Dynamic problem of coupled thermoelasticity for a thin composite structure

Y. A. Antipov[1] and B. T. Dickinson[2]
[1]Department of Mathematics, Louisiana State University, USA
[2] USAF AFMC AFRL/RWWN, USA

Abstract

A dynamic problem of coupled thermoelasticity for a composite body is analyzed. The body is modeled as a composite beam in response to elastic deformation. It consists of two parts of different densities chosen such that the center of mass is the junction point. The beam is clamped at the center of mass and free at its ends. The body acts as a two-dimensional structure in response to the heat flow. The problem reduces to three partial integro-differential equations, and the two beam equations are coupled by the heat equation. A method based on integral transformations and expansion of the transforms of the beam deflection in terms of shape and vibration modes is proposed. Ultimately, it leads to an infinite system of linear algebraic equations with respect to the Laplace transforms of the vibration modes solved by the method of successive approximations. In addition, expressions for the generalized free energy and the dissipation function are derived. By employing the variational principle it is shown that this formulation deduces the governing differential equations of thermoelastic vibration of the thin structure that follow from dynamic thermoelasticity and the beam theory.

Keywords

Dynamic coupled thermoelasticity, Laplace integral transform, successive approximations, free energy, dissipation function

1 Introduction

It is known [1] that at high speed hypersonic regimes, aerodynamic heating becomes a major and extremely complicated problem. For example, an accurate modeling of high-speed flight during the atmospheric reentry that captures the effects of high temperatures in the shock layer, large aerodynamic heating of the vehicle and the chemically reacting gas surrounding the vehicle in addition to the classical aerodynamics requires employing several disciplines, including those of gas dynamics, thermodynamics, thermochemistry, molecular physics, and statistical mechanics.

By making use of mean axes, Lagrange's equations and the principle of virtual work nonlinear equations of motion for uncoupled elastic and rigid body modes were developed in [2], [3], [4]. Analysis of coupled elastic and rigid body modes of a maneuvering flexible aircraft was carried out in [5], [6], [7]. Nonlinear equations of longitudinal dynamics for an elastic aircraft were derived in [8]. The aircraft was modeled as a beam, and the bending vibration modes of the aircraft and rigid body dynamics were coupled in the resulting dynamical system. Thermal effects as a part of a generalized Lagrange force acting upon a beam in the framework of uncoupled thermoelasticity were studied in [9]. A model problem for a single simply supported beam acting as a two-dimensional structure in response to heat by the averaging method for integral equations was approximately solved in [10].

Motivated by applications of the model problem on vibration of elastic and thermoelastic beams to aircraft dynamics in this paper we analyze a model dynamic problem of coupled thermoelasticity for a thin structure. It works as a one-dimensional body (a composite beam) in response to elastic deformation and a two-dimensional rectangular body in response to heat flow. The beam is clamped at the center of mass, has free ends, and the elastic equations for the two parts of the beam are coupled by the heat equation. This model problem is different from [10] since it concerns a composite beam, and the beam ends are assumed to be free, not simply supported. The change of the boundary conditions makes the method [10] inapplicable. Also, instead of the averaging procedure with respect to time employed in [10] our method hinges on the exact direct and inverse Laplace transforms.

In Section 2, we derive a governing boundary value problem for partial integro-differential equations. Section 3 describes the method of solution that requires applying the Laplace transform, two finite sine-integral transforms and expanding the deflection of the composite beam in terms of the shape and vibration modes. Eventually, the procedure deduces an infinite system of linear algebraic equations for the Laplace transforms of the vibration modes. It is solved by the method of successive approximations. In Section 4, similarly to [11], we derive the generalized free energy and the dissipation function associated with the structure considered in the previous sections. The aim of this section is to derive an alternative variational formulation of the model problem and show its equivalence to the system of governing partial integro-differential equations employed in Section 3. Also, the expression of the isothermal strain energy obtained may be used for derivation of a dynamical system associated with the flight dynamics of a thermoelastic aircraft.

2 Governing boundary value problem

The problem under consideration is one of dynamic coupled thermoelasticity for a structure acting as a beam $\{-l_- < x < l_+, -b/2 < y < b/2, -h/2 < z < h/2\}$ ($b << l_\pm, h << l_\pm$) in response to deformation in the z-direction and as a two-dimensional body (a rectangle) in response to heat flow. The beam consists of two parts, $B_- = \{-l_- < x < 0\}$ and $B_+ = \{0 < x < l_+\}$, of mass densities ρ_- and $\rho_+ = \rho_- l_-/l_+$, respectively, and the point $x = 0$ is the center of mass. It is assumed that the beam is clamped at the center of mass

$$w(0^\pm, t) = \frac{\partial w}{\partial x}(0^\pm, t) = 0, \quad t \ge 0, \tag{2.1}$$

and its ends are free

$$\frac{\partial^2 w}{\partial x^2}(\pm l_\pm, t) = \frac{\partial^3 w}{\partial x^3}(\pm l_\pm, t) = 0, \quad t \ge 0, \tag{2.2}$$

while its upper side is subjected to a load of density $p(x,t)$. Here, $w(x,t)$ is the deflection of a point x of the beam at time t. The initial deflection and its rate vanish at all points of the structure, $w(x,0) = w_t(x,0) = 0$, $-l_- \le x \le l_+$, and the initial temperature of the beam is prescribed and constant, $T = T_0$. It is assumed that the surface temperature $T(x,z,t)$ does not change with time everywhere on the surface except at the upper side, $T(x,h/2,t) = \theta^+(x,t) + T_0$, with $\theta^+(x,t)$ being a prescribed function.

To write down the governing equations, first we employ the strain-stress relations of the beam theory

$$\varepsilon_x^{(z)} = -z\frac{\partial^2 w}{\partial x^2} = \frac{\sigma_x}{E} + \alpha_T\theta, \quad \varepsilon_y^{(z)} = \varepsilon_z^{(z)} = -\frac{\nu\sigma_x}{E} + \alpha_T\theta, \tag{2.3}$$

where ν and E are the Poisson ratio and the Young modulus, respectively, measured under the conditions of isothermal deformation when $T = T_0$, and α_T is the linear thermal expansion coefficient. The variation of temperature $\theta(x,z,t) = T - T_0$ and the beam deflection $w(x,t)$ are to be determined.

By multiplying the first relation in (2.3) by z, integrating it with respect to z in $(-h/2, h/2)$, using the bending moment representation

$$M(x) = b\int_{-h/2}^{h/2} \sigma_x z dz, \tag{2.4}$$

and the beam equilibrium equation

$$\frac{d^2 M(x)}{dx^2} + bp(x,t) = \rho_\pm hb\frac{\partial^2 w}{\partial t^2}, \quad x \in B_\pm, \quad t > 0, \tag{2.5}$$

we eventually arrive at the two integro-differential equations which govern thermoelastic bending of the two parts of the beam

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho_+ hb}{EJ}\frac{\partial^2 w}{\partial t^2} + \frac{12\alpha_T}{h^3}\frac{\partial^2}{\partial x^2}\int_{-h/2}^{h/2}\theta z dz = \frac{bp(x,t)}{EJ}, \quad 0 < x < l_+, \quad t > 0,$$

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho_- hb}{EJ}\frac{\partial^2 w}{\partial t^2} + \frac{12\alpha_T}{h^3}\frac{\partial^2}{\partial x^2}\int_{-h/2}^{h/2}\theta z dz = \frac{bp(x,t)}{EJ}, \quad -l_- < x < 0, \quad t > 0, \tag{2.6}$$

that is a generalization of the homogeneous beam equation [10], p.201 to the case when the beam is composed of two parts and is subjected to loading. Here, $J = \frac{1}{12}bh^3$ is the moment of inertia of a rectangular area.

