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A GENERALIZATION OF THE MADER-HELGASON INVERSION FORMULAS FOR RADON TRANSFORMS

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Abstract. In 1927, Philomena Mader derived elegant inversion formulas for the hyperplane Radon transform on $\mathbb{R}^n$. These formulas differ from the original ones by Radon and seem to be forgotten. We generalize Mader's formulas to totally geodesic Radon transforms in any dimension on arbitrary constant curvature space. Another new interesting inversion formula for the k-plane transform was presented in the recent book "Integral geometry and Radon transform" by S. Helgason. We extend this formula to arbitrary constant curvature space. The paper combines tools of integral geometry and complex analysis.

1. Introduction

Let $g(\theta, s) = \int f(x)\, dm(x)$ be the Radon transform of a sufficiently good function f on the Euclidean space $\mathbb{R}^n$. Here, the integral is taken over a hyperplane $x \cdot \theta = s$, where $x \cdot \theta = x_1\theta_1 + \cdots + x_n\theta_n$ is the usual inner product, θ is a point of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$, and $dm(x)$ stands for the volume element on the hyperplane $x \cdot \theta = s$. Mader's inversion formulas [5] have the following elegant form:

$$f(x) = A_0 \frac{\partial^n}{\partial t^n}\, F_0(x,t)\Big|_{t=0} \qquad \text{if } n \text{ is even,} \tag{1.1}$$

$$f(x) = A_1 \frac{\partial^n}{\partial t^n}\, F_1(x,t)\Big|_{t=0} \qquad \text{if } n \text{ is odd,} \tag{1.2}$$

where

$$A_0 = \frac{(-1)^{(n-2)/2}}{\pi (n-2)!\, \sigma_{n-2}}, \quad A_1 = \frac{(-1)^{(n-1)/2}}{2(n-2)!\, \sigma_{n-2}},$$

$$F_0(x,t) = \int_{-\infty}^{\infty} G(x,s) \log|s-t|\, ds, \quad F_1(x,t) = \int_{-\infty}^{\infty} G(x,s)\, \mathrm{sgn}(s-t)\, ds,$$

$$G(x,s) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} g(\theta, s + x\cdot\theta)\, d\sigma(\theta),$$

$d\sigma(\theta)$ is the surface element, and σ_{n-1} is the surface area of $\mathbb{S}^{n-1}$.

An interesting feature of Mader's result is that she used tools of complex analysis (unlike most of the other authors in the area).

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In the present paper we derive more general inversion formulas, having the same nice structure as (1.1) and (1.2) and applicable to totally geodesic Radon transforms in *any* dimension on *arbitrary* constant curvature space. We also generalize one recent inversion formula by S. Helgason, which has a similar flavor and is described below.

Let X be either the Euclidean space $\mathbb{R}^n$, the n-dimensional hyperbolic space $\mathbb{H}^n$, or the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. We denote by Ξ the set of all k-dimensional totally geodesic submanifolds of X, $1 \le k \le n-1$. The totally geodesic Radon transform of a function f on X is defined by

$$(Rf)(\xi) = \int_\xi f(x)\, d_\xi x, \qquad \xi \in \Xi, \tag{1.3}$$

where $d_\xi x$ stands for the corresponding canonical measure; see [4] for details. Let $d(x,\xi)$ be the geodesic distance between $x \in X$ and $\xi \in \Xi$. We introduce the *distance function*

$$\rho(x,\xi) = \begin{cases} d(x,\xi) & \text{if } X = \mathbb{R}^n, \\ \sin d(x,\xi) & \text{if } X = \mathbb{S}^n, \\ \sinh\, d(x,\xi) & \text{if } X = \mathbb{H}^n, \end{cases} \tag{1.4}$$

and the corresponding shifted dual Radon transform

$$(R_r^*\varphi)(x) = \int_{\rho(x,\xi)=r} \varphi(\xi)\, d\mu(\xi), \qquad x \in X, \quad r > 0. \tag{1.5}$$

The last integral averages $\varphi(\xi)$ over all $\xi \in \Xi$ satisfying $\rho(x,\xi)=r$. After pioneering papers by P. Funk and J. Radon, it is known [3, 4, 7, 8, 9] that f can be explicitly reconstructed from $\varphi = Rf$ by applying differentiation of order $k/2$ (if k is odd this differentiation is fractional and interpreted in a suitable sense) to $R_r^*\varphi$ in the r^2-variable. However, it was recently proved by Helgason [4, p. 116] that in the case $X = \mathbb{R}^n$ and k even one can differentiate in r, rather than in r^2, and obtain the following result:

$$f(x) = c_k \left[\partial_r^k (R_r^* Rf)(x)\right]_{r=0}, \tag{1.6}$$

where c_k is a constant depending only on k. The constant c_k was not evaluated in [4].

We compute this constant explicitly and generalize (1.6) for an arbitrary constant curvature space X and arbitrary $1 \le k \le n-1$.

Notation. For the sake of simplicity, throughout the paper, we assume f to be infinitely differentiable. If X is noncompact, we also assume that f is rapidly decreasing together with derivatives of all orders. The letter c stands for a constant to be specified in every occurrence; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

2. Main results

Let $\rho = \rho(x,\xi)$ be the distance function (1.4), $1 \le k \le n-1$. For $r > 0$ we denote

$$(L_r^*\varphi)(x) = \int_\Xi \varphi(\xi)\, \rho^{k+1-n}\, \mathrm{sgn}(\rho - r)\, d\xi, \tag{2.1}$$

$$(\tilde{L}_r^*\varphi)(x) = \int_\Xi \varphi(\xi)\, \rho^{k+1-n}\, \log|\rho^2 - r^2|\, d\xi. \tag{2.2}$$

Theorem 2.1. *Let $\varphi = Rf$.*

(i) *If k is even, then*

$$\partial_r^{k+1}(L_r^*\varphi)(x)\Big|_{r=0} = d_X\, f(x), \tag{2.3}$$

where

$$d_X = \begin{cases} 2(-1)^{(k+2)/2}\sigma_{n-k-1}\sigma_{k-1}\,(k-1)! & \text{if } X = \mathbb{R}^n, \mathbb{H}^n, \\ 2\,\sigma_{n-k-1}\,\sigma_k\,\sigma_{k-1}\,(k-1)!/\sigma_n & \text{if } X = \mathbb{S}^n. \end{cases}$$

(ii) *If k is odd, then*

$$\partial_r^{k+1}(\tilde{L}_r^*\varphi)(x)\Big|_{r=0} = \tilde{d}_X\, f(x), \tag{2.4}$$

where

$$\tilde{d}_X = \begin{cases} \pi\,(-1)^{(k-1)/2}\sigma_{n-k-1}\,\sigma_{k-1}\,(k-1)! & \text{if } X = \mathbb{R}^n, \mathbb{H}^n, \\ 2\pi(-1)^{(k-1)/2}\,\sigma_{n-k-1}\,\sigma_k\,\sigma_{k-1}\,(k-1)!/\sigma_n & \text{if } X = \mathbb{S}^n. \end{cases}$$

