_1|_I + \frac{1}{2}\omega_1 \tilde{I} \omega_1^t + \frac{1}{2}\int_{\mathcal{V}} \dot{\mathbf{r}}_1'|_M \cdot \dot{\mathbf{r}}_1'|_M \rho d\mathcal{V}$$

$$+ \dot{\mathbf{R}}_1|_I \cdot \int_{\mathcal{V}} \dot{\mathbf{r}}_1'|_M \rho d\mathcal{V} + \dot{\mathbf{R}}_1|_I \cdot \left(\omega_1 \times \int_{\mathcal{V}} (\mathbf{r}_1 + \mathbf{u})\rho d\mathcal{V}\right) + \omega_1 \cdot \int_{\mathcal{V}} \mathbf{r}_1' \times \dot{\mathbf{r}}_1'|_M \rho d\mathcal{V}, \tag{16}$$

where $\omega_1 = P_1\mathbf{i}_M + Q_1\mathbf{j}_M + R_1\mathbf{k}_M$ is the angular velocity of frame M. Due to the six mean-axes conditions (15) the fourth and six terms in this formula vanish, and we arrive at the formula

$$\mathcal{T} = \frac{m}{2}\dot{\mathbf{R}}_1|_I \cdot \dot{\mathbf{R}}_1|_I + \frac{1}{2}\omega_1\tilde{I}\omega_1^t + \frac{1}{2}\int_{\mathcal{V}} \dot{\mathbf{r}}_1'|_M \cdot \dot{\mathbf{r}}_1'|_M \rho d\mathcal{V} + \dot{\mathbf{R}}_1|_I \cdot \left(\omega_1 \times \int_{\mathcal{V}} (\mathbf{r}_1 + \mathbf{u})\rho d\mathcal{V}\right). \tag{17}$$

It is seen that the expressions (12) and (17) for the kinetic energy in the undeformed body frame and the mean axes frame share some terms and have also their own terms. This is because the six scalar conditions (4) are satisfied in the frame B and not met in the frame M, while the six scalar conditions (15) are valid in the frame M and not fulfilled in the case of the frame B.

Although the expression of the kinetic energy is slightly simpler in the mean axes frame, it requires determining the location of the origin O_M and the direction of the axes ξ_1, ξ_2, and ξ_3. The main difficulty is to satisfy the conditions (15). These conditions might be fulfilled by choosing the three components of the vector $\mathbf{R}_1$ and the three Euler angles Φ_1, Θ_1, and Ψ_1 associated with the frames B and M. However, since $\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}$, the conditions (15) have the vector $\mathbf{u} = u_1'\mathbf{i}_M + u_2'\mathbf{j}_M + u_3'\mathbf{k}_M$. To determine the elastic displacement in the mean axes frame, one needs to solve the corresponding dynamic problem of elasticity. Its setup requires fixing the axes first or, equivalently, to determine the Lagrangian coordinates of the body. However, these axes are unknown at this stage and depend on the displacement vector $\mathbf{u}$. On the other hand, in the undeformed body frame, the conditions (4) are automatically met, and it is natural to formulate the associated dynamic boundary value problem of elasticity in the undeformed body frame B and stick to this reference frame all the time. In what follows we proceed with the undeformed body frame.

III. Coupled flight dynamics and continuum mechanics equations of motion of a thermoelastic body

In this section we will apply the variational formulation to derive equations of motion of a thermoelastic body.

A. Four invariants

Our derivation will be based on four invariants, the kinetic energy

$$\mathcal{T} = \frac{m}{2}\dot{\mathbf{R}}|_I \cdot \dot{\mathbf{R}}|_I + \frac{1}{2}\omega\tilde{I}\omega^t + \dot{\mathbf{R}}|_I \cdot \left(\int_{\mathcal{V}} \dot{\mathbf{u}}\rho d\mathcal{V} + \omega \times \int_{\mathcal{V}} \mathbf{u}\rho d\mathcal{V}\right) + \omega \cdot \int_{\mathcal{V}} (\mathbf{r} + \mathbf{u}) \times \dot{\mathbf{u}}\rho d\mathcal{V} + \frac{1}{2}\int_{\mathcal{V}} \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\rho d\mathcal{V} \tag{18}$$

in the undeformed body frame $Ox_1x_2x_3$, the gravitational potential energy, $\mathcal{U}$, thermoelastic potential, $\mathcal{W}$, and the dissipation function, $\mathcal{D}$. Denote by $H = H(\Phi, \Theta, \Psi)$ the direction cosine matrix for transformation from the frame I

to the frame B

$$H = \begin{pmatrix} \cos\Theta\cos\Psi & \cos\Theta\sin\Psi & -\sin\Theta \\ \sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi & \sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi & \sin\Phi\cos\Theta \\ \cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi & \cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi & \cos\Phi\cos\Theta \end{pmatrix} \tag{19}$$

and express the unit basis vector $\mathbf{k}$ of the inertial system I through the Euler angles and the basis vectors $\mathbf{e}_i$ of the body frame B [10]

$$\mathbf{k} = -\sin\Theta\mathbf{e}_1 + \sin\Phi\cos\Theta\mathbf{e}_2 + \cos\Phi\cos\Theta\mathbf{e}_3. \tag{20}$$

Then the gravitational potential energy $\mathcal{U}$ due to the second relation in (4) becomes

$$\begin{aligned} \mathcal{U} &= -g\mathbf{k}\cdot\int_{\mathcal{V}}\mathbf{R}\rho d\mathcal{V} - g\mathbf{k}|_B\cdot\int_{\mathcal{V}}(\mathbf{r}+\mathbf{u})\rho d\mathcal{V} \\ &= -gmZ + g(b_1\sin\Theta - b_2\cos\Theta\sin\Phi - b_3\cos\Theta\cos\Phi), \end{aligned} \tag{21}$$

where g is the gravitational constant and $\mathbf{b} = (b_1, b_2, b_3)$ is the vector defined in (13).

Next, following [22] we express the thermoelastic potential $\mathcal{W}$ through the isothermal mechanical energy w_T

$$2w_T = (\lambda+2\mu)\epsilon^2 + \mu(\epsilon_{23}^2 + \epsilon_{31}^2 + \epsilon_{12}^2 - 4\epsilon_{22}\epsilon_{33} - 4\epsilon_{33}\epsilon_{11} - 4\epsilon_{11}\epsilon_{22}) \tag{22}$$

and the entropy density $s = \beta_T\epsilon + c_T T/T_a$ as

$$\mathcal{W} = \int_{\mathcal{V}}\left(w_T + \frac{c_T T^2}{2T_a}\right)d\mathcal{V}. \tag{23}$$

Here, $\epsilon_{ii} = u_{i,i}$ and $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are the strain components, $u_{i,j}$ is the derivative of u_i with respect to x_j, $\epsilon = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$, T is an increment of temperature above a reference absolute temperature T_a, and λ and μ are the Lamé constants measured under the conditions of isothermal deformation when $T = 0$. The parameter $\beta_T = (3\lambda+2\mu)\alpha_T$ [39] is the coefficient of thermoelastic expansion, α_T is the coefficient of thermal expansion, and c_T is the specific heat of the unit volume in the absence of deformation. The entropy density s is expressible through a vector $\mathbf{S}$, the entropy flow vector, that represents the amount of heat divided by T_a which has flown in a given direction, $s = -\nabla\cdot\mathbf{S}$.

The fourth invariant is the dissipation function [22]

$$\mathcal{D} = \frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}\frac{T_a}{k_T}\mathbf{S}\cdot\mathbf{S}d\mathcal{V}, \tag{24}$$

where k_T is the thermal conductivity. The two invariants, the thermoelastic potential $\mathcal{W}$ and the dissipation function $\mathcal{D}$, represent the Biot's [22], [23] generalization of the concept of free energy. They are chosen such that when the flight dynamics effects are ignored the variational principles lead to the classical thermoelasticity equations.

Our derivations will be based on the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{T}}{\partial \mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}(\mathcal{U} + \mathcal{W} + \mathcal{D}) = \boldsymbol{Q}, \tag{25}$$

where $\boldsymbol{Q}$ is the vector of generalized forces acting on the body,

$$\mathbf{q} = \{X, Y, Z, \Phi, \Theta, \Psi, u_1, u_2, u_3, S_1, S_2, S_3\} \tag{26}$$

is the vector of 12 generalized coordinates. The first six entries represent the coordinates of the origin of the inertial system and the Euler angles. The remaining six coordinates are the three elastic displacements and the three components of the entropy flow vector $\mathbf{S}$. They are distributed parameter coordinate functions [41] (Section 6.6) of time t and spatial position A or, equivalently, $\mathbf{r}$.

B. Translational equations of motion

To derive the translational motion equations, we consider the first three generalized coordinates, the vector $\mathbf{R} = (X, Y, Z)$ whose components are the inertial coordinates. Note that the invariants $\mathcal{W}$ and $\mathcal{D}$ are independent of $\mathbf{R}$. This results in the following Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{R}}} - \frac{\partial \mathcal{T}}{\partial \mathbf{R}} + \frac{\partial \mathcal{U}}{\partial \mathbf{R}} = \boldsymbol{Q}_{\mathbf{R}}, \tag{27}$$

where $\boldsymbol{Q}_{\mathbf{R}}$ is the associated generalized force. We aim to transform this vector equation written in the inertial system I into the body frame B. We have

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{R}}}\dot{\mathbf{R}}|_I \cdot \dot{\mathbf{R}}|_I = \dot{\mathbf{V}} + \omega \times \mathbf{V}, \\ \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{R}}}\dot{\mathbf{R}}|_I \cdot (\dot{\mathbf{b}} + \omega \times \mathbf{b}) = \ddot{\mathbf{b}} + \frac{d(\omega \times \mathbf{b})}{dt}|_B + \omega \times (\dot{\mathbf{b}} + \omega \times \mathbf{b}),\end{gathered} \tag{28}$$

and

$$\left.\frac{\partial \mathcal{U}}{\partial \mathbf{R}}\right|_I = (0\ 0\ -mg)H^t \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = -m\tilde{\mathbf{g}}, \tag{29}$$

where

$$\tilde{\mathbf{g}} = \tilde{g}_1\mathbf{e}_1 + \tilde{g}_2\mathbf{e}_2 + \tilde{g}_3\mathbf{e}_3, \quad \tilde{g}_1 = -g\sin\Theta, \quad \tilde{g}_2 = g_1\sin\Phi\cos\Theta, \quad \tilde{g}_3 = g\cos\Phi\cos\Theta. \tag{30}$$

Now, the generalized force $\boldsymbol{Q}_{\mathbf{R}} = Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k}$ associated with the generalized coordinates vector $\mathbf{R}$ is equal to the total resultant force $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ acting on the vehicle due to aerodynamic and propulsive effects in the body frame B [9]. The Euler-Lagrange equation (27) reduces to

$$m(\dot{\mathbf{V}} + \omega \times \mathbf{V}) + \ddot{\mathbf{b}} + \frac{d(\omega \times \mathbf{b})}{dt}|_B + \omega \times (\dot{\mathbf{b}} + \omega \times \mathbf{b}) = m\tilde{\mathbf{g}} + \mathbf{F}. \tag{31}$$

This is the translational vector equation of motion written in the undeformed body frame centered at the body center of mass. It is seen that in comparison to the corresponding equation of motion of a rigid body there are three new terms which possess the vector $\mathbf{b}$ given by (13). In the rigid body case, $\mathbf{u} = 0$, $\mathbf{b} = 0$, and equations (31) coincide with the classical vector equation of rigid body translational motion

$$m(\dot{\mathbf{V}} + \omega \times \mathbf{V}) = m\tilde{\mathbf{g}} + \mathbf{F}. \tag{32}$$

In addition to the vector equation (31), we write down the corresponding set of scalar equations. They are

$$m(\dot{U} + QW - RV) + \ddot{b}_1 + \dot{Q}b_3 + 2Q\dot{b}_3 - \dot{R}b_2 - 2R\dot{b}_2 + (Qb_2 + Rb_3)P - (Q^2 + R^2)b_1 = -mg\sin\Theta + F_1,$$

$$m(\dot{V} + RU - PW) + \ddot{b}_2 + \dot{R}b_1 + 2R\dot{b}_1 - \dot{P}b_3 - 2P\dot{b}_3 + (Rb_3 + Pb_1)Q - (R^2 + P^2)b_2 = mg\sin\Phi\cos\Theta + F_2,$$

$$m(\dot{W} + PV - QU) + \ddot{b}_3 + \dot{P}b_2 + 2P\dot{b}_2 - \dot{Q}b_1 - 2Q\dot{b}_1 + (Pb_1 + Qb_2)R - (P^2 + Q^2)b_3 = mg\cos\Phi\cos\Theta + F_3. \tag{33}$$

C. Rotational equations of motion

To derive the rotational equations of motion, we follow the derivation [9] of such equations for a rigid body and make the changes needed due to the presence of the extra terms in (12). Select the vector of the Euler angles $\mathbf{E} = (\Phi, \Theta, \Psi)$ as the vector of generalized coordinates and analyze the Euler-Lagrange equation. The independence of the invariants $\mathcal{W}$, and $\mathcal{D}$ of the Euler angles yields

