METHOD OF ANALYTIC CONTINUATION FOR THE INVERSE SPHERICAL MEAN TRANSFORM IN CONSTANT CURVATURE SPACES

By

YURI A. ANTIPOV,* RICARDO ESTRADA,† AND BORIS RUBIN‡

Abstract. The following problem arises in thermoacoustic tomography and has intimate connection with PDEs and integral geometry. *Reconstruct a function f supported in an n-dimensional ball B given the spherical means of f over all geodesic spheres centered on the boundary of B.* We propose a new approach to this problem, which yields explicit reconstruction formulas in arbitrary constant curvature space, including euclidean space $\mathbb{R}^n$, the n-dimensional sphere, and hyperbolic space. The main idea is analytic continuation of the corresponding operator families. The results are applied to inverse problems for a large class of Euler-Poisson-Darboux equations in constant curvature spaces of arbitrary dimension.

1 Introduction

This paper deals with the spherical mean operator, which is also called the spherical mean Radon transform. The importance of this transformation in analysis and geometry was pointed out by many authors; see, e.g., [11, p. 699], [32, 62]. In recent years, interest in this object has grown tremendously in view of the rapidly developing field of thermoacoustic tomography (TAT); see [1]-[3], [19, 20], [21]-[18], [31, 36], [35]-[37], [39, 46], [51]-[55], [66, 67]. Another series of challenging problems is connected with sets of injectivity of this transform; see [15, 4, 5, 61] and references therein.

Setting of the problem. Let f be an infinitely differentiable function with compact support in the open ball $B = \{x \in \mathbb{R}^n : |x| < R\}$ and let ∂B denote the

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boundary of B. The **spherical mean Radon transform** Mf, which integrates f over spheres centered on ∂B, is defined by

$$(Mf)(\xi, t) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(\xi - t\theta) d\theta, \tag{1.1}$$

where $\xi \in \partial B$, $t \in \mathbb{R}_+ = (0, \infty)$, S^{n-1} is the unit sphere in $\mathbb{R}^n$ with the area σ_{n-1}, and $d\sigma$ stands for the usual Lebesgue measure on S^{n-1}. For the classical Radon transform, its modifications, and applications, see, e.g., [12, 14, 26, 13, 30, 44, 63, 68]. The problem of reconstructing f from known data $(Mf)(\xi, t)$ on the cylinder $\partial B \times \mathbb{R}_+$ stems from the following commonly accepted mathematical model of TAT in $\mathbb{R}^3$; see, e.g., [37, 66, 31].

Given the speed $c(x)$ of ultrasound propagation in the tissue and the measured value $g(\xi, t)$, of the pressure at time t at the transducer's location $\xi \in S^2$, find a function $f(x)$ and initial pressure distribution $p(x, 0)$ (the TAT image) such that

$$\begin{aligned} p_{tt} &= c^2(x)\Delta p && \text{for all } t \geq 0, x \in \mathbb{R}^3, \\ p(x, 0) &= f(x), p_t(x, 0) = 0 && \text{for all } x \in \mathbb{R}^3, \\ p(\xi, t) &= g(\xi, t) && \text{for all } \xi \in S^2(\subset \mathbb{R}^3), t \geq 0. \end{aligned} \tag{1.2}$$

Here, p_t and p_{tt} denote the first and second time derivatives, respectively, of p, and Δ is the Laplace operator with respect to the spatial variable x. In the important particular case of constant speed $c(x)$, a solution of this problem is equivalent to the reconstruction of f from its spherical mean (1.1).

This problem admits natural generalization to arbitrary dimensions, more general Riemannian spaces, and a variety of differential equations of the Euler-Poisson-Darboux (EPD) type. A thorough discussion of main inversion methods for Mf in the euclidean case can be found in [37, 66].

Explicit inversion formulas for Mf are of particular interest. For odd n, such formulas were obtained by Finch, Patch, and Rakesh in [20]. Another derivation was suggested by Palamodov [52, Section 7.5]; see also [22, 23, 67, 59]. The corresponding formulas for n even were obtained by Finch, Haltmeier, and Rakesh in [19]. An alternative approach that covers both odd and even cases simultaneously was developed by Kunyansky [39]. A general method, yielding all the variety of aforementioned formulas, was suggested by L. V. Nguyen [46].

In view of practical importance, it is challenging to develop a new inversion procedure which not only yields known formulas but is also applicable to more general geometric and analytic settings, in particular, to arbitrary spaces of constant curvature. The latter is the main objective of the present article.

Regarding more general Riemannian spaces, some comments are in order. The corresponding wave equations and their EPD generalizations were studied in

[41, 33, 34], [48]-[50]. For example, the wave equation on the n-dimensional sphere S^n has the form [41]

$$\delta_x u = u_{\omega\omega} + \left(\frac{n-1}{2}\right)^2 u, \quad (x, \omega) \in S^n \times (0, \pi), \tag{1.3}$$

where δ_x denotes the Beltrami-Laplace operator. The Cauchy problem for the relevant EPD equation

$$\tilde{\Box}_\alpha u = 0, \quad u(x, 0) = f(x), \quad u_\omega(x, 0) = 0, \tag{1.4}$$

where

$$\tilde{\Box}_\alpha u = \delta_x u - u_{\omega\omega} - (n - 1 + 2\alpha) \cot \omega u_\omega + \alpha(n - 1 + \alpha)u, \tag{1.5}$$

and various modifications were discussed in [9, 10, 24, 33, 34, 49]. The problem (1.4) for the case $\alpha = 0$, corresponding to the usual Darboux equation, was studied by Olevskii [48] and also by Kipriyanov and Ivanov [33]. Our definition of the EPD-equation on S^n differs from that in [33] and agrees with [49]. The particular case $\alpha = (1 - n)/2$, corresponding to the wave equation (1.3), can be regarded as the spherical analogue of the TAT model (1.2) for constant speed.

Our inversion formulas (and the proofs) look similar in a certain sense for different constant curvature spaces. This phenomenon is known in integral geometry. In this connection, we mention the paper [28] by Gindikin, which provides a view that inversions in the three constant curvature cases are not only similar, but are essentially equivalent and can be done for all cases simultaneously using the language of differential forms and generalized functions (see analogous considerations in [26, 52]). It would be interesting to extend this point of view to the spherical mean transforms of the present article. Another important open problem is to understand why closed form inversion formulas for such transforms are available only (with a few polyhedral exceptions, as in [40]) when centers are located on a sphere or a hyperplane; see, e.g., [52, 45] and references therein.

Plan of the paper and main results. Section 2 contains preliminaries. There, the main statement is Lemma 2.2. For convenience of the reader and for better treatment of the subject, we supply this lemma with alternative proofs, which are based on different ideas while leading to the same result. All these are presented in the Appendix. Section 3 contains derivation of inversion formulas for Mf in the euclidean case. The main inversion results are presented in Theorems 3.4 and 3.7; see also the modified inversion formulas (3.23), (3.24). The results of

Section 3 are applied in Section 4 to the Cauchy problem for the Euler-Poisson-Darboux equation

$$\Box_\alpha u \equiv \Delta u - u_{tt} - \frac{n+2\alpha-1}{t} u_t = 0, \quad u(x,0) = f(x), u_t(x,0) = 0, \tag{1.6}$$

where f is a smooth function with compact support in the ball B. Using the results of Section 3, combined with known properties of Erdélyi-Kober fractional integrals, we give an explicit solution (Theorem 4.1) to the following inverse problem.

Given the trace $u(\xi,t)$ of the solution of (1.6) *for all* (ξ,t) *on the cylindrical surface $\partial B \times \mathbb{R}_+$, reconstruct $f(x)$.*

The case $\alpha = (1-n)/2$ agrees with the TAT problem (1.2) for $c(x) \equiv 1$.

The spherical mean Radon transform on the n-dimensional unit sphere S^n in $\mathbb{R}^{n+1}$ is studied in Section 5. Inversion formulas for this transform are given in Theorems 5.3 and 5.5. The concomitant inverse problem for the EPD equation on S^n is solved in Section 6. Section 7 contains the derivation of inversion formulas for the spherical mean Radon transform in n-dimensional hyperbolic space. There, the main results are given by Theorems 7.3 and 7.5.

After the present paper was publicized in arXiv:1107.5992 and submitted for publication, the authors became aware of the work by Nguyen [47] devoted to the range description for the spherical mean transform in n-dimensional hyperbolic space and on the two-dimensional sphere.

2 Auxiliary statements

Notation. We abbreviate analytic continuation as $a.c.$ and denote the area of the unit sphere S^{n-1} in $\mathbb{R}^n$ by $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$. We write $d\theta$ $(d\xi)$ for the usual Lebesgue measure on S^{n-1} (on ∂B, respectively); $[a]$ denotes the integer part of a real number a; $(\cdot)_+^\lambda$ means $(\cdot)^\lambda$ if the expression in parentheses is positive and zero, otherwise.

We need the following result.

Lemma 2.1. *Let $\varphi \in C_c^\infty(\mathbb{R})$.*

(i) *If $m = 0, 1, 2, \ldots$, then*

$$\underset{\alpha=-2m}{a.c.} \int_{\mathbb{R}} \frac{|t|^{\alpha-1}}{\Gamma(\alpha/2)} \varphi(t)\,dt = c_{m,1}\varphi^{(2m)}(0), \quad c_{m,1} = \frac{(-1)^m m!}{(2m)!}. \tag{2.1}$$

(ii) *If $m = 1, 2, \ldots$, then*

$$\underset{\alpha=1-2m}{a.c.} \int_{\mathbb{R}} \frac{|t|^{\alpha-1}}{\Gamma(\alpha/2)} \varphi(t)\,dt = c_{m,2} \int_{\mathbb{R}} \frac{\varphi^{(2m-1)}(t)}{t}\,dt \tag{2.2}$$

$$= -c_{m,2} \int_{\mathbb{R}} \varphi^{(2m)}(t) \log|t|\,dt, \tag{2.3}$$

where $c_{m,2} = (\Gamma(1/2-m)(2m-1)!)^{-1}$ *and the integral on the right-hand side of* (2.2) *is understood in the sense of principal value.*

Proof. Both statements summarize known facts from [27, Chapter 1, Sec. 3]. For instance, (ii) can be proved as follows. Using the equality

$$|t|^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+2m-1)}(|t|^{\alpha+2m-2}\text{sgn}t)^{(2m-1)},$$

we write the left-hand side of (2.2) in the form

$$-\underset{\alpha=1-2m}{a.c.}\frac{\Gamma(\alpha)}{\Gamma(\alpha+2m-1)\Gamma(\alpha/2)}(|t|^{\alpha+2m-2}\text{sgn}t, \varphi^{(2m-1)}(t)).$$

The latter yields the principal value integral

$$\frac{1}{\Gamma(1/2-m)(2m-1)!}\int_{\mathbb{R}}\frac{\varphi^{(2m-1)}(t)}{t}dt,$$

which coincides with (2.3). □

Lemma 2.2. *Let* $n \geq 2$, $|h| < 1$.