Theorem 2.2. *Let $\varphi = Rf$. If k is even, then*

$$\partial_r^k \lambda_X(r)(R_r^*\varphi)(x)\Big|_{r=0} = c_X\, f(x), \tag{2.5}$$

where

$$\lambda_X(r) = \begin{cases} 1 & \text{if } X = \mathbb{R}^n, \\ (1-r^2)^{(k-1)/2} & \text{if } X = \mathbb{S}^n, \\ (1+r^2)^{(k-1)/2} & \text{if } X = \mathbb{H}^n, \end{cases}$$

$$c_X = \begin{cases} (-1)^{k/2}(k-1)!\,\sigma_{k-1} & \text{if } X = \mathbb{R}^n, \mathbb{H}^n, \\ 2(-1)^{k/2}(k-1)!\,\sigma_{k-1} & \text{if } X = \mathbb{S}^n. \end{cases}$$

Theorem 2.1 includes Mader's result for $X = \mathbb{R}^n$, $k = n-1$. Theorem 2.2 gives a precise form to the aforementioned result by S. Helgason and extends it to any constant curvature space X.

The following lemma plays a key role in our consideration.

Lemma 2.3. *Let k be an odd positive integer,*

$$\psi_k(u) = \int_0^1 (1-v^2)^{k/2-1}\log|u^2 - v^2|\, dv. \tag{2.6}$$

Then

$$\psi_k(u) = \begin{cases} P_{k-1}(u) & \text{if } 0 < u < 1, \\ P_{k-1}(u) + \pi(-1)^{(k-1)/2}\Theta(u) & \text{if } u > 1, \end{cases} \tag{2.7}$$

where $P_{k-1}(u)$ is a polynomial of degree $k-1$, $\Theta(u) = \int_1^u (v^2-1)^{k/2-1}\, dv$.

This statement is presented in Mader's paper [5] in a slightly different form. We obtain (2.7) as a consequence of a more general result, which might be of independent interest.

Sections 3, 4, 5 contain the proof of Theorems 2.1 and 2.2 for the cases $X = \mathbb{R}^n, \mathbb{S}^n$, and $\mathbb{H}^n$, respectively. Lemma 2.3 is proved in Section 6.

3. The case $X = \mathbb{R}^n$

We recall basic definitions. Let $G_{n,k}$ $(1 \le k \le n-1)$ be the Grassmann manifold of k-dimensional linear subspaces ζ of $\mathbb{R}^n$ and let $\Xi = \mathcal{G}_{n,k}$ be the affine Grassmann manifold of all nonoriented k-planes ξ in $\mathbb{R}^n$. Each k-plane ξ is parameterized by the pair (ζ, u), where $\zeta \in G_{n,k}$ and $u \in \zeta^\perp$ (the orthogonal complement to ζ in $\mathbb{R}^n$). The manifold $\mathcal{G}_{n,k}$ will be endowed with the product measure $d\xi = d\zeta du$, where $d\zeta$ is the $SO(n)$-invariant measure on $G_{n,k}$ of total mass 1, and du denotes the usual volume element on $\zeta^\perp$.

The k-plane transform $(Rf)(\xi)$ of a function f on $X = \mathbb{R}^n$ is defined by

$$(Rf)(\xi) = \int_\zeta f(u+v)dv, \qquad \xi = (\zeta, u) \in \mathcal{G}_{n,k}. \tag{3.1}$$

Throughout this section we assume that f is a rapidly decreasing infinitely differentiable function.

The shifted dual k-plane transform is defined by

$$(R_r^*\varphi)(x) = \int_{SO(n)} \varphi(\gamma\zeta_0 + x + r\gamma e_n)\, d\gamma, \tag{3.2}$$

where ζ_0 is an arbitrary fixed k-dimensional subspace of $\mathbb{R}^n$, and $e_n = (0, \dots, 0, 1)$. This integral averages φ over all k-planes at distance r from x. The case $r = 0$ corresponds to the usual dual Radon transform [4].

The spherical mean of a locally integrable function f at a point $x \in \mathbb{R}^n$ is defined by

$$(\mathcal{M}_t f)(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} f(x + t\theta)d\,\theta, \qquad t > 0, \tag{3.3}$$

so that $\lim\limits_{t\to 0}(\mathcal{M}_t f)(x) = f(x)$.

Lemma 3.1. *The following equalities hold:*

$$(R_r^* Rf)(x) = \sigma_{k-1} \int_r^\infty (\mathcal{M}_t f)(x)(t^2 - r^2)^{k/2-1} t\, dt; \tag{3.4}$$

$$\int_{\mathcal{G}_{n,k}} \varphi(\xi)\, a(\rho(x,\xi))\, d\xi = \sigma_{n-k-1} \int_0^\infty r^{n-k-1}\, a(r)\, (R_r^*\varphi)(x)\, dr. \tag{3.5}$$

It is assumed that f, φ and a are measurable functions, for which either side of the corresponding equality exists in the Lebesgue sense.

Equality (3.4) can be found in [4, p. 115, formula (8)]. Both equalities were proved in [9, formulas (5.2), (2.29)].

Proof of Theorem 2.1 (the case $X = \mathbb{R}^n$). We apply (3.5) to $\varphi = Rf$ and replace $R_r^*\varphi = R_r^* Rf$ according to (3.4). Changing the order of integration, we obtain the following expression:

$$\sigma_{n-k-1}\sigma_{k-1} \int_0^\infty (\mathcal{M}_t f)(x)\, t^k\, dt \int_0^1 (tv)^{n-k-1} a(tv)(1 - v^2)^{k/2-1} dv. \tag{3.6}$$

If k is even, we choose $a(\cdot)$ in the form $a(\rho) = \rho^{k+1-n}\,\mathrm{sgn}(\rho^2 - r^2)$. Then (3.6) yields

$$(L_r^*\varphi)(x) = \sigma_{n-k-1}\sigma_{k-1}\int_0^\infty (\mathcal{M}_t f)(x)\, t^k\, \psi(r/t)\, dt, \tag{3.7}$$

$$\psi(u) = \int_0^1 \mathrm{sgn}(v-u)\,(1-v^2)^{k/2-1} dv. \tag{3.8}$$

Setting

$$c_k = \int_0^1 (1-v^2)^{k/2-1} dv, \qquad \Theta(u) = \int_1^u (v^2-1)^{k/2-1}\, dv,$$

we have

$$\psi(u) = \begin{cases} -c_k + 2(-1)^{k/2}\,\Theta(u) & \text{if } 0 < u < 1, \\ -c_k & \text{if } u > 1. \end{cases} \tag{3.9}$$