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{E}}} - \frac{\partial \mathcal{T}}{\partial \mathbf{E}} + \frac{\partial \mathcal{U}}{\partial \mathbf{E}} = \boldsymbol{Q}_{\mathbf{E}}, \tag{34}$$

where $\boldsymbol{Q}_{\mathbf{E}} = (Q_\Phi, Q_\Theta, Q_\Psi)$ is the vector of generalized forces associated with the vector $\mathbf{E}$. On employing the transformation matrix H one can express the vector $\boldsymbol{Q}_{\mathbf{E}}$ though the resultant external moment, $\mathbf{M}$, acting on the body and described in the undeformed body frame $\mathbf{M} = \mathcal{L}\mathbf{e}_1 + \mathcal{M}\mathbf{e}_2 + \mathcal{N}\mathbf{e}_3$ as

$$Q_\Phi = \mathcal{L}, \quad Q_\Theta = \mathcal{M}\cos\Phi - \mathcal{N}\sin\Phi,$$

$$Q_\Psi = -\mathcal{L}\sin\Theta + \mathcal{M}\sin\Phi\cos\Theta + \mathcal{N}\cos\Phi\cos\Theta. \tag{35}$$

We next employ the kinematic relations

$$\dot\Phi = P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta, \quad \dot\Theta = Q\cos\Phi - R\sin\Phi,$$

$$\dot\Psi = (Q\sin\Phi + R\cos\Phi)\sec\Theta. \tag{36}$$

Inversion of these relations brings us to the expressions of the rotational velocity $\boldsymbol{\omega}$ components through the vector $\mathbf{E}$ components

$$P = \dot\Phi - \dot\Psi\sin\Theta, \quad Q = \dot\Theta\cos\Phi + \dot\Psi\sin\Phi\cos\Theta,$$

$$R = -\dot\Theta\sin\Phi + \dot\Psi\cos\Phi\cos\Theta. \tag{37}$$

Now, since the matrix $\tilde{I}$ is symmetric, the following differentiation formula is valid:

$$\frac{1}{2}\frac{\partial}{\partial\boldsymbol{\omega}}(\boldsymbol{\omega}\tilde{I}\boldsymbol{\omega}^t) = \boldsymbol{\omega}\tilde{I}, \tag{38}$$

and therefore

$$\frac{1}{2}\frac{\partial}{\partial\mathbf{E}}(\boldsymbol{\omega}\tilde{I}\boldsymbol{\omega}^t) = \boldsymbol{\omega}\tilde{I}\frac{\partial\boldsymbol{\omega}^t}{\partial\mathbf{E}}. \tag{39}$$

Here, $\boldsymbol{\omega} = (P, Q, R)$ and $\boldsymbol{\omega}^t$ is the corresponding column-vector.

Substitution of the expression of the kinetic energy (12) into the Euler-Lagrange equation (34), formula (39), and simple algebra give us

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot\Phi} - \frac{\partial\mathcal{T}}{\partial\Phi} = \mathcal{G}_1,$$

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot\Theta} - \frac{\partial\mathcal{T}}{\partial\Theta} = \mathcal{G}_2\cos\Phi - \mathcal{G}_3\sin\Phi,$$

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot\Psi} - \frac{\partial\mathcal{T}}{\partial\Psi} = -\sin\Theta\mathcal{G}_1 + \sin\Phi\cos\Theta\mathcal{G}_2 + \cos\Phi\cos\Theta\mathcal{G}_3, \tag{40}$$

where $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are the components of the vector

$$\boldsymbol{\mathcal{G}} = \frac{d}{dt}[\tilde{I}\boldsymbol{\omega}^t] + \boldsymbol{\omega}\times\tilde{I}\boldsymbol{\omega}^t + \dot{\mathbf{c}} - \mathbf{c}\times\boldsymbol{\omega} + \frac{d}{dt}(\mathbf{b}\times\mathbf{V}) - (\mathbf{b}\cdot\boldsymbol{\omega})\mathbf{V} + (\mathbf{V}\cdot\boldsymbol{\omega})\mathbf{b}. \tag{41}$$

Evaluate now the derivative vector $\partial\mathcal{U}/\partial\mathbf{E} = (\partial\mathcal{U}/\partial\Phi, \partial\mathcal{U}/\partial\Theta, \partial\mathcal{U}/\partial\Psi)$. From (21) we obtain

$$\frac{\partial\mathcal{U}}{\partial\mathbf{E}} = -g(b_2\cos\Phi - b_3\sin\Phi)\cos\Theta\mathbf{e}_1 + g(b_1\cos\Theta + b_2\sin\Phi\sin\Theta + b_3\cos\Phi\sin\Theta)\mathbf{e}_2 + 0\mathbf{e}_3. \tag{42}$$

It will be convenient to rewrite the derivative in the form

$$\frac{\partial\mathcal{U}}{\partial\mathbf{E}} = \hat{g}_1\mathbf{e}_1 + (\hat{g}_2\cos\Phi - \hat{g}_3\sin\Phi)\mathbf{e}_2 + (-\hat{g}_1\sin\Theta + \hat{g}_2\sin\Phi\cos\Theta + \hat{g}_3\cos\Phi\cos\Theta)\mathbf{e}_3, \tag{43}$$

where

$$\hat{g}_1 = -g(b_2\cos\Phi - b_3\sin\Phi)\cos\Theta, \quad \hat{g}_2 = g(b_1\cos\Phi\cos\Theta + b_3\sin\Theta), \quad \hat{g}_3 = -g(b_1\sin\Phi\cos\Theta + b_2\sin\Theta). \tag{44}$$

The expressions for $\hat{g}_1$, $\hat{g}_2$, and $\hat{g}_3$ are determined as follows. On comparing (42) and (43) we immediately recover $\hat{g}_1$. As for the other two parameters, they solve the system of two equations

$$\hat{g}_2\cos\Phi - \hat{g}_3\sin\Phi = g(b_1\cos\Theta + b_2\sin\Phi\sin\Theta + b_3\cos\Phi\sin\Theta),$$

$$\hat{g}_2\sin\Phi\cos\Theta + \hat{g}_3\cos\Phi\cos\Theta = \hat{g}_1\sin\Theta. \tag{45}$$

Upon substituting formulas (35), (40), and (43) into the Euler-Lagrange equation (34) we deduce the rotational equations of motion of an elastic body. In the vector form, they are

$$\frac{d}{dt}[\tilde{I}\omega^t] + \omega\times\tilde{I}\omega^t + \dot{\mathbf{c}} - \mathbf{c}\times\omega + \frac{d}{dt}(\mathbf{b}\times\mathbf{V}) - (\mathbf{b}\cdot\omega)\mathbf{V} + (\mathbf{V}\cdot\omega)\mathbf{b} + \hat{\mathbf{g}} = \mathbf{M}, \tag{46}$$

where $\hat{\mathbf{g}} = (\hat{g}_1, \hat{g}_2, \hat{g}_3)^t$. The time-derivative $\frac{d}{dt}\tilde{I}$ of the elastic mass moment of inertia is the matrix $\{\tilde{I}'_{ij}\}_{i,j=1,2,3}$, given by

$$I'_{ii} = 2\int_{\mathcal{V}}\sum_{j=1, j\neq i}^{3}(x_j + u_j)\dot{u}_j\rho d\mathcal{V},$$

$$I'_{ij} = \int_{\mathcal{V}}[(x_j + u_j)\dot{u}_i + (x_i + u_i)\dot{u}_j]\rho d\mathcal{V}, \quad i \neq j. \tag{47}$$

In addition to the vector rotational equation of motion, we write down the set of corresponding scalar equations. They have the form

$$\frac{d}{dt}(\tilde{I}_{11}P - \tilde{I}_{12}Q - \tilde{I}_{13}R) - R(\tilde{I}_{22}Q - \tilde{I}_{12}P - \tilde{I}_{23}R) + Q(\tilde{I}_{33}R - \tilde{I}_{13}P - \tilde{I}_{23}Q) + \dot{c}_1 + c_3Q - c_2R$$

$$-b_3\dot{V} + b_2\dot{W} - \dot{b}_3V + \dot{b}_2W - (b_3R + b_2Q)U + b_1(VQ + WR) - g(b_2\cos\Phi - b_3\sin\Phi)\cos\Theta = \mathcal{L},$$

$$\frac{d}{dt}(\tilde{I}_{22}Q - \tilde{I}_{23}R - \tilde{I}_{12}P) - P(\tilde{I}_{33}R - \tilde{I}_{23}Q - \tilde{I}_{13}P) + R(\tilde{I}_{11}P - \tilde{I}_{12}Q - \tilde{I}_{13}R) + \dot{c}_2 + c_1R - c_3P$$

$$-b_1\dot{W} + b_3\dot{U} - \dot{b}_1W + \dot{b}_3U - (b_1P + b_3R)V + b_2(WR + UP) + g(b_1\cos\Phi\cos\Theta + b_3\sin\Theta) = \mathcal{M},$$

$$\frac{d}{dt}(\tilde{I}_{33}R - \tilde{I}_{13}P - \tilde{I}_{23}Q) - Q(\tilde{I}_{11}P - \tilde{I}_{13}R - \tilde{I}_{12}Q) + P(\tilde{I}_{22}Q - \tilde{I}_{23}R - \tilde{I}_{12}P) + \dot{c}_3 + c_2P - c_1Q$$

$$-b_2\dot{U} + b_1\dot{V} - \dot{b}_2U + \dot{b}_1V - (b_2Q + b_1P)W + b_3(UP + VQ) - g(b_1\sin\Phi\cos\Theta + b_2\sin\Theta) = \mathcal{N}. \tag{48}$$

D. Thermoelastic equations

Our next step in the derivation of the complete set of the thermoelastic flight equations is to choose the elastic displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ and the entropy flow vector $\mathbf{S} = (S_1, S_2, S_3)$ as vectors of generalized coordinates. We can write the variational principle in the form of the Euler-Lagrange equations as

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{u}_j} - \frac{\partial\mathcal{T}}{\partial u_j} + \frac{\partial}{\partial u_j}(\mathcal{U} + \mathcal{W} + \mathcal{D}) = \int_{\mathcal{A}} f_j d\mathcal{A}, \quad j = 1, 2, 3,$$

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{\mathbf{S}}} - \frac{\partial\mathcal{T}}{\partial\mathbf{S}} + \frac{\partial}{\partial\mathbf{S}}(\mathcal{U} + \mathcal{W} + \mathcal{D}) = -\int_{\mathcal{A}} T\mathbf{n}_j d\mathcal{A} + \int_{\mathcal{V}} \mathbf{\Xi} d\mathcal{V}, \tag{49}$$

where $\mathbf{n}$ is the unit normal pointing outward at the boundary $\mathcal{A}$ of the body $\mathcal{V}$, $\mathbf{\Xi}$ is the heat source whose divergence is defined through the heat capacity Ω by $\nabla \cdot \mathbf{\Xi} = -\Omega/k_T$, and f_j is the density of the surface force F_j.

To write an alternative form of the left-hand sides of equations in (49), we note that

$$\frac{\partial\mathcal{T}}{\partial\mathbf{S}} = \frac{\partial\mathcal{T}}{\partial\dot{\mathbf{S}}} = 0 \tag{50}$$

and compute the derivative of the kinetic energy associated with the displacement u_j and its derivative

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{u}_j} - \frac{\partial\mathcal{T}}{\partial u_j} = m\dot{\mathbf{V}}\cdot\mathbf{e}_j - m\mathbf{V}\cdot(\omega\times\mathbf{e}_j) - \frac{1}{2}\omega\tilde{I}^{(j)}\omega^t + \int_{\mathcal{V}}[2\omega\cdot(\dot{\mathbf{u}}\times\mathbf{e}_j) + \ddot{\mathbf{u}}_j]\rho d\mathcal{V}, \quad j = 1, 2, 3. \tag{51}$$

Here, $\tilde{I}^{(j)}$ is the derivative of the elastic mass moment of inertia $\tilde{I}$ with respect to the displacement u_j,

$$\tilde{I}^{(j)} = \int_{\mathcal{V}} I_j\rho d\mathcal{V}, \tag{52}$$

and the integrands I_j are 3×3 symmetric matrices given by

$$I_1 = \begin{pmatrix} 0 & x_2 + u_2 & x_3 + u_3 \\ x_2 + u_2 & 2(x_1 + u_1) & 0 \\ x_3 + u_3 & 0 & 2(x_1 + u_1) \end{pmatrix}, \quad I_2 = \begin{pmatrix} 2(x_2 + u_2) & x_1 + u_1 & 0 \\ x_1 + u_1 & 0 & x_3 + u_3 \\ 0 & x_3 + u_3 & 2(x_2 + u_2) \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 2(x_3 + u_3) & 0 & x_1 + u_1 \\ 0 & 2(x_3 + u_3) & x_2 + u_2 \\ x_1 + u_1 & x_2 + u_2 & 0 \end{pmatrix}. \tag{53}$$