(i) *The integral*

$$g_\alpha(h) = \frac{1}{\Gamma(\alpha/2)}\int_{-1}^{1}|t-h|^{\alpha-1}(1-t^2)^{(n-3)/2}dt, \quad Re\,\alpha > 0, \tag{2.4}$$

extends as an entire function of α *and this extension represents a* C^∞ *function of* h *uniformly in* $\alpha \in K$ *for any compact subset* K *of the complex plane.*

(ii) *Moreover,*

$$\underset{\alpha=3-n}{a.c.}\, g_\alpha(h) = \Gamma\big((n-1)/2\big). \tag{2.5}$$

The proof of this lemma is given in the Appendix.

3 The euclidean case. Derivation of the inversion formula

Recall that our aim is to reconstruct a C^∞ function f supported in the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, provided that the spherical means

$$(Mf)(\xi, t) = \frac{1}{\sigma_{n-1}}\int_{S^{n-1}} f(\xi - t\sigma)d\sigma, \quad (\xi, t) \in \partial B \times \mathbb{R}_+,$$

are known for all spheres centered on the boundary ∂B of B (Fig. 1).

We introduce the "back-projection" operator P, which sends a function $F(\xi, t)$ on $\partial B \times \mathbb{R}_+$ to a function $(PF)(x)$ on B, by the formula

$$(PF)(x) = \frac{1}{|\partial B|}\int_{\partial B} F(\xi, |x-\xi|)d\xi, \quad x \in B, \tag{3.1}$$

where $d\xi$ stands for the surface element of ∂B and $|\partial B|$ denotes the area of ∂B.

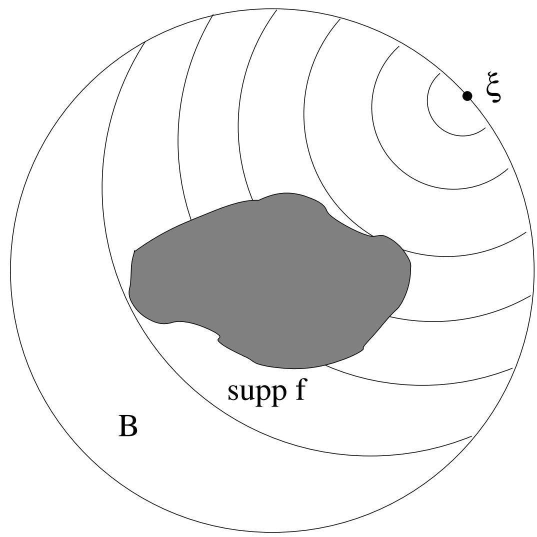

Figure 1. The euclidean case.

3.1 The case $n > 2$. Consider the analytic family of operators

$$(N^{\alpha} f)(\xi, t) = \int_B \frac{|t^2 - |y-\xi|^2|^{\alpha-1}}{\Gamma(\alpha/2)} f(y)dy, \tag{3.2}$$

$$(\xi, t) \in \partial B \times \mathbb{R}_+, \quad Re\,\alpha > 0. \tag{3.3}$$

Lemma 3.1. *Let f be an infinitely differentiable function supported in $B = \{x \in \mathbb{R}^n : |x| < R\}$. Then*

$$\underset{\alpha=3-n}{a.c.} (PN^{\alpha} f)(x) = \lambda_n \int_B \frac{f(y)}{|x-y|^{n-2}} dy, \tag{3.4}$$

$$\lambda_n = (2R)^{2-n} \pi^{-1/2} \Gamma(n/2). \tag{3.5}$$

Proof. For $Re\,\alpha > 0$, changing the order of integration, we obtain

$$(PN^{\alpha} f)(x) = \int_B f(y) k_{\alpha}(x, y)dy,$$

where

$$\begin{aligned} k_{\alpha}(x, y) &= \frac{1}{|\partial B|\Gamma(\alpha/2)} \int_{\partial B} ||x-\xi|^2 - |y-\xi|^2|^{\alpha-1} d\xi \\ &= \frac{1}{\sigma_{n-1}\Gamma(\alpha/2)} \int_{S^{n-1}} ||x|^2 - |y|^2 - 2R\theta \cdot (x-y)|^{\alpha-1} d\theta \\ &= \frac{(2R|x-y|)^{\alpha-1}}{\sigma_{n-1}} \int_{S^{n-1}} \frac{|\theta \cdot \sigma - h|^{\alpha-1}}{\Gamma(\alpha/2)} d\theta, \end{aligned} \tag{3.6}$$

$$\sigma = \frac{x-y}{|x-y|}, \qquad h = \frac{|x|^2-|y|^2}{2R|x-y|}.$$

By the rotation invariance of the inner product, the integral in (3.6) is independent of σ and can be written as

$$\frac{\sigma_{n-2}}{\Gamma(\alpha/2)}\int_{-1}^{1}|t-h|^{\alpha-1}(1-t^2)^{(n-3)/2}dt = \sigma_{n-2}g_\alpha(h);$$

cf. (2.4). Note that $|h| < 1-\delta$ for some $\delta > 0$, because x and y belong to the support of f and the latter is separated from the boundary of B. Hence Lemma 2.2 yields

$$\begin{aligned}\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) &= \frac{(2R)^{2-n}\sigma_{n-2}}{\sigma_{n-1}}\int_B \frac{f(y)}{|x-y|^{n-2}}\underset{\alpha=3-n}{a.c.}g_\alpha(h)dy\\ &= \lambda_n\int_B \frac{f(y)}{|x-y|^{n-2}}dy, \qquad \lambda_n = (2R)^{2-n}\pi^{-1/2}\Gamma(n/2).\end{aligned}$$

(To justify interchange of integration and analytic continuation, the reader may consult, e.g., [57, Lemma 1.17]). □

Let us obtain another representation of the analytic continuation of $PN^\alpha f$, now, in terms of the spherical means of f.

Lemma 3.2. *Let f be an infinitely differentiable function supported in $B = \{x \in \mathbb{R}^n : |x| < R\}$, and let*

$$D = \frac{1}{2t}\frac{d}{dt}, \qquad \delta_n = \frac{(-1)^{[n/2-1]}\Gamma((n-1)/2)}{(n-3)!}. \tag{3.7}$$

(i) *If $n = 3, 5, \ldots$, then*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \frac{\delta_n}{2R^{n-1}}\int_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi,t)]\Big|_{t=|x-\xi|}d\xi. \tag{3.8}$$

(ii) *If $n = 4, 6, \ldots$, then*

$$\begin{aligned}\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = -\frac{\delta_n}{\pi R^{n-1}}\int_{\partial B}d\xi\int_0^{2R} tD^{n-2}[t^{n-2}(Mf)(\xi,t)]\\ \times \log|t^2-|x-\xi|^2|dt.\end{aligned} \tag{3.9}$$

Proof. Passing to polar coordinates, we have

$$\begin{aligned}(N^\alpha f)(\xi,t) &= \sigma_{n-1}\int_0^{2R}\frac{|t^2-r^2|^{\alpha-1}}{\Gamma(\alpha/2)}(Mf)(\xi,r)r^{n-1}dr\\ &= \int_0^{4R^2}\frac{|t^2-\tau|^{\alpha-1}}{\Gamma(\alpha/2)}\varphi_\xi(\tau)d\tau,\\ \varphi_\xi(\tau) &= \frac{\sigma_{n-1}}{2}\tau^{n/2-1}(Mf)(\xi,\tau^{1/2}).\end{aligned}$$

Since the support of f is separated from the boundary of B, there is an $\varepsilon > 0$ such that $\varphi_\xi(\tau) \equiv 0$ when $\tau \notin (\varepsilon, 4R^2 - \varepsilon)$. Hence $\varphi_\xi(\tau)$ can be regarded as a function in $C_c^\infty(\mathbb{R})$, and we can write

$$(N^\alpha f)(\xi, t) = \int_{\mathbb{R}} \frac{|\tau|^{\alpha-1}}{\Gamma(\alpha/2)} \varphi_\xi(\tau + t^2) d\tau.$$

Now, Lemma 2.1 yields the following equalities. For $n = 3, 5, \ldots,$

$$\underset{\alpha=3-n}{a.c.} (N^\alpha f)(\xi, t) = \delta_{n,1} \varphi_\xi^{(n-3)}(t^2), \qquad \delta_{n,1} = \frac{(-1)^{(n-3)/2}((n-3)/2)!}{(n-3)!}.$$

For $n = 4, 6, \ldots,$

$$\underset{\alpha=3-n}{a.c.} (N^\alpha f)(\xi, t) = \delta_{n,2} \int_{\mathbb{R}} \varphi_\xi^{(n-2)}(\tau) \log|\tau - t^2| d\tau,$$
$$\delta_{n,2} = -\frac{1}{\Gamma((3-n)/2)(n-3)!}.$$

Combining these formulas with the back-projection P and noting that the operations $a.c.$ and P commute, we obtain the following. For $n = 3, 5, \ldots,$

$$\underset{\alpha=3-n}{a.c.} (PN^\alpha f)(x) = \frac{\delta_{n,1}}{|\partial B|} \int_{\partial B} \varphi_\xi^{(n-3)}(|x-\xi|^2) d\xi.$$

For $n = 4, 6, \ldots,$

$$\underset{\alpha=3-n}{a.c.} (PN^\alpha f)(x) = \frac{\delta_{n,2}}{|\partial B|} \int_{\partial B} d\xi \int_0^{4R^2} \varphi_\xi^{(n-2)}(\tau) \log|\tau - |x-\xi|^2| d\tau.$$

These formulas give the desired result. □

Comparing different forms of the analytic continuation in Lemmas 3.1 and 3.2, we obtain the following statement.

Lemma 3.3. *Let f be an infinitely differentiable function supported in the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, and $D = \frac{1}{2t}\frac{d}{dt}$. Then*

$$\lambda_n \int_B \frac{f(y)}{|x-y|^{n-2}} dy = \begin{cases} T_{odd} & \text{if } n = 3, 5, \ldots, \\ T_{even} & \text{if } n = 4, 6, \ldots, \end{cases} \tag{3.10}$$

where

$$T_{odd} = \frac{\delta_n}{2R^{n-1}} \int_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi, t)]\Big|_{t=|x-\xi|} d\xi,$$
$$T_{even} = -\frac{\delta_n}{\pi R^{n-1}} \int_{\partial B} d\xi \int_0^{2R} t D^{n-2}[t^{n-2}(Mf)(\xi, t)] \log|t^2 - |x-\xi|^2| dt,$$

and λ_n and δ_n are defined by (3.5) *and* (3.7)*, respectively.*

The left-hand side of (3.10) is a constant multiple of the Riesz potential of order 2 defined by

$$(I^2 f)(x) = \frac{\Gamma(n/2-1)}{4\pi^{n/2}} \int_B \frac{f(y)dy}{|x-y|^{n-2}} \tag{3.11}$$

and satisfying

$$-\Delta I^2 f = f, \quad \Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}. \tag{3.12}$$

Thus, we arrive at the following result.