This allows us to represent $\sigma_{n-k-1}^{-1}\sigma_{k-1}^{-1}(L_r^*\varphi)(x)$ as

$$\begin{aligned} &-c_k \int_0^r (\mathcal{M}_t f)(x)\, t^k\, dt + \int_r^\infty (\mathcal{M}_t f)(x)\, t^k\, [-c_k + 2(-1)^{k/2}\Theta(r/t)]\, dt \\ &= 2(-1)^{(k+2)/2}\int_0^r (\mathcal{M}_t f)(x)\, t^k \Theta(r/t)\, dt \\ &+ \int_0^\infty (\mathcal{M}_t f)(x)\, t^k\, [-c_k + 2(-1)^{k/2}\Theta(r/t)]\, dt. \end{aligned}$$

Since $\int_0^\infty$ is a polynomial of degree $k-1$, then

$$\begin{aligned} \partial_r^k (L_r^*\varphi)(x) &= 2\sigma_{n-k-1}\sigma_{k-1}(-1)^{(k+2)/2}\partial_r^k \int_0^r (\mathcal{M}_t f)(x)\, t^k \Theta(r/t)\, dt \\ &= 2\sigma_{n-k-1}\sigma_{k-1}(-1)^{(k+2)/2}\partial_r^{k-1} \int_0^r (\mathcal{M}_t f)(x)\, t^k\, [\partial_r \Theta(r/t)]\, dt \\ &= 2\sigma_{n-k-1}\sigma_{k-1}(-1)^{(k+2)/2}\partial_r^{k-1}(\Lambda_r f)(x), \end{aligned}$$

$$(\Lambda_r f)(x) = \int_0^r (\mathcal{M}_t f)(x)\, (r^2 - t^2)^{k/2-1} t\, dt. \tag{3.10}$$

We write the last integral as $\Lambda_1 + \Lambda_2$, where

$$\Lambda_1 = f(x)\int_0^r (r^2-t^2)^{k/2-1} t\, dt = \frac{r^k}{k}\, f(x),$$

$$\Lambda_2 = \int_0^r (r^2-t^2)^{k/2-1}[(\mathcal{M}_t f)(x) - f(x)]\, t\, dt = r^k h(r),$$

$$h(r) = \int_0^1 (1-t^2)^{k/2-1}[(\mathcal{M}_{rt} f)(x) - f(x)]\, t\, dt.$$

This gives

$$\lim_{r\to 0} \partial_r^k (\Lambda_r f)(x) = (k-1)!\, f(x), \tag{3.11}$$

and therefore,

$$\lim_{r\to 0} \partial_r^{k+1}(L_r^*\varphi)(x) = 2(-1)^{(k+2)/2}\sigma_{n-k-1}\sigma_{k-1}\,(k-1)!\, f(x), \tag{3.12}$$

as desired.

If k is odd, we write (3.6) with $a(\rho) = \rho^{k+1-n}\log|\rho^2 - r^2|$ in the form $\sigma_{n-k-1}\sigma_{k-1}(A+B(r))$, where

$$A = 2\int_0^\infty (\mathcal{M}_t f)(x)\, t^k \log t\, dt \int_0^1 (1-v^2)^{k/2-1} dv,$$

$$B(r) = \int_0^\infty (\mathcal{M}_t f)(x)\, t^k \psi_k(r/t)\, dt, \tag{3.13}$$

where ψ_k is the function in (2.6). By Lemma 2.3,

$$\begin{aligned} B(r) &= \int_0^r (\mathcal{M}_t f)(x)\, t^k\, [P_{k-1}(r/t) + \pi(-1)^{(k-1)/2}\Theta(r/t)]\, dt \\ &+ \int_r^\infty (\mathcal{M}_t f)(x)\, t^k\, P_{k-1}(r/t)\, dt \\ &= \int_0^\infty (\mathcal{M}_t f)(x)\, t^k\, P_{k-1}(r/t)\, dt + \pi(-1)^{(k-1)/2}\int_0^r (\mathcal{M}_t f)(x)\, t^k\, \Theta(r/t)]\, dt. \end{aligned}$$

The integral $\int_0^\infty$ is a polynomial in r of degree $k-1$. Denoting

$$(\tilde{L}_r^*\varphi)(x) = \int_{\mathcal{G}_{n,k}} \varphi(\xi)\, \rho^{k+1-n} \log|\rho^2 - r^2|\, d\xi, \qquad \rho \equiv \rho(x,\xi), \tag{3.14}$$

and differentiating $k+1$ times, we obtain

$$\partial_r^{k+1}(\tilde{L}_r^*\varphi)(x) = \pi(-1)^{(k-1)/2}\sigma_{n-k-1}\sigma_{k-1}\, \partial_r^k \int_0^r (\mathcal{M}_t f)(x)\, (r^2 - t^2)^{k/2-1} t\, dt.$$

The last integral is already known; see (3.10). Hence, by (3.11),

$$\lim_{r\to 0} \partial_r^{k+1}(\tilde{L}_r^*\varphi)(x) = c\, f(x), \qquad c = \pi(-1)^{(k-1)/2}\sigma_{n-k-1}\sigma_{k-1}\,(k-1)!,$$

which completes the proof. □

Proof of Theorem 2.2 (the case $X = \mathbb{R}^n$). For $\varphi = Rf$, equality (3.4) yields $(R_r^*\varphi)(x) = \sigma_{k-1}[(-1)^{k/2}(\Lambda_r f)(x) + (\Delta_r f)(x)]$, where $(\Lambda_r f)(x)$ is the function in (3.10) and

$$(\Delta_r f)(x) = \int_0^\infty (\mathcal{M}_t f)(x)(t^2 - r^2)^{k/2-1} t\, dt.$$

Since $(\Delta_r f)(x)$ is a polynomial in the r-variable of degree $k-2$, owing to (3.11), we get

$$\lim_{r\to 0} \partial_r^k (R_r^*\varphi)(x) = (-1)^{k/2}(k-1)!\, \sigma_{k-1} f(x).$$

□

4. The case $X = \mathbb{S}^n$

We recall that $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$, $d(\cdot,\cdot)$ denotes the geodesic distance on $\mathbb{S}^n$, and Ξ stands for the set of all k-dimensional totally geodesic submanifolds ξ of $\mathbb{S}^n$. Each $\xi \in \Xi$ is an intersection of $\mathbb{S}^n$ with the relevant $(k+1)$-plane through the origin. Thus Ξ can be identified with the Grassmann manifold $G_{n+1,k+1}$. The totally geodesic Radon transform $(Rf)(\xi)$ of a function f on $\mathbb{S}^n$ is defined by

$$(Rf)(\xi) = \int_\xi f(x)\, d_\xi x, \qquad \xi \in \Xi, \tag{4.1}$$

where $d_\xi x$ stands for the usual Lebesgue measure on ξ. Throughout this section we assume that f is an even C^∞ function.