The system of equations (2.6) should be complemented by a thermoconductivity equation. Under the assumption that $\theta = T - T_0$ is small and $T_0 = \text{const}$ it can be linearized to the form [11], [12]

$$\nabla^2\theta - \frac{c_\varepsilon}{\lambda_q}\frac{\partial\theta}{\partial t} - \frac{E\alpha_T T_0}{\lambda_q(1-2\nu)}\frac{\partial\varepsilon_{kk}}{\partial t} = -\frac{\Omega(x,z,t)}{\lambda_q}, \tag{2.7}$$

where c_ε is the specific heat capacity at $T = T_0$ when there is no deformation, λ_q is the thermoconductivity coefficient, Ω is the heat flow capacity, and

$$\varepsilon_{kk} = \varepsilon_x^{(z)} + \varepsilon_y^{(z)} + \varepsilon_z^{(z)} = -(1-2\nu)z\frac{\partial^2 w}{\partial x^2} + 2(1+\nu)\alpha_T\theta. \tag{2.8}$$

Due to this formula for ε_{kk} the thermoconductivity equation can be transformed to the form

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial z^2} - \frac{1}{a}\frac{\partial\theta}{\partial t} + \frac{ET_0\alpha_T}{\lambda_q}\frac{z\partial^3 w}{\partial x^2\partial t} = -\frac{\Omega(x,z,t)}{\lambda_q},$$

$$-l_- < x < l_+, \quad -h/2 < z < h/2, \quad t > 0, \tag{2.9}$$

where

$$a = \lambda_q \left(c_\varepsilon + \frac{2E(1+\nu)T_0\alpha_T^2}{1-2\nu} \right)^{-1}. \tag{2.10}$$

For simplicity, we assume that, apart from the density, the beam parts B_- and B_+ share all the parameters involved in equations (2.6) and (2.9). Note that the two equations in (2.6) for the parts B_- and B_+ are coupled by the heat equation (2.9). The function θ satisfies the Dirichlet boundary conditions and vanishes at the initial time

$$\theta|_{z=h/2} = \theta^+(x,t), \quad \theta|_{z=-h/2} = 0, \quad -l_- < x < l_+, \quad t > 0,$$

$$\theta|_{x=\pm l_\pm} = 0, \quad |z| < h/2, \quad t > 0; \qquad \theta|_{t=0} = 0, \quad -l_- < x < l_+, \quad |z| < h/2. \tag{2.11}$$

It is convenient to rewrite the governing initial-boundary value problem in dimensionless coordinates. Denote

$$\xi = \frac{x}{l}, \quad \zeta = \frac{z}{h}, \quad \tau = \frac{at}{h^2},$$

$$\hat{h} = \frac{h}{l}, \quad c_\pm = \frac{l_\pm}{l}, \quad D_\pm^4 = \frac{E\hat{h}^4 h^2}{12\rho_\pm a^2}, \quad \beta = \frac{12E\alpha_T^2 a T_0}{\lambda_q}, \tag{2.12}$$

where $l = l_- + l_+$ (then, apparently, $c_+ + c_- = 1$). Notice that β is the dimensionless thermoelastic diffusivity, $D_\pm$ are the dimensionless thermoelastic bending rigidities, and the generalized coordinate τ is an analogue of the Fourier number $\mathrm{Fo} = a_1 t/l^2$, where $a_1 = \lambda_q/c_\varepsilon$. Introduce the functions

$$\hat{w}(\xi,\tau) = \frac{\hat{h}w(x,t)}{12l\alpha_T}, \quad \hat{\theta}(\xi,\zeta,\tau) = \theta(x,z,t),$$

$$\hat{\theta}^+(\xi,\tau) = \theta^+(x,t), \quad \hat{p}(\xi,\tau) = \frac{p(x,t)}{\hat{h}^2 E\alpha_T}, \quad \hat{\Omega}(\xi,\zeta,\tau) = \frac{l^2\Omega(x,z,t)}{\lambda_q}. \tag{2.13}$$

It is directly verified that the functions $\hat{w}$, $\hat{\theta}$, $\hat{p}$, and $\hat{\Omega}$ are dimensionless. The new functions $\hat{w}$ and $\hat{\theta}$ solve the following initial-boundary value problem.

Find the functions $\hat{w}(\xi,\tau)$ and $\hat{\theta}(\xi,\zeta,\tau)$, the solutions of the system of differential equations

$$\frac{\partial^4 \hat{w}}{\partial \xi^4} + \frac{1}{D_+^4}\frac{\partial^2 \hat{w}}{\partial \tau^2} + \frac{\partial^2}{\partial \xi^2}\int_{-1/2}^{1/2} \hat{\theta}\zeta d\zeta = \hat{p}, \quad 0 < \xi < c_+, \quad \tau > 0,$$

$$\frac{\partial^4 \hat{w}}{\partial \xi^4} + \frac{1}{D_-^4}\frac{\partial^2 \hat{w}}{\partial \tau^2} + \frac{\partial^2}{\partial \xi^2}\int_{-1/2}^{1/2} \hat{\theta}\zeta d\zeta = \hat{p}, \quad -c_- < \xi < 0, \quad \tau > 0,$$

$$\hat{h}^2\frac{\partial^2 \hat{\theta}}{\partial \xi^2} + \frac{\partial^2 \hat{\theta}}{\partial \zeta^2} - \frac{\partial \hat{\theta}}{\partial \tau} + \beta\zeta\frac{\partial^3 \hat{w}}{\partial \xi^2 \partial \tau} = -\hat{\Omega}, \quad -c_- < \xi < c_+, \quad |\zeta| < 1/2, \quad \tau > 0, \tag{2.14}$$

subject to the boundary conditions

$$\hat{w}|_{\xi=0^\pm} = 0, \quad \left.\frac{\partial \hat{w}}{\partial \xi}\right|_{\xi=0^\pm} = 0, \quad \left.\frac{\partial^2 \hat{w}}{\partial \xi^2}\right|_{\xi=\pm c_\pm} = 0, \quad \left.\frac{\partial^3 \hat{w}}{\partial \xi^3}\right|_{\xi=\pm c_\pm} = 0, \quad \tau > 0,$$

$$\hat{\theta}|_{\zeta=1/2}=\hat{\theta}^{+},\quad \hat{\theta}|_{\zeta=-1/2}=0,\ -c_{-}<\xi<c_{+};\quad \hat{\theta}|_{\xi=\pm c_{\pm}}=0,\ |\zeta|<1/2,\ \tau>0,\quad (2.15)$$

and the initial conditions

$$\hat{w}|_{\tau=0}=\left.\frac{\partial\hat{w}}{\partial\tau}\right|_{\tau=0}=0,\ -c_{-}<\xi<c_{+};\quad \hat{\theta}|_{\tau=0}=0,\ -c_{-}<\xi<c_{+},\ |\zeta|<1/2.\quad (2.16)$$

3 Solution method

3.1 Infinite system of linear algebraic equations

We apply the operational method to the problem to reduce its dimensionality and eventually to convert it into an infinite system of linear algebraic transforms with respect to the Laplace transforms of the vibration modes. Introduce the Laplace transforms of the functions $\hat{\theta}$, $\hat{w}$, $\hat{\Omega}$, and $\hat{p}$ as

$$(\theta_s(\xi,\zeta),w_s(\xi),\Omega_s(\xi,\zeta),p_s(\xi))=\int_0^\infty e^{-s\tau}(\hat{\theta},\hat{w},\hat{\Omega},\hat{p})d\tau.\quad (3.1)$$

In virtue of the initial conditions the system of equations (2.14) becomes

$$\hat{h}^2\frac{\partial^2\theta_s}{\partial\xi^2}+\frac{\partial^2\theta_s}{\partial\zeta^2}-s\theta_s+\beta s\zeta\frac{d^2w_s}{d\xi^2}=-\Omega_s,\quad -c_{-}<\xi<c_{+},\quad |\zeta|<1/2,$$

$$\frac{d^4w_s}{d\xi^4}-\frac{s^2w_s}{D_\pm^4}+\frac{d^2}{d\xi^2}\int_{-1/2}^{1/2}\theta_s\zeta d\zeta=p_s,\quad a_\pm<\xi<b_\pm,\quad (3.2)$$

where $a_-=-c_-$, $a_+=0$, $b_-=0$, $b_+=c_+$. We consider first the heat equation. Applying the finite sine-transforms

$$\theta_{sk}(\xi)=\int_{-1/2}^{1/2}\theta_s(\xi,\zeta)\sin\pi k(\zeta+1/2)d\zeta,$$

$$\theta_{skm}=\int_{-c_-}^{c_+}\theta_{sk}(\xi)\sin\pi m(\xi+c_-)d\xi\quad (3.3)$$

to the first equation in (3.2) and taking into account the boundary and initial conditions for the function θ we find

$$\theta_{skm}=\frac{\Omega_{skm}-\pi k(-1)^k\theta_{sm}^{+}+\beta s\mu_k w_{sm}^{*}}{\pi^2(\hat{h}^2m^2+k^2)+s}.\quad (3.4)$$

Here

$$w_{sm}^{*}=\int_{-c_-}^{c_+}\frac{d^2w_s(\xi)}{d\xi^2}\sin\pi m(\xi+c_-)d\xi,\quad \mu_k=\left\{\begin{array}{ll}0, & k=1,3,\dots,\\ -(k\pi)^{-1}, & k=2,4,\dots,\end{array}\right.$$

$$\theta_{sm}^{+}=\int_{-c_-}^{c_+}\sin\pi m(\xi+c_-)d\xi\int_0^\infty\theta^{+}(\xi,\tau)e^{-s\tau}d\tau,$$

$$\Omega_{skm}=\int_{-c_-}^{c_+}\sin\pi m(\xi+c_-)d\xi\int_{-1/2}^{1/2}\Omega_s(\xi,\zeta)\sin\pi k(\zeta+1/2)d\zeta.\quad (3.5)$$

Notice that the terms w^*_{sm} are unknown. Integration by parts in the integral representation of w^*_{sm} does not simplify the expression since the function $w_s(\xi)$ is unknown at the ends $\xi = \pm c_\pm$.