Theorem 3.4. *Let f be an infinitely differentiable function supported in the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, and let $D = \frac{1}{2t}\frac{d}{dt}$.*

(i) *If $n = 3, 5, \ldots$, then*

$$\begin{aligned} f(x) &= d_{n,1}\Delta \int_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi,t)]\Big|_{t=|x-\xi|} d\xi, \\ d_{n,1} &= \frac{(-1)^{(n-1)/2}\pi^{1-n/2}}{4R\Gamma(n/2)}. \end{aligned} \tag{3.13}$$

(ii) *If $n = 4, 6, \ldots$, then*

$$\begin{aligned} f(x) &= d_{n,2}\Delta \int_{\partial B} d\xi \int_0^{2R} tD^{n-2}[t^{n-2}(Mf)(\xi,t)] \log|t^2 - |x-\xi|^2| dt, \\ d_{n,2} &= \frac{(-1)^{n/2-1}\pi^{-n/2}}{2R(n/2-1)!}. \end{aligned} \tag{3.14}$$

3.2 The case $n = 2$ Let $\mathcal{D}$ be the open disk in $\mathbb{R}^2$ of radius R centered at the origin. In this section, for the sake of completeness, we reproduce (with minor changes) the argument from [19], keeping in mind that the Riesz potential of order 2 in the previous section is replaced with the logarithmic potential

$$(I_* f)(x) = \frac{1}{2\pi} \int_{\mathcal{D}} f(y) \log|x-y| dy, \tag{3.15}$$

which satisfies $\Delta I_* f = f$.

The following statement is a substitute for Lemma 2.2.

Lemma 3.5. *Let $-1 < h < 1$ and $\sigma \in S^1$. Then*

$$g_* \equiv \int_{S^1} \log|\theta \cdot \sigma - h| d\theta = -2\pi \log 2. \tag{3.16}$$

Proof. Owing to rotational invariance, we can write

$$g_* = 2\int_{-1}^{1} \frac{\log|t-h|}{\sqrt{1-t^2}}dt. \tag{3.17}$$

This integral is known; see, e.g., [17, p. 296], [19].[1] □

Lemma 3.6. *Let f be a C^∞ function supported in $\mathcal{D}$. Then*

$$(I_* f)(x) = \frac{1}{2\pi R}\int_{\partial\mathcal{D}}\int_0^{2R} (Mf)(\xi,t)\log\left|t^2-|x-\xi|^2\right| t\,dt\,d\xi + c_f, \tag{3.18}$$

$$c_f = -\frac{\log R}{2\pi}\int_{\mathcal{D}} f(y)dy. \tag{3.19}$$

Proof. Let $(N_* f)(\xi,t) = \int_{\mathcal{D}} f(y)\log|t^2-|y-\xi|^2|dy$. Changing the order of integration and making use of (3.16) together with $\sigma = (x-y)/|x-y|$ and $h = (|x|^2-|y|^2)/2R|x-y|$, we obtain $(PN_* f)(x) = \int_{\mathcal{D}} f(y)k_*(x,y)dy$, where

$$\begin{aligned} k_*(x,y) &= \frac{1}{2\pi R}\int_{\partial\mathcal{D}} \log\left||x-\xi|^2-|y-\xi|^2\right| d\xi \\ &= \frac{1}{2\pi R}\int_{\partial\mathcal{D}} \log\left||x|^2-|y|^2-2\xi\cdot(x-y)\right| d\xi \\ &= \frac{1}{2\pi}\int_{S^1} (\log(2R\,|x-y|)+\log|h-\theta\cdot\sigma|)\,d\theta \\ &= \log(2R\,|x-y|)+\frac{g_*}{2\pi} = \log R+\log|x-y|. \end{aligned}$$

This gives

$$(PN_* f)(x) = \int_{\mathcal{D}} f(y)\log|x-y|dy + \log R\int_{\mathcal{D}} f(y)dy. \tag{3.20}$$

On the other hand, $(PN_* f)(x)$ can be expressed in terms of the spherical means. Indeed, passing to polar coordinates, we have

$$(N_* f)(\xi,t) = 2\pi\int_0^{2R} (Mf)(\xi,r)\log|r^2-t^2|r\,dr,$$

and therefore,

$$(PN_* f)(x) = \frac{1}{R}\int_{\partial\mathcal{D}}\int_0^{2R} (Mf)(\xi,r)\log\left|r^2-|x-\xi|^2\right| r\,dr\,d\xi. \tag{3.21}$$

Comparing (3.20) and (3.21), we arrive at (3.18). □

Lemma 3.6 allows us to complete Theorem 3.4 as follows.

[1] A more general integral was evaluated in [7, Lemma 6.1].

Theorem 3.7. *Let f be an infinitely differentiable function supported in the disk $\mathcal{D} = \{x \in \mathbb{R}^2 : |x| < R\}$. Then*

$$f(x) = \Delta \left(\frac{1}{2\pi R} \int_{\partial \mathcal{D}} \int_0^{2R} (Mf)(\xi, t) \log \left| t^2 - |x-\xi|^2 \right| t dt d\xi \right). \tag{3.22}$$

Formula (3.22) can be obtained formally from (3.14) by setting $n = 2$. It coincides with [19, (1.4)].

3.3 Modified inversion formulas.

We can replace f by Δf in (3.10). Since f is smooth and $\operatorname{supp} f$ is separated from the boundary of B, $I^2 \Delta f = -f$. Furthermore, since $u(x,t) \equiv (Mf)(x,t)$ satisfies the Darboux equation

$$\Box u \equiv \Delta u - u_{tt} - \frac{n-1}{t} u_t = 0$$

and Δ commutes with rotations and translations,

$$(M\Delta f)(x,t) = (\Delta M f)(x,t) = L[(Mf)(x,\cdot)](t) \quad \text{for all } x \in \mathbb{R}^n,\ t > 0,$$

where

$$L = \frac{d^2}{dt^2} + \frac{n-1}{t}\frac{d}{dt};$$

see, e.g., [30, p. 17]. This reasoning and its obvious analogue for $n = 2$ give the following modifications of inversion formulas (3.13), (3.14), and (3.22) with the same constant factors.

(i) If $n = 3, 5, \dots$, then

$$f(x) = d_{n,1} \int_{\partial B} D^{n-3}[t^{n-2}(LMf)(\xi,t)]\Big|_{t=|x-\xi|} d\xi. \tag{3.23}$$

(ii) If $n = 2, 4, 6, \dots$, then

$$f(x) = d_{n,2} \int_{\partial B} d\xi \int_0^{2R} t D^{n-2}[t^{n-2}(LMf)(\xi,t)] \log |t^2 - |x-\xi|^2| dt. \tag{3.24}$$

Formula (3.23) agrees with [59, formula (3.8)].

4 Spherical means and EPD equations

Consider the Cauchy problem for the Euler-Poisson-Darboux equation:

$$\Box_\alpha u \equiv \Delta u - u_{tt} - \frac{n+2\alpha-1}{t} u_t = 0, \tag{4.1}$$

$$u(x,0) = f(x), \qquad u_t(x,0) = 0. \tag{4.2}$$

As in the previous section, we assume that f is a smooth function with compact support in the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$. If $\alpha \geq (1-n)/2$, then (4.1)-(4.2) has the unique solution

$$u(x,t) = (M^\alpha f)(x,t), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}_+, \tag{4.3}$$

where $M^\alpha f$ is defined as analytic continuation of the integral

$$(M^\alpha f)(x,t) = \frac{\Gamma(\alpha+n/2)}{\pi^{n/2}\Gamma(\alpha)} \int_{|y|<1} (1-|y|^2)^{\alpha-1} f(x-ty)dy, \quad Re\,\alpha > 0;$$

see [8] for details. If $\alpha = 0$, then $M^0 f \equiv \underset{\alpha=0}{a.c.} M^\alpha f$ yields the spherical mean (1.1).

Consider the following problem.

Given the trace $u(\xi, t)$ *of the solution of* (4.1) - (4.2) *for all* $(\xi, t) \in \partial B \times \mathbb{R}_+$, *reconstruct* $f(x)$.

To solve this problem, we need some facts from fractional calculus; see, e.g., [60, Sec. 18.1] or [17, Sec. 9.6]. For $Re\,\alpha > 0$ and $\eta \geq -1/2$, the Erdélyi-Kober fractional integral of a function φ on $\mathbb{R}_+$ is defined by

$$(I_\eta^\alpha \varphi)(t) = \frac{2t^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^t (t^2 - r^2)^{\alpha-1} r^{2\eta+1} \varphi(r)dr, \quad t > 0. \tag{4.4}$$

In our case, it suffices to assume that φ is infinitely smooth and supported away from the origin. Then $I_\eta^\alpha \varphi$ extends as an entire function of α and η, so that

$$I_\eta^0 \varphi = \varphi, \quad \left(I_\eta^\alpha\right)^{-1} \varphi = I_{\eta+\alpha}^{-\alpha}\varphi,$$
$$(I_\eta^{-m}\varphi)(t) = t^{-2(\eta-m)} D^m t^{2\eta}\varphi(t), \quad D = \frac{1}{2t}\frac{d}{dt}.$$

Assuming $x = \xi \in \partial B$ and passing to polar coordinates, we obtain

$$u_\xi(t) = (M^\alpha f)(\xi, t) = \frac{\Gamma(\alpha+n/2)}{\Gamma(n/2)} \left(I_\eta^\alpha \varphi_\xi\right)(t), \tag{4.5}$$

where $u_\xi(t) = u(\xi,t)$, $\varphi_\xi(t) = (Mf)(\xi,t)$, and $\eta = n/2-1$. This gives

$$\varphi_\xi = \frac{\Gamma(n/2)}{\Gamma(\alpha+n/2)} \left(I_\eta^\alpha\right)^{-1} u_\xi = \frac{\Gamma(n/2)}{\Gamma(\alpha+n/2)} I_{\eta+\alpha}^{-\alpha} u_\xi. \tag{4.6}$$

Now, since $\varphi_\xi(t) = (Mf)(\xi,t)$ is known, we can use Theorem 3.4 to reconstruct f by the formulas given in the following theorem.

Theorem 4.1. *Let f be an infinitely differentiable function supported in the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, $D = (2t)^{-1}d/dt$.*

(i) *If $n = 3, 5, \ldots$, then*

$$f(x) = \widetilde{d}_{n,1}\Delta \int_{\partial B} D^{n-3}[t^{n-2}(I^{-\alpha}_{\eta+\alpha}u_\xi)(t)]\Big|_{t=|x-\xi|} d\xi,$$
$$\widetilde{d}_{n,1} = \frac{(-1)^{(n-1)/2}\pi^{1-n/2}}{4R\Gamma(\alpha+n/2)}.$$

(ii) *If $n = 2, 4, 6, \ldots$, then*

$$f(x) = \widetilde{d}_{n,2}\Delta \int_{\partial B} d\xi \int_0^{2R} tD^{n-2}[t^{n-2}(I^{-\alpha}_{\eta+\alpha}u_\xi)(t)]\log|t^2 - |x-\xi|^2|dt,$$
$$\widetilde{d}_{n,2} = \frac{(-1)^{n/2-1}\pi^{-n/2}}{2R\Gamma(\alpha+n/2)}.$$

The case $\alpha = (1-n)/2$ in this theorem gives an explicit solution to the TAT problem (see Introduction) with constant speed $c(x) \equiv 1$. Moreover, once f has been found, we can reconstruct $u(x,t)$ in the whole space by setting $u(x,t) = (M^\alpha f)(x,t)$, giving an explicit solution of the Cauchy problem for the generalized EPD equation (4.1) with initial data on the cylinder $\partial B \times \mathbb{R}_+$.

