A Fracture Criterion of "Barenblatt" Type for an Intersonic Shear Crack

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Abstract

Steady-state intersonic propagation of a shear crack is considered, with the admission of cohesion across the crack faces. The asymptotic limit of "small-scale cohesion", which occurs when the magnitude of the cohesive stress far exceeds that of the applied stress, is developed explicitly, to obtain a criterion of "Barenblatt" type. The application of this criterion requires only the calculation of the "applied" stress intensity coefficient with cohesion disregarded; an equation of motion follows by equating this coefficient to a "modulus of cohesion" which depends on the cohesive model that is employed. An explicit formula for the "modulus of cohesion" is given for the special case of a cohesive zone of Dugdale type.

1 Introduction

Interest in the propagation of cracks in the intersonic speed range under shear loading has intensified since such propagation was observed experimentally (Rosakis, Samudrala and Coker 1999). It is well-known (e.g. Freund 1990) that, if cohesive forces are neglected, the singularity in stress at the tip of an intersonically-propagating crack has order r^{-q}, where r is distance from the crack tip. The exponent q depends on the speed of the crack but is in the range $0 \leq q \leq 1/2$. It takes the extreme value $q = 0$ when the speed V of propagation equals either of the elastic wave speeds, and the value $q = 1/2$ when $V = \sqrt{2}$ times the speed of shear waves. Hence, except for one particular speed, the flux of energy into the crack tip is zero, and the conventional "Griffith energy balance" cannot be applied. The development of a fracture criterion therefore requires the explicit admission of cohesion in the vicinity of the crack tip. Analyses of this type have been performed (e.g. Broberg 1989, 1999, Huang and Gao 2001) but they have tended to focus on the flux of energy into the crack tip region. In general, the results depend not only on the model of cohesion but also on the configuration of the applied loading. Other analyses have been performed with allowance for cohesion (e.g. Kubair, Geubelle and Huang 2002) but no clear separation has been made between the loading configuration and some measure of the "dynamic toughness" of the material. Here, such separation is achieved by performing an asymptotic analysis of the case in which the magnitude of the cohesive forces far exceeds a typical magnitude of the applied loading. In the case of statics (and, more generally, in the case of subsonic propagation), this asymptotic limit leads to the Barenblatt formulation (e.g. Barenblatt 1962) in terms of a "modulus of cohesion". Physically, surface energy represents the work done against cohesive forces in separating two surfaces. A demonstration of the equivalence of Barenblatt's formulation to the usual Griffith energy balance was provided (also in the case of subsonic propagation) by Willis (1967).

The main result of the present work is the demonstration that it is possible to define a "modulus of cohesion" for propagation in the intersonic range. This has physical dimensions that depend on crack speed but the corresponding "applied" stress intensity coefficient has exactly the same dimensional dependence on crack speed and the formulation can be placed in a correct dimensionless form. Explicit formulae are given for the widely-used model of cohesion first introduced by Dugdale (1960): we were somewhat surprised to observe that the formulation could be developed without the need for numerical quadratures in this case. Although the analysis here is based on a "steady-state" formulation, the resulting asymptotic formulae have much wider applicability, to non-steady propagation and to propagation through a viscoelastic medium. Such developments will be reported elsewhere.

2 Problem specification

The problem under consideration is one of plane strain, in which the component of displacement in the x_3-direction is zero. No further mention will be made of the coordinate x_3.

A crack propagates in an infinite isotropic elastic medium at uniform speed V so that, at time t, it occupies

$$S(t) := \{(x_1, x_2) : -\infty < x_1 < Vt, \ \ x_2 = 0\}. \tag{2.1}$$

The medium is loaded in such a way that it would experience the stress field $\sigma_{ij}^{\mathrm{nc}}$ and displacement field u_i^{nc} if the crack were not present. This field has two special features. First, it corresponds to antisymmetric shear loading, so that $\sigma_{22}^{\mathrm{nc}} = 0$ and $u_1^{\mathrm{nc}} = 0$ when $x_2 = 0$. Second, it follows the crack by depending on t, x_1, x_2 only through $x_1 - Vt, x_2$. It is convenient to define