To proceed with the solution of the second and third equations in the system (3.2) we invert the two sine-transforms

$$\theta_s(\xi,\zeta) = 4\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \sin\pi m(\xi + c_-)\sin\pi k(\zeta + 1/2)\theta_{skm} \tag{3.6}$$

and write down the expression for the integral in (3.2)

$$\int_{-1/2}^{1/2} \theta_s(\xi,\zeta)\zeta d\zeta = E_s(\xi) + F_s(\xi) + G_s(\xi), \tag{3.7}$$

where

$$E_s(\xi) = -\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{m=1}^{\infty}\frac{\sin\pi m(\xi + c_-)\Omega_{s2km}}{\pi^2(\hat{h}^2m^2 + 4k^2) + s},$$

$$F_s(\xi) = 4\sum_{m=1}^{\infty}\sin\pi m(\xi + c_-)\theta^+_{sm}\sum_{k=1}^{\infty}\frac{1}{\pi^2(\hat{h}^2m^2 + 4k^2) + s},$$

$$G_s(\xi) = \frac{\beta s}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{m=1}^{\infty}\frac{\sin\pi m(\xi + c_-)w^*_{sm}}{\pi^2(\hat{h}^2m^2 + 4k^2) + s}, \tag{3.8}$$

The double sums can be simplified by summing the terms depending on k [13]. This yields us

$$E_s(\xi) = -\frac{1}{4\pi^2}\sum_{m=1}^{\infty}\int_{-1/2}^{1/2}\Omega_{sm}(\zeta)\frac{\sin\pi m(\xi + c_-)}{a^2_{sm}}\left(\frac{\sinh 2\pi a_{sm}\zeta}{\sinh\pi a_{sm}} - 2\zeta\right)d\zeta$$

$$F_s(\xi) = \frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{\sin\pi m(\xi + c_-)\theta^+_{sm}}{a_{sm}}\left(\coth\pi a_{sm} - \frac{1}{\pi a_{sm}}\right),$$

$$G_s(\xi) = \frac{\beta s}{4\pi^4}\sum_{m=1}^{\infty}\frac{\sin\pi m(\xi + c_-)w^*_{sm}}{a^2_{sm}}\left(\frac{\pi^2}{6} - \frac{\pi}{2a_{sm}}\coth\pi a_{sm} + \frac{1}{2a^2_{sm}}\right), \tag{3.9}$$

where

$$a_{sm} = \frac{1}{2\pi}\sqrt{\pi^2\hat{h}^2m^2 + s} \tag{3.10}$$

is a fixed branch of the two-valued function $v^2(s) = (4\pi^2)^{-1}(\pi^2\hat{h}^2m^2 + s)$.

Rewrite now the second and third equations of the system (3.2) in the form

$$\frac{d^4w_s(\xi)}{d\xi^4} - \frac{s^2}{D^4_+}w_s(\xi) = p_s(\xi) - \frac{d^2}{d\xi^2}[E_s(\xi) + F_s(\xi) + G_s(\xi)], \quad 0 < \xi < c_+,$$

$$\frac{d^4w_s(\xi)}{d\xi^4} - \frac{s^2}{D^4_-}w_s(\xi) = p_s(\xi) - \frac{d^2}{d\xi^2}[E_s(\xi) + F_s(\xi) + G_s(\xi)], \quad -c_- < \xi < 0. \tag{3.11}$$

The function $w_s(\xi)$ has to satisfy the boundary conditions

$$w_s(0) = w'_s(0) = 0, \quad w''_s(\pm c_\pm) = w'''_s(\pm c_\pm) = 0. \tag{3.12}$$

It would be a simple matter to solve this boundary value problem if the right-hand side did not have the unknown function $G_s(\xi)$. Initially, we assume that the right-hand side is known and expand the solution of the problem (3.11) to (3.12) in terms of the eigenfunctions (the shape modes) $\phi_{m\pm}(\xi)$ as

$$w_s(\xi) = \sum_{m=1}^{\infty} \eta_{sm\pm}\phi_{m\pm}(\xi), \quad a_\pm < \xi < b_\pm, \tag{3.13}$$

with the functions $\eta_{m\pm}(\tau)$ to be determined. Here, $a_- = -c_-$, $a_+ = 0$, $b_- = 0$, $b_+ = c_+$. The Laplace inversion of these representations give expansions of the dimensionless displacement $\hat{w}(\xi,\tau)$ in terms of the shape modes $\phi_{m\pm}(\xi)$ and the vibration modes $\eta_{m\pm}(\tau)$

$$\hat{w}(\xi,\tau) = \sum_{m=1}^{\infty} \eta_{m\pm}(\tau)\phi_{m\pm}(\xi), \quad a_\pm < \xi < b_\pm. \tag{3.14}$$

The functions $\phi_{m\pm}(\xi)$ solve the eigenvalue problems

$$\frac{d^4}{d\xi^4}\phi_{m+}(\xi) - \omega_{m+}^4\phi_{m+}(\xi) = 0, \quad 0 < \xi < c_+,$$

$$\phi_{m+}(0) = \phi'_{m+}(0) = 0, \quad \phi''_{m+}(c_+) = \phi'''_{m+}(c_+) = 0, \quad \int_0^{c_+} \phi_{m+}^2(\xi)d\xi = 1, \tag{3.15}$$

and

$$\frac{d^4}{d\xi^4}\phi_{m-}(\xi) - \omega_{m-}^4\phi_{m-}(\xi) = 0, \quad -c_- < \xi < 0,$$

$$\phi_{m-}(0) = \phi'_{m-}(0) = 0, \quad \phi''_{m-}(-c_-) = \phi'''_{m-}(-c_-) = 0, \quad \int_{-c_-}^{0} \phi_{m-}^2(\xi)d\xi = 1, \tag{3.16}$$

and form orthonormal bases of the associated functional spaces. The eigenvalues $\omega_{m\pm} = c_\pm^{-1}\gamma_m$ $(m = 1, 2, \ldots)$ are expressed through the roots γ_m of the transcendental equation

$$\cos\gamma_m \cosh\gamma_m + 1 = 0, \quad m = 1, 2\ldots, \tag{3.17}$$

The eigenfunctions $\phi_{m\pm}(\xi)$ admit the following representation

$$\phi_{m\pm}(\xi) = c_\pm^{-1/2}\nu_m\{\pm(\sin\gamma_m - \sinh\gamma_m)[\sin(\omega_{m\pm}\xi) - \sinh(\omega_{m\pm}\xi)]$$
$$+(\cos\gamma_m + \cosh\gamma_m)[\cos(\omega_{m\pm}\xi) - \cosh(\omega_{m\pm}\xi)]\} \tag{3.18}$$

with ν_m being determined as

$$\nu_m = -\frac{\cos\gamma_m}{\sin^2\gamma_m} > 0, \quad m = 1, 2\ldots. \tag{3.19}$$

The numerical values of the roots of the transcendental equation (3.17) can be easily determined by the iterative procedure

$$\gamma_m^{(n)} = (-1)^{m-1}\cos^{-1}\left(\frac{-1}{\cosh\gamma_m^{(n-1)}}\right) + 2\pi(m-1), \quad n = 1, 2, \ldots,$$

$$\gamma_m^{(0)} = (-1)^{m-1}\frac{\pi}{2} + 2\pi(m-1), \quad m = 1, 2, \ldots. \tag{3.20}$$

For $m \geq 4$, the roots are determined asymptotically by

$$\gamma_m \sim (-1)^{m-1}\frac{\pi}{2} + 2\pi(m-1) \tag{3.21}$$

with accuracy 10^{-7}. As for the eigenfunctions, for numerical purposes, it will be convenient to use their alternative representation

$$\phi_{m\pm}(\xi) = \frac{1}{\sqrt{c_\pm}\sin\gamma_m}\{\sin[\gamma_m(1 \mp \xi/c_\pm)] \pm (-1)^m \sin[\gamma_m\xi/c_\pm]$$

$$+\frac{\mp\sinh(\gamma_m\xi/c_\pm) + (-1)^m\sinh[\gamma_m(1\mp\xi/c_\pm)]}{\cosh\gamma_m}\Bigg]. \tag{3.22}$$

Here, we employed the following identities for the eigenvalues:

$$\cosh\gamma_m = -\frac{1}{\cos\gamma_m}, \quad \sinh\gamma_m = (-1)^m\tan\gamma_m, \quad \sin\gamma_m = (-1)^{m-1}\tanh\gamma_m. \tag{3.23}$$