Compute now the derivatives of the gravitational potential energy $\mathcal{U}$ and the sum of the thermoelastic potential $\mathcal{W}$ and the dissipation function $\mathcal{D}$ with respect to the displacement vector $\mathbf{u}$ and the the entropy flow $\mathbf{S}$. From (21) and (30) we immediately obtain

$$\frac{\partial \mathcal{U}}{\partial u_j} = -m\tilde{g}_j, \quad \frac{\partial \mathcal{U}}{\partial \mathbf{S}} = 0. \tag{54}$$

On using the representations of the thermoelastic potential and the dissipative function (22), (23), and (24) and integration by parts in (23) we derive

$$\frac{\partial}{\partial \mathbf{u}}(\mathcal{W} + \mathcal{D}) = -\int_{\mathcal{V}} [\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\nabla\epsilon - \beta_T \nabla T] d\mathcal{V} + \int_{\mathcal{A}} f_j d\mathcal{A},$$

$$\frac{\partial}{\partial \mathbf{S}}(\mathcal{W} + \mathcal{D}) = \int_{\mathcal{V}} \left(\nabla T + \frac{T_a}{k}\dot{\mathbf{S}}\right) d\mathcal{V} - \int_{\mathcal{A}} T\mathbf{n}_j d\mathcal{A}. \tag{55}$$

Substituting formulas (50), (51), (54), and (55) into the Euler-Lagrange equations (49) we deduce the following equations:

$$-\int_{\mathcal{V}} \left[\mu \nabla^2 u_j + (\lambda + \mu)\frac{\partial \epsilon}{\partial x_j} - \beta_T \frac{\partial T}{\partial x_j}\right] d\mathcal{V} + m\dot{\mathbf{V}} \cdot \mathbf{e}_j - m\mathbf{V} \cdot (\omega \times \mathbf{e}_j) - \frac{1}{2}\omega \tilde{I}^{(j)} \omega^t$$

$$+ \int_{\mathcal{V}} [2\omega \cdot (\dot{\mathbf{u}} \times \mathbf{e}_j) + \ddot{u}_j]\rho d\mathcal{V} = m\tilde{g}_j, \quad j = 1, 2, 3,$$

$$\int_{\mathcal{V}} \left(\nabla T + \frac{T_a}{k}\dot{\mathbf{S}}\right) d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\Xi} d\mathcal{V}. \tag{56}$$

To write local partial differential equations for the displacements u_j and the entropy flow $\mathbf{S}$, we choose an arbitrary small volume within the body, apply the mean value theorem and let the volume go to zero. Then, at an arbitrary point (x_1, x_2, x_3) of the body, we have

$$\mu \nabla^2 u_j + (\lambda + \mu)\frac{\partial \epsilon}{\partial x_j} - \beta_T \frac{\partial T}{\partial x_j} - \rho\dot{\mathbf{V}} \cdot \mathbf{e}_j + \rho\mathbf{V} \cdot (\omega \times \mathbf{e}_j) + \frac{\rho}{2}\omega I_j \omega^t - 2\rho\omega \cdot (\dot{\mathbf{u}} \times \mathbf{e}_j) = \rho\ddot{u}_j - \rho\tilde{g}_j, \ j = 1, 2, 3,$$

$$\nabla T + \frac{T_a}{k_T}\dot{\mathbf{S}} = \mathbf{\Xi}. \tag{57}$$

The last vector equation for the entropy flow reduces to the standard equation of coupled thermoelasticity by applying to it the divergence operator and using the formula $\nabla \cdot \mathbf{S} = -c_T T/T_a - \beta_T \epsilon$. The resulting equation is

$$\frac{c_T}{k_T}\dot{T} - \nabla^2 T + \frac{T_a \beta_T}{k_T}\dot{\epsilon} = \frac{\Omega}{k_T}, \quad (x_1, x_2, x_3) \in \mathcal{V}, \quad t > 0. \tag{58}$$

It is of interest to write down the elastic equations in (57) in the explicit form as a system of three scalar equations

$$\frac{\mu}{\rho}\nabla^2 u_1 + \frac{\lambda+\mu}{\rho}\frac{\partial \epsilon}{\partial x_1} - \frac{\beta_T}{\rho}\frac{\partial T}{\partial x_1} - \ddot{u}_1 = -\tilde{g}_1 + \dot{U} - VR + WQ$$

$$+2\dot{u}_3 Q - 2\dot{u}_2 R - (x_1 + u_1)(Q^2 + R^2) + P[(x_2 + u_2)Q + (x_3 + u_3)R],$$

$$\frac{\mu}{\rho}\nabla^2 u_2 + \frac{\lambda+\mu}{\rho}\frac{\partial \epsilon}{\partial x_2} - \frac{\beta_T}{\rho}\frac{\partial T}{\partial x_2} - \ddot{u}_2 = -\tilde{g}_2 + \dot{V} - WP + UR$$

$$+2\dot{u}_1 R - 2\dot{u}_3 P - (x_2 + u_2)(R^2 + P^2) + Q[(x_3 + u_3)R + (x_1 + u_1)P],$$

$$\frac{\mu}{\rho}\nabla^2 u_3 + \frac{\lambda+\mu}{\rho}\frac{\partial \epsilon}{\partial x_3} - \frac{\beta_T}{\rho}\frac{\partial T}{\partial x_3} - \ddot{u}_3 = -\tilde{g}_3 + \dot{W} - UQ + VP + 2\dot{u}_2 P - 2\dot{u}_1 Q$$

$$-(x_3 + u_3)(P^2 + Q^2) + R[(x_1 + u_1)P + (x_2 + u_2)Q], \quad (x_1, x_2, x_3) \in \mathcal{V}, \quad t > 0. \tag{59}$$

These three equations couple the elastic displacements u_1, u_2, and u_3 with the translational and rotational velocities U, V, W and P, Q, R and the temperature T, while the fourth thermoelasticty equation (58) couples the temperature with the elastic displacements by means of the divergence $\dot{\epsilon}$ of the vector $\dot{\mathbf{u}}$. Notice that the left-hand sides of the three equations (59) coincide with those of dynamic Navier's equations of a thermoelastic body in which the translational and rotational velocities equal zero.

IV. Longitudinal and lateral flight dynamics of a thermoelastic body

In this section we wish to investigate if it is possible to decouple the system of six flight dynamics equations (33) and (48) into two systems, the longitudinal and lateral sets of equations, as it can be done in rigid body flight dynamics.

A. Longitudinal flight dynamics equations

Assume first that the elastic body $\mathcal{V}$ is symmetric with respect to the plane $x_1 O x_3$ and subjected to the forces F_1 and F_3 and the moment $\mathcal{M}$, while the force F_2 and the moments $\mathcal{L}$ and $\mathcal{N}$ equal zero. This implies that the velocity component V, the roll and yaw rates P and R, and the Euler angle Φ vanish, while $u_2(x_1, x_2, x_3) = -u_2(x_1, -x_2, x_3)$, it is small and neglected. By direct calculation from (11) and (13) we have $b_2 = 0$, $c_1 = c_3 = 0$, $\tilde{I}_{12} = \tilde{I}_{23} = 0$. One of the translational and two out of the rotational equations (33) and (48) are satisfied automatically, while the elastic equations (59) are simplified. The resulting three longitudinal flight dynamics and the associated three thermoelastic equations have the form

$$m(\dot{U} + QW) + \ddot{b}_1 + \dot{Q}b_3 + 2Q\dot{b}_3 - Q^2 b_1 = -mg\sin\Theta + F_1,$$

$$m(\dot{W} - QU) + \ddot{b}_3 - \dot{Q}b_1 - 2Q\dot{b}_1 - Q^2 b_3 = mg\cos\Theta + F_3,$$

$$\frac{d}{dt}(\tilde{I}_{22}Q) + \dot{c}_2 - b_1\dot{W} + b_3\dot{U} - \dot{b}_1 W + \dot{b}_3 U + g(b_1\cos\Theta + b_3\sin\Theta) = \mathcal{M},$$

$$\frac{\mu}{\rho}\nabla^2 u_1 + \frac{\lambda+\mu}{\rho}\frac{\partial\epsilon}{\partial x_1} - \frac{\beta_T}{\rho}\frac{\partial T}{\partial x_1} - \ddot{u}_1 = g\sin\Theta + \dot{U} + WQ + 2\dot{u}_3 Q - (x_1 + u_1)Q^2,$$

$$\frac{\mu}{\rho}\nabla^2 u_3 + \frac{\lambda+\mu}{\rho}\frac{\partial\epsilon}{\partial x_3} - \frac{\beta_T}{\rho}\frac{\partial T}{\partial x_3} - \ddot{u}_3 = -g\cos\Theta + \dot{W} - UQ - 2\dot{u}_1 Q - (x_3 + u_3)Q^2,$$

$$\frac{c_T}{k_T}\dot{T} - \nabla^2 T + \frac{T_a\beta_T}{k_T}\dot{\epsilon} = \frac{\Omega}{k_T}, \quad (x_1, x_2, x_3) \in \mathcal{V}, \quad t > 0. \tag{60}$$

Here, $\epsilon = u_{1,1} + u_{3,3}$. We will employ these governing equations in the following sections.

B. Lateral flight dynamics equations

To derive the lateral dynamics equations, we assume that the elastic body $\mathcal{V}$ is symmetric with respect to the plane x_1Ox_3, the velocity W equals zero, and the body is subjected to zero pitch moment, $\mathcal{M} = 0$. As a consequence, the pitch rate Q vanishes, $Q = 0$. Because the body is elastic and since the x_2-component of the displacement vector is not symmetric with respect to the plane x_1Ox_3, in general, $\tilde{I}_{12}$ and $\tilde{I}_{13}$ do not equal zero. For the same reason, in general, $b_i \neq 0$ and $c_i \neq 0$, $i = 1, 2, 3$.

In the case of a rigid body, $\tilde{I}_{12} = I_{12} = 0$, $\tilde{I}_{23} = I_{23} = 0$, $b_i = 0$, and $c_i = 0$, $i = 1, 2, 3$. This makes possible to simplify the translational and rotational equations in (33) and (48) as

$$m(\dot{U} - RV) = -mg\sin\Theta + F_1, \quad m(\dot{V} + RU) = mg\sin\Phi\cos\Theta + F_2, \quad mPV = mg\cos\Phi\cos\Theta + F_3,$$

$$I_{11}\dot{P} - I_{13}\dot{R} = \mathcal{L}, \quad -P(I_{33}R - I_{13}P) + R(I_{11}P - I_{13}R) = 0, \quad I_{33}\dot{R} - I_{13}\dot{P} = \mathcal{N}. \tag{61}$$

It is customary in the classical flight dynamics [6] to neglect the small terms RV, PV, PR, P^2, and R^2. If the flight regime is such that these terms are negligible indeed, then the first and the third equations are the drag and lift equations, while the fifth equation is an identity. The remaining three equations are the classical lateral dynamics equations [6]

$$m(\dot{V} + RU) = mg\sin\Phi\cos\Theta + F_2, \quad I_{11}\dot{P} - I_{13}\dot{R} = \mathcal{L}, \quad I_{33}\dot{R} - I_{13}\dot{P} = \mathcal{N}. \tag{62}$$

Now, switch to the elastic case. If we follow the pattern and want to confine ourselves to the elastic counterpart of the system (62), we need to disregard the first and third equations in (33) and the second equation in (48). As in the rigid body case we neglect the small terms RV, PV, PR, P^2, and R^2. However, even after that the resulting three equations cannot be neglected unless we disregard the effect of elastic displacements. We write down the simplified six flight dynamic equations

$$m\dot{U} + \ddot{b}_1 - \dot{R}b_2 - 2R\dot{b}_2 = -mg\sin\Theta + F_1,$$

$$m(\dot{V} + RU) + \ddot{b}_2 + \dot{R}b_1 + 2R\dot{b}_1 - \dot{P}b_3 - 2P\dot{b}_3 = mg\sin\Phi\cos\Theta + F_2,$$

$$\ddot{b}_3 + \dot{P}b_2 + 2P\dot{b}_2 = mg\cos\Phi\cos\Theta + F_3,$$

$$\frac{d}{dt}(\tilde{I}_{11}P - \tilde{I}_{13}R) + \dot{c}_1 - c_2R - b_3\dot{V} - \dot{b}_3V - RUb_3 - g(b_2\cos\Phi - b_3\sin\Phi)\cos\Theta = \mathcal{L},$$

$$\frac{d}{dt}(-\tilde{I}_{23}R - \tilde{I}_{12}P) + \dot{c}_2 + c_1R - c_3P + b_3\dot{U} + \dot{b}_3U + b_2PU + g(b_1\cos\Phi\cos\Theta + b_3\sin\Theta) = 0,$$

$$\frac{d}{dt}(\tilde{I}_{33}R - \tilde{I}_{13}P) + \dot{c}_3 + c_2P - b_2\dot{U} + b_1\dot{V} - \dot{b}_2U + \dot{b}_1V + PUb_3 - g(b_1\sin\Phi\cos\Theta + b_2\sin\Theta) = \mathcal{N}. \qquad (63)$$

Here, due to the symmetry of the body $\mathcal{V}$ with respect to the plane x_1Ox_3,

$$\tilde{I}_{12} = \int_{\mathcal{V}} u_2(x_1 + u_1)\rho d\mathcal{V}, \quad \tilde{I}_{23} = \int_{\mathcal{V}} u_2(x_3 + u_3)\rho d\mathcal{V},$$

$$c_1 = \int_{\mathcal{V}} [u_2\dot{u}_3 - (x_3 + u_3)\dot{u}_2]\rho d\mathcal{V}, \quad c_2 = \int_{\mathcal{V}} [(x_1 + u_3)\dot{u}_1 - (x_1 + u_1)\dot{u}_3]\rho d\mathcal{V},$$

$$c_3 = \int_{\mathcal{V}} [(x_1 + u_1)\dot{u}_2 - u_2\dot{u}_1]\rho d\mathcal{V}. \qquad (64)$$

The flight dynamics equations (63) need to be complemented by the thermoelasticity equations (58), (59) which can be further simplified by identifying small quadratic terms and neglecting them. In certain regimes, some of the equations in (63) might be simplified. However, in general, under the assumptions made in lateral rigid body dynamics, in the termoelastic flight dynamic equations, it is impossible to decouple the six equations into three equations which can be disregarded and the other three equations which need to be solved. In what follows we analyze the longitudinal regime.