$$x = x_1 - Vt. \tag{2.2}$$

All fields will depend on t, x_1, x_2 only in the combination x, x_2. The speeds of dilatational and shear waves in the elastic medium are a, b respectively, and the elastic shear modulus is μ. The primary concern of this work is for intersonic propagation of the crack. This means that

$$b < V < a. \tag{2.3}$$

Most of the formulae also apply, suitably specialised, to subsonic propagation, and this case will also receive brief mention, for completeness. The faces of the crack experience no external load, but they display cohesion so that

$$\sigma_{12}(x, x_2) \to f \text{ as } x_2 \to \pm 0 \text{ with } x < 0, \tag{2.4}$$

where f depends on the relative displacement of the crack faces and tends to zero rapidly as the relative displacement increases. The only quantities of concern in the present work are the crack face traction and the relative displacement. It is convenient, therefore, to drop the suffixes and write $\sigma(x)$ for $\sigma_{12}(x_1 - Vt, 0)$ and $u(x)$ for $u_1(x_1 - Vt, +0)$. Then, by symmetry, $u_1(x_1 - Vt, -0) = -u(x)$ and the relative displacement of the crack faces $[u_1] = 2u(x)$. There is no jump in u_2. The property of rapid decay of f now translates into the statement that $f \to 0$ as $2u/\delta_0 \to \infty$, where δ_0 is a decay length, characteristic of the cohesion law. In the general case, f could be a functional but more usually it is assumed to be a function of $2u$, or perhaps a function of $2u$ and $2\dot{u}$. With this understanding, we shall write $f(2u)$ to indicate the dependence on $2u$. The cohesion f also has a characteristic magnitude, σ_0. This could be (but does not have to be) its maximum value.

3 Integral equation formulation

The problem as specified can be solved by standard methods. The solution to follow is included for completeness; it derives the formulae upon which the subsequent discussion is based.

It is helpful to express σ in the form

$$\sigma = \sigma^{\rm nc} + \sigma^{\rm sc} \tag{3.1}$$

so that $\sigma^{\rm sc}$ is the additional, or scattered field, generated by the presence of the crack. Since $u_1^{\rm nc} = 0$ along the line of the crack, it follows that u and $\sigma^{\rm sc}$ must be related via the relevant component $G = G_{11}$ of the Green's function for the half-plane $x_2 > 0$:

$$u = -G * \sigma^{\rm sc}, \tag{3.2}$$

where the symbol $*$ denotes the operation of convolution. For the steady-state problem under consideration, u and $\sigma^{\rm sc}$ are functions only of $x = x_1 - Vt$. Correspondingly, G can be taken as the Green's function for this restricted problem. Since both "source" and "receiver" are on the surface $x_2 = 0$, G is also just a function of x. It is easily expressed in terms of its Fourier transform $\tilde{G}(\xi)$, where

$$\tilde{G}(\xi) := \int G(x) {\rm e}^{{\rm i}\xi x}\, {\rm d}x. \tag{3.3}$$

The exact expression for $\tilde{G}$ is given in the Appendix. For the present, it suffices to note that it is a homogeneous function of degree -1 in ξ. It follows by Fourier transforming (3.2) that

$$-{\rm i}\xi\tilde{u} = -\tilde{K}^{-1}\tilde{\sigma}^{\rm sc}, \tag{3.4}$$

where

$$\tilde{K}(\xi) = 1/(-{\rm i}\xi\tilde{G}(\xi)) \tag{3.5}$$

is a homogeneous function of degree zero. Hence,

$$\sigma^{\rm sc} = -K * u', \tag{3.6}$$

where

$$\begin{aligned} K(x) &= (2\pi)^{-1} \int \tilde{K}(\xi){\rm e}^{-{\rm i}\xi x}\, {\rm d}\xi \\ &= (2\pi)^{-1} \left\{ \int_{-\infty}^{0} \tilde{K}(-1){\rm e}^{-{\rm i}\xi x}\, {\rm d}\xi + \int_0^\infty \tilde{K}(1){\rm e}^{-{\rm i}\xi x}\, {\rm d}\xi \right\} \\ &= \frac{1}{2\pi{\rm i}} \left\{ \frac{\tilde{K}(1)}{x - 0{\rm i}} - \frac{\tilde{K}(-1)}{x + 0{\rm i}} \right\}, \end{aligned} \tag{3.7}$$

having exploited the homogeneity of $\tilde{K}$ in inverting the transform.