On expanding the functions p_s, $E''_s(\xi)$, $F''_s(\xi)$, and $G''_s(\xi)$ in terms of the eigenfunctions $\phi_\pm(\xi)$

$$p_s(\xi) = \sum_{m=1}^{\infty} p_{sm\pm}\phi_{m\pm}(\xi), \quad E''_s(\xi) = \sum_{m=1}^{\infty} E_{sm\pm}\phi_{m\pm}(\xi), \quad a_\pm < \xi < b_\pm,$$

$$F''_s(\xi) = \sum_{m=1}^{\infty} F_{sm\pm}\phi_{m\pm}(\xi), \quad G''_s(\xi) = \sum_{m=1}^{\infty} G_{sm\pm}\phi_{m\pm}(\xi), \quad a_\pm < \xi < b_\pm, \tag{3.24}$$

we can express the coefficients $\eta_{sm\pm}$ through the known numbers $p_{sm\pm}$, $E_{sm\pm}$, and $F_{sm\pm}$, and the unknown numbers $G_{sm\pm}$ which depend on $\eta_{sm\pm}$. Here,

$$p_{sm\pm} = \int_{a_\pm}^{b_\pm} p_s(\xi)\phi_{m\pm}(\xi)d\xi, \quad E_{sm\pm} = \int_{a_\pm}^{b_\pm} E''_s(\xi)\phi_{m\pm}(\xi)d\xi,$$

$$F_{sm\pm} = \int_{a_\pm}^{b_\pm} F''_s(\xi)\phi_{m\pm}(\xi)d\xi, \quad G_{sm\pm} = \int_{a_\pm}^{b_\pm} G''_s(\xi)\phi_{m\pm}(\xi)d\xi. \tag{3.25}$$

The expansions (3.24), when substituted into equations (3.11), give

$$\eta_{sm\pm} = \frac{p_{sm\pm} - E_{sm\pm} - F_{sm\pm} - G_{sm\pm}}{(\gamma_m/c_\pm)^4 - s^2/D_\pm^4}. \tag{3.26}$$

Now, the unknown coefficients $G_{s\nu\pm}$ can be represented in the following series form:

$$G_{s\nu\pm} = -\frac{\beta s}{4\pi^2}\sum_{m=1}^{\infty}\frac{m^2 d_{m\nu}^{\pm}}{a_{sm}^2}\left(\frac{\pi^2}{6} - \frac{\pi}{2a_{sm}}\coth\pi a_{sm} + \frac{1}{2a_{sm}^2}\right)$$

$$\times\left(\int_{-c_-}^{0}\sin\pi m(\xi_1 + c_-)\sum_{j=1}^{\infty}\eta_{sj-}\phi''_{j-}(\xi_1)d\xi_1 + \int_0^{c_+}\sin\pi m(\xi_1 + c_-)\sum_{j=1}^{\infty}\eta_{sj+}\phi''_{j+}(\xi_1)d\xi_1\right). \tag{3.27}$$

Here, $d^{\pm}_{m\nu\pm}$ are the sine-transforms of the eigenfunctions $\phi_{m\pm}(\xi)$

$$d^{\pm}_{m\nu} = \int_{a_\pm}^{b_\pm} \sin \pi m(\xi + c_-)\phi_{\nu_\pm}(\xi) d\xi. \tag{3.28}$$

These coefficients can be explicitly evaluated. Remembering that $c_+ + c_- = 1$ we have

$$d^{+}_{m\nu} = \frac{(-1)^m \hat{d}^{+}_{m\nu}}{\sqrt{c_+}}, \quad d^{-}_{m\nu} = -\frac{\hat{d}^{-}_{m\nu}}{\sqrt{c_-}}, \tag{3.29}$$

where

$$\hat{d}^{\pm}_{m\nu} = -\frac{2\pi m(\gamma_\nu/c_\pm)^2}{\sin^2\gamma_\nu(\gamma_\nu^4/c_\pm^4 - m^4\pi^4)} \left[\sin\gamma_\nu \left(\sin\gamma_\nu \cos \pi m c_\pm - \frac{\gamma_\nu}{\pi m c_\pm} \cos\gamma_\nu \sin \pi m c_\pm\right)\right.$$
$$\left. -(-1)^\nu \sin\gamma_\nu \left(\frac{\pi^2 m^2 c_\pm^2}{\gamma_\nu^2} \sin\gamma_\nu - \frac{\gamma_\nu}{\pi m c_\pm} \sin \pi m c_\pm\right)\right]. \tag{3.30}$$

Analysis of asymptotics of the functions $\phi''_{j\pm}(\xi)$ and the coefficients of the series in (3.27) and numerical tests show that due to the nonuniform convergence of the series involved it is infeasible to rewrite the infinite system of equations (3.26), (3.27) with respect to the coefficients $\eta_{sm\pm}$ in the standard form

$$\eta_{s\nu\pm} + \sum_{m=1}^{\infty} (c^{+}_{\nu m\pm}\eta_{sm+} + c^{-}_{\nu m\pm}\eta_{sm-}) = g_{s\nu\pm}, \quad \nu = 1, 2, \ldots. \tag{3.31}$$

We propose another way for finding the coefficients $\eta_{sm\pm}$. It is based on the method of successive approximations, integration by parts, and summation of the weakly convergent series exactly.

3.2 Method of successive approximations

Considerable simplifications result if formula (3.26) for the Laplace transforms of the vibration modes is rewritten in the form

$$\eta^{(n)}_{s\nu\pm} = \frac{p_{s\nu\pm} - E_{s\nu\pm} - F_{s\nu\pm} - G^{(n)}_{s\nu\pm}}{(\gamma_\nu/c_\pm)^4 - s^2/D^4_\pm}, \quad n = 1, 2, \ldots, \quad G^{(1)}_{s\nu\pm} = 0, \quad \nu = 1, 2, \ldots. \tag{3.32}$$

This transformation itself does not improve the converge of the series representation of the coefficients (3.27). However, in each step of the iterative procedure (3.32) the coefficients $G^{(n)}_{s\nu\pm}$ are known, and the convergence may be improved significantly. It is convenient to represent the terms $G^{(n)}_{s\nu\pm}$ as

$$G^{(n)}_{s\nu\pm} = \int_{-c_-}^{0} \Sigma^{(n)}_s(\xi)\phi_{\nu-}(\xi)d\xi + \int_0^{c_+} \Sigma^{(n)}_s(\xi)\phi_{\nu+}(\xi)d\xi, \tag{3.33}$$

where

$$\Sigma^{(n)}_s(\xi) = -\frac{\beta s}{4\pi^2} \sum_{m=1}^{\infty} \frac{m^2 \sin \pi m(\xi + c_-)}{a^2_{sm}} \left(\frac{\pi^2}{6} - \frac{\pi}{2a_{sm}} \coth \pi a_{sm} + \frac{1}{2a^2_{sm}}\right) \Psi^{(n-1)}_{sm},$$

$$\Psi_{sm}^{(n-1)} = \int_{-c_-}^{c_+} \sin \pi m(\xi + c_-) \frac{d^2 w_s^{(n-1)}(\xi)}{d\xi^2} d\xi. \tag{3.34}$$

On integrating by parts two times, taking into account the boundary conditions, the discontinuity at the point $\xi = 0$ of the second and third derivatives of the function $w_s(\xi)$, the formula

$$\frac{d^4 w_s^{(n-1)}(\xi)}{d\xi^4} = \sum_{m=1}^{\infty} \eta_{sm\pm}^{(n-1)} \omega_{m\pm}^4 \phi_{m\pm}(\xi), \quad a_\pm < \xi < b_\pm, \tag{3.35}$$

and also the relations (3.28) we deduce

$$\Psi_{sm}^{(n-1)} = \frac{\cos \pi m c_-}{\pi m} w_{s0}^{*(n-1)} - \frac{\sin \pi m}{\pi^2 m^2} w_{s1}^{*(n-1)} - \frac{\lambda_{sm}^{(n-1)}}{\pi^2 m^2}, \tag{3.36}$$

where

$$w_{sj}^{*(n-1)} = \frac{d^{j+2} w_s^{(n-1)}}{d\xi^{j+2}}(0^+) - \frac{d^{j+2} w_s^{(n-1)}}{d\xi^{j+2}}(0^-), \quad j = 0, 1,$$

$$\lambda_{sm}^{(n-1)} = \sum_{\nu=1}^{\infty} \left(d_{m\nu}^+ \frac{p_{s\nu+} - E_{s\nu+} - F_{s\nu+} - G_{s\nu+}^{(n-1)}}{1 - s^2 (c_+/D_+/\gamma_\nu)^4} \right.$$
$$\left. + d_{m\nu}^- \frac{p_{s\nu-} - E_{s\nu-} - F_{s\nu-} - G_{s\nu-}^{(n-1)}}{1 - s^2 (c_-/D_-/\gamma_\nu)^4} \right). \tag{3.37}$$