C. Linearized longitudinal equations

Consider the longitudinal flight dynamics and choose an undeformed body frame such that the axis Ox_1 is aligned with the equilibrium direction of the velocity vector, and the equilibrium values of the components of the translational and rotational velocities and the pitch angle of interest are $U_0 = \text{const}$, $W_0 = 0$, $Q_0 = 0$ (the body is initially in unaccelerated flight), and Θ_0, respectively. Denote the disturbed values of U, W, Q, and Θ by U_0u, U_0w, q, and θ. Then, approximately, $U = U_0(1 + u)$, $W = U_0w$, $Q = q$, $\Theta = \Theta_0 + \theta$, and also $u_2 \approx 0$. We can write down the three longitudinal equations for the variations U_0u, U_0w, and q

$$mU_0(\dot{u} + qw) + \ddot{b}_1 + \dot{q}b_3 + 2q\dot{b}_3 - q^2 b_1 = -mg\theta\cos\Theta_0 + \delta F_1,$$

$$mU_0(\dot{w} - q - qu) + \ddot{b}_3 - \dot{q}b_1 - 2q\dot{b}_1 - q^2 b_3 = -mg\theta\sin\Theta_0 + \delta F_3,$$

$$\tilde{I}_{22}\dot{q} + \tilde{I}'_{22}q + \dot{c}_2 - b_1U_0\dot{w} + b_3U_0\dot{u} - \dot{b}_1U_0w + \dot{b}_3U_0(1+u) + g\theta(-b_1\sin\Theta_0 + b_3\cos\Theta_0) = \delta\mathcal{M}, \tag{65}$$

Here, F_1, F_3, and M are the equilibrium forces and moment due to lift, drag, and thrust, and δF_1, δF_3, and $\delta\mathcal{M}$ are their variations. Considering quasi-steady flow [6] and therefore neglecting the dependence of the forces and the moment on $\dot{U}$ we replace the variations by the linear expansions

$$\delta F_j = \left.\frac{\partial F_j}{\partial u}\right|_0 u + \left.\frac{\partial F_j}{\partial w}\right|_0 w + \left.\frac{\partial F_j}{\partial \dot{w}}\right|_0 \dot{w} + \left.\frac{\partial F_j}{\partial \dot{\theta}}\right|_0 \dot{\theta}, \quad j = 1, 3,$$

$$\delta\mathcal{M} = \left.\frac{\partial \mathcal{M}}{\partial u}\right|_0 u + \left.\frac{\partial \mathcal{M}}{\partial w}\right|_0 w + \left.\frac{\partial \mathcal{M}}{\partial \dot{w}}\right|_0 \dot{w} + \left.\frac{\partial \mathcal{M}}{\partial \dot{\theta}}\right|_0 \dot{\theta}. \tag{66}$$

The subscript 0 means the value of the corresponding derivative in the initial state conditions. Confine ourselves to small perturbations about the equilibrium state, neglect the small quantities qw, q^2, and qu, substitute the expansions (66) into equations (65) and notice that $q = \dot{\theta}$. This gives the linearized longitudinal equations of motion of a thermoelastic body

$$\frac{mU_0}{Sp}\dot{u} - C_{1u}u - \frac{c}{2U_0}C_{1\dot{w}}\dot{w} - C_{1w}w + \frac{b_3}{Sp}\ddot{\theta} + \left(-\frac{c}{2U_0}C_{1q} + \frac{2\dot{b}_3}{Sp}\right)\dot{\theta} - C_g\cos\Theta_0\theta = -\frac{\ddot{b}_1}{Sp} + \frac{F_{ia}}{Sp},$$

$$-C_{3u}u + \left(\frac{mU_0}{Sp} - \frac{c}{2U_0}C_{3\dot{w}}\right)\dot{w} - C_{3w}w - \frac{b_1}{Sp}\ddot{\theta} - \left(\frac{mU_0}{Sp} + \frac{c}{2U_0}C_{3q} + \frac{2\dot{b}_1}{Sp}\right)\dot{\theta} - C_g\sin\Theta_0\theta = -\frac{\ddot{b}_3}{Sp} + \frac{F_{3a}}{Sp},$$

$$\begin{aligned}&\frac{U_0b_3}{Spc}\dot{u} + \left(-C_{\mathcal{M}u} + \frac{U_0\dot{b}_3}{Spc}\right)u - \left(\frac{c}{2U_0}C_{\mathcal{M}\dot{w}} + \frac{U_0b_1}{Spc}\right)\dot{w} - \left(C_{\mathcal{M}w} + \frac{U_0\dot{b}_1}{Spc}\right)w + \frac{\tilde{I}_{22}}{Spc}\ddot{\theta}\\ &+ \left(\frac{\tilde{I}'_{22}}{Spc} - \frac{c}{2U_0}C_{\mathcal{M}q}\right)\dot{\theta} + \frac{g}{Spc}(-b_1\sin\Theta_0 + b_3\cos\Theta_0)\theta = -\frac{\dot{c}_2 + U_0\dot{b}_3}{Spc} + \frac{\mathcal{M}_a}{Spc}.\end{aligned} \tag{67}$$

Here, the following standard notations [6] for the dimensionless longitudinal stability derivatives are used:

$$C_{ju} = \frac{1}{Sp}\left.\frac{\partial F_j}{\partial u}\right|_0, \quad C_{jw} = \frac{1}{Sp}\left.\frac{\partial F_j}{\partial w}\right|_0, \quad C_{j\dot{w}} = \frac{2U_0}{Spc}\left.\frac{\partial F_j}{\partial \dot{w}}\right|_0, \quad C_{jq} = \frac{2U_0}{Spc}\left.\frac{\partial F_j}{\partial q}\right|_0, \quad j = 1, 3,$$

$$C_g = -\frac{mg}{Sp},$$

$$C_{\mathcal{M}u} = \frac{1}{Spc}\left.\frac{\partial \mathcal{M}}{\partial u}\right|_0, \quad C_{\mathcal{M}w} = \frac{1}{Spc}\left.\frac{\partial \mathcal{M}}{\partial w}\right|_0, \quad C_{\mathcal{M}\dot{w}} = \frac{2U_0}{Spc^2}\left.\frac{\partial \mathcal{M}}{\partial \dot{w}}\right|_0, \quad C_{\mathcal{M}q} = \frac{2U_0}{Spc^2}\left.\frac{\partial \mathcal{M}}{\partial q}\right|_0, \tag{68}$$

S is the wing area, $p = \frac{1}{2}\rho_a U_0^2$ is the dynamic pressure, ρ_a is the air density, and c is the mean aerodynamic chord, and F_{1a}, F_{3a}, and $\mathcal{M}_a$ are the applied aerodynamic forces and moment [6]. The three thermoelastic equations (60) need to be transformed accordingly and become

$$\frac{\mu}{\rho}\nabla^2 u_1 + \frac{\lambda+\mu}{\rho}\frac{\partial\epsilon}{\partial x_1} - \frac{\beta_T}{\rho}\frac{\partial T}{\partial x_1} - \ddot{u}_1 + u_1\dot{\theta}^2 - 2\dot{u}_3\dot{\theta} = g\cos\Theta_0\theta + U_0\dot{u} + U_0 w\dot{\theta} - x_1\dot{\theta}^2,$$

$$\frac{\mu}{\rho}\nabla^2 u_3 + \frac{\lambda+\mu}{\rho}\frac{\partial\epsilon}{\partial x_3} - \frac{\beta_T}{\rho}\frac{\partial T}{\partial x_3} - \ddot{u}_3 + u_3\dot{\theta}^2 + 2\dot{u}_1\dot{\theta} = g\sin\Theta_0\theta + U_0\dot{w} - U_0 u\dot{\theta} - x_3\dot{\theta}^2,$$

$$\frac{c_T}{k_T}\dot{T} - \nabla^2 T + \frac{T_a\beta_T}{k_T}\dot{\epsilon} = \frac{\Omega}{k_T}, \quad (x_1, x_2, x_3) \in \mathcal{V}, \quad t > 0. \tag{69}$$

To be consistent with the assumptions made in (67), we can neglect the terms $w\dot{\theta}$, $u\dot{\theta}$, and $\dot{\theta}^2$ in equations (69) as well. Notice that even in the linearized longitudinal system of equations the flight dynamics velocities u, w, and $q = \dot{\theta}$ are coupled with the elastic displacements u_1 and u_3 which are coupled with the temperature distribution T. To solve the system of equations (67), (69), it is convenient to apply the method of successive approximations which, at the first step, assumes that the elastic displacements and temperature are known (say, their initial values at time $t = 0$) and solves the linear system (67). At the second step, it uses the values of u, w, and $q = \dot{\theta}$ and solves the linear system of the thermoelastic equations This procedure enables us to preserve the high order terms $u\dot{\theta}$, $u\dot{\theta}$, and $\dot{\theta}^2$ in the thermoelastic system (69).

D. Blakelock's model of longitudinal motion of a transport

To analyze the impact of the thermoelastic displacements on the linearized longitudinal equations (67), we consider the homogeneous equations (67) and find the associated quartic equation. Its roots determine the frequencies and damping ratios of short-period and phugoid oscillations. Without loss $\Theta_0 = 0$, and we neglect u_1, u_2, and $\dot{u}_3$. This means that the thermoelastic displacement vector $\mathbf{u}$ is $(0, 0, u_3)$, $\dot{\mathbf{u}} = 0$ and therefore $d\tilde{I}_{22}/dt = 0$. The homogeneous system of differential equations reads

$$\frac{mU_0}{Sp}\dot{u} - C_{1u}u - \frac{c}{2U_0}C_{1\dot{w}}\dot{w} - C_{1w}w + \frac{b_3}{Sp}\ddot{\theta} - \frac{c}{2U_0}C_{1q}\dot{\theta} - C_g\theta = 0,$$

$$-C_{3u}u + \left(\frac{mU_0}{Sp} - \frac{c}{2U_0}C_{3\dot{w}}\right)\dot{w} - C_{3w}w - \left(\frac{mU_0}{Sp} + \frac{c}{2U_0}C_{3q}\right)\dot{\theta} = 0,$$

$$\frac{U_0 b_3}{Spc}\dot{u} - C_{\mathcal{M}u}u - \frac{c}{2U_0}C_{\mathcal{M}\dot{w}}\dot{w} - C_{\mathcal{M}w}w + \frac{\tilde{I}_{22}}{Spc}\ddot{\theta} - \frac{c}{2U_0}C_{\mathcal{M}q}\dot{\theta} + \frac{gb_3}{Spc}\theta = 0. \tag{70}$$

Blakelock [6] analyzed the homogeneous system of a rigid four-engine jet transport flying in straight and level flight at 40,000 ft at speed 600 ft/sec. To be consistent with the Nomenclature dimensions, we transform the Imperial system units used in [6] to the SI system. The following data are used [6]: $m = 84,645$ kg, $U_0 = 183$ m sec^{-1}, $S = 223$ m^2 ,

$p = 5032.2$ Pa, $c = 6.16$ m, and $I_{22} = 3.552 \cdot 10^6$ kg m^2. The dimensionless longitudinal stability derivatives needed for the analysis of system (70) are [6]

$$C_{1u} = -0.088, \quad C_{1w} = 0.392, \quad C_g = -0.74, \quad C_{3u} = -1.48, \quad C_{3\dot{w}} = -1.13,$$

$$C_{3w} = -4.46, \quad C_{3q} = -3.94, \quad C_{\mathcal{M}\dot{w}} = -3.27, \quad C_{\mathcal{M}w} = -0.619, \quad C_{\mathcal{M}q} = -11.4. \tag{71}$$

On applying the Laplace transform to the system (70) and accepting the zero initial conditions we obtain a homogeneous system of three linear algebraic equations. Its characteristic equation is the quartic equation

$$\alpha_4\sigma^4 + \alpha_3\sigma^3 + \alpha_2\sigma^2 + \alpha_1\sigma + \alpha_0 = 0. \tag{72}$$

where σ is the Laplace transform parameter,

$$\alpha_0 = 0.6779 + 1.5492\hat{b}_3, \quad \alpha_1 = 0.9407 + 1.9366\hat{b}_3, \quad \alpha_2 = 129.342 + 0.9726\hat{I}_{22} + 160.783\hat{b}_3,$$

$$\alpha_3 = 37.646 + 62.7637\hat{I}_{22} + 0.00814\hat{b}_3 - 132.475\hat{b}_3^2, \quad \alpha_4 = 190.647\hat{I}_{22} - 410.347\hat{b}_3^2. \tag{73}$$

and

$$\hat{I}_{22} = \frac{\tilde{I}_{22}}{Spc}, \quad \hat{b}_3 = \frac{b_3}{Sp}. \tag{74}$$

For realistic values of the parameters, the quartic equations (72) has two complex conjugate roots $s_{1,2} = \sigma_1 \pm i\sigma_2$ and $s_{3,4} = \sigma_3 \pm i\sigma_4$, where $\sigma_1 < 0$, $\sigma_3 < 0$, and $|\sigma_1| < |\sigma_3|$. Equation (72) can be represented in the form

$$(\sigma^2 + 2\zeta_p\omega_{np}\sigma + \omega_{np})(\sigma^2 + 2\zeta_s\omega_{ns}\sigma + \omega_{ns}) = 0. \tag{75}$$

Here, $\omega_{np} = |s_1|$, $\zeta_p = -\sigma_1/|s_1|$ and $\omega_{ns} = |s_3|$, $\zeta_s = -\sigma_3/|s_3|$ are the frequencies and damping ratios of phugoid and short-period oscillations, respectively.

