Computational Methods and Function Theory
Volume 00 (0000), No. 0, 1–?

CMFT-MS XXYYYZZ

A genus-3 Riemann-Hilbert problem and diffraction of a wave by two orthogonal resistive half-planes

Yuri A. Antipov

Abstract. Diffraction of a plane electromagnetic wave (E-polarization) by two orthogonal electrically resistive half-planes is analyzed. The physical problem reduces to a Riemann-Hilbert problem in the real axis for four pairs of analytic functions $\Phi_j^+(\eta)$ ($\eta \in \mathbb{C}^+$) and $\Phi_j^-(\eta) = \Phi_j^+(-\eta)$ ($\eta \in \mathbb{C}^-$), $j = 1, 2, 3, 4$, and $\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and lower half-planes. It is shown that the problem is equivalent to two scalar Riemann-Hilbert problems on a plane and a Riemann-Hilbert problem on a genus-3 hyperelliptic surface subject to a certain symmetry condition. A closed-form solution is derived in terms of singular integrals and the genus-3 Riemann Theta function.

Keywords. Riemann-Hilbert problem, Riemann surfaces, matrix factorization, electromagnetic diffraction.

2000 MSC. Primary 30Exx; Secondary 35P05, 45E05, 78A45.

1. Introduction

Many model problems of elasticity, fluid mechanics and diffraction theory require the solution of a scalar Riemann-Hilbert problem on the Riemann surface of an algebraic function. The Riemann-Hilbert problem on a hyperelliptic surface associated with a static mixed boundary-value problem for an elastic isotropic plane with a system of cuts along a line is analyzed and solved by quadratures in [21]. This problem generalizes the formulation [11] by removing any restriction on the location of the points where the boundary conditions are changed. This approach is applied in [18], [19], [17] for some other contact problems of fracture mechanics. Method of conformal mappings and the Riemann-Hilbert problem on a Riemann surface is developed in [8], [9] for nonlinear free-boundary problems of supercavitating flow in multiply connected domains. An approach for the Riemann-Hilbert problem on a hyperelliptic surface based on the use of the Baker-Akhieser function is employed in the theory of finite gap integration [10].

Version January 3, 2011.
The work was partly funded by National Science Foundation through grant DMS0707724.

ISSN 1617-9447/\$ 2.50

Another application of the theory of the Riemann-Hilbert problem on a Riemann surface arises in the Wiener-Hopf factorization of matrices of the following structure:

$$G(t) = f_0(t)I + f_1(t)A + \ldots + f_{n-1}(t)A^{n-1}, \tag{1}$$

$f_j(t)$ $(j = 0, 1, \ldots, n-1)$ are Hölder functions, I is the $n \times n$ unit matrix, and A is an $n \times n$ polynomial matrix with the characteristic polynomial $\varphi(z, w) = \det(wI - A(z))$ irreducible over the field of rational functions. The idea [15] of reducing the vector Riemann-Hilbert problem with the matrix coefficient (1) to a scalar Riemann-Hilbert problem on a Riemann surface of the algebraic function $\varphi(z, w) = 0$ is developed and implemented for the hyperelliptic case in [16], [2] to solve the governing systems of singular integral equations arising in contact and fracture mechanics.

Diffraction problems in acoustics and electromagnetics for a half-plane and a wedge can be treated by means of the Sommerfeld integral and require the solution of a system of difference equations with periodic coefficients and, ultimately, a Riemann-Hilbert problem on two segments in a Riemann surface [7], [4], [5], [6]. Another approach for solving difference equations with periodic coefficients in diffraction theory using the Riemann bilinear relations is proposed in [14]. It has recently been shown [1] that the use of the method [12] and the Laplace integral substantially simplifies the solution procedure. Instead of a system of Maliuzhinets-type difference equations this technique leads to a system of symmetric Wiener-Hopf equations. This method was applied [1] to the diffraction problem for two orthogonal half-planes, an electrically resistive screen and a perfectly magnetically conductive one, when the structure was illuminated by a line electric current of uniform excitation. The problem was reduced to a inhomogeneous Riemann-Hilbert problem on an elliptic surface subject to a certain symmetry condition. The condition was verified for the homogeneous case. In this paper we generalize the result [1] by considering the case when both screens are electrically resistive half-planes with the surface resistivity being neither zero, nor infinity. The structure is illuminated by a plane wave $E_z^i = e^{-ikx\cos\varphi_0 - iky\sin\varphi_0}$ (φ_0 is the angle of incidence). We show that the physical problem is equivalent to a certain homogeneous Riemann-Hilbert problem for four pairs of functions subject to a symmetry condition and solve it exactly by reducing the problem to a scalar Riemann-Hilbert problem on a genus-3 hyperelliptic surface.

2. Formulation

Let W be a planar structure consisting of two orthogonal electrically resistive half-planes, $W_1 = \{(x, y) : 0 < x < +\infty, y = \pm 0\}$ and $W_2 = \{(x, y) : x = \pm 0, 0 < y < +\infty\}$, of resistivity R_1 and R_2, respectively. On these half-planes, the electric and magnetic fields, $\mathbf{E} = (E_x, E_y, E_z)^\top$ and $\mathbf{H} = (H_x, H_y, H_z)^\top$, satisfy the transition conditions

$$\mathbf{n}\times\mathbf{E}=R_j\mathbf{n}\times[\mathbf{n}\times\mathbf{H}]^+_-,\quad [\mathbf{n}\times\mathbf{E}]^+_-=0,\quad (x,y)\in W_j,\quad j=1,2, \tag{2}$$

where $\mathbf{n}$ is the unit normal external to the positive side of the half-planes, and $[f]^+_-$ denotes the jump of a function f. Let μ be the magnetic permeability, ε the electric permittivity, ω the angular frequency, $k=\omega\sqrt{\varepsilon\mu}$ the wave number, and $Z=\sqrt{\mu/\varepsilon}$ the intrinsic impedance of the medium. It is assumed that μ and ν are complex numbers and the branches $\sqrt{\varepsilon\mu}$ and $\sqrt{\mu/\varepsilon}$ are chosen such that $k=k_1+ik_2$, $k_2>0$, $kZ=\omega\mu$, and $k/Z=\omega\varepsilon$. The structure is illuminated by a plane wave (the factor $e^{-i\omega t}$ is suppressed)

$$E_z^i=e^{-ikx\cos\varphi_0-iky\sin\varphi_0}, \tag{3}$$

where the angle φ_0 defines the direction of incidence.