We set $\mathbb{R}^{n+1} = \mathbb{R}^{k+1} \oplus \mathbb{R}^{n-k}$, $\mathbb{R}^{k+1} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{k+1}, \mathbb{R}^{n-k} = \mathbb{R}e_{k+2} \oplus \cdots \oplus \mathbb{R}e_{n+1}, e_i$ being the coordinate unit vectors.

Let $\xi_0 = \mathbb{S}^k$ be the unit sphere in $\mathbb{R}^{k+1}$. Given $x \in \mathbb{S}^n$, $\xi \in \Xi$, we denote by r_x an arbitrary rotation satisfying $r_x e_{n+1} = x$ and set $\varphi_x(\xi) = \varphi(r_x \xi)$.

Let $K = SO(n)$ be the group of rotations about the x_{n+1}-axis. For $\theta \in [0, \pi/2]$, let g_θ be the rotation in the plane (e_{k+1}, e_{n+1}) with the matrix

$$\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}.$$

The shifted dual Radon transform of a function φ on Ξ is defined by

$$(R^*_\theta \varphi)(x) = \int_K \varphi_x(\rho g_\theta^{-1} \xi_0)\, d\rho. \tag{4.2}$$

The case $\theta = 0$ corresponds to the usual dual Radon transform [4].

A geometric meaning of $(R^*_\theta \varphi)(x)$ is as follows. Let

$$x_\theta = g_\theta e_{n+1} = e_{k+1} \cos\theta + e_{n+1} \sin\theta$$

so that $d(x_\theta, \xi_0) = \theta$. Hence,

$$(R^*_\theta \varphi)(x) = \int_{d(x,\xi)=\theta} \varphi(\xi)\, d\mu(\xi) = \int_{\rho(x,\xi)=r} \varphi(\xi)\, d\mu(\xi) \equiv (R^*_r \varphi)(x), \tag{4.3}$$

where $r = \sin\theta$, $\rho(x,\xi) = \sin d(x,\xi)$, and $d\mu(\xi)$ is the relevant normalized measure; cf. (1.5).

We need one more averaging operator. Given $x \in \mathbb{S}^n$ and $t \in (-1, 1)$, denote

$$(\mathbb{M}_t f)(x) = \frac{(1-t^2)^{(1-n)/2}}{\sigma_{n-1}} \int_{\{y \in \mathbb{S}^n :\, x \cdot y = t\}} f(y)\, d\sigma(y). \tag{4.4}$$

The integral (4.4) is the mean value of f on the planar section of $\mathbb{S}^n$ by the hyperplane $x \cdot y = t$, and $d\sigma(y)$ stands for the induced Lebesgue measure on this section. Thus, $\lim\limits_{t \to 1} (\mathbb{M}_t f)(x) = f(x)$.

Lemma 4.1. *Let $\rho(x,\xi) = \sin[d(x,\xi)]$. The following equalities hold:*

$$(R^*_\theta R f)(x) = 2\sigma_{k-1} \int_0^1 (1-\tau^2)^{k/2-1} (\mathbb{M}_{\tau \cos\theta} f)(x)\, d\tau; \tag{4.5}$$

$$\int_\Xi a(\rho(x,\xi)) \varphi(\xi)\, d\xi = \frac{\sigma_{n-k-1}\sigma_k}{\sigma_n} \int_0^1 (1-r^2)^{(k-1)/2} r^{n-k-1} a(r) (R^*_{\sin^{-1} r} \varphi)(x)\, dr. \tag{4.6}$$

It is assumed that f, φ and a are measurable functions, for which either side of the corresponding equality exists in the Lebesgue sense.

These equalities were proved in [8, pp. 485, 480].

Proof of Theorem 2.1 (the case $X = \mathbb{S}^n$). Let $\varphi = Rf$. We write (4.5) in the form

$$(R^*_\theta \varphi)(x) = \frac{2\sigma_{k-1}}{\rho^{k-1}} \int_0^\rho (\rho^2 - s^2)^{k/2-1} (\mathbb{M}_s f)(x)\, ds, \quad \rho = \cos\theta \in (0,1).$$

Setting $r = \sin\theta = \sqrt{1-\rho^2}$, $s = \sqrt{1-t^2}$, and denoting

$$(\tilde{\mathbb{M}}_t f)(x) = (1-t^2)^{-1/2} (\mathbb{M}_{\sqrt{1-t^2}} f)(x),$$

so that $\lim_{t\to 0}(\tilde{\mathbb{M}}_t f)(x) = f(x)$, we obtain

$$(4.7)\qquad (R^*_\theta\varphi)(x) = \frac{2\sigma_{k-1}}{(1-r^2)^{(k-1)/2}}\int_r^1 (\tilde{\mathbb{M}}_t f)(x)(t^2-r^2)^{k/2-1}t\,dt.$$

Then we apply (4.6) to $\varphi = Rf$ and replace $R^*_{\sin^{-1} r}\varphi$ according to (4.7). Changing the order of integration, we obtain

(4.8)

$$\int_\Xi a(\rho(x,\xi))\varphi(\xi)\,d\xi = c\int_0^1 (\tilde{\mathbb{M}}_t f)(x)t^k\,dt\int_0^1 (tv)^{n-k-1}a(tv)(1-v^2)^{k/2-1}dv,$$
$$c = \frac{2\,\sigma_{n-k-1}\,\sigma_k\,\sigma_{k-1}}{\sigma_n}.$$

This mimics (3.6) and we continue as in Section 3.

If k is even we choose $a(\cdot)$ in the form $a(\eta) = \eta^{k+1-n}\text{sgn}(\eta^2-r^2)$. Then for $\varphi = Rf$, (4.8) yields

$$(4.9)\qquad (L^*_r\varphi)(x) = c\int_0^1 (\tilde{\mathbb{M}}_t f)(x)\,t^k\,\psi(r/t)\,dt,$$

where ψ is a function from the previous section; see (3.8), (3.9). Hence, as in Section 3, $(L^*_r\varphi)(x)/c$ can be written as

$$-c_k\int_0^r (\tilde{\mathbb{M}}_t f)(x)\,t^k\,dt + \int_r^1 (\tilde{\mathbb{M}}_t f)(x)\,t^k\,[-c_k + 2(-1)^{k/2}\Theta(r/t)]\,dt$$
$$= 2(-1)^{(k+2)/2}\int_0^r (\tilde{\mathbb{M}}_t f)(x)\,t^k\Theta(r/t)\,dt$$
$$+\int_0^1 (\tilde{\mathbb{M}}_t f)(x)\,t^k\,[-c_k + 2(-1)^{k/2}\Theta(r/t)]\,dt.$$