Now define

$$F(z) := \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{0} \frac{u'(s)\,\mathrm{d}s}{s-z} \tag{3.8}$$

(since $u = 0$ when $x \geq 0$). Then (3.6) gives

$$\sigma^{\mathrm{sc}}(x) = -\tilde{K}(-1)F_+(x) + \tilde{K}(1)F_-(x), \tag{3.9}$$

where

$$F_\pm(x) := F(x \pm 0\mathrm{i}). \tag{3.10}$$

It may be noted that $G(x)$ is real, and hence that $\tilde{G}(1)$ and $\tilde{G}(-1)$ are complex conjugates. We define

$$\tilde{G}(1) := \frac{-\mathrm{i}(g_1 + \mathrm{i}g_2)}{\mu} \equiv \frac{-\mathrm{i}\Lambda}{\mu}\mathrm{e}^{\mathrm{i}\pi q}, \tag{3.11}$$

where

$$\Lambda = (g_1^2 + g_2^2)^{1/2} \text{ and } q = \pi^{-1}\tan^{-1}(g_2/g_1). \tag{3.12}$$

Correspondingly,

$$\tilde{K}(1) = -\mu \mathrm{e}^{-\mathrm{i}\pi q}/\Lambda \tag{3.13}$$

and $\tilde{K}(-1)$ is its complex conjugate.

The relation (3.9), restricted to $x < 0$, defines a Hilbert problem for $F(z)$, if σ^{sc} is regarded as known. The corresponding homogeneous Hilbert problem has solution z^{-q} and, as will be seen below, $0 < q \leq 1/2$. The Hilbert problem for $F(z)$ therefore has unique solution

$$F(z) = \frac{\Lambda}{2\pi \mathrm{i}\mu z^q} \int_{-\infty}^{0} \frac{(-s)^q \sigma^{\mathrm{sc}}(s)\,\mathrm{d}s}{s-z} \tag{3.14}$$

amongst the class of F that are locally (square-) integrable and tend to zero as $|z| \to \infty$. It follows now from the Plemelj formulae that

$$u'(x) = -\frac{\Lambda \sin(\pi q)}{\mu\pi(-x)^q} \int_{-\infty}^{0} \frac{(-s)^q \sigma^{\mathrm{sc}}(s)\,\mathrm{d}s}{s-x} + \frac{\Lambda}{\mu}\cos(\pi q)\sigma^{\mathrm{sc}}(x) \quad (x < 0). \tag{3.15}$$

Here, the singular integral is interpreted as a Cauchy principal value.

Also, from (3.9),

$$\sigma^{\mathrm{sc}}(x) = -\frac{\sin(\pi q)}{\pi x^q} \int_{-\infty}^{0} \frac{(-s)^q \sigma^{\mathrm{sc}}(s)\,\mathrm{d}s}{x-s} \quad (x > 0). \tag{3.16}$$

Now recall that $\sigma^{\mathrm{sc}} = \sigma - \sigma^{\mathrm{nc}} = f(2u) - \sigma^{\mathrm{nc}}$, and note that $f(2u)$ is always finite. For consistency of the formulation, $\sigma(x)$ for $x > 0$ must not exceed the value of f at the crack tip. Therefore, the speed of the crack must adjust itself so that σ^{sc} remains finite as $x \to +0$. (If this cannot be achieved then no solution exists in the speed range that is envisaged). Thus, it is required that

$$0 = \int_{-\infty}^{0} \frac{\sigma^{\mathrm{sc}}(s)\,\mathrm{d}s}{(-s)^{1-q}}. \tag{3.17}$$

This requirement can be expressed in the form

$$k^{\mathrm{appl}} = k_c(V), \tag{3.18}$$

where

$$k^{\mathrm{appl}} = \frac{\sin(\pi q)}{\pi} \int_{-\infty}^{0} \frac{\sigma^{\mathrm{nc}}(s)\,\mathrm{d}s}{(-s)^{1-q}} \tag{3.19}$$

is the coefficient of x^{-q} in the stress ahead of the crack, that *would* be produced in the absence of cohesion, and

$$k_c(V) = \frac{\sin(\pi q)}{\pi} \int_{-\infty}^{0} \frac{f(2u)\,\mathrm{d}s}{(-s)^{1-q}}. \tag{3.20}$$