The Maxwell equations imply

$$E_x=-\frac{Z}{ik}\frac{\partial H_z}{\partial y},\quad E_y=\frac{Z}{ik}\frac{\partial H_z}{\partial x},\quad E_z=-\frac{Z}{ik}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right), \tag{4}$$

$$H_x=\frac{1}{ikZ}\frac{\partial E_z}{\partial y},\quad H_y=-\frac{1}{ikZ}\frac{\partial E_z}{\partial x},\quad H_z=\frac{1}{ikZ}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right),$$

and reduce to the Helmholtz equation for the component E_z

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)E_z=0,\quad (x,y)\in\mathbb{R}^2\setminus W. \tag{5}$$

For E-polarization, the transition conditions (2) read

$$E_z|_{y=0^\pm}=-R_1[H_x|_{y=0^+}-H_x|_{y=0^-}],\quad 0<x<+\infty, \tag{6}$$

$$E_z|_{x=0^\pm}=R_2[H_y|_{x=0^+}-H_y|_{x=0^-}],\quad 0<y<+\infty.$$

To solve the problem, we convert it to a vector Riemann-Hilbert problem. First, we introduce the following Laplace transforms:

$$\begin{pmatrix}\tilde{\mathbf{E}}_+\\ \tilde{\mathbf{H}}_+\end{pmatrix}(x,\eta)=\int_0^\infty e^{i\eta y}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(x,y)dy, \tag{7}$$

$$\begin{pmatrix}\tilde{\mathbf{E}}_-\\ \tilde{\mathbf{H}}_-\end{pmatrix}(x,\eta)=\int_{-\infty}^0 e^{i\eta y}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(x,y)dy,$$

$$\begin{pmatrix}\hat{\mathbf{E}}_+\\ \hat{\mathbf{H}}_+\end{pmatrix}(\zeta,y)=\int_0^\infty e^{i\zeta x}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(x,y)dx,$$

and obtain from equations (4) and (5) [1]

$$kZ\tilde{H}_{y+}(0^+,\eta)+i\xi\tilde{E}_{z+}(0^+,\eta)=kZ\hat{H}_{x+}(i\xi,0^+)-\eta\hat{E}_{z+}(i\xi,0^+), \tag{8}$$

$$kZ\tilde{H}_{y+}(0^+,-\eta)+i\xi\tilde{E}_{z+}(0^+,-\eta)=kZ\hat{H}_{x+}(i\xi,0^+)+\eta\hat{E}_{z+}(i\xi,0^+).$$

$$kZ\tilde{H}_{y-}(0^+,\eta)+i\xi\tilde{E}_{z-}(0^+,\eta)=-kZ\hat{H}_{x+}(i\xi,0^-)+\eta\hat{E}_{z+}(i\xi,0^-),$$
$$kZ\tilde{H}_{y-}(0^+,-\eta)+i\xi\tilde{E}_{z-}(0^+,-\eta)=-kZ\hat{H}_{x+}(i\xi,0^-)-\eta\hat{E}_{z+}(i\xi,0^-),$$
$$-ikZ[\tilde{H}_{y+}(0^-,\eta)+\tilde{H}_{y-}(0^-,\eta)]=\xi\tilde{E}_{z+}(0^-,\eta)+\xi\tilde{E}_{z-}(0^-,\eta),$$
$$-ikZ[\tilde{H}_{y+}(0^-,-\eta)+\tilde{H}_{y-}(0^-,-\eta)]=\xi\tilde{E}_{z+}(0^-,-\eta)+\xi\tilde{E}_{z-}(0^-,-\eta).$$

Here, the function $\xi(\eta)=(\eta^2-k^2)^{1/2}$ is the branch of the two-valued function $\xi^2=\eta^2-k^2$ satisfying the condition $\xi(0)=-ik$. It is defined in the η-plane cut along a line joining the branch points $\eta=k$ and $\eta=-k$ and passing through the infinite point. Applying the Laplace transformations (7) to the the boundary conditions (6) yields

$$\hat{E}_{z+}(i\xi,0^\pm)=-R_1[\hat{H}_{x+}(i\xi,0^+)-\hat{H}_{x+}(i\xi,0^-)],\tag{9}$$
$$\tilde{E}_{z+}(0^\pm,\eta)=R_2[\tilde{H}_{y+}(0^+,\eta)-\tilde{H}_{y+}(0^-,\eta)].$$

We next use these conditions and equations (8) to eliminate the functions $\hat{E}_{z+}(i\xi,0^\pm)$ and $\hat{H}_{x+}(i\xi,0^\pm)$. This leads to the following result.