Since $\int_0^1(\dots)$ is a polynomial of degree $k-1$, then the corresponding reasoning from Section 3 yields verbatim

$$\lim_{r\to 0}\partial_r^{k+1}(L^*_r\varphi)(x) = c\,(k-1)!\,f(x) = \frac{2\,\sigma_{n-k-1}\,\sigma_k\,\sigma_{k-1}}{\sigma_n}\,(k-1)!\,f(x).$$

If k is odd, we choose $a(\cdot)$ in the form $a(\eta) = \eta^{k+1-n}\log|\eta^2-r^2|$ and write (4.8) as $c\,(A+B(r))$, where

$$A = 2\int_0^1 (\tilde{\mathbb{M}}_t f)(x)\,t^k\,\log t\,dt\int_0^1 (1-v^2)^{k/2-1}dv = \text{const},$$
$$B(r) = \int_0^1 (\tilde{\mathbb{M}}_t f)(x)\,t^k\psi_k(r/t)\,dt,$$

where ψ_k is the function in (2.6); cf. (3.13). By Lemma 2.3,

$$B(r) = \int_0^r (\tilde{\mathbb{M}}_t f)(x)\,t^k\,[P_{k-1}(r/t) + \pi(-1)^{(k-1)/2}\Theta(r/t)]\,dt$$
$$+\int_r^1 (\tilde{\mathbb{M}}_t f)(x)\,t^k\,P_{k-1}(r/t)\,dt$$
$$= \int_0^1 (\tilde{\mathbb{M}}_t f)(x)\,t^k\,P_{k-1}(r/t)\,dt + \pi(-1)^{(k-1)/2}\int_0^r (\tilde{\mathbb{M}}_t f)(x)\,t^k\,\Theta(r/t)\,dt.$$

The integral $\int_0^1$ is a polynomial in r of degree $k-1$. Denoting

$$(\tilde{L}_r^*\varphi)(x) = \int_\Xi \varphi(\xi)\, \rho^{k+1-n} \log|\rho^2 - r^2|\, d\xi, \qquad \rho \equiv \rho(x,\xi) \tag{4.10}$$

(cf. (3.14)) and differentiating $k+1$ times, we obtain

$$\partial_r^{k+1}(\tilde{L}_r^*\varphi)(x) = c\,\pi(-1)^{(k-1)/2}\,\partial_r^k \int_0^r (\tilde{\mathbb{M}}_t f)(x)\,(r^2-t^2)^{k/2-1} t\,dt.$$

As in Section 3, this gives

$$\lim_{r\to 0} \partial_r^{k+1}(\tilde{L}_r^*\varphi)(x) = c\,f(x), \tag{4.11}$$

$$c = \frac{2\pi(-1)^{(k-1)/2}\,\sigma_{n-k-1}\,\sigma_k\,\sigma_{k-1}\,(k-1)!}{\sigma_n}\ (= \tilde{d}_X). \qquad \square$$

Proof of Theorem 2.2 (the case $X = \mathbb{S}^n$). Let $\varphi = Rf$ and write (4.7) in the form

$$R^*_{\sin^{-1} r}\varphi = \frac{2\sigma_{k-1}}{(1-r^2)^{(k-1)/2}}\,[I_1(r) + (-1)^{k/2}\, I_2(r)],$$

where

$$I_1(r) = \int_0^1 (\tilde{\mathbb{M}}_t f)(x)(t^2-r^2)^{k/2-1} t\,dt, \quad I_2(r) = \int_0^r (\tilde{\mathbb{M}}_t f)(x)(r^2-t^2)^{k/2-1} t\,dt,$$

$I_1(r)$ being a polynomial of degree $k-2$. Hence, as in (3.11), we obtain

$$\lim_{r\to 0} \partial_r^k[(1-r^2)^{(k-1)/2}\,(R^*_{\sin^{-1} r}\varphi)(x)] = 2(-1)^{k/2}\sigma_{k-1}\,(k-1)!\,f(x). \qquad \square$$

5. The case $X = \mathbb{H}^n$

Let $E^{n,1}$, $n \geq 2$, be the pseudo-Euclidean space of points $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ with the inner product

$$[x,y] = -x_1y_1 - \cdots - x_ny_n + x_{n+1}y_{n+1}. \tag{5.1}$$

We realize the n-dimensional hyperbolic space $X = \mathbb{H}^n$ as the upper sheet of the two-sheeted hyperboloid

$$\mathbb{H}^n = \{x \in \mathbb{E}^{n,1} : [x,x] = 1, x_{n+1} > 0\}.$$

Let Ξ be the set of all k-dimensional totally geodesic submanifolds $\xi \subset \mathbb{H}^n$, $1 \leq k \leq n-1$. As usual, $e_1, \dots, e_{n+1}$ denote the coordinate unit vectors. We set $\mathbb{R}^{n+1} = \mathbb{R}^{n-k} \oplus \mathbb{R}^{k+1}$, where

$$\mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-k}, \qquad \mathbb{R}^{k+1} = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_{n+1},$$

and identify $\mathbb{R}^{k+1}$ with the pseudo-Euclidean space $E^{k,1}$.

In the following, $x_0 = (0, \dots, 0, 1)$ and $\xi_0 = \mathbb{H}^n \cap \mathbb{E}^{k,1} = \mathbb{H}^k$ denote the origins in X and Ξ respectively; $G = SO_0(n,1)$ is the identity component of the pseudo-orthogonal group $O(n,1)$ preserving the bilinear form (5.1); $K = SO(n)$ and $H = SO(n-k) \times SO_0(k,1)$ are the isotropy subgroups of x_0 and ξ_0, so that $X = G/K$, $\Xi = G/H$. One can write $f(x) \equiv f(gK)$, $\varphi(\xi) \equiv \varphi(gH)$, $g \in G$. The geodesic distance between points x and y in X is defined by $d(x,y) = \cosh^{-1}[x,y]$.

Given $x \in X, \xi \in \Xi$, we denote by $r_x, r_\xi (\in G)$ arbitrary pseudo-rotations such that $r_x x_0 = x$, $r_\xi \xi_0 = \xi$, and write

$$f_\xi(x) = f(r_\xi x), \qquad \varphi_x(\xi) = \varphi(r_x \xi).$$

The totally geodesic Radon transform $(Rf)(\xi)$ of a rapidly decreasing C^∞ function f on $\mathbb{H}^n$ is defined by

$$(Rf)(\xi)=\int_\xi f(x)\,d_\xi x=\int_{SO_0(k,1)} f_\xi(\gamma x_0)d\gamma, \qquad \xi\in\Xi. \tag{5.2}$$

As in [1], let

$$g_\theta = \begin{bmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & I_{n-1} & 0 \\ \sinh\theta & 0 & \cosh\theta \end{bmatrix},$$

where I_{n-1} is the unit matrix of dimension $n-1$. The shifted dual Radon transform of a function φ on Ξ is defined by

$$(R^*_\theta\varphi)(x) = \int_K \varphi_x(\gamma g_\theta^{-1}\xi_0)\,d\gamma. \tag{5.3}$$

The case $\theta=0$ corresponds to the usual dual Radon transform.