5 Spherical means on S^n

The suggested method of analytic continuation enables us to study the spherical mean Radon transform Mf on an arbitrary space X of constant curvature. In this setting, $\operatorname{supp} f \subset B$, where B is a geodesic ball centered at the origin, and the spherical means of f are evaluated over geodesic spheres, the centers of which are located on the boundary of B. In this section, we consider the case when $X = S^n$ is the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.

Given $x \in S^n$, $f \in C^\infty(S^n)$, and $t \in (-1, 1)$, let

$$(5.1) \qquad (Mf)(x,t) = \frac{(1-t^2)^{(1-n)/2}}{\sigma_{n-1}} \int_{x\cdot y=t} f(y)d\sigma(y)$$

be the mean value of f over the planar section $\{y \in S^n : x \cdot y = t\}$; see Figure 2.

Our aim is to reconstruct f under the following assumptions.

(a) The support of f lies on the spherical cap

$$(5.2) \qquad B_\theta = \{x \in S^n : x \cdot e_{n+1} > \cos\theta\},$$

(the geodesic ball of radius θ), where $e_{n+1} = (0, \ldots, 0, 1)$ is the north pole of S^n and $\theta \in (0, \pi/2]$ is fixed.

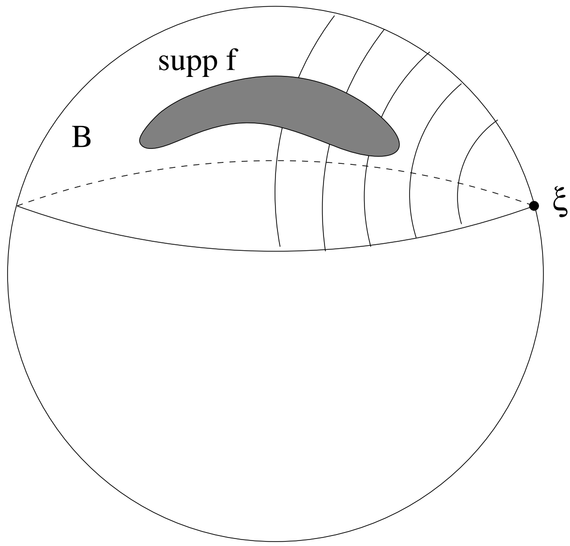

Figure 2. The spherical case.

(b) The mean values (5.1) are known for all $x = \xi \in \partial B_\theta$ and all $t \in (-1, 1)$, where ∂B_θ is the boundary of B_θ.

This problem can be solved using the method of the previous section. Let us fix our notation. In the following, $S^n_+ = \{x \in S^n : x_{n+1} \geq 0\}$ is the upper hemisphere, B_0 denotes the unit ball in the hyperplane $x_{n+1} = 0$; S^{n-1} stands for the boundary of B_0, which is also the boundary of S^n_+. For $x \in S^n_+$, we write

$$x = (x', \sqrt{1 - |x'|^2}), \quad x' = (x_1, \dots, x_n, 0) \in B_0,$$

so that

$$\int_{S^n_+} f(x)dx = \int_{B_0} f(x', \sqrt{1 - |x'|^2}) \sqrt{1 + \left(\frac{\partial x_{n+1}}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial x_{n+1}}{\partial x_n}\right)^2} dx'$$
$$= \int_{B_0} f(x', \sqrt{1 - |x'|^2}) \frac{dx'}{\sqrt{1 - |x'|^2}}.$$

We introduce the back-projection operator P, which sends functions on $\partial B_\theta \times (-1, 1)$ to functions on B_θ by the formula

$$(PF)(x) = \frac{1}{|\partial B_\theta|} \int_{\partial B_\theta} F(\xi, \xi \cdot x) d\xi, \quad x \in B_\theta. \tag{5.3}$$

Here, $d\xi$ and $|\partial B_\theta|$ denote the surface element and the area of ∂B_θ, respectively. We denote by $\tilde{B}_\theta$ the orthogonal projection of B_θ onto the hyperplane $x_{n+1} = 0$.

5.1 The case $n > 2$. Assuming $(\xi, t) \in \partial B_\theta \times (-1, 1)$ and $Re\,\alpha > 0$, consider the following analytic family of operators

$$(N^\alpha f)(\xi, t) = \int_{B_\theta} \frac{|\xi \cdot y - t|^{\alpha - 1}}{\Gamma(\alpha/2)} f(y) dy. \tag{5.4}$$

Lemma 5.1. *Let $f \in C^\infty(S^n)$, $\text{supp}\, f \subset B_\theta$. Then*

$$\underset{\alpha = 3-n}{a.c.} (PN^\alpha f)(x) = \frac{\Gamma(n/2)(\sin\theta)^{2-n}}{\pi^{1/2}} \int_{\tilde{B}_\theta} \frac{\tilde{f}(y') dy'}{|x' - y'|^{n-2}}, \tag{5.5}$$

$$\tilde{f}(y') = (1 - |y'|^2)^{-1/2} f(y', (1 - |y'|^2)^{1/2}). \tag{5.6}$$

Proof. For $Re\,\alpha > 0$, changing the order of integration, we obtain[2]

$$(PN^\alpha f)(x) = \int_{B_\theta} f(y) k_\alpha(x, y), dy, \quad k_\alpha(x, y) = \frac{1}{|\partial B_\theta|} \int_{\partial B_\theta} \frac{|\xi \cdot (x - y)|^{\alpha - 1}}{\Gamma(\alpha/2)} d\xi.$$

Since ξ has the form $\xi = e_{n+1} \cos\theta + \omega \sin\theta$, $\omega \in S^{n-1}$,

$$\begin{aligned} |\xi \cdot (x - y)| &= |(x_{n+1} - y_{n+1}) \cos\theta + (x' - y') \cdot \omega \sin\theta| \\ &= |h - \omega \cdot \sigma|\, |x' - y'| \sin\theta, \end{aligned} \tag{5.7}$$

$$h = \frac{x_{n+1} - y_{n+1}}{|x' - y'|} \cot\theta, \qquad \sigma = \frac{x' - y'}{|x' - y'|}. \tag{5.8}$$

Hence

$$k_\alpha(x, y) = \frac{(|x' - y'| \sin\theta)^{\alpha - 1}}{\sigma_{n-1}} \int_{S^{n-1}} \frac{|h - \omega \cdot \sigma|^{\alpha - 1}}{\Gamma(\alpha/2)} d\omega. \tag{5.9}$$

The integral in (5.9) is independent of ξ and can be written as

$$\frac{\sigma_{n-2}}{\Gamma(\alpha/2)} \int_{-1}^{1} |t - h|^{\alpha - 1} (1 - t^2)^{(n-3)/2} dt = \sigma_{n-2} g_\alpha(h);$$

cf. (2.4). This gives

$$k_\alpha(x, y) = \frac{\sigma_{n-2} (|x' - y'| \sin\theta)^{\alpha - 1}}{\sigma_{n-1}} g_\alpha(h). \tag{5.10}$$

Let us show that $|h| < 1$. We write

$$x = e_{n+1} \cos\gamma + u \sin\gamma, \quad y = e_{n+1} \cos\delta + v \sin\delta,$$
$$\gamma, \delta \in (0, \theta); \quad u, v \in S^{n-1}; \quad \tilde{h} = \frac{x_{n+1} - y_{n+1}}{|x' - y'|}.$$

[2]For convenience, we use some notation which mimics analogous expressions in the euclidean case.

Then

$$\begin{aligned}|\tilde{h}|^2 &= \frac{(\cos\gamma - \cos\delta)^2}{|u\sin\gamma - v\sin\delta|^2}\\ &= \frac{(\cos\gamma - \cos\delta)^2}{\sin^2\gamma - 2(u\cdot v)\sin\gamma\sin\delta + \sin^2\delta}\\ &\le \frac{(\cos\gamma - \cos\delta)^2}{\sin^2\gamma - 2\sin\gamma\sin\delta + \sin^2\delta} = \frac{(\cos\gamma - \cos\delta)^2}{(\sin\gamma - \sin\delta)^2}.\end{aligned}$$

Without loss of generality, suppose that $\gamma \le \delta$. Then

$$|\tilde{h}| \le \frac{\cos\gamma - \cos\delta}{\sin\delta - \sin\gamma} = \tan\frac{\gamma+\delta}{2} < \tan\theta,$$

and therefore, $|h| = |\tilde{h}|\cot\theta < 1$.

Since $|h| < 1$, Lemma 2.2 yields

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = c_n\int_{B_\theta}\frac{f(y)dy}{|x'-y'|^{n-2}} = c_n\int_{\tilde{B}_\theta}\frac{\tilde{f}(y')dy'}{|x'-y'|^{n-2}},$$

$$\tilde{f}(y') = (1-|y'|^2)^{-1/2}f(y',(1-|y'|^2)^{1/2}), \quad c_n = \frac{\Gamma(n/2)(\sin\theta)^{2-n}}{\pi^{1/2}}. \qquad \square$$

As before, we need one more representation of $\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x)$, now in terms of the spherical means $(Mf)(\xi,t)$.

Lemma 5.2. *Let $f \in C^\infty(S^n)$, $\operatorname{supp} f \subset B_\theta$. Then*

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \frac{\delta_n}{(\sin\theta)^{n-1}}\int_{\partial B_\theta}(d/dt)^{n-3}[(Mf)(\xi,t)(1-t^2)^{n/2-1}]\Big|_{t=\xi\cdot x}d\xi,$$

if $n = 3, 5, \ldots$, and

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = -\frac{\delta_n}{\pi(\sin\theta)^{n-1}}\int_{\partial B_\theta}d\xi \times\int_{\cos 2\theta}^1(d/dt)^{n-2}[(Mf)(\xi,t)(1-t^2)^{n/2-1}]\log|t-\xi\cdot x|dt,$$

if $n = 4, 6, \ldots$, where δ_n is defined by (3.7).

Proof. For $Re\,\alpha > 0$, by making use of the formula

$$\int_{S^n} f(y)a(\xi\cdot y)dy = \sigma_{n-1}\int_{-1}^1 a(\tau)(Mf)(\xi,\tau)(1-\tau^2)^{n/2-1}d\tau, \tag{5.11}$$

we have

$$\begin{aligned}(N^\alpha f)(\xi,t) &= \frac{\sigma_{n-1}}{\Gamma(\alpha/2)}\int_{-1}^{1}(Mf)(\xi,\tau)|\tau-t|^{\alpha-1}(1-\tau^2)^{n/2-1}d\tau\\ &= \int_{\mathbb{R}}\frac{|\tau|^{\alpha-1}}{\Gamma(\alpha/2)}\varphi_\xi(\tau+t)d\tau,\\ \varphi_\xi(\tau) &= \sigma_{n-1}(Mf)(\xi,\tau)(1-\tau^2)_+^{n/2-1}.\end{aligned}$$

Since f is smooth and its support is separated from the boundary ∂B_θ, $(Mf)(\xi,\tau)$ is smooth in the τ-variable uniformly in ξ and vanishes identically in the respective neighborhoods of $\tau = \pm 1$. Thus, we can invoke Lemma 2.1, which yields the following equalities.