The relation (3.15) can be rearranged, using the identity (3.17), to give

$$\begin{aligned} u'(x) \quad &= \quad -\frac{\Lambda(-x)^{1-q}\sin(\pi q)}{\mu\pi} \int_{-\infty}^{0} \frac{\{f(2u) - \sigma^{\mathrm{nc}}(s)\}\mathrm{d}s}{(-s)^{1-q}(s-x)} \\ &\qquad + \frac{\Lambda\cos(\pi q)}{\mu}\{f(2u) - \sigma^{\mathrm{nc}}(x)\}. \end{aligned} \tag{3.21}$$

Equation (3.14) can similarly be rearranged to give

$$F(z) = -\frac{\Lambda z^{1-q}}{2\pi \mathrm{i}\mu} \int_{-\infty}^{0} \frac{\sigma^{\mathrm{sc}}(s)\,\mathrm{d}s}{(-s)^{1-q}(s-z)}. \tag{3.22}$$

The singularity in the integral when $z \to 0$ can be avoided by writing $\sigma^{\mathrm{sc}}(s) = (\sigma^{\mathrm{sc}}(s) - \sigma^{\mathrm{sc}}(0)) + \sigma^{\mathrm{sc}}(0)$ and noting that[1]

$$\int_{-\infty}^{0} \frac{\mathrm{d}s}{(-s)^{1-q}(s-z)} = -\frac{\pi z^{q-1}}{\sin(\pi q)}. \tag{3.23}$$

[1]The integral is easily evaluated from consideration of the contour integral $\int_C \frac{\mathrm{d}s}{(-s)^{1-q}(s-z)}$, where C encloses the branch cut which comprises the negative real axis.

Thus,

$$F(z) = -\frac{\Lambda z^{1-q}}{2\pi \mathrm{i}\mu} \int_{-\infty}^{0} \frac{(\sigma^{\mathrm{sc}}(s) - \sigma^{\mathrm{sc}}(0))\,\mathrm{d}s}{(-s)^{1-q}(s-z)} + \frac{\Lambda \sigma^{\mathrm{sc}}(0)}{2\mu \mathrm{i}\sin(\pi q)}. \tag{3.24}$$

Having solved the problem as posed, it is necessary to check that $\sigma(x)$ does not exceed the value of f at the crack tip, for all $x > 0$. A formula which is convenient for computation follows from (3.9) and (3.24): when $x > 0$,

$$\sigma^{\mathrm{sc}}(x) = -\frac{\sin(\pi q)}{\pi} x^{1-q} \int_{-\infty}^{0} \frac{(\sigma^{\mathrm{sc}}(s) - \sigma^{\mathrm{sc}}(0))\,\mathrm{d}s}{(-s)^{1-q}(s-x)} + \sigma^{\mathrm{sc}}(0), \tag{3.25}$$

which shows explicitly that $\sigma^{\mathrm{sc}}(x)$ is continuous at $x = 0$.

4 "Barenblatt" criterion

The relation (3.21) constitutes the exact equation which, together with the requirement (3.17), defines $u(x)$ and V. Here, however, an asymptotic solution will be sought, in the limit that the stress σ_0 characterising the cohesion is much greater than the applied stress, σ^{nc}. Then, to leading order, $u(x)$ satisfies the equation that results from (3.21) by neglecting the explicit contribution from σ^{nc}. Thus, to leading order,

$$u'(x) = -\frac{\Lambda}{\mu}\left\{ \frac{(-x)^{1-q}\sin(\pi q)}{\pi} \int_{-\infty}^{0} \frac{f(2u)\mathrm{d}s}{(-s)^{1-q}(s-x)} - \cos(\pi q) f(2u) \right\}. \tag{4.1}$$

This equation is independent of the applied loading, and hence defines an "autonomous end region". Equation (3.20) correspondingly defines a "modulus of cohesion" $k_c(V)$. In principle, it can be calculated for any chosen law of cohesion and then, subject to that law, the speed of the crack must be that for which the relation (3.18) is satisfied.