Combining equation (3.13) to the first formula in (3.37) gives the following expressions for the terms $w_{s0}^{*(n-1)}$ and $w_{s1}^{*(n-1)}$:

$$w_{sj}^{*(n-1)} = 2 \sum_{\nu=1}^{\infty} \left(\frac{\cos \gamma_\nu - (-1)^\nu}{\sin \gamma_\nu} \right)^j \left((-1)^{j+1} \frac{p_{s\nu+} - E_{s\nu+} - F_{s\nu+} - G_{s\nu+}^{(n-1)}}{\sqrt{c_+} (\omega_{\nu+}^{2-j} - s^2/D_+^4/\omega_{\nu+}^{2+j})} \right.$$
$$\left. + \frac{p_{s\nu-} - E_{s\nu-} - F_{s\nu-} - G_{s\nu-}^{(n-1)}}{\sqrt{c_-} (\omega_{\nu-}^{2-j} - s^2/D_-^4/\omega_{\nu-}^{2+j})} \right), \quad j = 0, 1. \tag{3.38}$$

Now we can sum the weakly convergent series [13] to obtain

$$\Sigma_s^{(n)}(\xi) = \sum_{j=1}^{3} \sigma_{sj}^{(n-1)}(\xi) \tag{3.39}$$

with $\sigma_{sj}(\xi)$ being defined as

$$\sigma_{s1}^{(n-1)}(\xi) = \frac{\beta s \sinh \pi \delta_s (\xi \mp c_\pm)}{12 \hat{h}^2 \sinh \pi \delta_s} \left[w_{s0}^{*(n-1)} \cosh \pi \delta_s c_\mp \mp w_{s1}^{*(n-1)} \frac{\sinh \pi \delta_s c_\mp}{\pi \delta_s} \right],$$

$$\sigma_{s2}^{(n-1)}(\xi) = -\frac{\beta s}{8\pi^3} \sum_{m=1}^{\infty} \frac{m \sin \pi m(\xi + c_-)}{a_{sm}^3} \left(-\pi \coth \pi a_{sm} + \frac{1}{a_{sm}} \right)$$
$$\times \left[\cos \pi m c_- w_{s0}^{*(n-1)} - \frac{\sin \pi m c_-}{\pi m} w_{s1}^{*(n-1)} - \frac{\lambda_{sm}^{(n-1)}}{\pi m} \right],$$

$$\sigma_{s3}^{(n-1)}(\xi) = \frac{\beta s}{24\pi^2} \sum_{m=1}^{\infty} \frac{\lambda_{sm}^{(n-1)} \sin \pi m(\xi + c_-)}{a_{sm}^2}, \quad a_\pm < \xi < b_\pm, \tag{3.40}$$

and $\delta_s = \sqrt{s}(\pi \hat{h})^{-1}$, $\sqrt{s}$ is a fixed branch of the function $v^2 = s$.

It become evident that the new series converge uniformly and absolutely. On multiplying the functions $\sigma_{sj}^{(n-1)}(\xi)$ by $\phi_{\nu\pm}(\xi)$ and integrating over the intervals $(a_\pm, b_\pm)$ we obtain formulas for the coefficients $G_{s\nu\pm}^{(n)}$ efficient for numerical purposes

$$G_{s\nu\pm}^{(n)} = \sum_{j=1}^{3} g_{s\nu j\pm}^{(n-1)}, \tag{3.41}$$

where

$$g_{s\nu 1\pm}^{(n-1)} = -\frac{\beta s}{12\hat{h}^2 \sinh \pi\delta_s} \left(-w_{s0}^{*(n-1)} \cosh \pi\delta_s c_\mp \pm \frac{1}{\pi\delta_s} w_{s1}^{*(n-1)} \sinh \pi\delta_s c_\mp \right) T_{s\nu\pm},$$

$$g_{s\nu 2\pm}^{(n-1)} = -\frac{\beta s}{8\pi^3} \sum_{m=1}^{\infty} \frac{m d_{m\nu}^{\pm}}{a_{sm}^3} \left(-\pi \coth \pi a_{sm} + \frac{1}{a_{sm}} \right)$$

$$\times \left[\cos \pi m c_- w_{s0}^{*(n-1)} - \frac{\sin \pi m c_-}{\pi m} w_{s1}^{*(n-1)} - \frac{\lambda_{sm}^{(n-1)}}{\pi m} \right],$$

$$g_{s\nu 3}^{(n-1)} = \frac{\beta s}{24\pi^2} \sum_{m=1}^{\infty} \frac{\lambda_{sm}^{(n-1)} d_{m\nu}^{\pm}}{a_{sm}^2}, \tag{3.42}$$

and

$$T_{s\nu\pm} = \pm \frac{2}{\sqrt{c_\pm}(\omega_{\nu\pm}^4 - \pi^4 \delta_s^4)} \Big[-\pi\delta_s \omega_{\nu\pm}^2 \cosh \pi\delta_s c_\pm - (-1)^\nu \pi^3 \delta_s^3$$

$$+ \frac{\omega_{\nu\pm}^3}{\sin \gamma_\nu} (\cos \gamma_\nu - (-1)^\nu) \sinh \pi\delta_s c_\pm \Big]. \tag{3.43}$$

Numerical tests implemented according to this procedure reveal a good convergence of the algorithm for $\beta \in [0, 50]$. As β grows further, the convergence rate slows down, and the method diverges as $\beta > 60$. After the Laplace transforms of the vibration modes are found, the function $w_s(\xi)$ can be found by (3.13), while the Laplace transform of the function $\hat{\theta}$ is determined by inversion of the sine-transforms. We have

$$\theta_s^{(n)}(\xi, \zeta) = -4 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \pi k \left(\zeta + \frac{1}{2}\right) \sin \pi m(\xi + c_-)}{\pi^2(\hat{h}^2 m^2 + k^2) + s} [\pi k(-1)^k \theta_{sm}^{+} - \beta s \mu_k \Psi_{sm}^{(n-1)}]. \tag{3.44}$$

This expression can be simplified by summing the k-series. We have

$$\theta_s^{(n)}(\xi, \zeta) = N_{1s}(\xi, \zeta) + N_{2s}^{(n-1)}(\xi, \zeta), \tag{3.45}$$

where

$$N_{1s}(\xi, \zeta) = 2 \sum_{m=1}^{\infty} \frac{\sin \pi m(\xi + c_-) \sinh[\pi(2\zeta + 1) a_{sm}]}{\sinh 2\pi a_{sm}} \theta_{sm}^{+},$$

$$N_{2s}^{(n-1)}(\xi, \zeta) = \frac{\beta s}{4\pi^2} \sum_{m=1}^{\infty} \frac{\sin \pi m(\xi + c_-)}{a_{sm}^2} \left(2\zeta + 1 - \frac{\sinh(2\pi a_{sm} \zeta)}{\sinh \pi a_{sm}} \right) \Psi_{sm}^{(n-1)}. \tag{3.46}$$

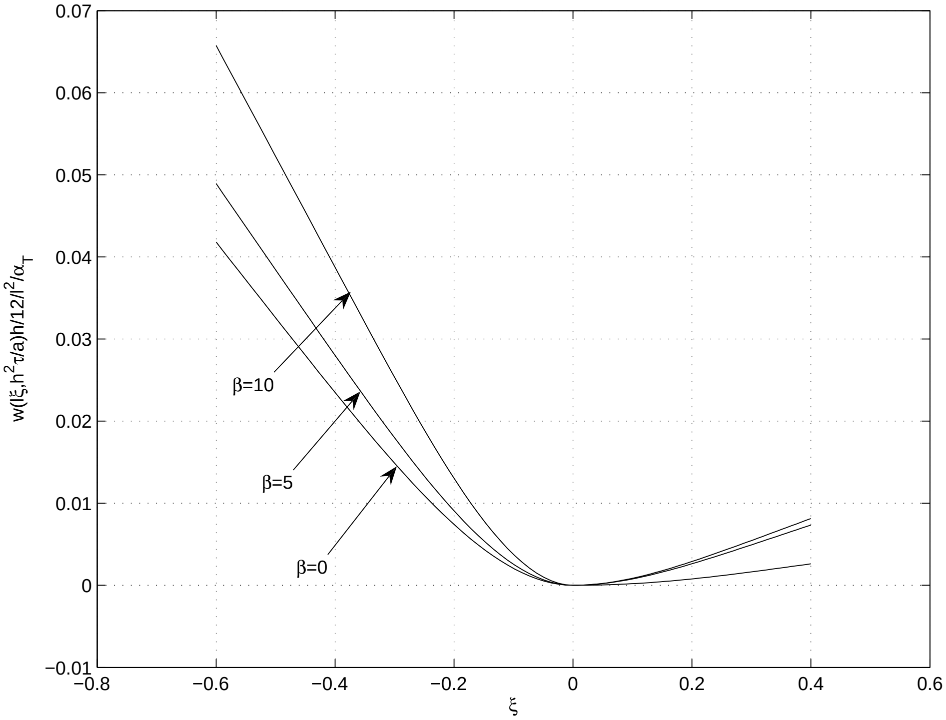

Figure 1: Variation of the dimensionless displacement $\hat{w}(\xi,\tau)$ with $\xi = x/l$ and for different values of the parameter β when $\tau = 1$.