For $n = 3, 5, \dots, \cos 2\theta < t < 1$,

$$\underset{\alpha=3-n}{a.c.}(N^\alpha f)(\xi,t) = \delta_n \varphi_\xi^{(n-3)}(t).$$

For $n = 4, 6, \dots$,

$$\underset{\alpha=3-n}{a.c.}(N^\alpha f)(\xi,t) = -\frac{\delta_n}{\pi}\int_{\cos 2\theta}^{1}\varphi_\xi^{(n-2)}(\tau)\log|\tau-t|d\tau,$$

δ_n being defined by (3.7). The above formulas mimic those in the proof of Lemma 3.2 and the result follows. □

Lemmas 5.1 and 5.2 imply the following inversion result for the spherical means on S^n. In the statement below, $\Delta_{x'} = \partial_1^2 + \dots + \partial_n^2$ is the usual Laplace operator in the x'-variable.

Theorem 5.3. *Let $f \in C^\infty(S^n)$,* supp $f \subset B_\theta$. *Then*

$$f(x) = \frac{d_n x_{n+1}}{\sin\theta}\Delta_{x'} f_0(x', \sqrt{1-|x'|^2}), \quad d_n = \frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)}, \tag{5.12}$$

where $f_0(x) \equiv f_0(x', \sqrt{1-|x'|^2})$ has the form

$$f_0(x) = -\int_{\partial B_\theta}(d/dt)^{n-3}[(Mf)(\xi,t)(1-t^2)^{n/2-1}]\Big|_{t=\xi\cdot x}d\xi$$

if $n = 3, 5, \dots$, and

$$f_0(x) = \frac{1}{\pi}\int_{\partial B_\theta}d\xi\int_{\cos 2\theta}^{1}(d/dt)^{n-2}[(Mf)(\xi,t)(1-t^2)^{n/2-1}]\log|t-\xi\cdot x|dt$$

if $n = 4, 6, \dots$.

5.2 The case $n = 2$. We keep the notation of Section 5.1. Let

$$(I_* f)(x) \equiv (I_* f)(x', \sqrt{1-|x'|^2}) = \frac{1}{2\pi} \int_{B_\theta} f(y) \log |x' - y'| dy, \tag{5.13}$$

so that

$$\Delta_{x'}(I_* f)(x) = (1 - |x'|^2)^{-1/2} f(x) = f(x)/x_3. \tag{5.14}$$

Lemma 5.4. *Let f be a C^∞ function supported in B_θ. Then*

$$\begin{aligned}(I_* f)(x) &= \frac{1}{|\partial B_\theta|} \int_{\partial B_\theta} \int_{-1}^{1} (Mf)(\xi, \tau) \log |\tau - \xi \cdot x| d\tau d\xi + c_f, \\ c_f &= -\frac{1}{2\pi}\Big(\log \frac{\sin\theta}{2}\Big) \int_{B_\theta} f(y) dy.\end{aligned} \tag{5.15}$$

Proof. Let

$$(N_* f)(\xi, t) = \int_{B_\theta} f(y) \log |\xi \cdot y - t| dy, \quad (\xi, t) \in \partial B_\theta \times (-1, 1).$$

Changing the order of integration, owing to (5.7), we obtain

$$(PN_* f)(x) = \int_{B_\theta} f(y) k_*(x, y) dy,$$

where

$$\begin{aligned} k_*(x, y) &= \frac{1}{|\partial B_\theta|} \int_{\partial B_\theta} \log |\xi \cdot (x - y)| d\xi \\ &= \frac{1}{2\pi} \int_{S^1} [\log |x' - y'| + \log \sin\theta + \log |h - \omega \cdot \sigma|] d\omega \\ &= \log |x' - y'| + \log \sin\theta + \frac{g_*}{2\pi}, \end{aligned}$$

$$g_* \equiv \int_{S^1} \log |h - \omega \cdot \sigma| d\omega = 2 \int_{-1}^{1} \frac{\log |t - h|}{\sqrt{1 - t^2}} dt = -2\pi \log 2;$$

cf. Lemma 3.5. This gives

$$(PN_* f)(x) = 2\pi (I_* f)(x) + \Big(\log \frac{\sin\theta}{2}\Big) \int_{B_\theta} f(y) dy. \tag{5.16}$$

On the other hand, by (5.11),

$$(PN_* f)(x) = \frac{1}{\sin\theta} \int_{\partial B_\theta} \int_{-1}^{1} (Mf)(\xi, \tau) \log |\tau - \xi \cdot x| d\tau d\xi. \tag{5.17}$$

Comparing (5.17) with (5.16), we obtain the result. □

Lemma 5.4 allows us to complete Theorem 5.3 in the following way.

Theorem 5.5. *Let f be an infinitely differentiable function supported in the spherical cap $B_\theta = \{x \in S^2 : x \cdot e_3 > \cos\theta\}$, $\theta \in (0, \pi/2]$. Then*

$$f(x) = \frac{x_3}{2\pi \sin\theta} \Delta_{x'} \int_{\partial B_\theta} \int_{-1}^{1} (Mf)(\xi, \tau) \log|\tau - \xi \cdot x| d\tau d\xi. \tag{5.18}$$

Formula (5.18) can be obtained formally from (5.12) by setting $n = 2$.

6 The inverse problem for the EPD equation on S^n

The Euler-Poisson-Darboux equation on S^n has the form

$$\tilde{\Box}_\alpha u \equiv \delta_x u - u_{\omega\omega} - (n - 1 + 2\alpha) \cot\omega u_\omega + \alpha(n - 1 + \alpha)u = 0. \tag{6.1}$$

Here, $x \in S^n$ is the space variable, $\omega \in (0, \pi)$ is the time variable, and δ_x is the relevant Beltrami-Laplace operator. For simplicity, we restrict ourselves to the case $Re\,\alpha > -n/2$. In this case the corresponding Cauchy problem

$$\tilde{\Box}_\alpha u = 0, \qquad u(x, 0) = f(x), \quad u_\omega(x, 0) = 0, \tag{6.2}$$

with $f \in C^\infty(S^n)$ has a solution $u(x, \omega) = (M^\alpha f)(x, \cos\omega)$, where $(M^\alpha f)(x, t)$ is defined as the analytic continuation of the integral

$$(M^\alpha f)(x, t) = \frac{c_{n,\alpha}}{(1 - t^2)^{\alpha - 1 + n/2}} \int_{x \cdot y > t} (x \cdot y - t)^{\alpha - 1} f(y) dy, \tag{6.3}$$

$$c_{n,\alpha} = 2^{\alpha-1} \pi^{-n/2} \Gamma(\alpha + n/2)/\Gamma(\alpha), \quad Re\alpha > 0, \quad t \in (-1, 1);$$

see [49], [58, p. 179], and references therein.

Let $B_\theta = \{x \in S^n : x \cdot e_{n+1} > \cos\theta\}$ be the spherical cap of a fixed radius $\theta \in (0, \pi/2]$, and let ∂B_θ be the boundary of B_θ. Our aim is to solve the following inverse problem.

Inverse problem. *Suppose that the values $g(\xi, \omega)$ of the solution of* (6.2) *are known for all $(\xi, \omega) \in \partial B_\theta \times (0, \pi)$. Reconstruct the initial function $f \in C^\infty(S^n)$, provided that the support of f lies in B_θ.*

This problem can be solved using the results of the previous section. Assuming $Re\,\alpha > 0$, we pass to spherical polar coordinates and write (6.3) as

$$(M^\alpha f)(\xi, t) = \frac{c_{n,\alpha}\sigma_{n-1}}{(1 - t^2)^{\alpha - 1 + n/2}} \int_t^1 (\tau - t)^{\alpha - 1} (Mf)(\xi, \tau)(1 - \tau^2)^{n/2 - 1} d\tau.$$

Then we set

$$F_\xi(t) = (Mf)(\xi, t)(1-t^2)^{n/2-1},$$
$$G_{\alpha,\xi}(t) = \frac{2^{1-\alpha}\pi^{n/2}}{\Gamma(\alpha+n/2)\sigma_{n-1}}(1-t^2)^{\alpha-1+n/2} g(\xi, \cos^{-1} t),$$

and invoke Riemann-Liouville fractional integrals [60]

$$(I^\alpha_- u)(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (\tau - t)^{\alpha-1} u(\tau) d\tau, \quad Re\, \alpha > 0. \tag{6.4}$$

Thus, if $Re\, \alpha > 0$,

$$(I^\alpha_- F_\xi)(t) = G_{\alpha,\xi}(t). \tag{6.5}$$

Since f is infinitely differentiable and the support of f is separated from the boundary ∂B_θ, F_ξ is infinitely differentiable on $(-1, 1)$ uniformly in ξ and supp$\,F_\xi$ does not meet the endpoints ± 1. It follows that (6.5) extends by analyticity to all complex α, and we have $(Mf)(\xi, t) = (1-t^2)^{1-n/2}(I^{-\alpha}_- G_{\alpha,\xi})(t)$, where $I^{-\alpha}_-$ is understood in the sense of analytic continuation. Now Theorem 5.3 yields the following explicit solution of our inverse problem.

Theorem 6.1. *Let B_θ be the spherical cap on S^n of radius $\theta \in (0, \pi/2]$,*

$$G_{\alpha,\xi}(t) = \frac{2^{1-\alpha}\pi^{n/2}}{\Gamma(\alpha+n/2)\sigma_{n-1}}(1-t^2)^{\alpha-1+n/2} g(\xi, \cos^{-1} t), \quad Re\, \alpha > -n/2,$$

where g is a given function on $\partial B_\theta \times (0, \pi)$. If the initial function f in the Cauchy problem (6.2) *is infinitely differentiable and the support of f lies in the interior of B_θ, then f can be reconstructed by the formula*

$$f(x) = \frac{d_n x_{n+1}}{\sin\theta} \Delta_{x'} f_0(x', \sqrt{1-|x'|^2}), \quad d_n = \frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)}, \tag{6.6}$$

where $f_0(x) \equiv f_0(x', \sqrt{1-|x'|^2})$ has the form

$$f_0(x) = -\int_{\partial B_\theta} (d/dt)^{n-3}[(I^{-\alpha}_- G_{\alpha,\xi})(t)]\Big|_{t=\xi\cdot x} d\xi$$

if $n = 3, 5, \ldots$, and

$$f_0(x) = \frac{1}{\pi}\int_{\partial B_\theta} d\xi \int_{\cos 2\theta}^1 (d/dt)^{n-2}[(I^{-\alpha}_- G_{\alpha,\xi})(t)] \log|t - \xi\cdot x| dt$$

if $n = 2, 4, 6, \ldots$.

We recall that the case $\alpha = (1-n)/2$ in this theorem gives a solution of the relevant inverse problem for the wave equation on S^n in the framework of the formal spherical TAT model.