Blakelock's model deals with the rigid body flight dynamics, and the parameter b_3 and the elastic moment of inertia $\tilde{I}_{22}$ do not appear in the corresponding equations in [6]: $b_3 = 0$ and $\tilde{I}_{22} = I_{22}$ if the elastic effects are disregarded. In the thermoelastic case, they have to be recovered by integrating over the body $\mathcal{V}$. To determine the values of ρ, b_3 and $\tilde{I}_{22}$ consistent with Blakelock's model, we employ the wings geometry of C-17 Globemaster III scaled such that S, c, and I_{22} from [6] are kept unchanged. The following data are employed for the computational test. The span is $2h_w = 40.84$ m, the leading edge sweep angle is 31.5°, the maximal wing thickness is 7/60 of the chord function $c(x_2)$, and the wing profile is chosen as the one in C-17. The wing root chord c_r and the wing tip chord c_t are selected to be $c_r = 8.84$ m

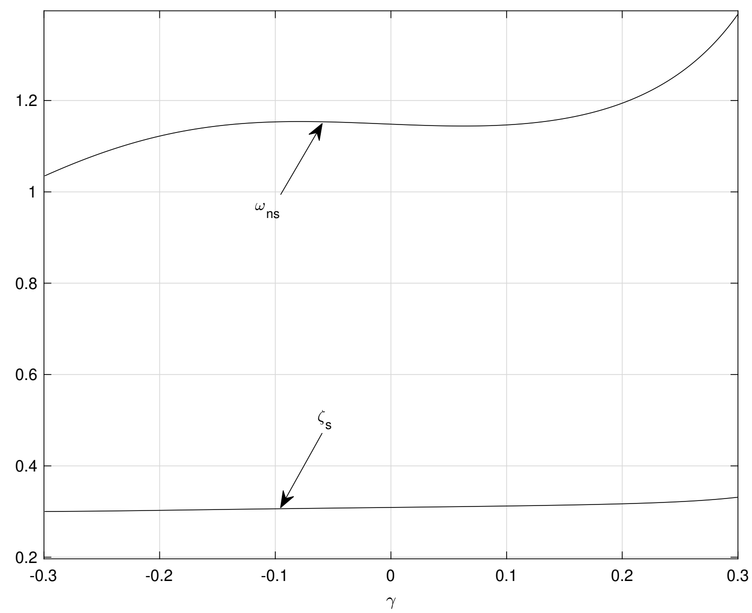

Fig. 3 Variation of the damping ratio ζ_s and frequency ω_{ns} of short-period oscillations with the thermoelastic deflection parameter γ.

and $c_t = 2.08$ m. We remark that for the parameters c_t, c_r, and h_w chosen, $c = \frac{2}{3}[c_t + c_r - c_t c_r/(c_t + c_r)] = 6.16$ m and $S = 223$ m^2 which are the Blakelock's values of the mean chord and wing area, respectively. These parameters and the sweep angle are also consistent with the wing geometry of C-17 scaled accordingly.

For the thermoelastic displacements, we assume that due to the temperature and aerodynamic and propulsive effects, the wings experience the following displacements (if $u_3 > 0$, then it is pointing downward):

$$- u_3(x_2) = \begin{cases} k_1 x_2^{k_2}, & 0 \le x_2 \le x_2^0, \\ m_1 x_2 + m_2, & x_2^0 \le x_2 \le h_w, \end{cases} \tag{76}$$

where the function $u_3(x_2)$ and its derivative are continuous at the point $x_2 = x_2^0$. Here, $x_2^0 = \delta_1 h_w$ and $u_3(x_2^0) = -\delta_2 h_w$, $u_3(h_w) = -\gamma h_w$, and γ is the deflection parameter. According to the solution of the thermoelastic problem in Section VB, we select $\delta_1 = 0.4$, $\delta_2 = 0.18$. Due to the continuity of u_2 and u_2' at $x_2 = x_2^0$,

$$m_1 = \frac{(1-\delta_2)\gamma}{1-\delta_1}, \quad m_2 = \frac{(\delta_2 - \delta_1)\gamma h_w}{1-\delta_1}, \quad k_2 = \frac{m_1 \delta_1}{\gamma \delta_2}, \quad k_1 = \frac{m_1}{k_2 (x_2^0)^{k_2 - 1}}.$$

As shown in Fig. 3, for short-period oscillations, the frequency and the damping ratio is increasing with the increase of the displacements directed upwards (the parameter γ is negative if the displacement is positive and pointing downward). At the same time, the behavior of the frequency ω_{ns} is not monotonic, and it has a local minimum and maximum at approximately $\gamma = -0.1$ and $\gamma = 0.1$, respectively. In the case of phugoid oscillations, for the range $[-0.3, 0.3]$ of the

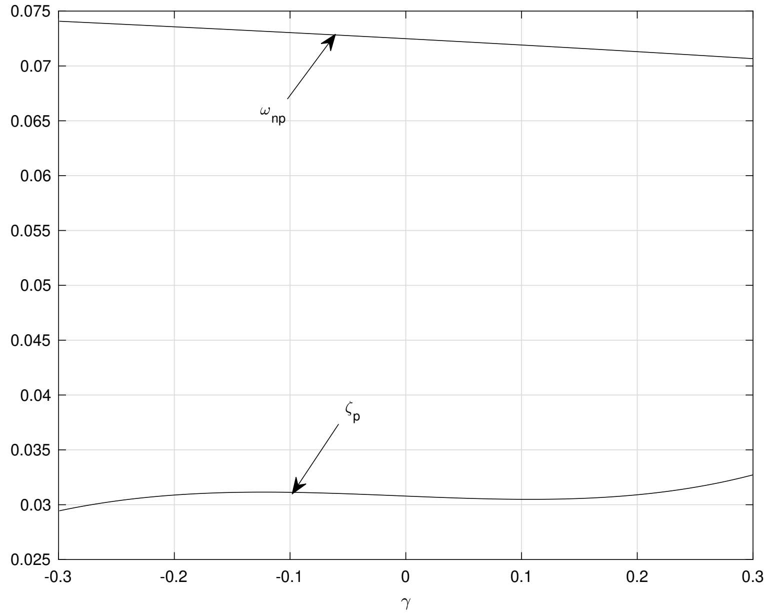

Fig. 4 Variation of the damping ratio ζ_p and frequency ω_{np} of phugoid oscillations with the thermoelastic deflection parameter γ.

parameter γ, the frequency has the largest value when $\gamma = -0.3$, $u_3(h_w) = 0.3h_w$ and is pointing downward, and then is decaying with the growth of γ (Fig. 4). The damping parameter is not a monotonic function. For sufficiently large $|\gamma|$, ζ_p is greater for the displacements pointing upwards.

V. Longitudinal dynamics of a thermoelastic and an elastic beam

For simplicity of analysis, we choose an aerodynamic thermoelastic missile as an air-breathing vehicle, model it as a thermoelastic beam and consider the longitudinal flight regime. Notice that by interchanging x_1 and x_2 and considering deflections of the beam in the x_3-direction as a function of x_2, not x_1, in the same fashion we can model thermoelastic aircraft wings and analyze their longitudinal dynamics.

A. Derivation of coupled flight dynamics and thermoelasticity equations

When it is undeformed, the beam, $\mathcal{V}$, is described by $\{-l/2 < x_1 < l/2, -b/2 < x_2 < b/2, -h/2 < x_3 < h/2\}$, and in the case of a wing, it is assumed that $b >> h$. In the longitudinal regime, $\Phi = 0$, $F_2 = 0$, and $\mathcal{L} = \mathcal{N} = 0$. This results in $V = 0$, $P = R = 0$. We neglect small values of the x_1- and x_2-components of the displacements vector $\mathbf{u}$, $u_1 = u_2 = 0$, assume that the beam density is constant, and take into account that due to the beam symmetry, $\tilde{I}_{12} = \tilde{I}_{23} = 0$. Then the two translational and one rotational equations may be simplified as

$$\dot{U} + QW + \dot{Q}[u_3] + 2Q[\dot{u}_3] = -g \sin\Theta + F_1/m,$$

$$\dot{W} - QU + [\ddot{u}_3] - Q^2[u_3] = g\cos\Theta + F_3/m,$$

$$\tilde{I}_{22}\dot{Q}/m + \tilde{I}'_{22}Q/m - [x_1\ddot{u}_3] + \dot{U}[u_3] + U[\dot{u}_3] + g[u_3]\sin\Theta = \mathcal{M}/m, \tag{77}$$

where $m = lbh\rho$ is the beam mass, $[f]$ is the average value of a function $f(x_1)$ in the interval $-l/2 < x_1 < l/2$,

$$[f] = \frac{1}{l}\int_{-l/2}^{l/2} f\,dx_1, \tag{78}$$

and $\tilde{I}_{22}$ is the corresponding component of the elastic mass moment of inertia given by (11). Since $u_1 = 0$ and due to the symmetry, it becomes

$$\tilde{I}_{22} = \frac{m}{12}(l^2 + h^2) + m[u_3^2], \tag{79}$$

and therefore, $\tilde{I}'_{22} = 2m[u_3\dot{u}_3]$. To write down the thermoelastic potential for the beam, we employ the strain-stress relations of the beam theory (for example, [42], [40])

$$\epsilon_{11} = -x_3\frac{\partial^2 u_3}{\partial x_1^2} = \frac{\sigma_{11}}{E} + \alpha_T T, \quad \epsilon_{22} = \epsilon_{33} = -\frac{\nu\sigma_{11}}{E} + \alpha_T T, \quad \epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0, \tag{80}$$

where σ_{11} is the normal stress in the x_1-direction and ν and E are the Poisson ratio and the Young modulus, respectively, measured under the conditions of isothermal deformation when $T = 0$; they are expressed through the Lamé constants by

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{81}$$

On eliminating σ_{11} from the relations (80) we can determine $\epsilon_{22} = \epsilon_{33}$ and therefore $\epsilon = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$,

$$\epsilon_{22} = \epsilon_{33} = \nu x_3\frac{\partial^2 u_3}{\partial x_1^2} + (1+\nu)\alpha_T T, \quad \epsilon = -(1-2\nu)x_3\frac{\partial^2 u_3}{\partial x_1^2} + 2(1+\nu)\alpha_T T. \tag{82}$$

Now, if we substitute the expressions (80) and (82) into formula (22) for isothermal mechanical energy w_T, we compute

$$\int_{\mathcal{V}} w_T\,d\mathcal{V} = bE\int_{-l/2}^{l/2}\left[\frac{h^3}{24}\left(\frac{\partial^2 u_3}{\partial x_1}\right)^2 + \frac{(1+\nu)\alpha_T^2}{1-2\nu}\int_{-h/2}^{h/2} T^2 dx_3\right]dx_1. \tag{83}$$

This enables us to transform the general formula (23) for the thermoelastic potential and write

$$\mathcal{W} = b\int_{-l/2}^{l/2}\left[\frac{Eh^3}{24}\left(\frac{\partial^2 u_3}{\partial x_1}\right)^2 + \left(\frac{(1+\nu)E\alpha_T^2}{1-2\nu} + \frac{c_T}{2T_a}\right)\int_{-h/2}^{h/2} T^2 dx_3\right]dx_1. \tag{84}$$

On applying the variational principle

$$\delta\mathcal{T} + \delta\mathcal{U} + \delta\mathcal{W} + \delta\mathcal{D} = \int_{\mathcal{A}} (-T\mathbf{n}\cdot\delta\mathbf{S} + f_3\delta u_3)d\mathcal{A} + \int_{\mathcal{V}} \mathbf{\Xi}\cdot\delta\mathbf{S}d\mathcal{V}, \tag{85}$$

we find the variation of the thermoelastic potential

$$\delta\mathcal{W} = \int_{-l/2}^{l/2}\left(\frac{bEh^3}{12}\frac{\partial^4 u_3}{\partial x_1^4} + bE\alpha_T\int_{-h/2}^{h/2}\frac{\partial^2 T}{\partial x_1^2}x_3dx_3\right)\delta u_3dx_1 - \int_{\mathcal{A}} T\mathbf{n}\cdot\delta\mathbf{S}d\mathcal{A} + \int_{\mathcal{V}}\nabla T\cdot\delta\mathbf{S}d\mathcal{V}. \tag{86}$$