A geometric meaning of $(R^*_\theta\varphi)(x)$ is as follows. Let $d(x,\xi)$ be the geodesic distance between $x\in X$ and $\xi\in\Xi$. We set

$$x_\theta=g_\theta e_{n+1}=e_1\sinh\theta+e_{n+1}\cosh\theta$$

so that $d(x_\theta,\xi_0)=d(x_\theta,e_{n+1})=\theta$. Then

$$d(x,r_x\gamma g_\theta^{-1}\xi_0)=\theta \tag{5.4}$$

for all $\gamma\in K$ and $x\in X$. Thus,

$$(R^*_\theta\varphi)(x)=\int_{d(x,\xi)=\theta}\varphi(\xi)\,d\mu(\xi)=\int_{\rho(x,\xi)=r}\varphi(\xi)\,d\mu(\xi)\equiv(R^*_r\varphi)(x), \tag{5.5}$$

where $r=\sinh\theta$, $\rho(x,\xi)=\sinh\, d(x,\xi)$, and $d\mu(\xi)$ is the relevant normalized measure; cf. (1.5). If $\varphi\in L^1(\Xi)$, then

$$\int_\Xi\varphi(\xi)\,d\xi=\sigma_{n-k-1}\int_0^\infty(R^*_\theta\varphi)(x)\,d\nu(\theta)\qquad\forall x\in X, \tag{5.6}$$

$$d\nu(\theta)=(\sinh\theta)^{n-k-1}(\cosh\theta)^k\,d\theta;$$

see [1, formula (2.30)]. Given $x\in\mathbb{H}^n$ and $t>1$, let

$$(M_tf)(x)=\frac{(t^2-1)^{(1-n)/2}}{\sigma_{n-1}}\int_{\{y\in\mathbb{H}^n:\,[x,y]=t\}}f(y)d\sigma(y) \tag{5.7}$$

be the spherical mean of f on $X=\mathbb{H}^n$, where $d\sigma(y)$ stands for the induced Lebesgue measure. Then $\lim\limits_{t\to1}(M_tf)(x)=f(x)$.

Lemma 5.1. *Let $\rho(x,\xi)=\sinh[d(x,\xi)]$. The following equalities hold:*

$$(R^*_\theta Rf)(x)=\frac{\sigma_{k-1}}{\tau^{k-1}}\int_\tau^\infty(M_sf)(x)(s^2-\tau^2)^{k/2-1}\,ds,\qquad\tau=\cosh\theta; \tag{5.8}$$

$$\int_\Xi a(\rho(x,\xi))\,\varphi(\xi)\,d\xi = \sigma_{n-k-1}\int_0^\infty(1+r^2)^{(k-1)/2}r^{n-k-1}a(r)(R^*_{\sinh^{-1}r}\varphi)(x)\,dr. \tag{5.9}$$

It is assumed that f, φ and a are measurable functions, for which either side of the corresponding equality exists in the Lebesgue sense.

Proof. Equality (5.8) can be found in [4]; see also [7, formula (2.3)]. Equality (5.9) follows from (5.6), (5.4), and (5.3). Indeed,

$$I \equiv \int_\Xi a(\rho(x,\xi))\,\varphi(\xi)\,d\xi = \int_\Xi \tilde\varphi(\xi)\,d\xi, \qquad \tilde\varphi(\xi) = a(\rho(e_{n+1},\xi))\,\varphi_x(\xi).$$

Hence,

$$\begin{aligned} I &= \sigma_{n-k-1}\int_0^\infty (R^*_\theta\tilde\varphi)(x)\,d\nu(\theta) \\ &= \sigma_{n-k-1}\int_0^\infty d\nu(\theta)\int_K a(\sinh d(e_{n+1},\gamma g_\theta^{-1}\xi_0))\,\varphi_x(\gamma g_\theta^{-1}\xi_0))\,d\gamma \\ &= \sigma_{n-k-1}\int_0^\infty a(\sinh\theta)\,(R^*_\theta\varphi)(x)\,d\nu(\theta). \end{aligned}$$

This gives (5.9). □

Proof of Theorem 2.1 (the case $X = \mathbb{H}^n$). Let $\varphi = Rf$. Setting $r = \sinh\theta = \sqrt{\tau^2-1}$, $s = \sqrt{1+t^2}$, and denoting

$$(\tilde M_t f)(x) = (1+t^2)^{-1/2}(M_{\sqrt{1+t^2}}f)(x), \tag{5.10}$$

so that $\lim\limits_{t\to 0}(\tilde M_t f)(x) = f(x)$, we obtain

$$(R^*_{\sinh^{-1} r}\varphi)(x) = \frac{\sigma_{k-1}}{(1+r^2)^{(k-1)/2}}\int_r^\infty (\tilde M_t f)(x)(t^2-r^2)^{k/2-1}t\,dt. \tag{5.11}$$

Then we apply (5.9) to $\varphi = Rf$ and replace $R^*_{\sinh^{-1} r}\varphi$ according to (5.11). Changing the order of integration, we obtain

$$\begin{aligned} &\int_\Xi a(\rho(x,\xi))\varphi(\xi)\,d\xi \\ &\quad = \sigma_{n-k-1}\,\sigma_{k-1}\int_0^\infty (\tilde M_t f)(x)\,t^k\,dt\int_0^1 (tv)^{n-k-1}a(tv)(1-v^2)^{k/2-1}dv. \end{aligned} \tag{5.12}$$

This mimics (3.6), and we proceed as in the previous sections. Specifically, choose $a(\cdot)$ in the form $a(\eta) = \eta^{k+1-n}\,\mathrm{sgn}(\eta^2-r^2)$. Then for $\varphi = Rf$, (5.12) yields

$$(L^*_r\varphi)(x) = \sigma_{n-k-1}\sigma_{k-1}\int_0^\infty (\tilde M_t f)(x)\,t^k\,\psi(r/t)\,dt; \tag{5.13}$$

cf. (3.7). Hence, as in Section 3,

$$\lim_{r\to 0}\partial_r^{k+1}(L^*_r\varphi)(x) = 2(-1)^{(k+2)/2}\sigma_{n-k-1}\sigma_{k-1}\,(k-1)!\,f(x). \tag{5.14}$$