Theorem 1. *Let $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$ be vector-functions whose components are*

$$\mathbf{\Phi}^+(\eta)=\begin{pmatrix}-ikZ\tilde{H}_{y+}(0^+,\eta)\\ -ikZ\tilde{H}_{y+}(0^-,\eta)\\ -ikZ\tilde{H}_{y-}(0^\pm,-\eta)\\ \tilde{E}_{z-}(0^\pm,-\eta)\end{pmatrix}.\tag{10}$$

and

$$\mathbf{\Phi}^-(\eta)=\mathbf{\Phi}^+(-\eta),\quad \eta\in\mathbb{C}^-.\tag{11}$$

Then the problem (4), (5), (6) is equivalent to the following vector Riemann-Hilbert problem in the real axis L:

Find two vectors, $\mathbf{\Phi}^+(\eta)$ and $\mathbf{\Phi}^-(\eta)$, analytic everywhere in the upper and lower half-planes, $\mathbb{C}^+$ and $\mathbb{C}^-$, apart from the simple poles (the geometric optics poles) η_0 and $-\eta_0$, respectively, where $\eta_0=k\sin\varphi_0$ if $-\frac{3\pi}{2}<\varphi_0<-\pi$ or $0<\varphi_0<\frac{\pi}{2}$, and $\eta_0=-k\sin\varphi_0$ if $-\pi<\varphi_0<0$. Furthermore, if $0<\varphi_0<\frac{\pi}{2}$, then

$$\operatorname*{res}_{\eta=k\sin\varphi_0}\Phi_1^+(\eta)=-k\cos\varphi_0.\tag{12}$$

If $-\frac{3\pi}{2}<\varphi_0<-\pi$, then

$$\operatorname*{res}_{\eta=k\sin\varphi_0}\Phi_2^+(\eta)=-k\cos\varphi_0.\tag{13}$$

If $-\pi<\varphi_0<0$, then

$$\operatorname*{res}_{\eta=k\sin\varphi_0}\Phi_3^-(\eta)=-k\cos\varphi_0.\tag{14}$$

As $\eta\to\infty$, the functions $\Phi_j^\pm(\eta)$ $(j=1,2,4)$ vanish, $\Phi_j^\pm(\eta)\sim N_j\eta^\alpha$ $(-1\le\alpha<0)$, while the functions $\Phi_3^\pm(\eta)$ may grow, $\Phi_3^\pm(\eta)\sim N_3\eta^{\alpha+1}$ (N_j $(j=1,2,3,4)$ are *constants). The vectors* $\mathbf{\Phi}^{\pm}(\eta)$ *are continuous in the half-planes* $\mathbb{C}^{\pm}$ *up to the boundary* L*, and their boundary values satisfy the linear relation*

$$\mathbf{\Phi}^{+}(\eta)=G(\eta)\mathbf{\Phi}^{-}(\eta),\quad \eta\in L. \tag{15}$$

Here,

$$G(\eta)=\frac{1}{\eta-\gamma}\begin{pmatrix} -\frac{(\gamma+i\nu\xi)^2}{\gamma+2i\nu\xi} & \frac{\nu\xi(i\gamma-\nu\xi)}{\gamma+2i\nu\xi} & -\frac{\eta\gamma+i\nu\xi(2\eta-\gamma)}{\gamma+2i\nu\xi} & -\frac{\gamma\xi(\eta+i\nu\xi)}{\gamma+2i\nu\xi} \\ -\frac{\nu\xi(i\gamma-\nu\xi)}{\gamma+2i\nu\xi} & -\frac{\nu^2\xi^2}{\gamma+2i\nu\xi} & \frac{\gamma(\gamma-\eta)-i\nu\xi(2\eta-\gamma)}{\gamma+2i\nu\xi} & \frac{\gamma\xi(\eta-\gamma-i\nu\xi)}{\gamma+2i\nu\xi} \\ -\frac{\eta+i\nu\xi}{2} & -\frac{\eta-\gamma-i\nu\xi}{2} & -\frac{\gamma}{2} & -\frac{\gamma\xi}{2} \\ \frac{i\gamma\nu\xi-\eta(\gamma+2i\nu\xi)}{2\gamma\xi} & -\frac{\gamma(\gamma-\eta)-i\nu\xi(2\eta-\gamma)}{2\gamma\xi} & -\frac{\gamma}{2\xi} & -\frac{\gamma}{2} \end{pmatrix}, \tag{16}$$

and

$$\gamma=\frac{kZ}{2R_1},\quad \nu=\frac{R_2}{2R_1}. \tag{17}$$

Note that the matrix is not singular everywhere in the real axis (it is assumed that $\operatorname{Im}\gamma>0$), and

$$\det G(\eta)=\frac{\eta+\gamma}{\eta-\gamma}. \tag{18}$$

The symmetry and boundary conditions, (11) and (15), imply that the matrix $G(\eta)$ has to satisfy the condition

$$G(\eta)G(-\eta)=I_4,\quad \eta\in L, \tag{19}$$

where $I_4=\operatorname{diag}\{1,1,1,1\}$. It has been verified that this condition is identically satisfied.

The conditions (12) for $0<\varphi_0<\frac{\pi}{2}$ follow from the formulas

$$E_z=E_z^i+E_z^\circ,\quad H_y=-\frac{1}{ikZ}\frac{\partial E_z}{\partial x}, \tag{20}$$

(7), (10) and (11). Here, E_z^i is the incident wave (3), and E_z° is the sum of the reflected, surface, transmitted and diffracted waves. On assuming that $\operatorname{Im}\eta>k_2\sin\varphi_0$ we compute the Laplace transform

$$\Phi_1^+(\eta)=\hat{\Phi}_1^+(\eta)-\frac{k\cos\varphi_0}{\eta-k\sin\varphi_0}, \tag{21}$$

where $\hat{\Phi}_1^+(\eta)$ is analytic in the upper half-plane $\mathbb{C}^+$. By analytic continuation we obtain that the function $\Phi_1^+(\eta)$ is analytic everywhere in $\mathbb{C}^+$ except at the simple pole $\eta=k\sin\varphi_0$, and its residue is equal to $-k\cos\varphi_0$.