7 Spherical means in hyperbolic space

Most of the facts listed below can be found in [65]. Let $\mathbb{E}^{n,1}$, $n \geq 2$, be the real pseudo-euclidean space of points $x = (x_1, \dots, x_{n+1})$ with the inner product

$$[x, y] = -x_1 y_1 - \cdots - x_n y_n + x_{n+1} y_{n+1}. \tag{7.1}$$

The hyperbolic space $\mathbb{H}^n$ is interpreted as the "upper" sheet of the two-sheeted hyperboloid

$$\mathbb{H}^n = \{x \in \mathbb{E}^{n,1} : [x, x] = 1, x_{n+1} > 0\}. \tag{7.2}$$

The hyperbolic coordinates of a point $x = (x_1, \dots, x_{n+1}) \in \mathbb{H}^n$ are defined by

$$\begin{cases} x_1 = \sinh r \sin \omega_{n-1} \dots \sin \omega_2 \sin \omega_1, \\ x_2 = \sinh r \sin \omega_{n-1} \dots \sin \omega_2 \cos \omega_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n = \sinh r \cos \omega_{n-1}, \\ x_{n+1} = \cosh r, \end{cases} \tag{7.3}$$

where $0 \leq \omega_1 < 2\pi$; $0 \leq \omega_j < \pi$, $1 < j \leq n-1$; and $0 \leq r < \infty$. By (7.3), each $x \in \mathbb{H}^n$ can be represented as

$$x = \omega \sinh r + e_{n+1} \cosh r \ = (\omega \sinh r, \cosh r) \tag{7.4}$$

where ω is a point of the unit sphere S^{n-1} in $\mathbb{R}^n$ with Euler angles $\omega_1, \dots, \omega_{n-1}$. We regard $\mathbb{R}^n$ as the hyperplane $x_{n+1} = 0$ in $E^{n,1}$. The invariant (with respect to hyperbolic motions) measure dx in $\mathbb{H}^n$ is given by $dx = \sinh^{n-1} r d\omega dr$, where $d\omega$ is the surface element of S^{n-1}. The geodesic distance between points x and y in $\mathbb{H}^n$ is defined by $\text{dist}(x, y) = \cosh^{-1}[x, y]$, (i.e., $\cosh \text{dist}(x, y) = [x, y]$). Given $x \in \mathbb{H}^n$ and $t > 1$, let

$$(Mf)(x, t) = \frac{(t^2 - 1)^{(1-n)/2}}{\sigma_{n-1}} \int_{[x,y]=t} f(y) d\sigma(y) \tag{7.5}$$

be the mean value of f over the planar section $\{y \in \mathbb{H}^n : [x, y] = t\}$.

As before, our aim is to reconstruct a function $f \in C^\infty(\mathbb{H}^n)$ under the following assumptions.

(a) The support of f lies in the geodesic ball (see Figure 3)

$$B = \{x \in \mathbb{H}^n : \text{dist}(x, e_{n+1}) < R\} = \{x \in \mathbb{H}^n : x_{n+1} < \cosh R\},$$

where $e_{n+1} = (0, \dots, 0, 1)$ is the origin of $\mathbb{H}^n$ and $R > 0$ is fixed.

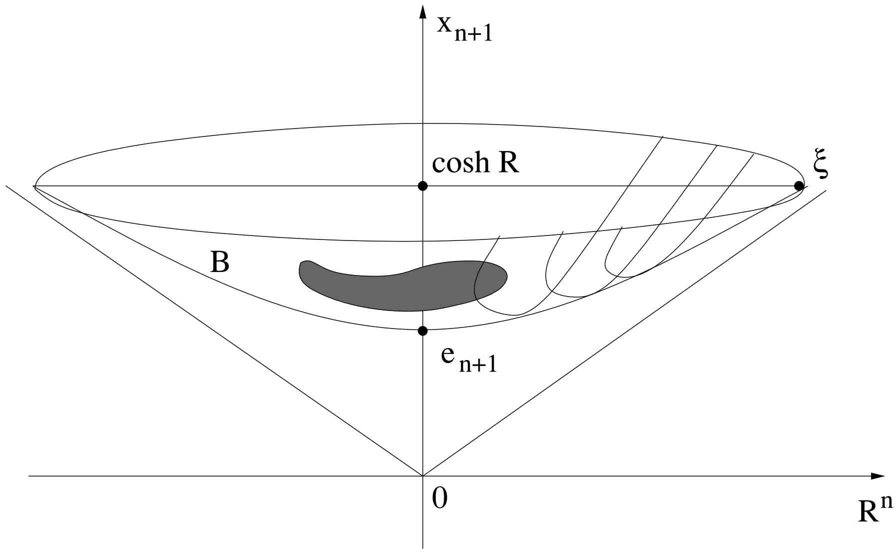

Figure 3. The hyperbolic case.

(b) The mean values (7.5) are known for all $x = \xi \in \partial B$ and all $t > 1$, where ∂B is the boundary of B.

Owing to (7.4), we can write $x \in \mathbb{H}^n$ as

$$x = (x', \sqrt{1 + |x'|^2}), \qquad x' = (x_1, \dots, x_n, 0) \in \mathbb{R}^n,$$

so that

$$\int_{\mathbb{H}^n} f(x)dx = \int_{\mathbb{R}^n} f(x', \sqrt{1 + |x'|^2})\rho(x')dx', \quad \rho(x') = \sqrt{\frac{1 + 2|x'|^2}{1 + |x'|^2}}. \tag{7.6}$$

We introduce the "back-projection" operator P, an operator which sends functions on $\partial B \times (1, \infty)$ to functions on B by the formula

$$(PF)(x) = \frac{1}{|\partial B|} \int_{\partial B} F(\xi, [\xi, x]) d\sigma(\xi), \quad x \in B. \tag{7.7}$$

Let $\tilde{B} = \{x' \in \mathbb{R}^n : |x'| < \sinh R\}$ be the orthogonal projection of B onto the hyperplane $x_{n+1} = 0$. If $\xi = e_{n+1} \cosh R + \omega \sinh R$, $\omega \in S^{n-1}$, and $x = (x', x_{n+1}) \in B$, then $[\xi, x] = \sqrt{1 + |x'|^2} \cosh R - (x' \cdot \omega) \sinh R$, and $(PF)(x)$ is actually a function of $x' \in \tilde{B}$. We denote this function by $(\tilde{P}F)(x')$.

7.1 The case $n > 2$. As above, let $\xi \in \partial B$ and $t > 1$. Consider the analytic family of operators

$$(N^\alpha f)(\xi, t) = \int_B \frac{|[\xi, y] - t|^{\alpha-1}}{\Gamma(\alpha/2)} f(y)dy, \quad Re\,\alpha > 0, \tag{7.8}$$

Lemma 7.1. *If $f \in C^\infty(\mathbb{H}^n)$, $\operatorname{supp} f \subset B$, then*

$$\underset{\alpha=3-n}{a.c.} (\tilde{P} N^\alpha f)(x') = \frac{\Gamma(n/2)(\sinh R)^{2-n}}{\pi^{1/2}} \int_{\tilde{B}} \frac{\tilde{f}(y')}{|x' - y'|^{n-2}} dy', \tag{7.9}$$

$$\tilde{f}(y') = f(y', \sqrt{1 + |y'|^2}) \sqrt{\frac{1 + 2|y'|^2}{1 + |y'|^2}}. \tag{7.10}$$

Proof. For $Re\,\alpha > 0$, changing the order of integration, we obtain

$$(PN^\alpha f)(x) = \int_B f(y) k_\alpha(x, y)dy, \quad k_\alpha(x, y) = \frac{1}{|\partial B|} \int_{\partial B} \frac{|[\xi, (x - y)]|^{\alpha-1}}{\Gamma(\alpha/2)} d\sigma(\xi).$$

Since ξ has the form $\xi = e_{n+1} \cosh R + \omega \sinh R$, $\omega \in S^{n-1}$,

$$\begin{aligned} |[\xi, (x - y)]| &= |(x_{n+1} - y_{n+1}) \cosh R - (x' - y') \cdot \omega \sinh R| \\ &= |h - \omega \cdot \sigma| \, |x' - y'| \sinh R, \end{aligned}$$

$$h = \frac{x_{n+1} - y_{n+1}}{|x' - y'|} \coth R, \quad \sigma = \frac{x' - y'}{|x' - y'|}. \tag{7.11}$$

Hence,

$$\begin{aligned} k_\alpha(x, y) &= \frac{(|x' - y'| \sinh R)^{\alpha-1}}{\sigma_{n-1}} \int_{S^{n-1}} \frac{|h - \omega \cdot \sigma|^{\alpha-1}}{\Gamma(\alpha/2)} d\omega \\ &= \frac{\sigma_{n-2}(|x' - y'| \sinh R)^{\alpha-1}}{\sigma_{n-1}} g_\alpha(h); \quad \text{cf. (5.10).} \end{aligned}$$

If $|h| < 1$, we can apply Lemma 2.2 and write the integral over B as that over $\tilde{B} \subset \mathbb{R}^n$. This gives the result.

It remains to show that $|h| < 1$. By symmetry, we may suppose that $\left|y'\right| \le \left|x'\right|$. Let $a = |y'|$, $b = |x'|$, and $b_0 = \sinh R$. Since $|x' - y'| \ge |x'| - |y'|$,

$$h \le f_a(b) \coth R, \quad f_a(b) = \frac{\sqrt{1 + b^2} - \sqrt{1 + a^2}}{b - a}.$$

For a fixed, $f_a(b)$ is increasing in (a, ∞), because

$$f_a'(b) = \frac{D(a, b)}{(b - a)^2 \sqrt{1 + b^2}}, \quad D(a, b) = \sqrt{\left(1 + b^2\right)\left(1 + a^2\right)} - 1 - ab > 0.$$

Hence $h \leq f_a(\sinh R)\coth R$. The right-hand side of this inequality is less than 1. Indeed, setting $b_0 = \sinh R$, we have

$$
\begin{aligned}
& f_a(\sinh R)\coth R = \frac{\sqrt{1+b_0^2} - \sqrt{1+a^2}}{b_0 - a}\frac{\sqrt{1+b_0^2}}{b_0} < 1, \\
& \Leftrightarrow \frac{1+b_0^2 - \sqrt{\left(1+a^2\right)\left(1+b_0^2\right)}}{(b_0 - a)\, b_0} < 1, \\
& \Leftrightarrow 1+b_0^2 - \sqrt{\left(1+a^2\right)\left(1+b_0^2\right)} < (b_0 - a)\, b_0, \\
& \Leftrightarrow 0 < D\,(a, b_0)\,.
\end{aligned}
$$

This completes the proof. □

Lemma 7.2. *Let* $f \in C^\infty(\mathbb{H}^n)$*,* $\operatorname{supp} f \subset B$*. Then*

$$
\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \frac{\delta_n}{(\sinh R)^{n-1}} \int_{\partial B} (d/dt)^{n-3}[(Mf)(\xi, t)(t^2-1)^{n/2-1}]\Big|_{t=[\xi,x]} d\xi
$$

if $n = 3, 5, \ldots$*, and*

$$
\begin{aligned}
\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = &-\frac{\delta_n}{\pi(\sinh R)^{n-1}} \int_{\partial B} d\xi \\
&\times \int_1^{\cosh 2R} (d/dt)^{n-2}[(Mf)(\xi, t)(t^2-1)^{n/2-1}] \log|t - [\xi, x]| dt
\end{aligned}
$$

if $n = 4, 6, \ldots$*, where* δ_n *is defined by* (3.7).