The sequence of reasoning just outlined was developed by Barenblatt (see, for instance, Barenblatt 1962) for the case of a stationary crack ($V = 0$) and f a function of $2u$. When $V = 0$ — and, in fact, for all subsonic V — the coefficient $g_1 = 0$ and $q = 1/2$. Equation (3.21) reduces to

$$u'(x) = -\frac{\Lambda(-x)^{1/2}}{\pi\mu} \int_{-\infty}^{0} \frac{f(2u(s))\,\mathrm{d}s}{(-s)^{1/2}(s-x)}. \tag{4.2}$$

Willis (1967) demonstrated that this equation implies (in present notation)

$$2\gamma = \frac{\pi\Lambda}{\mu} k_c^2, \tag{4.3}$$

where

$$\gamma := \int_0^\infty f(2u)\,\mathrm{d}u \tag{4.4}$$

is the specific surface energy. In the dynamic case ($V > 0$ but subsonic), this relation gives the dependence of k_c on V explicitly, the function $\Lambda(V)$ being given in the Appendix (equations (A.6), (A.7)).

There is no such general relation in the case of intersonic V: it appears necessary to make a more detailed analysis of equation (3.21), for a specifically chosen law of cohesion.

5 Example: Dugdale model of cohesion

We consider now the simple model of cohesion first introduced (in the context of plastic yielding ahead of a crack in a thin sheet) by Dugdale (1960) and developed into a fracture criterion by Bilby, Cottrell and Swinden (1963):

$$f(2u) = \begin{cases} \sigma_0, & 0 < 2u \le \delta_0, \\ 0, & \delta_0 < 2u < \infty. \end{cases} \tag{5.1}$$

(It is assumed that $2u \ge 0$ here.) The cohesion will correspondingly take the value σ_0 over an interval $-l < x < 0$ and will be zero elsewhere. The length l of the zone of cohesion is such that $2u(-l) = \delta_0$.

Note first that (3.24) gives

$$\begin{aligned} F(z) &= -\frac{\Lambda z^{1-q}}{2\pi \mathrm{i}\mu}\int_{-\infty}^{0}\frac{(f(2u)-f(0)-\sigma^{\mathrm{nc}}(s)+\sigma^{\mathrm{nc}}(0))\,\mathrm{d}s}{(-s)^{1-q}(s-z)} \\ &\qquad + \frac{\Lambda(f(0)-\sigma^{\mathrm{nc}}(0))}{2\mu \mathrm{i}\sin(\pi q)} \\ &\sim -\frac{\Lambda z^{1-q}}{2\pi \mathrm{i}\mu}\int_{-\infty}^{0}\frac{(f(2u)-f(0))\,\mathrm{d}s}{(-s)^{1-q}(s-z)} + \frac{\Lambda(f(0)-\sigma^{\mathrm{nc}}(0))}{2\mu \mathrm{i}\sin(\pi q)} \\ &= \frac{\Lambda\sigma_0 z^{1-q}}{2\pi \mathrm{i}\mu}\int_{-\infty}^{-l}\frac{\mathrm{d}s}{(-s)^{1-q}(s-z)} + \frac{\Lambda(\sigma_0-\sigma^{\mathrm{nc}}(0))}{2\mu \mathrm{i}\sin(\pi q)}. \end{aligned} \tag{5.2}$$

The first line of the above relations is exact. The "Barenblatt" asymptotic approximation is made in the second line by neglecting σ^{nc} in the integrand (it is retained in the other term so that continuity of σ is maintained exactly). The Dugdale form (5.1) for f is substituted to obtain the final line. When $|z| < l$, the final integral can be evaluated as a series in z, to give

$$F(z) \sim -\frac{\Lambda\sigma_0}{2\pi \mathrm{i}\mu}\sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1-q}}{l^{n+1-q}(n+1-q)} + \frac{\Lambda(\sigma_0-\sigma^{\mathrm{nc}}(0))}{2\mu \mathrm{i}\sin(\pi q)}. \tag{5.3}$$

Correspondingly, when $-l < x \leq 0$, by use of the Plemelj formula $u'(x) = F_+(x) - F_-(x)$,

$$u'(x) \sim -\frac{\Lambda\sigma_0 \sin(\pi q)}{\pi\mu} \sum_{n=0}^{\infty} \frac{(-x)^{n+1-q}}{l^{n+1-q}(n+1-q)}. \tag{5.4}$$