Finally, we note that the requirements $\Phi_j^{\pm}(\eta)\sim N_j\eta^\alpha$ as $\eta\to\infty$ ($j=1,2$, and the parameter $\alpha\in[-1,0)$ is to be determined), guarantees that the functions $H_y(0^\pm,y)$ are integrable as $y\to 0^+$. From the relations (8) we conclude that as $\eta\to\infty$, $\Phi_3^{\pm}(\eta)\sim N_3\eta^{\alpha+1}$ (the Laplace transform $\tilde{H}_{y-}(0^\pm,\eta)$ is understood in the generalized sense), and $\Phi_4^{\pm}(\eta)\sim N_4\eta^\alpha$.

3. Reduction of the Riemann-Hilbert problem dimension

We aim to reduce the Riemann-Hilbert problem for four pairs of functions to problems whose coefficients are either scalars or matrices of lower dimension. By following the scheme [1] we split the matrix $G(\eta)$ as follows:

$$G(\eta) = \frac{R(\eta)}{1 + 2i\nu\xi/\gamma}\left[I_4 + \frac{B(\eta)}{\gamma\xi(\gamma^2 - 16\nu^2\xi^2)}\right], \tag{22}$$

where
(23)

$$R(\eta) = \frac{1}{\gamma(\eta-\gamma)}\begin{pmatrix} -\gamma^2+\nu^2\xi^2 & -\nu^2\xi^2 & -\eta\gamma & -i\gamma\nu\xi^2 \\ \nu^2\xi^2 & -\nu^2\xi^2 & \gamma(\gamma-\eta) & -i\gamma\nu\xi^2 \\ \frac{1}{2}(-\eta\gamma+2\nu^2\xi^2) & \frac{1}{2}(-\eta\gamma+\gamma^2-2\nu^2\xi^2) & -\frac{\gamma^2}{2} & -i\gamma\nu\xi^2 \\ -\frac{i\nu}{2}(4\eta-\gamma) & \frac{i}{2}(4\eta\nu-3\gamma\nu) & -i\gamma\nu & -\frac{\gamma^2}{2} \end{pmatrix},$$

and $B(\eta)$ is a 4×4 polynomial matrix. Its elements are

$$\begin{gathered} b_{11} = 2i\nu\xi^2(\gamma^2 - 10\nu^2\xi^2), \quad b_{12} = -2i\nu\xi^2(\gamma^2 - 2\nu^2\xi^2), \\ b_{13} = -2i\nu\xi^2(\gamma^2 - 8\nu^2\xi^2), \quad b_{14} = 2\xi^2[6\gamma\nu^2\xi^2 + \eta(\gamma^2 - 8\nu^2\xi^2)], \\ b_{21} = -2i\nu\xi^2(\gamma^2 + 2\nu^2\xi^2), \quad b_{22} = 6i\nu\xi^2(\gamma^2 - 2\nu^2\xi^2), \\ b_{23} = 2i\nu\xi^2(\gamma^2 + 8\nu^2\xi^2), \quad b_{24} = -2\xi^2[\gamma(\gamma^2 + 2\nu^2\xi^2) + \eta(\gamma^2 + 8\nu^2\xi^2)], \\ b_{31} = -i\nu\xi^2(\gamma^2 - 4\nu^2\xi^2), \quad b_{32} = i\nu\xi^2(\gamma^2 + 12\nu^2\xi^2), \\ b_{33} = 2i\nu\xi^2(\gamma^2 - 8\nu^2\xi^2), \quad b_{34} = -\xi^2[\gamma^3 + 4\nu^2\xi^2(4\eta - \gamma)], \\ b_{41} = \eta(\gamma^2 - 4\nu^2\xi^2) - 8\nu^2\xi^2\left(\gamma + \frac{2}{\gamma}\nu^2\xi^2\right), \\ b_{42} = \gamma^2(\gamma - \eta) + 4\nu^2\xi^2\left(\gamma - 3\eta + \frac{4}{\gamma}\nu^2\xi^2\right), \\ b_{43} = \gamma^3 - 4\nu^2\xi^2(4\eta + \gamma), \quad b_{44} = -2i\nu\xi^2(8\eta^2 - 5\gamma^2 + 8\nu^2\xi^2). \end{gathered} \tag{24}$$

The possibility of the reduction of the dimension of the vector Riemann-Hilbert problem is determined by the algebraic structure of the characteristic polynomial of the matrix B. This polynomial, irreducible over the field of rational functions of η, can be factorized as

$$\det[B(\eta) - \mu I_4] = \phi_1(\mu)\phi_2(\mu), \tag{25}$$

where

$$\begin{gathered} \phi_1(\mu) = [\mu - 2i\nu\xi^2(\gamma^2 - 16\nu^2\xi^2)]^2, \\ \phi_2(\mu) = \mu^2 + 16i(\eta^2 - \gamma^2)\nu\xi^2\mu + \xi^2(\gamma^2 - 16\nu^2\xi^2)[3\gamma^4 - 4\eta^2(\gamma^2 - 4\nu^2\xi^2)]. \end{gathered} \tag{26}$$

Because of the structure of the characteristic polynomial there exists a rational matrix of transformation, $T(\eta)$, such that

$$\frac{1}{1 + 2i\nu\xi/\gamma}\left[I_4 + \frac{B(\eta)}{\gamma\xi(\gamma^2 - 16\nu^2\xi^2)}\right] = T(\eta)\Gamma(\eta)T^{-1}(\eta), \tag{27}$$

where $\Gamma(\eta)$ has a block-diagonal structure. The first two columns of the matrix T are eigenvectors of the matrix $B(\eta)$ corresponding to the multiplicity-2 eigenvalue $\mu = 2i\nu\xi^2(\gamma^2 - 16\nu^2\xi^2)$, while the other two columns comprise a basis of the kernel of the operator $\phi_2(B)$. The matrix $T(\eta)$ is not unique. By comparing the complexity of the matrix Γ and the symmetry condition (11) for a new Riemann-Hilbert problem for order-2 vectors to be derived, we have found that it is preferable to work with

$$T(\eta) = \begin{pmatrix} 1 & -\frac{i}{\nu}(\eta-\gamma) & 1 & 0 \\ 1 & -\frac{i\eta}{\nu} & 0 & 1 \\ -1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{i(\eta\gamma+\gamma^2+2\nu^2\xi^2)}{2\gamma\nu\xi^2} & \frac{i(\eta\gamma-2\nu^2\xi^2)}{2\gamma\nu\xi^2} \end{pmatrix}. \tag{28}$$