Proof. For $Re\, \alpha > 0$, making use of the formula

$$
\int_{\mathbb{H}^n} f(y) a([\xi, y]) dy = \sigma_{n-1} \int_1^\infty a(\tau)(Mf)(\xi, \tau)(\tau^2-1)^{n/2-1} d\tau, \tag{7.12}
$$

we have

$$
\begin{aligned}
(N^\alpha f)(\xi, t) &= \frac{\sigma_{n-1}}{\Gamma(\alpha/2)} \int_1^\infty (Mf)(\xi, \tau)|\tau - t|^{\alpha-1}(\tau^2-1)^{n/2-1} d\tau \\
&= \int_{\mathbb{R}} \frac{|\tau|^{\alpha-1}}{\Gamma(\alpha/2)} \varphi_\xi(\tau + t) d\tau, \quad \varphi_\xi(\tau) = \sigma_{n-1}(Mf)(\xi, \tau)(\tau^2-1)_+^{n/2-1}.
\end{aligned}
$$

Since f is smooth and the support of f is separated from the boundary ∂B, $(Mf)(\xi, \tau)$ is smooth in the τ-variable uniformly in ξ and vanishes identically in the respective neighborhood of $\tau = 1$. Thus Lemma 2.1 yields the following equalities. For $n = 3, 5, \ldots$,

$$
\underset{\alpha=3-n}{a.c.}(N^\alpha f)(\xi, t) = \delta_n \varphi_\xi^{(n-3)}(t).
$$

For $n = 4, 6, \ldots$,

$$\underset{\alpha=3-n}{a.c.} (N^\alpha f)(\xi, t) = -\frac{\delta_n}{\pi} \int_1^{\cosh 2R} \varphi_\xi^{(n-2)}(\tau) \log|\tau - t| d\tau,$$

δ_n being the constant from (3.7). Now the result follows; cf. Lemmas 3.2 and 5.2. □

Lemmas 7.1 and 7.2 imply the following inversion result for the spherical means on $\mathbb{H}^n$. We recall that $\Delta_{x'}$ denotes the usual Laplace operator in the x'-variable.

Theorem 7.3. *Let $n > 2$. An infinitely differentiable function f supported in the geodesic ball $B = \{x \in \mathbb{H}^n : \mathrm{dist}(x, e_{n+1}) < R\}$, can be reconstructed from its spherical means $(Mf)(\xi, \tau)$, $(\xi, t) \in \partial B \times (1, \infty)$, by the formula*

$$f(x) = \frac{d_n x_{n+1}}{|x| \sinh R} \Delta_{x'} f_0(x', \sqrt{1 + |x'|^2}), \quad d_n = \frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)},$$

where $|x| = \sqrt{|x'|^2 + x_{n+1}^2}$ and $f_0(x) \equiv f_0(x', \sqrt{1 + |x'|^2})$ has the form

$$f_0(x) = -\int_{\partial B} (d/dt)^{n-3}[(Mf)(\xi, t)(t^2 - 1)^{n/2-1}]\Big|_{t=[\xi,x]} d\xi,$$

if $n = 3, 5, \ldots$, and

$$f_0(x) = \frac{1}{\pi} \int_{\partial B} d\xi \int_1^{\cosh 2R} (d/dt)^{n-2}[(Mf)(\xi, t)(t^2 - 1)^{n/2-1}] \log|t - [\xi, x]| dt$$

if $n = 4, 6, \ldots$.

7.2 The case $n = 2$. The argument follows that in Subsection 5.2 almost verbatim. Let

$$(I_* f)(x) = \frac{1}{2\pi} \int_B f(y) \log|x' - y'| dy, \tag{7.13}$$

so that

$$\Delta_{x'}(I_* f)(x) = \tilde{f}(x') = |x| f(x)/x_3. \tag{7.14}$$

Lemma 7.4. *Let f be a C^∞ function supported in B. Then*

$$(I_* f)(x) = \frac{1}{|\partial B|} \int_{\partial B} \int_1^\infty (Mf)(\xi, \tau) \log|\tau - [\xi, x]| d\tau d\xi + c_f, \tag{7.15}$$
$$c_f = -\frac{1}{2\pi}\left(\log \frac{\sinh R}{2}\right) \int_B f(y) dy.$$

Proof. Let

$$(N_* f)(\xi, t) = \int_B f(y) \log |[\xi, y] - t| dy, \quad (\xi, t) \in \partial B \times (1, \infty).$$

Changing the order of integration, we obtain

$$(PN_* f)(x) = \int_B f(y) k_*(x, y) dy,$$

$k_*(x, y) = \log |x' - y'| + \log \sinh R - \log 2$; see the proof of Lemma 5.4. This gives

$$(PN_* f)(x) = 2\pi (I_* f)(x) + \left(\log \frac{\sinh R}{2} \right) \int_B f(y) dy. \tag{7.16}$$

On the other hand, by (7.12),

$$(PN_* f)(x) = \frac{1}{\sinh R} \int_{\partial B} \int_1^\infty (Mf)(\xi, \tau) \log |\tau - [\xi, x]| d\tau d\xi. \tag{7.17}$$

Comparing (7.17) with (7.16), we obtain (7.15)). □

Owing to (7.14), Lemma 7.4 allows us complete Theorem 7.3 as follows.

Theorem 7.5. *An infinitely differentiable function f supported in the geodesic ball $B = \{x \in \mathbb{H}^2 : \text{dist}(x, e_3) < R\}$ can be reconstructed from its spherical means $(Mf)(\xi, \tau)$, $(\xi, t) \in \partial B \times (1, \infty)$, by the formula*

$$f(x) = \frac{x_3}{2\pi |x| \sinh R} \Delta_{x'} \int_{\partial B} \int_1^\infty (Mf)(\xi, \tau) \log |\tau - [\xi, x]| d\tau d\xi. \tag{7.18}$$

Remark 7.6. As in Section 6, Theorems 7.3 and 7.5 can be applied to solve inverse problems for the EPD equation in the hyperbolic space. The reasoning follows the same lines as before. We leave the details to the interested reader.

8 Appendix: Proof of Lemma 2.2

It is convenient to split the proof in two parts.

(i) Recall that

$$g_\alpha(h) = \frac{1}{\Gamma(\alpha/2)} \int_{-1}^1 |t - h|^{\alpha - 1} (1 - t^2)^{(n-3)/2} dt, \quad Re\, \alpha > 0, \tag{8.1}$$

where $n \geq 2$ and $|h| \leq 1 - \delta$, $\delta > 0$. Changing variables $t = 2\tau - 1$, $h = 2\xi - 1$, we obtain $g_\alpha(h) \equiv G_\alpha((1 + h)/2)$, where

$$\begin{aligned} G_\alpha(\xi) &= \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)} \int_0^1 |\tau - \xi|^{\alpha-1} (1 - \tau)^{(n-3)/2} \tau^{(n-3)/2} dt \\ &= U_\alpha(\xi) + U_\alpha(1 - \xi), \quad \delta/2 \leq \xi \leq 1 - \delta/2, \end{aligned} \tag{8.2}$$

$$U_\alpha(\xi) = \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)} \int_0^\xi (\xi-\tau)^{\alpha-1}\tau^{(n-3)/2}(1-\tau)^{(n-3)/2}d\tau.$$

The last integral can be expressed using the Gauss hypergeometric function, so that $U_\alpha(\xi) = a_\xi(\alpha)b(\alpha)\zeta_\alpha(\xi)$, where

$$a_\xi(\alpha) = 2^{\alpha+n-3}\xi^{(n-3)/2+\alpha}\Gamma((n-1)/2), \quad b(\alpha) = \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)},$$
$$\zeta_\alpha(\xi) = \frac{1}{\Gamma(\alpha+(n-1)/2)} F\left(\frac{n-1}{2}, \frac{3-n}{2}; \frac{n-1}{2}+\alpha; \xi\right);$$

see, e.g., [56, 2.2.6(1)]. Owing to [16, 2.1.6], $\zeta_\alpha(\xi)$ extends as an entire function of α, which is represented by an absolutely convergent power series. Since $\delta/2 \le \xi \le 1-\delta/2$, this series converges uniformly in $\alpha \in K$ for any compact subset K of the complex plane. Furthermore, $a_\xi(\alpha)$ is also an entire function and $b(\alpha)$ is meromorphic with poles only at $-1, -3, -5, \ldots$. Since g_α is an even function, i.e., $g_\alpha(h) = g_\alpha(-h)$, these poles are removable. Hence, $G_\alpha(\xi)$ extends to all complex α as an entire function of α, and this extension represents a C^∞ function of ξ uniformly in $\alpha \in K$. This gives the desired result for $g_\alpha(h)$.

(ii) Let us compute the analytic continuation of g_α at $\alpha = 3-n$. For $n = 2$, the result is trivial. Consider the case $n > 2$. We first assume $1/2 < Re\,\alpha < 1$ and $|Im\alpha| < 1$, and represent $G_\alpha(\xi)$ as a Mellin convolution

$$G_\alpha(\xi) = \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)} f(\xi), \quad f(\xi) = \int_0^\infty f_1(\tau) f_2\left(\frac{\xi}{\tau}\right)\frac{d\tau}{\tau}, \tag{8.3}$$

where

$$f_1(\tau) = \begin{cases} \tau^{\alpha+(n-3)/2}(1-\tau)^{(n-3)/2} & \text{if } 0 < \tau < 1, \\ 0, & \text{if } 1 < \tau < \infty, \end{cases} \qquad f_2(\tau) = |1-\tau|^{\alpha-1}.$$

The Mellin transforms $\tilde{f}_j(s) = \int_0^\infty f_j(\tau)\tau^{s-1}d\tau$ $(j = 1, 2)$ are evaluated as

$$\tilde{f}_1(s) = \frac{\Gamma(s+\alpha+(n-3)/2)\Gamma((n-1)/2)}{\Gamma(s+\alpha+n-2)}, \quad Re\,s > \frac{3-n}{2} - Re\,\alpha,$$
$$\tilde{f}_2(s) = \frac{\Gamma(s)\Gamma(\alpha)}{\Gamma(s+\alpha)} + \frac{\Gamma(1-s-\alpha)\Gamma(\alpha)}{\Gamma(1-s)}, \qquad 0 < Re\,s < 1 - Re\,\alpha.$$