The value $u(-l)$ follows by termwise integration of the series in (5.4). The sum of the resulting series is elementary; the result is

$$u(-l) = \frac{\Lambda\sigma_0 l \sin(\pi q)}{\mu\pi(1-q)}. \tag{5.5}$$

The requirement that $\delta_0 = 2u(-l)$ now gives

$$\frac{\mu\delta_0}{\sigma_0} = \frac{2\Lambda \sin(\pi q) l}{\pi(1-q)}. \tag{5.6}$$

Therefore, equation (3.20) gives, for the Dugdale–Barenblatt model,

$$k_c = \frac{\sigma_0 l^q \sin(\pi q)}{\pi q} = \sigma_0 \left(\frac{\mu\delta_0}{\sigma_0}\right)^q \frac{1}{q} \left(\frac{1-q}{2\Lambda}\right)^q \left(\frac{\sin(\pi q)}{\pi}\right)^{1-q}. \tag{5.7}$$

A sample plot of k_c/μ against V/b is shown in Figure 1. This figure is somewhat unusual in that the physical dimension of k_c/μ depends on V. Thus, the plot is purely numerical, calculated from specific values of the physical parameters. The values chosen for the plot were $\sigma_0/\mu = 0.1$, $\mu\delta_0/\sigma_0 = 0.01$ (units of length – mm, for instance). The notion of "modulus of cohesion" loses validity close to either of the waves speeds a or b (because q approaches zero) but the figure nevertheless suggests that propagation at such speeds is likely to be difficult to achieve. More detailed implications of the fracture criterion (3.18) will be followed up elsewhere (Obrezanova and Willis, 2003). The fracture criterion (3.18) can be expressed in the non-dimensional form

$$\frac{k^{\text{appl}}}{\sigma_0} \left(\frac{\sigma_0}{\mu\delta_0}\right)^q = \frac{1}{q} \left(\frac{1-q}{2\Lambda}\right)^q \left(\frac{\sin(\pi q)}{\pi}\right)^{1-q}. \tag{5.8}$$

It is necessary finally to check that the stress ahead of the crack does not exceed σ_0. It follows from (3.9) and (5.3) that, for the Dugdale–Barenblatt model,

$$\sigma(x) \sim \sigma_0 \left\{1 - \frac{\sin(\pi q) x^{1-q}}{\pi} \int_{-\infty}^{-l} \frac{\mathrm{d}s}{(-s)^{1-q}(x-s)}\right\}. \tag{5.9}$$

Plots of $\sigma(x)$ for two values of V are shown in Figure 2. They are entirely representative: plots for other values confirm that the stress immediately

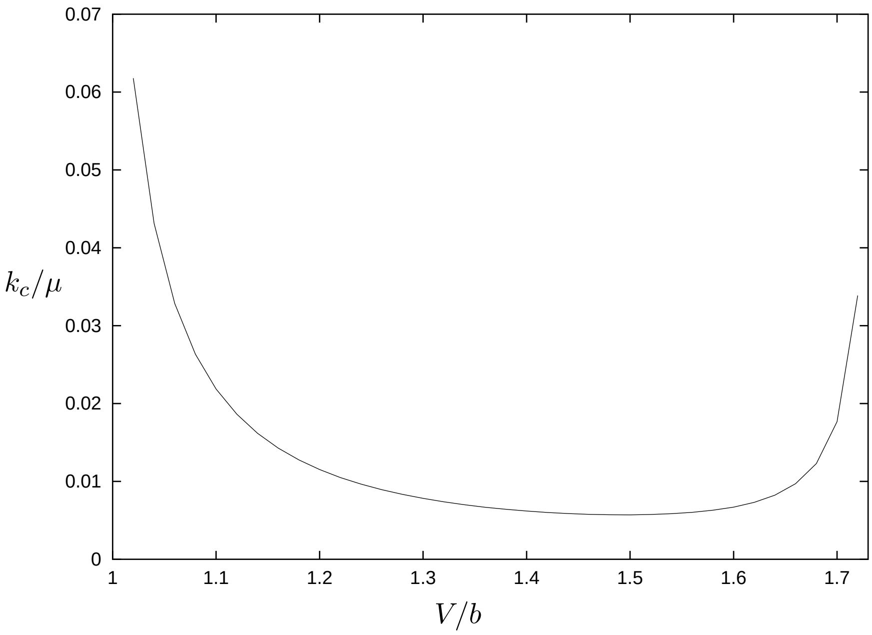

Figure 1: k_c/μ versus V/b.