This choice uniquely identifies the matrix $\Gamma(\eta)$

$$\Gamma(\eta) = \frac{1}{2\nu}\begin{pmatrix} 2\nu & 0 & 0 & 0 \\ 0 & 2\nu & 0 & 0 \\ 0 & 0 & b+cl_0 & cl_1 \\ 0 & 0 & cl_2 & b-cl_0 \end{pmatrix}, \tag{29}$$

where $b(\eta)$ and $c(\eta)$ are Hölder functions in the real axis, and $l_0(\eta)$, $l_1(\eta)$ and $l_2(\eta)$ are polynomials defined by

$$b(\eta) = 2\nu\frac{\gamma^3 - 8i\nu(\eta^2-\gamma^2)\xi - 16\gamma\nu^2\xi^2}{(\gamma+2i\nu\xi)(\gamma^2-16\nu^2\xi^2)}, \tag{30}$$

$$c(\eta) = -\frac{2i}{\gamma\xi(\gamma+2i\nu\xi)(\gamma^2-16\nu^2\xi^2)},$$

$$l_0 = \gamma^3(\nu^2\xi^2-\eta^2) - \eta(\gamma^4 - 16\nu^4\xi^4),$$

$$l_1(\eta) = \nu^2\xi^2(3\gamma^3 - 4\eta\gamma^2 - 16\nu^2\eta\xi^2) - \eta^2(\gamma^3 - 8\gamma\nu^2\xi^2),$$

$$l_2(\eta) = \gamma^3(\gamma^2+5\nu^2\xi^2) + \eta^2(\gamma^3+8\gamma\nu^2\xi^2) + 2\eta(\gamma^4+6\gamma^2\nu^2\xi^2+8\nu^4\xi^4).$$

To complete the reduction of the order-4 problem (15) to scalar and order-2 Riemann-Hilbert problems, introduce two new vector-functions

$$\mathbf{\Omega}^+(\eta) = U_0(\eta)\mathbf{\Phi}^+(\eta), \quad \eta\in\mathbb{C}^+, \tag{31}$$

$$\mathbf{\Omega}^-(\eta) = U_1(\eta)\mathbf{\Phi}^-(\eta), \quad \eta\in\mathbb{C}^-,$$

where

$$U_0(\eta) = \gamma d(\eta)T^{-1}(\eta)R^{-1}(\eta), \quad U_1(\eta) = \gamma d(\eta)T^{-1}(\eta), \tag{32}$$

$$d(\eta) = 3\gamma^2 - 4\eta^2 + 16\nu^2\xi^2.$$

Since the original vector-functions $\mathbf{\Phi}^\pm(\eta)$ have to satisfy the symmetry condition (11), we observe that

$$\mathbf{\Omega}^+(\eta) = S(\eta)\mathbf{\Omega}^-(-\eta), \quad \eta\in\mathbb{C}^+, \tag{33}$$

where

$$S(\eta) = T^{-1}(\eta)R^{-1}(\eta)T(-\eta) = \begin{pmatrix} 1 & \frac{i\eta}{\nu} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s_{33}(\eta) & s_{34}(\eta) \\ 0 & 0 & s_{43}(\eta) & s_{44}(\eta) \end{pmatrix}, \tag{34}$$

$$s_{33}(\eta) = -s_{44}(\eta) = -\frac{2(\eta-\gamma)(\gamma^2 - 4\nu^2\xi^2)}{\gamma(\gamma^2 - 16\nu^2\xi^2)},$$

$$s_{34}(\eta) = -\frac{8(\eta-\gamma)\nu^2\xi^2}{\gamma(\gamma^2 - 16\nu^2\xi^2)}, \quad s_{43}(\eta) = \frac{4(\eta-\gamma)(\gamma^2 + 2\nu^2\xi^2)}{\gamma(\gamma^2 - 16\nu^2\xi^2)}.$$

To study the other properties of the vectors $\mathbf{\Omega}^{\pm}(\eta)$, we explicitly express the elements of the vectors $\mathbf{\Phi}^{\pm}(\eta)$ through the functions $\Omega_j^{\pm}(\eta)$. We have

$$\Phi_1^+(\eta) = \frac{1}{\gamma d(\eta)}\left[\Omega_1^+(\eta) + \frac{i\gamma}{\nu}\Omega_2^+(\eta) + \frac{(4\nu^2\xi^2 - \gamma^2)\Omega_3^+(\eta) - 4\nu^2\xi^2\Omega_4^+(\eta)}{2\gamma(\eta-\gamma)}\right], \tag{35}$$

$$\Phi_2^+(\eta) = \frac{1}{\gamma d(\eta)}\left[\Omega_1^+(\eta) + \frac{2(2\nu^2\xi^2 + \gamma^2)\Omega_3^+(\eta) - (4\nu^2\xi^2 - \gamma^2)\Omega_4^+(\eta)}{2\gamma(\eta-\gamma)}\right],$$