3.3 Numerical results

To test the procedure proposed we assume that there is no external heat source, $\Omega(x,z,t) \equiv 0$. The composite structure is subjected to uniform loading applied at the moment $t = 0$, $p(x,t) = PH(t)$, where P is a constant load density, and $H(t)$ is the Heaviside function. Denote by $\hat{P}$ the dimensionless parameter $P(\hat{h}^2 E\alpha_T)^{-1}$. Then the coefficients $p_{sm\pm}$ introduced in (3.24) become

$$p_{sm\pm} = -\frac{2\hat{P}\sqrt{c_\pm}}{s\gamma_m \sin\gamma_m}[\cos\gamma_m - (-1)^m]. \tag{3.47}$$

The function $\theta^+(x,t)$ has to vanish at the ends $\mp l_\mp$. Select its dimensionless analogue as $\hat{\theta}^+(\xi,\tau) = A(c_+ - \xi)(\xi + c_-)$, A is a positive dimensionless parameter. Then

$$\theta^+_{sm} = -\frac{2A}{\pi^3 m^3 s}[(-1)^m - 1]. \tag{3.48}$$

Thus, the coefficients $E_{sm\pm}$ and $F_{sm\pm}$ have the following values:

$$E_{sm\pm} = 0, \quad F_{sm\pm} = -\frac{2A}{\pi^2 s}\sum_{m=1}^{\infty} \frac{d^{\pm}_{2m-1\nu}}{(2m-1)a_{s2m-1}}\left(\coth \pi a_{s2m-1} - \frac{1}{\pi a_{s2m-1}}\right). \tag{3.49}$$

The expression (3.46) for the functions $\theta^{(n)}_s$ possesses the coefficients θ^+_{sm}. By substituting their expression (3.48) into the first formula in (3.46) we find

$$N_{1s}(\xi,\zeta) = \frac{8A}{\pi^3 s}\sum_{m=1}^{\infty} \frac{\sin \pi m(\xi + c_-)\sinh[\pi(2\zeta+1)a_{sm}]}{(2m-1)^3 \sinh 2\pi a_{sm}}. \tag{3.50}$$

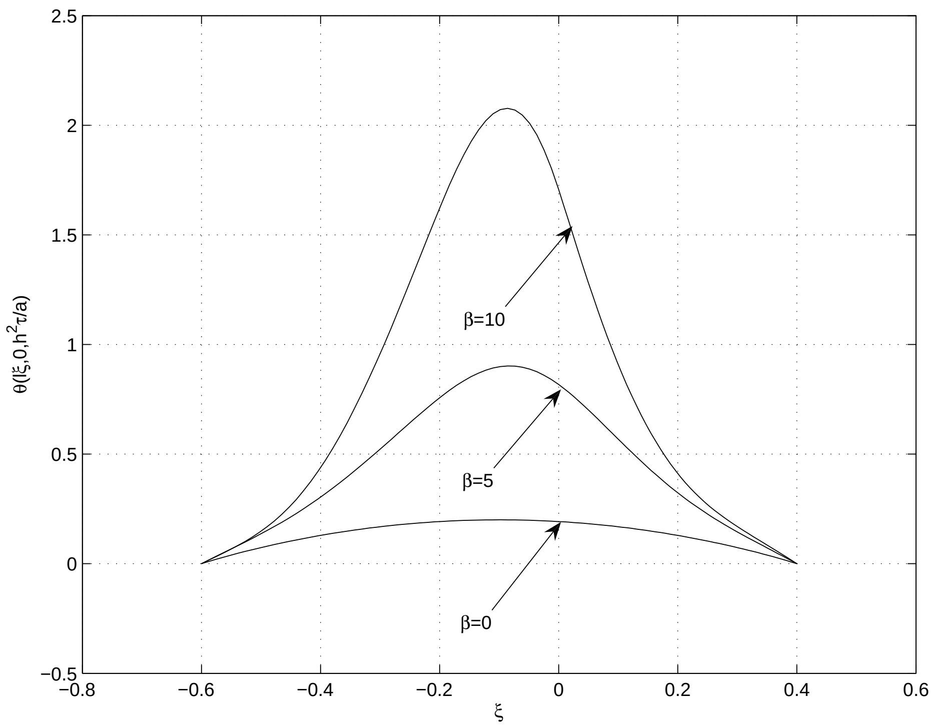

Figure 2: Variation of the temperature function $\hat{\theta}(\xi, \zeta, \tau)$ with $\xi = x/l$ and for different values of the parameter β when $\zeta = 0$ and $\tau = 1$.

The method of successive approximations employed enables us to determine the Laplace transforms of the vibration modes $\eta_{s\nu\pm}$, the dimensionless displacement $w_s(\xi)$, and the function $\theta_s(\xi, \zeta)$. To compute their originals, we compute the inverse Laplace transforms. For a function f_s it is given by

$$f(\tau) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} f_s e^{s\tau} ds, \quad c > 0. \tag{3.51}$$

Equivalently (see for example [14]), it can be rewritten as either the cosine inverse transform of the real part of the function $f_{c+i\sigma}$

$$f(\tau) = \frac{2e^{c\tau}}{\pi} \int_0^\infty \mathrm{Re}\, f_{c+i\sigma} \cos \sigma\tau d\sigma, \tag{3.52}$$

or as the sine transform of its imaginary part

$$f(\tau) = -\frac{2e^{c\tau}}{\pi} \int_0^\infty \mathrm{Im}\, f_{c+i\sigma} \sin \sigma\tau d\sigma. \tag{3.53}$$

These integrals were computed numerically by the Simpson rule.

For the numerical tests we chose $c_- = 0.6$ (then $c_+ = 0.4$), $\hat{h} = 0.1$, $D_+ = 1$, $D_- = D_+ \sqrt[4]{c_-/c_+}$, $\hat{P} = 1$, $A = 10$, The variation of the dimensionless displacement $\hat{w}(\xi, \tau)$ with $\xi \in (-c_-, c_+)$ for $\beta = 0$, $\beta = 5$, and $\beta = 10$ when $\tau = 1$ is shown in Fig. 1. It is seen that for small values of the parameter β there is no big difference between the results produced by the coupled and uncoupled thermoelasticity models.

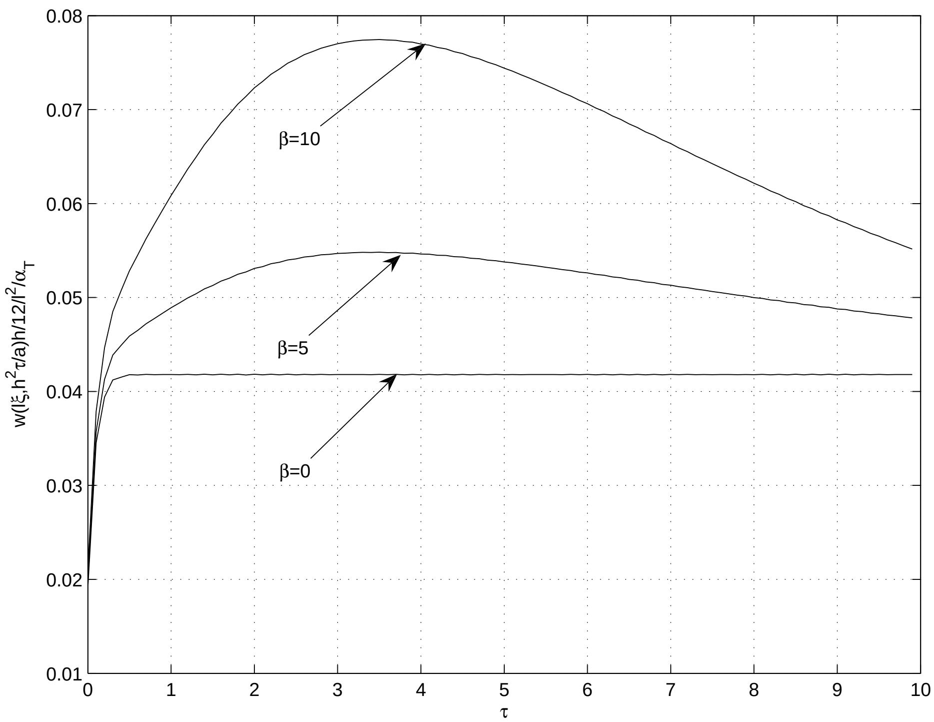

Figure 3: The dimensionless displacement $\hat{w}(\xi,\tau)$ at $\xi = -c_-$ versus the parameter $\tau = at/h^2$.