Applying the convolution theorem and the relevant Mellin inversion formula [64], we obtain

$$f(\xi) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \tilde{f}(s)\xi^{-s}ds, \qquad 0 < \kappa < 1 - Re\alpha,$$

where $\tilde{f}(s) = \tilde{f}_1(s)\tilde{f}_2(s)$. The function $\tilde{f}(s)$ has poles in the half-plane $Re\, s < \kappa$ at the points $s = -j$ and $s = -j - \alpha - (n-3)/2$, $j = 0, 1, 2, \ldots$. Since $1/2 < Re\, \alpha < 1$, all these poles are simple, and the Cauchy Residue Theorem yields

$$f(\xi) = \Gamma\left(\frac{n-1}{2}\right)\Gamma(\alpha)\sum_{j=0}^{\infty}\frac{(-\xi)^j}{j!}\left[\frac{\Gamma(\alpha - j + (n-3)/2)}{\Gamma(\alpha - j)\Gamma(\alpha - j + n - 2)}\right.$$
$$\left. + \frac{\xi^{\alpha+(n-3)/2}}{\Gamma((n-1)/2 - j)}\left(\frac{\Gamma(-\alpha + (3-n)/2 - j)}{\Gamma((3-n)/2 - j)} + \frac{\Gamma((n-1)/2 + j)}{\Gamma(\alpha + (n-1)/2 + j)}\right)\right].$$

Ultimately, we arrive at the following expression for $G_\alpha(\xi)$:

$$G_\alpha(\xi) = \lambda_1 F\left(1 - \alpha, 3 - \alpha - n; \frac{5-n}{2} - \alpha; \xi\right) + \lambda_2 F\left(\frac{3-n}{2}, \frac{n-1}{2}; \frac{n-1}{2} + \alpha; \xi\right), \tag{8.4}$$

where

$$\lambda_1 = \frac{\Gamma((n-1)/2)}{2^{3-\alpha-n}\Gamma(\alpha/2)}\frac{(-1)^n\Gamma(3 - \alpha - n)\sin\alpha\pi}{\Gamma((5-n)/2 - \alpha)\cos(\alpha + n/2)\pi},$$
$$\lambda_2 = \frac{\Gamma((n-1)/2)}{2^{3-\alpha-n}\Gamma(\alpha/2)}\frac{\xi^{\alpha+(n-3)/2}\Gamma(\alpha)}{\Gamma(\alpha + (n-1)/2)}\left(1 + \frac{\cos n\pi/2}{\cos(\alpha + n/2)\pi}\right).$$

We consider the following two cases.

Case 1: $n = 2m$, $m = 2, 3, \ldots$. In this case,

$$G_\alpha(\xi) = \frac{\pi\Gamma(m - 1/2)}{2^{3-\alpha-2m}\Gamma(\alpha/2)\cos\alpha\pi}[D_1(\xi;\alpha) + D_2(\xi;\alpha)], \tag{8.5}$$

where

$$D_1(\xi;\alpha) = \frac{F(1 - \alpha, 3 - \alpha - 2m; 5/2 - \alpha - m; \xi)}{(-1)^m\Gamma(\alpha + 2m - 2)\Gamma(5/2 - \alpha - m)},$$
$$D_2(\xi;\alpha) = \frac{\cot(\alpha\pi/2)F(3/2 - m, m - 1/2, \alpha + m - 1/2; \xi)}{\xi^{3/2-\alpha-m}\Gamma(1 - \alpha)\Gamma(\alpha + m - 1/2)}.$$

A simple computation yields $\underset{\alpha=3-2m}{a.c.} G_\alpha(\xi) = \Gamma(m - 1/2) = \Gamma((n-1)/2)$.

Case 2: $n = 2m + 1$, $m = 1, 2, \ldots$. In this case,

$$G_\alpha(\xi) = \frac{\Gamma(m)\Gamma(1 - \alpha/2)}{2^{2-\alpha-2m}\cos(\alpha\pi/2)}[E_1(\xi;\alpha) + E_2(\xi;\alpha)], \tag{8.6}$$

where

$$E_1(\xi;\alpha) = \frac{F(1 - \alpha, 2 - \alpha - 2m; 2 - \alpha - m; \xi)}{(-1)^{m+1}\Gamma(\alpha - 1 + 2m)\Gamma(2 - \alpha - m)},$$
$$E_2(\xi;\alpha) = \frac{\xi^{\alpha+m-1}F(1 - m, m; \alpha + m; \xi)}{\Gamma(1 - \alpha)\Gamma(\alpha + m)}.$$

Passing to the limit as $\alpha \to 2-2m$, we obtain $\underset{\alpha=2-2m}{a.c.} G_\alpha(\xi) = \Gamma(m) = \Gamma((n-1)/2)$. This completes the proof.

Remark 8.1. The basic equality (8.4) can be proved in a different way if we rearrange hypergeometric functions in (8.2) using known formulas. Specifically, the second term in (8.2) can be transformed by formulas (33), (6), and (21) from [16, Section 2.9]. This gives

$$\begin{aligned} U_\alpha(1-\xi) &= \frac{2^{\alpha+n-3}}{\Gamma(\alpha/2)} B\left(\frac{n-1}{2}, \alpha\right)(A(\xi)+B(\xi)), \\ A(\xi) &= \gamma_1(\alpha) F\left(1-\alpha, 3-n-\alpha; \frac{5-n}{2}-\alpha; \xi\right), \\ B(\xi) &= (\xi(1-\xi))^{\alpha+(n-3)/2}\gamma_2(\alpha) F\left(\alpha, \alpha+n-2; \frac{n-1}{2}+\alpha; \xi\right), \\ \gamma_1(\alpha) &= \frac{\Gamma(\alpha+(n-1)/2)\Gamma(\alpha+(n-3)/2)}{\Gamma(\alpha)\Gamma(\alpha+n-2)}, \\ \gamma_2(\alpha) &= \frac{\Gamma(\alpha+(n-1)/2)\Gamma((3-n)/2-\alpha)}{\Gamma((n-1)/2)\Gamma((3-n)/2)} = \frac{\sin\pi(n-1)/2}{\sin\pi((n-1)/2+\alpha)}. \end{aligned}$$

Owing to [16, 2.9(2)],

$$B(\xi) = \gamma_2(\alpha)\xi^{(n-3)/2+\alpha} F\left(\frac{n-1}{2}, \frac{3-n}{2}; \frac{n-1}{2}+\alpha; \xi\right).$$

This gives (8.4).

An alternative proof of (ii). The following alternative proof is instructive. In truth, it suffices to prove the equality

$$\underset{\alpha=3-n}{a.c.}\, g_\alpha(h) = \Gamma((n-1)/2) \tag{8.7}$$

in the weak sense. Indeed, suppose that

$$\underset{\alpha=3-n}{a.c.}\,(g_\alpha, \psi) \equiv \underset{\alpha=3-n}{a.c.} \int_{\mathbb{R}} g_\alpha(h)\psi(h)dh = \Gamma((n-1)/2)\int_{\mathbb{R}} \psi(h)dh \tag{8.8}$$

for any C^∞ function ψ with compact support in the interval $(-1, 1)$. By part (i), the analytic continuation of $g_\alpha(h)$ represents a C^∞ function of h uniformly in $\alpha \in K$ for any compact subset K of the complex plane. Hence (use [57, Lemma 1.17]) $\underset{\alpha=3-n}{a.c.}(g_\alpha, \psi) = (\underset{\alpha=3-n}{a.c.}\, g_\alpha, \psi)$ and (8.8) yields $(\underset{\alpha=3-n}{a.c.}\, g_\alpha, \psi) = \Gamma((n-1)/2)(1, \psi)$. This implies (8.7).

To prove (8.8), let $\rho_\alpha(t) = |t|^{\alpha-1}/\Gamma(\alpha/2)$ and $\omega(t) = (1-t^2)_+^{(n-3)/2}$. We interpret these functions as $\mathcal{D}'$-distributions on $\mathbb{R}$. Then g_α is a convolution of ρ_α with the compactly supported distribution ω, so that $(g_\alpha, \psi) = (\rho_\alpha(s), (\omega(t), \psi(s+t)))$; see [27, Ch. I, Sec. 4(2)].

If $n = 2m + 3$, $m = 0, 1, \ldots$, then (2.1) yields

$$\begin{aligned}\underset{\alpha=3-n}{a.c.} (g_\alpha, \psi) &= \underset{\alpha=-2m}{a.c.} (g_\alpha, \psi) = c_{m,1} \left(\frac{d}{ds}\right)^{2m} (\omega(t), \psi(s+t))\Big|_{s=0} \\ &= c_{m,1}(\omega, \psi^{(2m)}) = c_{m,1}([(1-t^2)^m]^{(2m)}, \psi(t)) \\ &= (-1)^m c_{m,1}(2m)! = m! = \Gamma((n-1)/2).\end{aligned}$$

Now let n be even. Since the convolution is commutative,

$$a.c.(g_\alpha, \psi) = a.c.(\omega(t), (\rho_\alpha(s), \psi(s+t))) = (\omega(t), a.c.(\rho_\alpha(s), \psi(s+t))).$$

If $n = 2m+2$ ($m = 1, 2, \ldots$), then, applying (2.2) and changing variables, we have

$$\begin{aligned}\underset{\alpha=3-n}{a.c.} (g_\alpha, \psi) &= \underset{\alpha=1-2m}{a.c.} (g_\alpha, \psi) = c_{m,2}\Big(\omega(t), \text{p.v.} \int_{\mathbb{R}} \frac{\psi^{(2m-1)}(s+t)}{s} ds\Big) \\ &= (-1)^{m+1} c_{m,2} \int_{-1}^{1} \psi^{(2m-1)}(h) q(h) dh \\ &= (-1)^m c_{m,2} \int_{-1}^{1} \psi(h) q^{(2m-1)}(h) dh,\end{aligned}$$

$$c_{m,2} = \frac{1}{\Gamma(1/2-m)(2m-1)!}, \quad q(h) = \text{p.v.} \int_{-1}^{1} \frac{(t^2-1)^m}{(t-h)\sqrt{1-t^2}} dt$$

(interchanging the order of integration is justified by completing $[-1, 1]$ to a closed contour and using [25, Section 7.1]). Now $q(h)$ is a polynomial with leading term πh^{2m-1}. This follows from the well-known relation for Chebyshev polynomials

$$\text{p.v.} \int_{-1}^{1} \frac{T_n(t)dt}{(t-h)\sqrt{1-t^2}} = \pi U_{n-1}(h), \quad -1 < h < 1, \tag{8.9}$$

and the fact that the leading terms of $T_n(h)$ and $U_n(h)$ are $2^{n-1}h^n$ and $2^n h^n$, respectively; see formulas 10.11(47), 10.11(22), and 10.11(23) in [16, vol. II)]. Integration by parts yields

$$\underset{\alpha=3-n}{a.c.} (g_\alpha, \psi) = (-1)^m c_{m,2}\pi(2m-1)! \int_{-1}^{1} \psi(h)dh = \Gamma(m+1/2) \int_{-1}^{1} \psi(h)dh,$$

where $\Gamma(m+1/2) = \Gamma((n-1)/2)$. This completes the proof.

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Yuri A. Antipov
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803, USA
email: antipov@math.lsu.edu

Ricardo Estrada
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803, USA
email: restrada@math.lsu.edu

Boris Rubin
Department of Mathematics
Louisiana State University
Baton Rouge, LA, 70803, USA
email: borisr@math.lsu.edu

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