$$\Phi_3^+(\eta) = \frac{1}{\gamma d(\eta)}\left[-\Omega_1^+(\eta) + \frac{i\eta}{\nu}\Omega_2^+(\eta) + \frac{(8\nu^2\xi^2 + \gamma^2)\Omega_3^+(\eta) - (8\nu^2\xi^2 - \gamma^2)\Omega_4^+(\eta)}{4\gamma(\eta-\gamma)}\right],$$

$$\Phi_4^+(\eta) = \frac{1}{\gamma d(\eta)}\left\{\Omega_2^+(\eta) - \frac{i}{4\gamma\nu\xi^2(\eta-\gamma)}\right.$$

$$\left.\times\left[(\gamma(\gamma^2 + 2\nu^2\xi^2) + \eta(8\nu^2\xi^2 + \gamma^2))\Omega_3^+(\eta) + (6\gamma\nu^2\xi^2 - \eta(8\nu^2\xi^2 - \gamma^2))\Omega_4^+(\eta)\right]\right\},$$

and

$$\Phi_1^-(\eta) = \frac{1}{\gamma d(\eta)}\left[\Omega_1^-(\eta) - \frac{i(\eta-\gamma)}{\nu}\Omega_2^-(\eta) + \Omega_3^-(\eta)\right], \tag{36}$$

$$\Phi_2^-(\eta) = \frac{1}{\gamma d(\eta)}\left[\Omega_1^-(\eta) - \frac{i\eta}{\nu}\Omega_2^-(\eta) + \Omega_4^-(\eta)\right],$$

$$\Phi_3^-(\eta) = \frac{1}{\gamma d(\eta)}\left[-\Omega_1^-(\eta) + \frac{1}{2}\Omega_3^-(\eta) + \frac{1}{2}\Omega_4^-(\eta)\right],$$

$$\Phi_4^-(\eta) = \frac{1}{\gamma d(\eta)}\left\{\Omega_2^-(\eta) + \frac{i}{2\gamma\nu\xi^2}[(\eta\gamma + \gamma^2 + 2\nu^2\xi^2)\Omega_3^-(\eta) - (2\nu^2\xi^2 - \eta\gamma)\Omega_4^-(\eta)]\right\}.$$

We are now ready to establish a one-to-one correspondence between the Riemann-Hilbert problem (15) and the problems for the components of the vectors $\mathbf{\Omega}^+(\eta)$ and $\mathbf{\Omega}^-(\eta)$.

Theorem 2. *Let $\mathbf{\Omega}^{\pm}(\eta)$ be the general form of dimension-4 vector-functions which*

(i) are analytic everywhere in the half-planes $\mathbb{C}^{\pm}$ apart from simple poles at the points $\pm\eta_0$,

(ii) satisfy the symmetry condition (33),

(iii) satisfy the boundary condition

$$\mathbf{\Omega}^{+}(\eta)=\Gamma(\eta)\mathbf{\Omega}^{-}(\eta),\quad \eta\in L, \tag{37}$$

(iv) grow at infinity as

$$\Omega_j^{\pm}(\eta)=K_j^{\pm}\eta^{3+\alpha}+O(\eta^{2+\alpha}),\quad j=1,3,4,\quad \eta\to\infty,\quad \eta\in\mathbb{C}^{\pm}, \tag{38}$$

$$\Omega_2^{\pm}(\eta)=K_2^{\pm}\eta^{1+\alpha}+O(\eta^{\alpha}),\quad \eta\to\infty,\quad \eta\in\mathbb{C}^{\pm},$$

where $-1\le\alpha<0$, $K_j^{\pm}$ are nonzero constants, and

$$K_3^{+}=K_4^{+}, \tag{39}$$

$$\lim_{\eta\to\infty}\left\{\frac{\Omega_1^{+}(\eta)}{\eta^2}+\frac{2\nu^2}{\eta\gamma}[\Omega_3^{+}(\eta)-\Omega_4^{+}(\eta)]\right\}=0, \tag{40}$$

(v) satisfy the following seven conditions:

$$\Omega_3^{+}(\gamma)=0,\quad \Omega_4^{+}(\gamma)=0, \tag{41}$$

$$(k+\gamma)\Omega_3^{+}(k)+k\Omega_4^{+}(k)=0, \tag{42}$$

$$\Omega_1^{-}(\eta^{-})-\frac{i(\eta^{-}-\gamma)}{\nu}\Omega_2^{-}(\eta^{-})+\Omega_3^{-}(\eta^{-})=0, \tag{43}$$

$$\Omega_1^{-}(\eta^{-})-\frac{i\eta^{-}}{\nu}\Omega_2^{-}(\eta^{-})+\Omega_4^{-}(\eta^{-})=0,$$

$$\Omega_1^{-}(\eta^{-})-\frac{1}{2}\Omega_3^{-}(\eta^{-})-\frac{1}{2}\Omega_4^{-}(\eta^{-})=0,$$

$$3\operatorname*{res}_{\eta=\gamma_0}\Omega_3^{+}(\eta)+\operatorname*{res}_{\eta=\gamma_0}\Omega_4^{+}(\eta)=0, \tag{44}$$

where

$$\eta^{-}=-\frac{1}{2}\sqrt{\frac{3\gamma^2-16k^2\nu^2}{1-4\nu^2}},\quad \gamma_0=\sqrt{k^2+\frac{\gamma^2}{16\nu^2}}, \tag{45}$$

and the branches of the square roots are chosen such that $\eta^{-}\in\mathbb{C}^{-}$ and $\gamma_0\in\mathbb{C}^{+}$.

Then the vector-functions $\mathbf{\Phi}^{+}(\eta)$ and $\mathbf{\Phi}^{-}(\eta)$ whose components are defined by (35) and (36) are the general solution of the problem (12) - (17) provided the conditions (12) - (14) are satisfied.