The discrepancies between the displacements grow with the growth of the parameter β. Fig. 2 presents variation of the temperature function $\hat{\theta}(\xi,\zeta,\tau)$ with ξ for $\zeta = 0$ and $\tau = 1$ for the same values of the parameter β. When β growth, the function θ also grows. The displacement function $\hat{w}(\xi,\tau)$ at the left end $\xi = -c_-$ is plotted versus τ in Fig. 3. It is seen that when $\beta = 0$, the displacement of the end point does not change with time after a certain moment t_*. If $\beta > 0$ and the coupling of the elastic and heat equations is not neglected, then the displacement differs from that for $\beta = 0$, and it attains its asymptotic limit significantly slower. Variation of the temperature function $\hat{\theta}(\xi,\zeta,\tau)$ at $\xi = -0.2$, $\zeta = 0$ with the parameter $\tau = at/h^2$ for different values of the parameter β is shown in Fig. 4.

4 Variational formulation

In this section we derive a variational formulation of the thermoelastic vibration of the thin structure analyzed in the previous section and show that it ultimately leads to the same governing boundary value problem. This formulation may be used as a basis for an alternative numerical method for the solution. Also, it can be employed for derivation of a dynamical system associated with the flight dynamics of a thermoelastic aircraft.

The variational formulation will be based on three invariants, the kinetic energy

$$K = \frac{1}{2}\frac{d^2}{dt^2}\int_{\mathcal{B}} \rho w^2 dV, \tag{4.1}$$

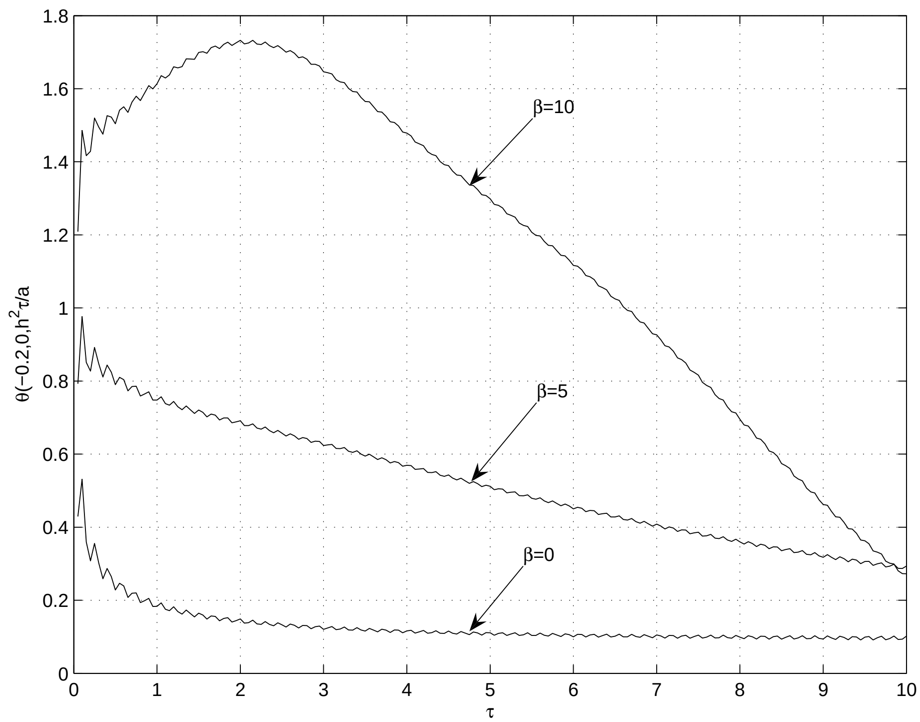

Figure 4: The temperature function $\hat{\theta}(\xi,\zeta,\tau)$ at $\xi=-0.2$, $\zeta=0$ versus the parameter $\tau=at/h^2$.

the generalized free energy

$$U=\frac{1}{2}\int_B\left[\alpha_1\left(z\frac{\partial^2 w}{\partial x^2}\right)^2+\alpha_2\theta^2\right]dV, \tag{4.2}$$

and the dissipation function

$$D=\frac{1}{2}\alpha_3\int_B \dot{\mathbf{S}}\cdot\dot{\mathbf{S}}dV. \tag{4.3}$$

Here, $B=B_+\cup B_-$ is the body (the beam analyzed in the previous section), $\rho(x)$ is the density of B, $\rho(x)=\rho_\pm$, $x\in B_\pm$, α_1, α_2, and α_3 are some factors to be determined from the equivalence of the corresponding variational equations and the governing equations (2.6), (2.9). The vector $\mathbf{S}$ is the entropy displacement field (entropy flow) (Biot, 1956)

$$\nabla\cdot\mathbf{S}=-s, \tag{4.4}$$

and s is the entropy density. The entropy flow $\mathbf{S}$ represents the amount of heat which has flown in a given direction divided by the absolute temperature. For small $\theta=T-T_0$ we have

$$\theta=\frac{T_0}{c_\varepsilon}\left(s-\frac{E\alpha_T\varepsilon_{kk}}{1-2\nu}\right). \tag{4.5}$$

On substituting the expression (2.8) and (4.4) into (4.5) we deduce for the function θ

$$\theta=-\frac{T_0}{c_\varepsilon\beta_1}\left(\nabla\cdot\mathbf{S}-E\alpha_T\frac{z\partial^2 w}{\partial x^2}\right), \tag{4.6}$$

where

$$\beta_1 = 1 + \frac{2ET_0(1+\nu)\alpha_T^2}{c_\varepsilon(1-2\nu)}. \tag{4.7}$$

Remembering the notations (2.10) and (2.12) we express β_1 through β

$$\beta_1 = \frac{12\alpha_T^2 T_0}{c_\varepsilon \beta}. \tag{4.8}$$

The variational principle reads

$$\delta K + \delta U + \delta D = \int_A (-\theta \mathbf{n} \cdot \delta \mathbf{S} + p\delta w) dA + \int_B \mathbf{Q} \cdot \delta \mathbf{S} dV, \tag{4.9}$$

where dA is an elementary surface of B, p is the load density, $\nabla \cdot \mathbf{Q} = -\lambda_q^{-1}\Omega$, and Ω is the heat flow capacity.

First we represent U as $U_1 + U_2$ and find the variation of U_1

$$\delta U_1 = \delta \frac{\alpha_1}{2} \int_B \left(z \frac{\partial^2 w}{\partial x^2} \right)^2 dV. \tag{4.10}$$

Choosing $\alpha_1 = E$ and integrating by parts in a usual manner yield

$$\delta U_1 = \frac{Eh^3 b}{12} \int_{-l_-}^{l_+} \frac{\partial^4 w}{\partial x^4} \delta w dx. \tag{4.11}$$

Compute now the variation of U_2, $\delta U_2 = \alpha_2 \int_B \theta \delta \theta dV$. From (4.6) we have

$$\delta U_2 = -\frac{\alpha_2 T_0}{c_\varepsilon \beta_1} \int_B \theta \delta \left(\nabla \cdot \mathbf{S} - E\alpha_T z \frac{\partial^2 w}{\partial x^2} \right) dV. \tag{4.12}$$

Implementing integration by parts and selecting $\alpha_2 = c_\varepsilon \beta_1 / T_0$ bring us

$$\delta U_2 = -\int_A \theta \mathbf{n} \cdot \delta \mathbf{S} dA + \int_B \nabla \theta \cdot \delta \mathbf{S} dV + E\alpha_T \int_B z \frac{\partial^2 \theta}{\partial x^2} \delta w dV. \tag{4.13}$$

Select next $\alpha_3 = T_0/\lambda_q$ and write down the variation of the kinetic energy and the dissipation function as

$$\delta K + \delta D = \int_B \rho \frac{\partial^2 w}{\partial t^2} \delta w dV + \frac{T_0}{\lambda_q} \int_B \dot{\mathbf{S}} \cdot \delta \mathbf{S} dV. \tag{4.14}$$

For the total variation of $K + U + D$ we find

$$\delta K + \delta U + \delta D = -\int_A \theta \mathbf{n} \cdot \delta \mathbf{S} dA + \int_B \left[\nabla \theta \cdot \delta \mathbf{S} + \frac{T_0}{\lambda_q} \dot{\mathbf{S}} \cdot \delta \mathbf{S} \right] dV$$

$$+ \frac{bh^3 E}{12} \int_{-l_-}^{l_+} \frac{\partial^4 w}{\partial x^4} \delta w dx + bE\alpha_T \int_{-l_-}^{l_+} \frac{\partial^2}{\partial x^2} \int_{-h/2}^{h/2} z\theta dz \delta w dx + hb \int_{-l_-}^{l_+} \rho \frac{\partial^2 w}{\partial t^2} \delta w dx. \tag{4.15}$$