This formula is derived by integration by parts and on using the natural homogeneous boundary conditions

$$\frac{\partial\delta u_3}{\partial x_1}\frac{\partial^2 u_3}{\partial x_1^2}\bigg|_{-l/2}^{l/2} = \delta u_3\frac{\partial^3 u_3}{\partial x_1^3}\bigg|_{-l/2}^{l/2} = 0,$$

$$\frac{\partial\delta u_3}{\partial x_1}T\bigg|_{-l/2}^{l/2} = \delta u_3\frac{\partial T}{\partial x_1}\bigg|_{-l/2}^{l/2} = 0. \tag{87}$$

The expression (86) makes possible to evaluate the derivatives of the potential $\mathcal{W}$ with respect to the displacement u_3 and the entropy flow $\mathbf{S}$. From (51) noticing that

$$\dot{\mathbf{V}}\cdot\mathbf{e}_3 = \dot{W}, \quad \mathbf{V}\cdot(\boldsymbol{\omega}\times\mathbf{e}_3) = UQ, \quad \boldsymbol{\omega}\cdot(\dot{\mathbf{u}}\times\mathbf{e}_3) = 0, \quad \frac{1}{2}\omega\tilde{I}^{(3)}\boldsymbol{\omega}^t = bh\rho Q^2\int_{-l/2}^{l/2}u_3dx_1, \tag{88}$$

we find

$$\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{u}_3} - \frac{\partial\mathcal{T}}{\partial u_3} = m(\dot{W} - UQ) + bh\rho\int_{-l/2}^{l/2}(\ddot{u}_3 - Q^2u_3)dx_1. \tag{89}$$

Now, we write down the Euler-Lagrange equations (49) with respect to u_3 and $\mathbf{S}$ to obtain

$$\int_{-l/2}^{l/2}\left[\frac{bEh^3}{12}\frac{\partial^4 u_3}{\partial x_1^4} + bE\alpha_T\frac{\partial^2}{\partial x_1^2}\int_{-h/2}^{h/2}Tx_3dx_3 + bh\rho(\dot{W} - UQ - Q^2u_3 + \ddot{u}_3 - g\cos\Theta) - bf_3\right]dx_1 = 0,$$

$$\int_{\mathcal{V}}\left(\nabla T + \frac{T_a}{k}\dot{\mathbf{S}} - \mathbf{\Xi}\right)d\mathcal{V} = 0. \tag{90}$$

We apply the mean value theorem to a small arbitrary segment in $(-l/2, l/2)$ in the first equation in (90) and a small volume in the body $\mathcal{V}$ in the second one, contract these segment and volume into a point and have the governing equations of a maneuvering thermoelastic beam

$$\ddot{u}_3 + D\frac{\partial^4 u_3}{\partial x_1^4} - Q^2u_3 + \frac{E\alpha_T}{\rho h}\frac{\partial^2}{\partial x_1^2}\int_{-h/2}^{h/2}x_3Tdx_3 = \frac{f_3}{\rho h} + g\cos\Theta - \dot{W} + UQ,$$

$$\nabla^2 T - a_0\dot{T} + \frac{ET_a\alpha_T x_3}{k_T}\frac{\partial^3 u_3}{\partial x_1^2\partial t} = -\frac{\Omega}{k_T}, \quad -l/2 < x_1 < l/2, \quad -h/2 < x_3 < h/2, \quad t > 0. \tag{91}$$

where

$$D = \frac{Eh^2}{12\rho}, \quad a_0 = \frac{1}{k_T}\left(c_T + \frac{2E(1+\nu)T_a\alpha_T^2}{1-2\nu}\right), \tag{92}$$

The coupled thermoelasticity equations (91) model the beam as a one-dimensional structure with respect to elastic deformations and two-dimensional body with respect to heat transfer.

B. Transient thermoelastic beam model of a wing: the longitudinal flight

To study the effect of the flight dynamics characteristics on the elastic deformation and temperature of a wing we consider the thermoelastic equations (91) of a beam $\{-b/2 < x_1 < b/2, 0 < x_2 < l/2, -h/2 < x_3 < h/2\}$. In equations (91), we interchange x_1 and x_2, note that now $u_3 = u_3(x_2,t)$ and $T = T(x_2,x_3,t)$ and assume that the upper side of the beam is subjected to nonzero temperature, $T_+(x_2,t)$, and U, W, Q, and Φ being prescribed. As $|x_3| < h/2$, at the end $x_2 = 0$, the heat flux is chosen to be zero, while at the end $x_2 = l/2$, temperature is zero. The end $x_2 = 0$ is clamped, while the end $x_2 = l/2$ is free. We can write the boundary and initial conditions as

$$u_3(0,t) = \frac{\partial u_3}{\partial x_2}(0,t) = 0, \quad \frac{\partial^2 u_3}{\partial x_2^2}\left(\frac{l}{2},t\right) = \frac{\partial^3 u_3}{\partial x_2^3}\left(\frac{l}{2},t\right) = 0, \quad t \geq 0,$$

$$\frac{\partial T}{\partial x_2}(0,x_3,t) = 0, \quad T\left(\frac{l}{2},x_3,t\right) = 0, \quad |x_3| \leq \frac{h}{2}, \quad t \geq 0,$$

$$T\left(x_2,\frac{h}{2},t\right) = T_+(x_2,t), \quad T\left(x_2,-\frac{h}{2},t\right) = 0, \quad 0 \leq x_2 \leq \frac{l}{2}, \quad t \geq 0,$$

$$u_3(x_2,0) = \frac{\partial u_3}{\partial t}(x_2,0) = 0, \quad T(x_2,x_3,0) = 0, \quad 0 \leq x_2 \leq \frac{l}{2}, \quad |x_3| \leq \frac{h}{2}. \tag{93}$$

It is convenient to rewrite the governing initial-boundary value problem in dimensionless coordinates. Denote

$$\xi = \frac{x_2}{l}, \quad \zeta = \frac{x_3}{h}, \quad \tau = \frac{t}{a_0 h^2}, \quad \hat{h} = \frac{h}{l}, \quad D_0^4 = \frac{E\hat{h}^4 h^2 a_0^2}{12\rho}, \quad \beta = \frac{12E\alpha_T^2 T_0}{k_T a_0}. \tag{94}$$

Here, β is the dimensionless thermoelastic diffusivity, D_0 is the dimensionless thermoelastic bending rigidity, and the generalized coordinate τ is an analogue of the Fourier number $\mathrm{Fo} = k_T c_T^{-1}(l/2)^{-2}t$. Introduce the functions

$$u_3^0(\xi,\tau) = \frac{\hat{h}u_3(x_2,t)}{12l\alpha_T}, \quad T^0(\xi,\zeta,\tau) = T(x_2,x_3,t), \quad T_+^0(\xi,\tau) = T_+(x_2,t),$$

$$p^0(\xi,\tau) = \frac{f_3(x_2,t) + \rho h(g\cos\Theta - \dot{W} + UQ)}{\hat{h}^2 E\alpha_T}, \quad \Omega^0(\xi,\zeta,\tau) = \frac{h^2\Omega(x_2,x_3,t)}{k_T}. \tag{95}$$

It is directly verified that the functions u_3^0, T^0, p^0, and Ω^0 are dimensionless. The new functions u_3^0 and T^0 solve the following initial-boundary value problem. Find the solutions $u_3^0(\xi,\tau)$ and $T^0(\xi,\zeta,\tau)$ of the system of differential equations

$$\frac{\partial^4 u_3^0}{\partial \xi^4}+\frac{1}{D_0^4}\frac{\partial^2 u_3^0}{\partial \tau^2}-Q_0^2 u_3^0+\frac{\partial^2}{\partial \xi^2}\int_{-1/2}^{1/2}T^0\zeta d\zeta=p^0, \quad 0<\xi<\frac{1}{2}, \quad \tau>0,$$

$$\hat{h}^2\frac{\partial^2 T^0}{\partial \xi^2}+\frac{\partial^2 T^0}{\partial \zeta^2}-\frac{\partial T^0}{\partial \tau}+\beta\zeta\frac{\partial^3 u_3^0}{\partial \xi^2\partial \tau}=-\Omega^0, \quad 0<\xi<\frac{1}{2}, \quad |\zeta|<1/2, \quad \tau>0, \tag{96}$$

subject to the boundary conditions

$$u_3^0(0,\tau)=\frac{\partial u_3^0}{\partial \xi}(0,\tau)=0, \quad \frac{\partial^2 u_3^0}{\partial \xi^2}\left(\frac{1}{2},\tau\right)=\frac{\partial^3 u_3^0}{\partial \xi^3}\left(\frac{1}{2},\tau\right)=0, \quad \tau\geq 0,$$

$$\frac{\partial T^0}{\partial \xi}(0,\zeta,t)=0, \quad T^0\left(\frac{1}{2},\zeta,t\right)=0, \quad |\zeta|\leq\frac{h}{2}, \quad t\geq 0,$$

$$T^0\left(\xi,\frac{1}{2},\tau\right)=T_+^0(\xi,\tau), \quad T^0\left(\xi,-\frac{1}{2},\tau\right)=0, \quad 0\leq\xi\leq\frac{1}{2}, \quad \tau\geq 0, \tag{97}$$

and the initial conditions

$$u_3^0(\xi,0)=\frac{\partial u_3^0}{\partial \tau}(\xi,0)=0, \quad T^0(\xi,\zeta,0)=0, \quad 0\leq\xi\leq\frac{1}{2}, \quad |\zeta|\leq\frac{1}{2}. \tag{98}$$

Here, Q_0 is the dimensionless pitch rate parameter

$$Q_0=\frac{2lQ}{\hat{h}}\sqrt{\frac{3\rho}{E}}. \tag{99}$$

The case $Q_0=0$ of the model problem (96) to (98) of a composite beam was analyzed in [40]. That solution can be utilized in the case $Q_0\neq 0$ as well if in the representation formulas [40] we accept $l_\pm=l/2$, $c_\pm=1/2$, w_s is the Laplace transform of u_3^0, and the term $Q_0^2 w_s$ is subtracted from the left-hand sides in (3.11) and from the denominator in formulas (3.26) and (3.32) in [40]. In addition, in formulas (3.37) and (3.38) in [40], the terms $s^2(c_\pm/D_\pm/\gamma_\nu)^4$ and $s^2/D_\pm^4$ need to be replaced by $(s^2/D_\pm^4+Q_0^2)(2\gamma_\nu)^{-4}$ and $s^2/D_\pm^4+Q_0^2$, respectively.

For numerical tests we take $T_+^0(\xi,\tau)=A_0(1/4-\xi^2)$, $A_0=10$, $\Omega^0=0$, $p^0(\xi,\tau)=PH(\tau)$, $H(\tau)$ is the Heaviside function, $P=1$, $\hat{h}=0.1$, $D_0=1$, and we assume that Q is constant. For $\tau=1$ and $\beta=5$, the elastic dimensionless displacements $u_3^0(\xi,\tau)$ for the values 0, 5 and 10 of the parameter Q_0 are displayed in Fig. 5. It is seen that when the dimensionless pitch rate Q_0 increases and the thermoelasticity coupling parameter (thermodiffusivity) β given by (94) is sufficiently large, the deflection of the wing tip increases significantly. The impact of the dimensionless pitch rate Q_0 on the temperature function T^0 for $\zeta=0$ and the same values of τ and β is illustrated in Fig. 6. Due to the

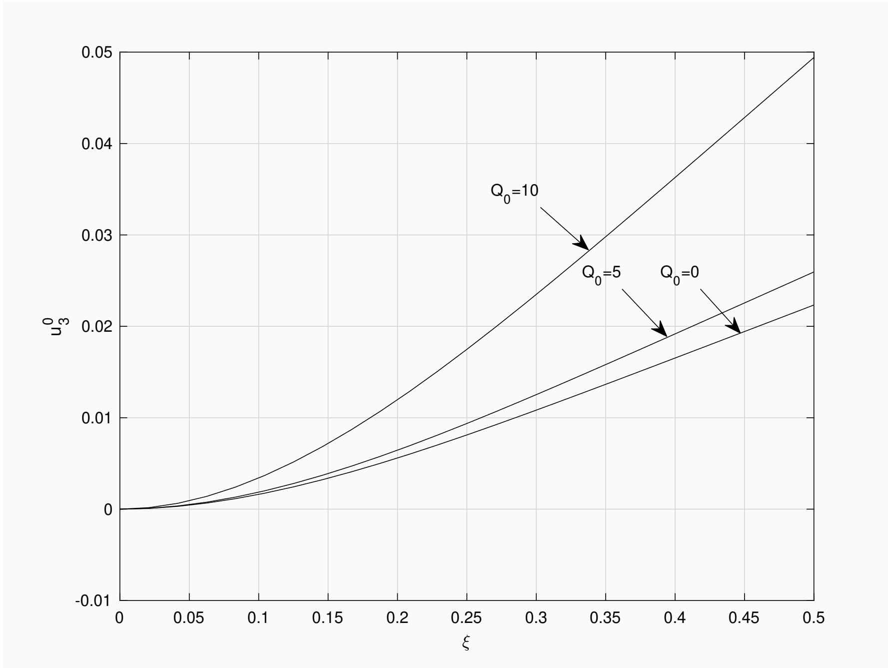

Fig. 5 Variation of the dimensionless displacement $u_3^0(\xi,\tau)$ with ξ for different values of the parameter Q_0 when $\tau = 1$ and $\beta = 5$.