Proof. It follows from (33), (37) and (31) that the functions (35) satisfy the boundary and symmetry conditions (15) and (11). Analysis of the expressions (35) shows that if the functions $\Omega_j^\pm(\eta)$ grow at infinity as in (38) and satisfy the conditions (39), then the functions $\Phi_j^\pm(\eta)$ $(j = 1, 2, 4)$ vanish, while $\Phi_3^\pm(\eta) \sim K^\pm \eta^{\alpha+1}$ ($K^\pm$ are constants) when $\eta \to \infty$, $\eta \in \mathbb{C}^\pm$.

According to (35), the functions $\Phi_j^\pm(\eta)$ $(j = 1, 2, 3, 4)$ have simple poles at the points $\mp\eta^- \in \mathbb{C}^\pm$. It is directly verified that the system

$$\underset{\eta=\eta^-}{\operatorname{res}} \Phi_j^-(\eta) = 0, \quad j = 1, 2, 3, 4, \tag{46}$$

with respect to $\Omega_s^-(\eta^-)$ $(s = 1, 2, 3, 4)$ has rank 3, and because of the symmetry condition (11), the conditions (43) are necessary and sufficient to these points being removable singularities of the vector-functions $\mathbf{\Phi}^\pm(\eta)$.

The point $\eta = \gamma \in \mathbb{C}^+$ is a simple pole of the functions $\Phi_j^+(\eta)$ $(j = 1, 2, 3, 4)$, and the points $\eta = \pm k \in \mathbb{C}^\pm$ are simple poles of the functions $\Phi_4^\pm(\eta)$. The conditions (41), (42) remove these singularities.

The elements of the matrix $U_1(\eta)$ are polynomials of η, and the vector $\mathbf{\Omega}^-(\eta)$ does not have poles in any finite part of the half-plane $\mathbb{C}^-$. As for $U_0(\eta)$, it is a rational matrix, and

$$\begin{aligned}
\Omega_1^+(\eta) &= 2[\eta\gamma(\gamma-\eta) + \nu^2\xi^2(\gamma+2\eta)]\Phi_1^+(\eta) + [\gamma(\gamma^2-2\eta^2) \\
&+2\nu^2\xi^2(3\gamma-2\eta)]\Phi_2^+(\eta) + 2\gamma(\gamma\eta-\gamma^2-4\nu^2\xi^2)\Phi_3^+(\eta) - 2i\gamma(\gamma-2\eta)\nu\xi^2\Phi_4^+(\eta), \\
\Omega_2^+(\eta) &= -i\nu(-2\eta\gamma+3\gamma^2+8\nu^2\xi^2)\Phi_1^+(\eta) + i\nu(2\eta\gamma+\gamma^2+8\nu^2\xi^2)\Phi_2^+(\eta) \\
&-2i\gamma\nu(\gamma-2\eta)\Phi_3^+(\eta) + 8\gamma\nu^2\xi^2\Phi_4^+(\eta), \\
\Omega_3^+(\eta) &= \frac{4(\eta-\gamma)}{\gamma^2-16\nu^2\xi^2}\{[\nu^2\xi^2(6\eta\gamma-3\gamma^2-8\eta^2+16\nu^2\xi^2)+\eta^2\gamma^2]\Phi_1^+(\eta) \\
&-[\nu^2\xi^2(5\gamma^2+16\nu^2\xi^2-8\eta^2-2\eta\gamma)+\eta\gamma^2(\eta+\gamma)]\Phi_2^+(\eta) \\
&-\gamma[8\nu^2\xi^2(\gamma-\eta)+\eta\gamma^2]\Phi_3^+(\eta) + i\gamma\nu\xi^2(2\eta\gamma+3\gamma^2-16\nu^2\xi^2)\Phi_4^+(\eta)\}, \\
\Omega_4^+(\eta) &= \frac{4(\eta-\gamma)}{\gamma^2-16\nu^2\xi^2}\{[\nu^2\xi^2(5\gamma^2+16\nu^2\xi^2-8\eta^2-2\eta\gamma)-\eta\gamma^2(\eta+\gamma)]\Phi_1^+(\eta) \\
&+[\nu^2\xi^2(10\eta\gamma+3\gamma^2+8\eta^2-16\nu^2\xi^2)+\gamma^2(\eta+\gamma)^2]\,\Phi_2^+(\eta) \\
&+\gamma(\eta+\gamma)(\gamma^2+8\nu^2\xi^2)\Phi_3^+(\eta) - i\gamma\nu\xi^2(6\eta\gamma+5\gamma^2+16\nu^2\xi^2)\Phi_4^+(\eta)\}.
\end{aligned} \tag{47}$$

Thus, the functions $\Omega_3^+(\eta)$ and $\Omega_4^+(\eta)$ have a simple pole at the point $\eta = \gamma_0$. This point is a removable singularity of the functions $\Phi_j^+(\eta)$ if

$$\Omega_j^\circ = \underset{\eta=\eta_0}{\operatorname{res}} \Omega_j^+(\eta), \quad j = 3, 4, \tag{48}$$

solve the system of the four equations

$$\underset{\eta=\eta_0}{\operatorname{res}} \Phi_j^+(\eta) = 0, \quad j = 1, 2, 3, 4, \tag{49}$$

It is a matter of simple algebra to show that its rank is 1, and it reduces to the equation $3\Omega_3^\circ + \Omega_4^\circ = 0$ that is the condition (44).