If k is odd, we choose $a(\eta) = \eta^{k+1-n}\log|\eta^2-r^2|$ and set

$$(L^*_r\varphi)(x) = \int_\Xi \varphi(\xi)\,\rho^{k+1-n}\,\log|\rho^2-r^2|\,d\xi, \qquad \rho\equiv\rho(x,\xi), \tag{5.15}$$

as in (4.10). This gives

$$\lim_{r\to 0}\partial_r^{k+1}(L^*_r\varphi)(x) = c\,f(x), \quad c = \pi(-1)^{(k-1)/2}\,\sigma_{n-k-1}\,\sigma_{k-1}\,(k-1)!.$$ □

Proof of Theorem 2.2 (the case $X = \mathbb{H}^n$). For $\varphi = Rf$, equality (5.8) yields

$$(R^*_{\sinh^{-1} r}\varphi)(x) = \frac{\sigma_{k-1}}{(1+r^2)^{(k-1)/2}}\,[(-1)^{k/2}\,(\Lambda_r f)(x) + (\Delta_r f)(x)], \tag{5.16}$$

where (cf. (5.10), (3.10))

$$(\Lambda_r f)(x) = \int_0^r (\tilde{M}_t f)(x)\,(r^2-t^2)^{k/2-1} t\,dt,$$
$$(\Delta_r f)(x) = \int_0^\infty (\tilde{M}_t f)(x)(t^2-r^2)^{k/2-1} t\,dt.$$

Hence, as in (3.11), we obtain

$$\lim_{r\to 0} \partial_r^k [(1+r^2)^{(k-1)/2}\,(R^*_{\sinh^{-1} r}\varphi)(x)] = (-1)^{k/2}\sigma_{k-1}\,(k-1)!\,f(x). \qquad \square$$

6. Proof of Lemma 2.3

The statement will be obtained as a consequence of the more general result, which generalizes the reasoning from [5]. Our aim is to evaluate the following integral:

$$\phi(u) = \int_{-1}^1 (1+\xi)^\alpha (1-\xi)^{m-\alpha} \log|\xi-u|\,d\xi, \tag{6.1}$$
$$m\in\mathbb{Z}, \qquad -1<\alpha<m+1, \qquad \alpha\notin\mathbb{Z}, \qquad 0<u<\infty.$$

For $u=1$,

$$\phi(1) = \frac{2^{m+1}\Gamma(\alpha+1)\Gamma(m-a+1)}{(m+1)!}[\log 2 + \psi(m-\alpha+1) - \psi(m+2)], \tag{6.2}$$

$\psi(x) = \frac{d}{dx}\log\Gamma(x)$. To prove (6.2) it suffices to differentiate an obvious equality,

$$\int_{-1}^1 (1+\xi)^\alpha (1-\xi)^{\beta-1}\,d\xi = \frac{2^{\alpha+\beta}\,\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)},$$

in the β-variable and then set $\beta = m-\alpha+1$.

In the general case we denote

$$P_{m+1}(u) = \phi(1) - \begin{cases} (-1)^m \pi\,\csc\alpha\pi \displaystyle\sum_{r=1}^{m+1} \lambda_r\,(u^r-1) & \text{if } m+1>0, \\ 0 & \text{if } m+1=0, \end{cases} \tag{6.3}$$

$$\lambda_r = \begin{cases} \dfrac{1}{r}\displaystyle\sum_{\ell=1}^{m+1-r} (-1)^\ell \binom{m-\alpha}{\ell}\binom{\alpha}{m+1-r-\ell} & \text{if } r<m+1, \\ \dfrac{1}{m+1} & \text{if } r=m+1. \end{cases} \tag{6.4}$$

Lemma 6.1. *The integral* (6.1) *can be evaluated as follows:*

$$\phi(u) = P_{m+1}(u) + \mu_\alpha(u)\,\Theta_\alpha(u), \tag{6.5}$$

where $P_{m+1}(u)$ *is a polynomial* (6.3),

$$\mu_\alpha(u) = \begin{cases} -\pi \cot \alpha\pi, & \text{if } 0 < u < 1, \\ (-1)^m \pi \csc \alpha\pi, & \text{if } 1 < u < \infty, \end{cases}$$

$$\Theta_\alpha(u) = \int_1^u (1+\xi)^\alpha |1-\xi|^{m-\alpha} d\xi. \tag{6.6}$$

Proof. The result will be obtained using the classical tools of complex analysis. Consider the auxiliary functions

$$\zeta(z) = (z+1)^\alpha (z-1)^{m-\alpha}, \qquad h(z,u) = \zeta(z) \log \frac{z-u}{z-1}; \qquad z \in \mathbb{C}.$$

To fix a single branch of $\zeta(z)$, we cut the z-plane along the segment $-1 \le \xi \le 1$ and choose $\arg(z \pm 1) \in [0, 2\pi]$. Then $\zeta(z) \sim z^m$, $z \to \infty$, and

$$\zeta(\xi \pm i0) = \begin{cases} (-1)^m e^{\mp i\alpha\pi} |1+\xi|^\alpha |1-\xi|^{m-\alpha}, & \text{if } -1 < \xi < 1, \\ \\ |1+\xi|^\alpha |1-\xi|^{m-\alpha}, & \text{if } 1 < \xi < \infty. \end{cases} \tag{6.7}$$

Case 1. Let $1 < u < \infty$. Continue the cut $[-1, 1]$ along the real axis up to the point u and fix a branch of the logarithmic function so that $\arg(z-u) \in [0, 2\pi]$ and, as before, $\arg(z-1) \in [0, 2\pi]$. Then

$$\log \frac{z-u}{z-1} \sim 0, \quad z \to \infty,$$

and

$$\log \frac{\xi \pm i0 - u}{\xi \pm i0 - 1} = \log \left| \frac{\xi - u}{\xi - 1} \right| \pm \begin{cases} i\pi, & 1 < \xi < u, \\ 0, & 0 < \xi < 1. \end{cases} \tag{6.8}$$

At the point $z = \infty$, the function $h(z,u)$ has an integer order, and its residue is

$$-c_{-1} = \operatorname*{res}_{z=\infty} h(z,u) = \frac{1}{2\pi i} \int_{C^-} h(z,u) dz. \tag{6.9}$$

Here c_{-1} is the coefficient at z^{-1} in the Laurent series of $h(z,u)$ in a neighborhood of the infinite point, and C^- is a circle of radius R centered at $z = 0$. It is assumed that R is big enough, so that $z = \infty$ is the only singular point outside of C^-. The direction of integration on C^- is chosen to be clockwise. Deforming the contour C^-, we obtain

$$\operatorname*{res}_{z=\infty} h(\xi,u) = \frac{1}{2\pi i} \int_L h(\xi,u)\, d\xi, \tag{6.10}$$

where $L = [u,1]^- \cup [1,-1]^- \cup [-1,1]^+ \cup [1,u]^+$, and $[a,b]^+$ and $[a,b]^-$ are the upper and lower sides of a cut $[a,b]$, respectively. Owing to (6.7) and (6.8), (6.10) yields

$$\operatorname*{res}_{z=\infty} h(z,u) = \Theta_\alpha(u) - \frac{(-1)^m \sin \alpha\pi}{\pi} \int_{-1}^1 (1+\xi)^\alpha (1-\xi)^{m-\alpha} \log \left| \frac{\xi-u}{\xi-1} \right| d\xi.$$