form of the function $T_+^0(\xi,\tau) = A_0(1/4 - \xi^2)$ prescribed in the boundary condition (97, the qualitative behavior of the temperature function T^0 in the middle line $\zeta = 0$ along the ξ-line was expected. However, this figure clarifies how it varies quantitatively when the pitch parameter increases. The results of evaluating the dimensionless displacements $u_3^0(\xi,\tau)$ versus ξ for $\tau = 1$, $\beta = 0, 3$, and 10 when Q_0 is either 0 or 5 are summarized in Fig. 7. The curves show the effect of the thermodiffusivity β and the flight dynamics parameter Q_0 acting in concert. When both parameters are significant, the deflection of the wing tip is also noticeable. As seen in Fig.8, when $\beta = 0$, temperature T^0 is practically invariant when the dimensionless pitch rate Q_0 varies the two indistinguishable curves for $\beta = 0$ and $Q_0 = 0$ and $Q_0 = 5$) . When the diffusivity β does not vanish, temperature varies with Q_0. However, even in this case the variation is not significant. The dimensionless displacement u_3^0 of the beam end point $\xi = 0.5$ versus the dimensionless time $\tau = t/a_0/h^2$ for some values of β and Q_0 is plotted in Fig. 9. It is seen that the displacement grows when β or Q_0 are growing. It is also becomes evident that, in comparison to the case when both coupling parameters vanish, the relaxation time is notably larger for their nonzero values.

C. Longitudinal flight of an elastic beam

Assume now that there is no variation of temperature and we are dealing with an elastic beam. In this case the number of equations reduces to four, and we have the following longitudinal equations of motion of an elastic beam:

$$\dot{U} + QW + \dot{Q}[u_3] + 2Q[\dot{u}_3] = -g\sin\Theta + F_1/m,$$

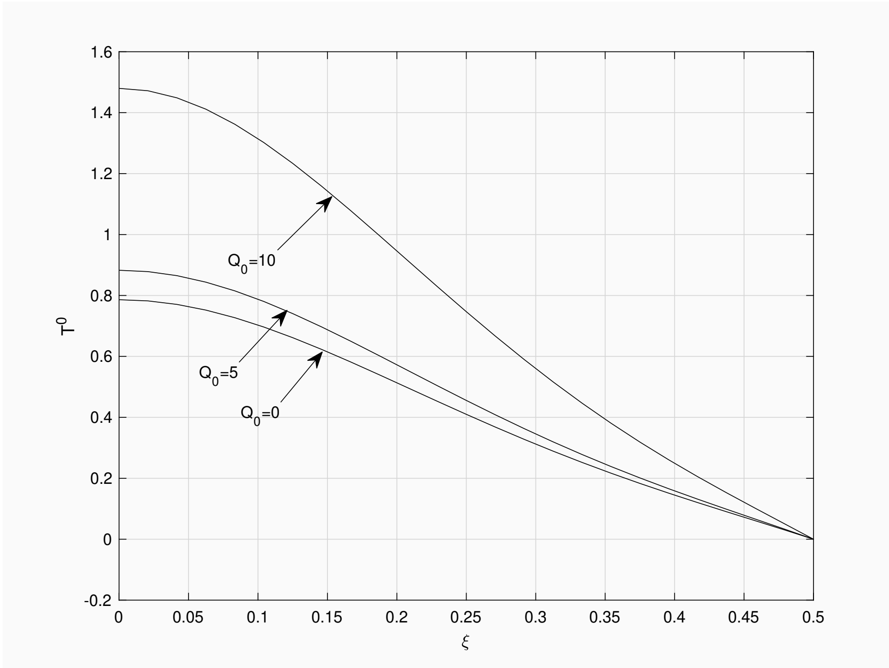

Fig. 6 Variation of temperature $T^0(\xi, \zeta, \tau)$ with ξ for different values of the parameter Q_0 when $\zeta = 0$, $\tau = 1$, and $\beta = 5$.

$$\dot{W} - QU + [\ddot{u}_3] - Q^2[u_3] = g\cos\Theta + F_3/m,$$

$$\left(\frac{l^2+h^2}{12} + [u_3^2]\right)\dot{Q} + 2[u_3\dot{u}_3]Q - [x_2\ddot{u}_3] + \dot{U}[u_3] + U[\dot{u}_3] + g[u_3]\sin\Theta = \frac{\mathcal{M}}{m},$$

$$\ddot{u}_3 + D\frac{\partial^4 u_3}{\partial x_2^4} - Q^2 u_3 = \frac{f_3}{\rho h} + g\cos\Theta - \dot{W} + UQ, \quad -l/2 < x_2 < l/2, \quad t > 0. \tag{100}$$

Analyze the fourth equation. As in the thermoelastic case, we consider the right half of the beam whose left end $x_2 = 0$ is clamped, while the right end $x_2 = l/2$ is free. The initial conditions are taken to be homogeneous. We have

$$\ddot{u}_3 + D\frac{\partial^4 u_3}{\partial x_2^4} - Q^2 u_3 = f, \quad 0 < x_2 < l/2, \quad t > 0,$$

$$u_3(0,t) = \frac{\partial u_3}{\partial x_2}(0,t) = 0, \quad \frac{\partial^2 u_3}{\partial x_2^2}\left(\frac{l}{2}, t\right) = \frac{\partial^3 u_3}{\partial x_2^3}\left(\frac{l}{2}, t\right) = 0, \quad t \geq 0,$$

$$u_3(x_2, 0) = \dot{u}_3(x_2, 0) = 0, \quad 0 < x_2 < l/2, \tag{101}$$

where

$$f = f_3/(\rho h) + g\cos\Theta - \dot{W} + UQ. \tag{102}$$

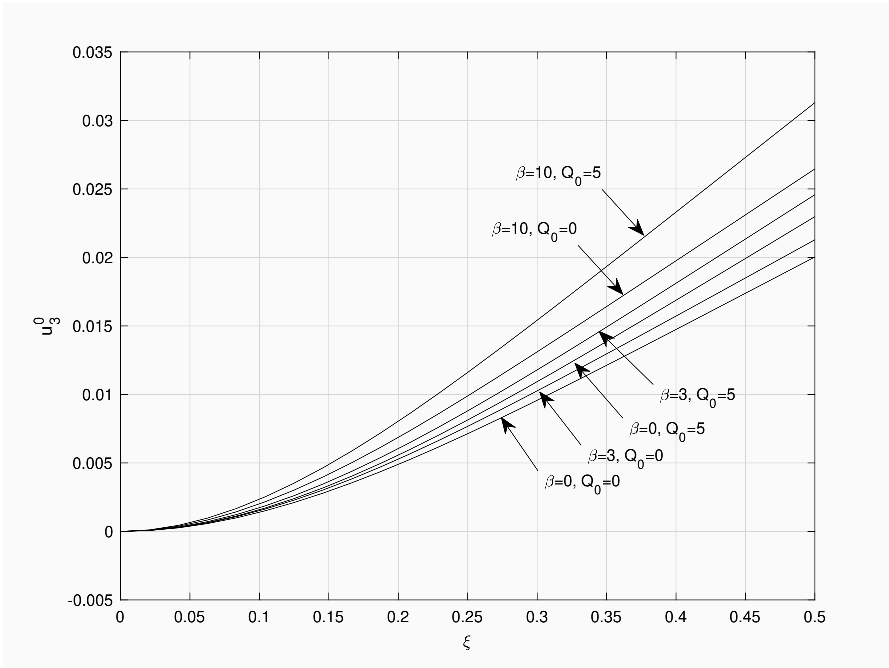

Fig. 7 Variation of the dimensionless displacement $u_3^0(\xi,\tau)$ with ξ for different values of the parameter β when $\tau = 1$ and $Q_0 = 0$ or $Q_0 = 5$.

The problem can be easily solved by the separation of variables method. To find the eigenfunctions, write $u_3(x_2,t) = \tilde{X}(x_2)\tilde{T}(t)$. The eigenvalue problem has the standard form

$$\tilde{X}^{IV} - \kappa^2\tilde{X} = 0, \quad 0 < x_2 < l/2, \quad \tilde{X}(0) = \tilde{X}'(0) = 0, \quad \tilde{X}''(l/2) = \tilde{X}'''(l/2) = 0. \tag{103}$$

The eigenvalues, κ_i, are the roots of the equation $\cosh\sqrt{\kappa}l/2\cos\sqrt{\kappa}l/2 = -1$, $0 < \kappa_1 < \kappa_2 < \dots$. The associated eigenfunctions are

$$\tilde{X}_i(x_2) = \sinh\sqrt{\kappa_i}x_2 - \sin\sqrt{\kappa_i}x_2 - \frac{\sinh\sqrt{\kappa_i}l/2 + \sin\sqrt{\kappa_i}l/2}{\cosh\sqrt{\kappa_i}l/2 + \cos\sqrt{\kappa_i}l/2}(\cosh\sqrt{\kappa_i}x_2 - \cos\sqrt{\kappa_i}x_2). \tag{104}$$

The functions $\tilde{T}_i(t)$ are the solutions of the equation

$$\frac{d^2}{dt^2}\tilde{T}_i(t) + (\kappa_i^2 D - Q^2)\tilde{T}_i(t) = 0, \quad t > 0. \tag{105}$$

If $Q > \kappa_1\sqrt{D}$, then the function $T_1(t)$ is growing exponentially as $t \to \infty$, and the motion becomes unstable. We remark that the stability analysis should be modified accordingly if the aerodynamic force f_3 in (102) depends on at least one of the three functions u_3, $\dot{u}_3$, and $\ddot{u}_3$ that is

$$f_3 = f_3|_0 + \left.\frac{\partial f_3}{\partial u_3}\right|_0 u_3 + \left.\frac{\partial f_3}{\partial \dot{u}_3}\right|_0 \dot{u}_3 + \left.\frac{\partial f_3}{\partial \ddot{u}_3}\right|_0 \ddot{u}_3, \tag{106}$$

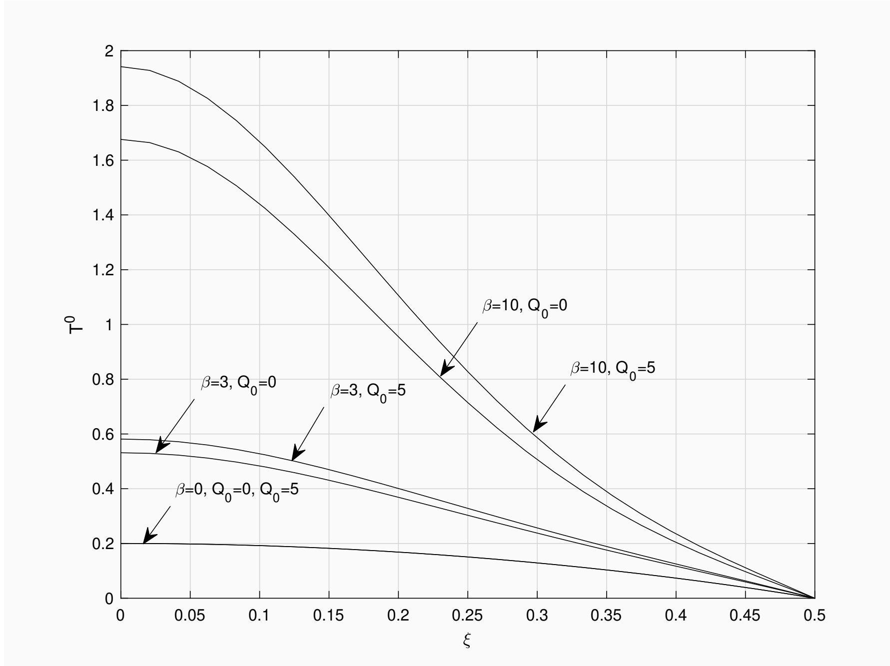

Fig. 8 Variation of temperature $T^0(\xi,\zeta,\tau)$ with ξ for different values of the parameter β when $\zeta = 0$, $\tau = 1$, and $Q_0 = 0$ or $Q_0 = 5$.

where, as in Section IVC, the subscript 0 means the value of the corresponding function in the initial state conditions. At the same time, in real situations, the coefficients $\left.\frac{\partial f_3}{\partial u_3}\right|_0$, $\left.\frac{\partial f_3}{\partial \dot{u}_3}\right|_0$, and $\left.\frac{\partial f_3}{\partial \ddot{u}_3}\right|_0$ are insignificant and may be dropped.