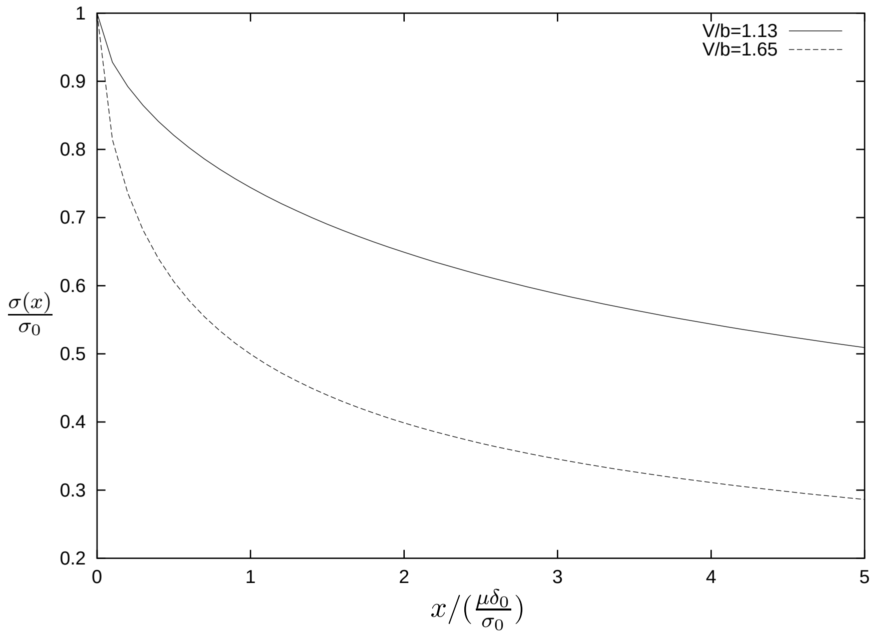

Figure 2: The stress ahead of the crack $\sigma(x)/\sigma_0$ versus $x/(\frac{\mu\delta_0}{\sigma_0})$ for $V/b = 1.13,\ 1.65$; for both of these values $q(V) = 0.4$.

ahead of the crack tip starts at the value σ_0 and decays monotonically. Since the integral in (5.9) defines a continuous function of x for all $x > -l$, its evaluation when $x = 0$ gives the result

$$\sigma(x) \sim \sigma_0 \left[1 - \frac{\sin \pi q}{\pi(1-q)} \left(\frac{x}{l} \right)^{1-q} \right] < \sigma_0$$

as $x \to +0$. Of course, far from the crack tip the asymptotic representation breaks down and the detail of the applied loading influences the stress. However, the assumption upon which the asymptotic formulation was developed ensures that these "far-field" stresses are smaller than σ_0 in magnitude.

6 Concluding Remarks

The main contribution of this work is the development of an asymptotic formulation for intersonic crack propagation, in the spirit of the Barenblatt formulation for statics. It becomes inaccurate when $q(V)$ is small, and hence applies at best qualitatively when V is close to either of the wave speeds a, b. Its virtue is its universal applicability, independent of the detail of the applied loading. Analysis of the governing equation (4.1) could, in principle, be performed for any chosen model of cohesion with allowance, for instance, for rate-dependence, friction, etc. The result would always be a "modulus of cohesion", for use in conjunction with the fracture criterion (3.18). The example of the Dugdale–Barenblatt model is distinctive because the fracture criterion can be given in explicit form.

Although the analysis here was performed in the context of steady-state loading and uniform propagation, it is evident that the equations that follow from the *asymptotic* development would apply equally well in the case of general, transient loading, and even non-uniform propagation: the only requirement would be to calculate k^{appl} for the case in question. This has been pursued in relation to a small time-dependent perturbation of the speed of the crack, to investigate the stability of steady-state propagation, by Obrezanova and Willis (2003). Equally, in the asymptotic limit, strain-rates in the vicinity of the cohesive zone are high, and even a viscoelastic medium will respond, locally, as though it is elastic. The criterion has been employed in the context of viscoelastic propagation by Antipov and Willis (2003).