In other words,

$$\phi(u) = \phi(1) + \mu_\alpha(u)\, \Theta_\alpha(u) - \mu_\alpha(u) \operatorname*{res}_{z=\infty} h(z,u), \tag{6.11}$$

and it remains to evaluate the residue on the right-hand side. We set

$$h(z,u) = -W(z,u) + W(z,1), \tag{6.12}$$

where

$$\begin{aligned} W(z,u) &= -z^m \left(1+\frac{1}{z}\right)^{\alpha} \left(1-\frac{1}{z}\right)^{m-\alpha} \log\left(1-\frac{u}{z}\right) \\ &= \sum_{p=0}^{\infty} a_p\, z^{m-p} \sum_{q=0}^{\infty} b_q\, z^{-q} \sum_{r=1}^{\infty} c_r\, z^{-r}, \end{aligned}$$

$$a_p = \binom{\alpha}{p}, \qquad b_q = (-1)^q \binom{m-\alpha}{q}, \qquad c_r = \frac{u^r}{r}.$$

A simple calculation gives $W(z,u) = \gamma z^{-1} +$ terms of degree $\neq -1$, with

$$\gamma = \sum_{r=1}^{m+1} \lambda_r\, u^r, \tag{6.13}$$

where λ_r has the form (6.4). Hence,

$$\operatorname*{res}_{z=\infty} h(z,u) = \sum_{r=1}^{m+1} \lambda_r\, (u^r - 1) = Q_{m+1}(u), \tag{6.14}$$

and, by (6.11), we finally get $\phi(u) = \mu_\alpha(u)\,\Theta_\alpha(u) + P_{m+1}(u)$,

$$P_{m+1}(u) = \phi(1) - \mu_\alpha(u)\, Q_{m+1}(u) = \phi(1) - \frac{(-1)^m \pi}{\sin \alpha\pi}\, Q_{m+1}(u). \tag{6.15}$$

Case 2. Let $0 < u < 1$. Then the substitute for (6.10) has the form

$$\operatorname*{res}_{z=\infty} h(z,u) = \frac{1}{2\pi i} \int_{L'} h(\xi,u)\, d\xi, \tag{6.16}$$

where $L' = [1,u]^- \cup [u,-1]^- \cup [-1,u]^+ \cup [u,1]^+$, and the residue of $h(z,u)$ at the infinite point remains unchanged. Now, (6.6), (6.7), the counterpart of formula (6.8) for this case, i.e.,

$$\log \frac{\xi \pm i0 - u}{\xi \pm i0 - 1} = \log \left| \frac{\xi - u}{\xi - 1} \right| \pm \begin{cases} i\pi, & u < \xi < 1, \\ 0, & 0 < \xi < u, \end{cases} \tag{6.17}$$

and (6.14) yield

$$\begin{aligned} Q_{m+1}(u) &= (-1)^{m-1} \cos\, \alpha\pi\, \Theta_\alpha(u) \\ &\qquad - \frac{(-1)^m \sin \alpha\pi}{\pi} \int_{-1}^{1} (1+\xi)^{\alpha} (1-\xi)^{m-\alpha} \log \left| \frac{\xi - u}{\xi - 1} \right| d\xi. \end{aligned}$$

This gives

$$\begin{aligned} \phi(u) &= \phi(1) - \pi \cot \alpha\pi\, \Theta_\alpha(u) - \frac{(-1)^m \pi}{\sin \alpha\pi}\, Q_{m+1}(u) \\ &= \mu_\alpha(u)\, \Theta_\alpha(u) + P_{m+1}(u), \end{aligned}$$

as desired. □

Let us turn to the particular case $\alpha = m - \alpha = \frac{k}{2} - 1$, where k is an odd positive integer. Then

$$\phi(u) = \int_{-1}^{1} (1-\xi^2)^{k/2-1} \log|\xi - u|\, d\xi, \tag{6.18}$$

and Lemma 6.1 yields

$$\phi(u) = P_{k-1}(u) + \begin{cases} 0, & \text{if } 0 < u < 1, \\ \pi(-1)^{(k-1)/2}\Theta(u), & \text{if } 1 < u < \infty, \end{cases} \tag{6.19}$$

where $P_{k-1}(u)$ is a polynomial of degree $k-1$, and

$$\Theta(u) = \int_1^u (\xi^2 - 1)^{k/2-1}\, d\xi.$$

Since

$$\phi(u) = \Big(\int_{-1}^0 + \int_0^1\Big)(\dots) = \int_0^1 (1-\xi^2)^{k/2-1} \log|\xi^2 - u^2|\, d\xi,$$

we arrive at Lemma 2.3.

References

1. C. A. Berenstein and B. Rubin, *Totally geodesic Radon transform of L^p-functions on real hyperbolic space*. Fourier analysis and convexity, 37–58, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2004. MR2087237 (2005f:44003)
2. S.R. Deans, *The Radon transform and some of its applications*, Dover Publ. Inc., Mineola, New York, 2007. MR1274701 (95a:44003)
3. I.M. Gel'fand, S.G. Gindikin and M.I. Graev, *Selected topics in integral geometry*, Translations of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 2003. MR2000133 (2004f:53092)
4. S. Helgason, *Integral geometry and Radon transform*, Springer, New York-Dordrecht-Heidelberg-London, 2011. MR2743116
5. Ph. Mader, *Über die Darstellung von Punktfunktionen im n-dimensionalen euklidischen Raum durch Ebenenintegrale*, Math. Zeit. **26** (1927), 646–652. MR1544879
6. J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. - Nat. Kl., **69** (1917), 262–277 (Russian translation is in the Russian edition of S. Helgason, The Radon transform, Moscow, Mir, 1983, pp. 134–148; English translation is in [2]). MR692055 (84f:01040)
7. B. Rubin, *Helgason-Marchaud inversion formulas for Radon transforms*, Proc. Amer. Math. Soc. **130** (2002), 3017–3023. MR1908925 (2003f:44003)
8. B. Rubin, *Inversion formulas for the spherical Radon transform and the generalized cosine transform*. Advances in Appl. Math. **29** (2002), 471–497. MR1942635 (2004c:44006)
9. B. Rubin, *Reconstruction of functions from their integrals over k-dimensional planes*, Israel J. of Math. **141** (2004), 93–117. MR2063027 (2005b:44004)

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