We also note that if the deflections are large, then instead of the Euler-Bernoulli beam model the nonlinear model [43] of large deflections of beams and plates can be employed.

VI. Flight dynamics of a thermoelastic body in terms of structural modes and nodal temperature

Represent the elastic and entropy displacement vectors as [22]

$$\mathbf{u} = \sum_{i=1}^{n} \mathbf{u}_i(x_1,x_2,x_3)q_i(t), \quad \mathbf{S} = \sum_{i=1}^{n} \mathbf{S}_i(x_1,x_2,x_3)q_i(t), \tag{107}$$

where $q_i(t)$ are n generalized coordinates. According to Onsager's theorem the thermoelastic potential and the dissipative function are expressible through positive definite quadratic forms [22], and

$$\mathcal{W} = \frac{1}{2}\sum_{i,j}^{n} a_{ij}q_iq_j, \quad \mathcal{D} = \frac{1}{2}\frac{d}{dt}\sum_{i,j}^{n} b_{ij}q_iq_j. \tag{108}$$

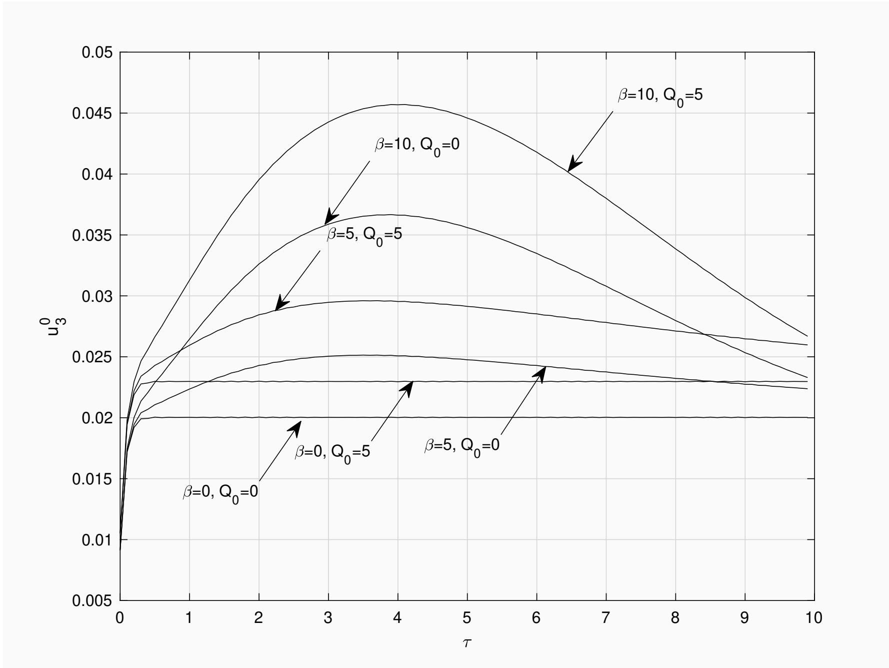

Fig. 9 Variation of the dimensionless displacement u_3^0 at the right endpoint $\xi = 0.5$ with the dimensionless time parameter τ for different values of the parameters β and Q_0.

Denote by $q_{n+1} = X$, $q_{n+2} = Y$, $q_{n+3} = Z$, the inertial coordinates, and $q_{n+4} = \Phi$, $q_{n+5} = \Theta$, $q_{n+6} = \Psi$, the Euler angles. Write down the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}_i} - \frac{\partial \mathcal{T}}{\partial q_i} + \frac{\partial \mathcal{U}}{\partial q_i} + \frac{\partial \mathcal{W}}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial q_i} = Q_i, \quad i = 1, \dots, n, \dots, n+6, \tag{109}$$

where Q_i are generalized forces. To derive the longitudinal and rotational equations, we use formulas (18) and (21) for the kinetic and gravitational energies and (108) for the thermoelastic potential and the dissipative function, employ the variational principle with respect to two vectors $(q_{n+1}, q_{n+2}, q_{n+3})$ and $(q_{n+4}, q_{n+5}, q_{n+6})$. The first vector brings us to the three translational equations. The use of the second vector of the generalized coordinates yields the three rotational equations. We write down the resulting translational and rotational equations in the vector form

$$m(\dot{\mathbf{V}} + \omega \times \mathbf{V}) + \sum_{i=1}^{n} \{\hat{\mathbf{u}}_i \ddot{q}_i + 2(\omega \times \hat{\mathbf{u}}_i)\dot{q}_i + [\dot{\omega} \times \hat{\mathbf{u}}_i + \omega \times (\omega \times \hat{\mathbf{u}}_i)]q_i\} = m\tilde{\mathbf{g}} + \mathbf{F},$$

$$\frac{d}{dt}[\tilde{I}\omega^t] + \omega \times \tilde{I}\omega^t + \sum_{i=1}^{n} \{\hat{\mathbf{c}}_i \ddot{q}_i + (\hat{\mathbf{u}}_i \times \mathbf{V} - \hat{\mathbf{c}}_i \times \omega)\dot{q}_i + [\hat{\mathbf{u}}_i \times \dot{\mathbf{V}} - (\hat{\mathbf{u}}_i \cdot \omega)\mathbf{V} + (\mathbf{V} \cdot \omega)\hat{\mathbf{u}}_i]q_i\}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} [\tilde{\mathbf{c}}_{ij}(\dot{q}_i \dot{q}_j + q_i \ddot{q}_j) - (\tilde{\mathbf{c}}_{ij} \times \omega) q_i \dot{q}_j] = \mathbf{M}. \tag{110}$$

Here,

$$\hat{\mathbf{u}}_i = \int_{\mathcal{V}} \mathbf{u}_i(\mathbf{r})\rho d\mathcal{V}, \quad \hat{\mathbf{c}}_i = \int_{\mathcal{V}} \mathbf{r} \times \mathbf{u}_i \rho d\mathcal{V}, \quad \tilde{\mathbf{c}}_{ij} = \int_{\mathcal{V}} \mathbf{u}_i \times \mathbf{u}_j \rho d\mathcal{V}, \tag{111}$$

Finally, we take the n generalized coordinates q_k, $k = 1, 2, \ldots, n$, use the Euler-Lagrange equation and the expressions (108) for the thermoelastic potential and the dissipative function, introduce the notations

$$J_k = \begin{pmatrix} d'_{23k} & -d''_{12k} & -d''_{13k} \\ -d''_{12k} & d'_{13k} & -d''_{23k} \\ -d''_{13k} & -d''_{23k} & d'_{12k} \end{pmatrix}, \quad K_{ki} = \begin{pmatrix} e'_{23ki} & -e''_{12ki} & -e''_{13ki} \\ -e''_{12ki} & e'_{13ki} & -e''_{23ki} \\ -e''_{13ki} & -e''_{23ki} & e'_{12ki} \end{pmatrix}, \quad \mathbf{u} = \sum_{i=1}^{n} \begin{pmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \end{pmatrix} q_i,$$

$$d'_{ijk} = \int_{\mathcal{V}} (x_i u_{ik} + x_j u_{jk}) \rho d\mathcal{V}, \quad d''_{ijk} = \frac{1}{2} \int_{\mathcal{V}} (x_i u_{jk} + x_j u_{ik}) \rho d\mathcal{V},$$

$$e'_{ijkl} = \int_{\mathcal{V}} (u_{ik} u_{il} + u_{jk} u_{jl}) \rho d\mathcal{V}, \quad e''_{ijkl} = \frac{1}{2} \int_{\mathcal{V}} (u_{il} u_{jk} + u_{jl} u_{ik}) \rho d\mathcal{V}, \tag{112}$$

and apply the differentiation formula

$$\frac{\partial}{\partial q_k} \frac{1}{2} \omega \tilde{I} \omega^t = \omega J_k \omega^t + \sum_{i=1}^{n} \omega K_{ki} \omega^t q_i, \quad k = 1, \ldots, n. \tag{113}$$

Eventually, we have the following n structural dynamics equations coupled with the flight dynamics equations:

$$\sum_{i=1}^{n} c^*_{ik} \ddot{q}_i + \sum_{i=1}^{n} b_{ik} \dot{q}_i + \sum_{i=1}^{n} (\dot{\omega} \cdot \tilde{\mathbf{c}}_{ik} - \omega K_{ik} \omega^t + a_{ik}) q_i = Q_k - \dot{\mathbf{V}} \cdot \hat{\mathbf{u}}_k - \dot{\omega} \cdot \hat{\mathbf{c}}_k + \omega J_k \omega^t + \mathbf{V} \cdot (\omega \times \hat{\mathbf{u}}_k),$$

$$k = 1, 2, \ldots, n, \quad c^*_{ik} = \int_{\mathcal{V}} \mathbf{u}_i \cdot \mathbf{u}_k \rho d\mathcal{V}. \tag{114}$$

VII. Conclusion

In this paper an expression for the kinetic energy of aircraft has been obtained in an undeformed body frame under the assumptions that the aircraft is a thermoelastic body, the mass of the aircraft is constant, and the earth is the inertial reference. The origin of the orthogonal body frame is the center of mass, the x_1-axis is chosen along the fuselage forward (in the linearized longitudinal flight dynamics model, the x_1-axis is aligned with the equilibrium direction of the translational velocity vector), the x_2-axis is out in the direction of the right wing, and the x_3-axis is directed downward. The location of this reference frame does not vary with respect to the original undeformed body. The kinetic energy has also been determined in the mean axes frame defined as a right-hand orthogonal system whose axes have to satisfy certain six additional conditions. It has been shown that the employment of the mean axes frame requires satisfying six conditions which determine the location of the origin and the direction of the axes. These conditions might be fulfilled by choosing the three components of the rigid translation and the rigid rotation vectors. However, since the elastic displacements are involved in these conditions, it is not a trivial task: to determine the elastic displacement in the mean axes frame one needs to solve the associated dynamic problem of elasticity. Its setup requires fixing the axes first, and its location is unknown at this stage. On the other hand, the undeformed body frame is a natural choice for the reference frame in the associated dynamic problem of elasticity.

In the undeformed body frame, a set of six translational and rotational flight dynamics equations of thermoelastic aircraft coupled with the elastic displacements and a set of four thermoelasticity partial differential equations coupled with the translational and rotational velocities have been derived. At certain regimes, the scale of the terms involved might differ, and simplifications of the ten equations derived are possible. Following the classical decoupling procedure of the six rigid body flight dynamics equations into the longitudinal and lateral systems their analogs in the thermoelastic case were considered. The longitudinal dynamics regime is governed by three flight dynamics and three thermoelastic equations. We have shown that in the lateral dynamics regime all the six flight dynamics equations are coupled unless the elastic displacements are disregarded. In addition, in terms of the longitudinal stability derivatives, linearized longitudinal dynamics equations have been derived. Note that disregarding the quadratic terms RV, PV, PR, P^2, and R^2 not completely linearizes the system of equations for the presence of products of the flight dynamics modes and the integral characteristics of the elastic displacements in the flight dynamics equations and products of the rotational velocities and elastic displacements in the thermoelasticity equations. A method of successive approximations can be used to approximately solve the system. At the first step, the elastic displacements are assumed to be known (their initial values for example), and the linearized flight dynamics equations are solved. At the second step, with the values of the rotational velocities recovered, the linear thermoelasticity equations are solved, and the values of the elastic displacements and temperature are updated. Termination of this process after a reasonable number of steps approximates the solution to the system. The first step of this procedure, when the thermoelastic displacements are obtained from the thermoelasticity equations and considered as prescribed, was implemented for a thermoelastic analog of Bakelock's model for a transport. We analyzed the homogeneous system associated with the linearized longitudinal equations and studied the effects of the wing deflection on the frequencies and damping ratios of short-period and phugoid oscillations.

We emphasize that the equations we have derived couple the flight dynamics equations with the thermoelasticity equations. The heat equation does not include the flight dynamics modes explicitly. However, temperature and the translational and rotational velocities are coupled by means of the elastic displacements which are present in the flight dynamics, elasticity and the heat equations. We also note that the characteristics of the aerodynamic flow surrounding aircraft are not taken into account by this model. Coupling the governing fluid dynamics equations with the flight dynamics and thermoelasticity equations is a challenging and interesting problem.

To develop a simple model of a missile or a wing of aircraft, a thermoelastic beam-like body in the longitudinal flight regime was considered. The body was modeled as an elastic one-dimensional beam in response to elastic deformation and as a two-dimensional body in response to heat flow. In this model, the governing thermoelastic integro-differential equation for the beam deflection is coupled with the heat equation and the flight dynamics equations. Our numerical tests revealed that both coupling parameters, the dimensionless pitch rate and the dimensionless thermoelastic diffusivity, may strongly affect the elastic deflections. We also analyzed the coupled equations of elasticity and the flight dynamics for a purely elastic beam and established a simple instability criterion in terms of the first eigenvalue of the associated eigenvalue problem and the pitch rate.

Also, in this paper, a system of $n+6$ equations of structural flight dynamics of a thermoelastic body in terms of the structural modes and nodal temperature used in the finite element method have been derived. The system was written for the generalized coordinates used in the representations of the elastic displacements and the entropy flow vector. the have been derived. The system was written for the generalized coordinates used in the representations of the elastic displacements and the entropy flow vector.

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