Substituting this expression in equation (4.9) gives the following two equations

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho(x) hb}{EJ} \frac{\partial^2 w}{\partial t^2} + \frac{12\alpha_T}{h^3} \int_{-h/2}^{h/2} \theta z dz = \frac{bp}{EJ}, \quad -l_- < x < l_+, \ t > 0,$$

$$\nabla\theta + \frac{T_0}{\lambda_q}\dot{\mathbf{S}} = \mathbf{Q}. \tag{4.16}$$

The first equation coincides with (2.6). To write the second equation in (4.16) in the form (2.9), we apply the divergence operator and use the relation

$$\nabla\dot{\mathbf{S}} = -\frac{c_\varepsilon\beta_1}{T_0}\frac{\partial\theta}{\partial t} + E\alpha_T z\frac{\partial^3 w}{\partial x^2\partial t}. \tag{4.17}$$

that follows from (4.5). Since $\nabla\cdot\mathbf{Q} = -\lambda_q^{-1}\Omega$, after a simple rearrangement, we derive

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial z^2} - \frac{c_\varepsilon\beta_1}{\lambda_q}\frac{\partial\theta}{\partial t} + \frac{ET_0\alpha_T}{\lambda_q}\frac{z\partial^3 w}{\partial x^2\partial t} = -\frac{\Omega(x,z,t)}{\lambda_q}. \tag{4.18}$$

Because of the relation $c_\varepsilon\beta_1/\lambda_q = a^{-1}$ this equation is identical to (2.9).

Note that if we employ the strain-stress relations of the beam theory (2.3), then the generalized free energy

$$U = \frac{1}{2}\int_B \left[E\left(z\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{c_\varepsilon\beta_1}{T_0}\theta^2\right]dV \tag{4.19}$$

derived in this section coincides with the one deduced from the expression [11]

$$U = \int_{\mathcal{B}}\left(W + \frac{c_\varepsilon}{2T_0}\theta^2\right)dV \tag{4.20}$$

proposed for any three-dimensional body $\mathcal{B}$. Here, W is the isothermal strain energy

$$2W = \lambda\varepsilon_{kk}^2 + 2\mu(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 2\varepsilon_{23}^2 + 2\varepsilon_{31}^2 + 2\varepsilon_{12}^2). \tag{4.21}$$

In what follows we show the equivalence of formulas (4.19) and (4.20). For the thin thermoelastic structure B of consideration the strain components have the form

$$\varepsilon_{11} = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{22} = \varepsilon_{33} = \nu z\frac{\partial^2 w}{\partial x^2} + (1+\nu)\alpha_T\theta, \quad \varepsilon_{ij} = 0, \quad i \neq j. \tag{4.22}$$

Upon computing the integrals

$$\begin{aligned}
&\frac{\lambda}{2}\int_{-h/2}^{h/2}\varepsilon_{kk}^2 dz = \frac{\lambda(1-2\nu)^2 h^3}{24}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \\
&+2(1+\nu)^2\lambda\alpha_T^2\int_{-h/2}^{h/2}\theta^2 dz - 2\lambda(1-2\nu)(1+\nu)\alpha_T\frac{\partial^2 w}{\partial x^2}\int_{-h/2}^{h/2} z\theta dz, \\
&\mu\int_{-h/2}^{h/2}(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2)dz = \frac{\mu(1+2\nu^2)h^3}{12}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \\
&+2(1+\nu)^2\mu\alpha_T^2\int_{-h/2}^{h/2}\theta^2 dz + 4\mu\nu(1+\nu)\alpha_T\frac{\partial^2 w}{\partial x^2}\int_{-h/2}^{h/2} z\theta dz,
\end{aligned} \tag{4.23}$$

and adding these expressions we obtain the integral of the energy W

$$\int_B W dV = bE \int_{-l_-}^{l_+} \left[\frac{h^3}{24} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{(1+\nu)\alpha_T^2}{1-2\nu} \int_{-h/2}^{h/2} \theta^2 dz \right] dx. \tag{4.24}$$

This gives us the isothermal strain energy for the thin structure B

$$W = \frac{E}{2} \left[\left(\frac{z\partial^2 w}{\partial x^2} \right)^2 + \frac{2(1+\nu)\alpha_T^2}{1-2\nu} \theta^2 \right]. \tag{4.25}$$

It is directly verified that formula (4.20) with W being defined by (4.25) coincides with the expression (4.19).

5 Concluding remarks

We have analyzed a dynamic model of coupled thermoelasticity for a structure that acts as a composite beam in response to elastic deformation and as a two-dimensional body in response to heat flow. The beam is clamped at the center of mass and free at its ends. The two governing thermoelastic integro-differential equations for the deflection of the beam parts are coupled by the thermoconductivity equation. We have proposed a method for the boundary value problem for these three integro-differential equations that is based on two integral sine transforms applied to the heat equation and one integral sine transform applied to the elastic equations. The consequent expansion of the deflections in terms of the shape and vibration modes leads the problem to an infinite system of linear algebraic equations with respect to the Laplace transforms of the vibration modes. For its solution we proposed a method that comprises successive approximations, integration by parts and summation of weakly and nonuniformly convergent series. Our numerical tests revealed that unless the parameter β is small the coupling of the heat bending equations effects are essential on the deflection and the temperature distribution.

We have also derived the generalized free energy and the dissipation function expressions. Employing the variational formulation we have shown that this formulation deduces the governing differential equations of thermoelastic vibration of the thin structure analyzed in the first part of the paper. The expressions of the generalized free energy, the kinetic energy, and the dissipation function, when expanded as quadratic forms, in conjunction with the variational principle can be employed to obtain the corresponding Lagrangian equations and ultimately the dynamical system associated with flight of a thermoelastic aircraft modeled as the two-dimensional structure studied in the paper.

Conflict of interest. We have no competing interests.

Funding. Research of the first author was sponsored by U.S. Air Force Research Lab Summer Faculty Fellowship Program and by the Army Research Office and was accomplished under Grant Number **W911NF-17-1-0157**. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Office or the U.S. Government. The U.S. Government is authorized to reproduce and distribute

reprints for Government purposes notwithstanding any copyright notation herein. A part of this research was conducted at University of Florida REEF.

References

[1] Anderson, J.D. *Hypersonic and high-temperature gas dynamics*. Reston: AIAA, 2006.

[2] Waszak, M. R. and Schmidt, D. K. Fight dynamics of aeroelastic vehicles. *J. Aircraft* 1988; 25: 563-571.

[3] Schmidt, D. K. *Modern Flight Dynamics*. New York: McGraw-Hill, 2011.

[4] Schmidt, D. K. Discussion: "The lure of the mean axes" (Meirovitch, L,, and Tuzcu, L., ASME J. Appl. Mech,. 74 (3), pp. 497-504). *ASME J. Appl. Mech.* 2015; 82, 125501: 1-5.

[5] Meirovitch, L. and Tuzcu, I. Time simulations of the response of maneuvering, flexible aircraft, *J. Guidance Control Dynamics* 2004; 27: 814-828.

[6] Meirovitch, L. and Tuzcu, I. Control of flexible aircraft executing time-dependent maneuvers, *J. Guidance Control Dynamics* 2005; 28: 1291-1300.

[7] Meirovitch, L. and Tuzcu, I. The lure of the mean axes. *ASME J. Appl. Mech.* 2007; 74: 497-504.

[8] Bolender, M. A. and Doman, D. B. Nonlinear longitudinal dynamical model of an air-breathing hypersonic vehicle. *J. Spacecraft Rockets* 2007; 44: 374-387.

[9] Culler, A. J., Williams, T., and Bolender, M.A. Aerothermal modeling and dynamic analysis of a hypersonic vehicle, AIAA Atmospheric Flight Mechanics Conf. Exhibit, Hilton Head, South Carolina, 2007: 1-21.

[10] Kovalenko, A. D. *Thermoelasticity.* Kiev: Vischa Shkola, 1975,

[11] Biot, M.A. Thermoelasticity and irreversible thermodynamics. *J. Appl. Phys.* 1956; 27: 240-253.

[12] Nowacki, W. Dynamic problems of thermoelasticity, Leyden: Noordhoff International Publ., 1976.

[13] Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products.* Orlando: Academic Press, 1980.

[14] Antipov, Y.A. and Willis, J.R. Transient loading of a rapidly-advancing Mode-II crack in a viscoelastic medium. *Mechanics of Materials* 2003; 35: 415-431.