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Appendix: Green's function

The Green's function employed in the main text can be found straightforwardly by Fourier transforming the governing equations, written in terms of the moving coordinate $x = x_1 - Vt$. A more general alternative, however, is to start from the Fourier transform

$$\overline{G}^{\rm dyn}(\omega,\xi) = \int {\rm d}t \int {\rm d}x_1\, G^{\rm dyn}(t,x_1){\rm e}^{{\rm i}(\omega t+\xi x_1)} \tag{A.1}$$

of the dynamic Green's function $G^{\rm dyn}(t,x_1)$. It is easily verified that the required "steady-state" Green's function $G(x)$ has Fourier transform, relative to the moving coordinate x,

$$\tilde{G}(\xi) = \overline{G}^{\rm dyn}(-V\xi,\xi). \tag{A.2}$$

Now

$$\overline{G}^{\rm dyn}(\omega,\xi) = \frac{{\rm i}\omega^2/b^2(\omega^2/b^2-\xi^2)^{1/2}}{\mu[4\xi^2(\omega^2/a^2-\xi^2)^{1/2}(\omega^2/b^2-\xi^2)^{1/2}+(\omega^2/b^2-2\xi^2)^2]}, \tag{A.3}$$

in which the branches of the square root functions are chosen so that the functions have positive imaginary parts when ξ is real and ω has positive imaginary part. Correspondingly,

$$\tilde{G}(\xi) = \frac{{\rm i}V^2/b^2[(\frac{-V{\rm sgn}(\xi)}{b}+0{\rm i})^2-1]^{1/2}}{\mu\xi\{4[(\frac{-V{\rm sgn}(\xi)}{b}+0{\rm i})^2-1]^{1/2}[(\frac{-V{\rm sgn}(\xi)}{b}+0{\rm i})^2-1]^{1/2}+(V^2/b^2-2)^2\}}. \tag{A.4}$$

Subsonic case

When V is subsonic ($0 \leq V < b$), it follows that

$$\tilde{G}(1) = \frac{-(V^2/b^2)(1-V^2/b^2)^{1/2}}{\mu[(V^2/b^2-2)^2 - 4(1-V^2/a^2)^{1/2}(1-V^2/b^2)^{1/2}]}. \tag{A.5}$$

Thus,

$$g_1 = 0, \quad g_2 = \frac{(V^2/b^2)(1-V^2/b^2)^{1/2}}{4(1-V^2/a^2)^{1/2}(1-V^2/b^2)^{1/2} - (V^2/b^2-2)^2}. \tag{A.6}$$

Correspondingly,

$$\Lambda = g_2. \tag{A.7}$$

Subsonic propagation is physically possible, however, only in the range $0 \leq V < c$ which gives $g_2 > 0$, where c is the speed of Rayleigh waves, for which

$$4(1-c^2/a^2)^{1/2}(1-c^2/b^2)^{1/2} - (c^2/b^2-2)^2 = 0. \tag{A.8}$$

Intersonic case

When $b < V < a$,

$$\tilde{G}(1) = \frac{-\mathrm{i}(V^2/b^2)(V^2/b^2-1)^{1/2}}{\mu[(V^2/b^2-2)^2 - 4\mathrm{i}(V^2/b^2-1)^{1/2}(1-V^2/a^2)^{1/2}]} \tag{A.9}$$

and hence

$$g_1 = \frac{(V^2/b^2-1)^{1/2}(V^2/b^2-2)^2}{R(V)}, \quad g_2 = \frac{4(V^2/b^2-1)(1-V^2/a^2)^{1/2}}{R(V)}, \tag{A.10}$$

where

$$R(V) = V^6/b^6 - 8V^4/b^4 + 8(3V^2/b^2 - 2V^2/a^2) - 16(1-b^2/a^2). \tag{A.11}$$

Correspondingly,

$$\Lambda = \frac{(V/b)(V^2/b^2-1)^{1/2}}{[R(V)]^{1/2}}, \tag{A.12}$$

$$q = \frac{1}{\pi}\tan^{-1}\left\{\frac{4(V^2/b^2-1)^{1/2}(1-V^2/a^2)^{1/2}}{(V^2/b^2-2)^2}\right\}. \tag{A.13}$$

(It is perhaps worth noting that $R(c) = 0$.)

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