■

4. Vectors $\mathbf{\Omega}^+(\eta)$ and $\mathbf{\Omega}^-(\eta)$

It follows from (37) that the first two components of the vectors $\mathbf{\Omega}^\pm(\eta)$, the functions $\Omega_1(\eta)$ and $\Omega_2(\eta)$, are rational function which have simple poles at the points η_0 and $-\eta_0$. Because of the asymptotic relations (38) these functions have the form ($\alpha=-1$)

$$\Omega_1(\eta) = C_1 + C_2\eta + C_3\eta^2 + \frac{D_1}{\eta-\eta_0} + \frac{D_2}{\eta+\eta_0}, \tag{50}$$

$$\Omega_2(\eta) = C_1' + \frac{D_1'}{\eta-\eta_0} + \frac{D_2'}{\eta+\eta_0}.$$

where C_1, C_2, C_3, D_1 and D_2 are constants to be determined *a posteriori*. The other constants are fixed by the symmetry condition (33) as

$$C_1' = -2i\nu C_2, \quad D_1' = -D_2' = -\frac{i\nu}{\eta_0}(D_1+D_2). \tag{51}$$

4.1. Riemann-Hilbert problem on a genus-3 Riemann surface. In what follows we shall define the vectors $\hat{\mathbf{\Omega}}^\pm(\eta) = (\Omega_3^\pm(\eta), \Omega_4^\pm(\eta))^\top$ from the Riemann-Hilbert problem:

$$\hat{\mathbf{\Omega}}^+(\eta) = \frac{1}{2\nu}\hat{\Gamma}(\eta)\hat{\mathbf{\Omega}}^-(\eta), \quad \eta \in L, \tag{52}$$

subject to the symmetry condition

$$\hat{\mathbf{\Omega}}^+(\eta) = \hat{S}(\eta)\hat{\mathbf{\Omega}}^-(-\eta), \quad s \in \mathbb{C}^+, \tag{53}$$

where

$$\hat{\Gamma}(\eta) = \begin{pmatrix} b+cl_0 & cl_1 \\ cl_2 & b-cl_0 \end{pmatrix}, \quad \hat{S}(\eta) = \begin{pmatrix} s_{33}(\eta) & s_{34}(\eta) \\ s_{43}(\eta) & s_{44}(\eta) \end{pmatrix}. \tag{54}$$

Notice that

$$\det\hat{\Gamma}(\eta) = \frac{\eta^2-\gamma^2}{\eta^2-\gamma_0^2}. \tag{55}$$

Since the matrix coefficient $\hat{\Gamma}(\eta)$ has the structure (54), and $f(\eta) = l_0^2 + l_1 l_2$ is a degree-8 polynomial:

$$f(\eta) = \nu_0^2(\eta^2-k^2)\prod_{j=1}^{3}(\eta^2-\eta_j^2), \tag{56}$$

the problem of matrix factorization

$$\hat{\Gamma}(\eta) = X^+(\eta)[X^-(\eta)]^{-1}, \quad \eta \in L, \tag{57}$$

can be solved in terms of the solution of a Riemann-Hilbert problem on a genus-3 Riemann surface. Here,

$$\nu_0 = 8\nu^2\gamma\sqrt{1-4\nu^2}, \quad \eta_1 = i\sqrt{-\frac{k^2}{2}+\sqrt{\frac{k^4}{4}+\frac{\gamma^4}{16\nu^2}}}, \tag{58}$$

$$\eta_2 = \sqrt{\frac{k^2}{2}+\sqrt{\frac{k^4}{4}+\frac{\gamma^4}{16\nu^2}}}, \quad \eta_3 = \frac{1}{2}\sqrt{\frac{16k^2\nu^2-3\gamma^2}{4\nu^2-1}},$$

and the branches of the square roots are chosen such that $\eta_j \in \mathbb{C}^+$, $j=1,2,3$.

Consider the algebraic function $w^2 = f(\eta)$. To fix its branch, $\sqrt{f(\eta)}$, we cut the extended η-plane along the segments $l_1^\pm = [\pm\eta_1, \pm k] \in \mathbb{C}^\pm$ and $l_2^\pm = [\pm\eta_2, \pm\eta_3] \in \mathbb{C}^\pm$ and enforce the condition $f^{1/2}(\eta) \sim -\nu_0\eta^4$, $\eta \to \infty$.

Let $\mathfrak{R}$ be the hyperelliptic surface generated by the algebraic function $w^2 = f(\eta)$ and formed by gluing two copies, $\mathbb{C}_1$ and $\mathbb{C}_2$, of the extended η-plane $\mathbb{C}\cup\{\infty\}$ cut along the segments $l_1^\pm$ and $l_2^\pm$ according to the rule

$$w = \begin{cases} \sqrt{f(\eta)}, & \eta \in \mathbb{C}_1, \\ -\sqrt{f(\eta)}, & \eta \in \mathbb{C}_2. \end{cases} \tag{59}$$

On following the results in [15], [3], allows us to reduce the problem of matrix factorization to a scalar Riemann-Hilbert problem on the surface $\mathfrak{R}$.

Theorem 3. *Let $\lambda(\eta, w)$ be a function defined on the surface $\mathfrak{R}$ as*

$$\lambda(\eta, w) = \begin{cases} \lambda_1(\eta), & \eta \in \mathbb{C}_1, \\ \lambda_2(\eta), & \eta \in \mathbb{C}_2, \end{cases} \tag{60}$$

where $\lambda_j(\eta) = b(\eta) + (-1)^{j-1}c(\eta)\sqrt{f(\eta)}$ $(j=1,2)$ are the eigenvalues of the matrix $\hat{\Gamma}(\eta)$, and let $\chi(\eta, w)$ be a solution to the Riemann-Hilbert problem

$$\chi^+(\eta, w) = \lambda(\eta, w)\chi^-(\eta, w), \quad (\eta, w) \in \mathfrak{L}, \tag{61}$$

where $\mathfrak{L} = (L \subset \mathbb{C}_1) \cup (L \subset \mathbb{C}_2)$. Then the factors

$$X^\pm(\eta) = e^{\chi^\pm(\eta,w)}Y(\eta, w) + e^{\chi^\pm(\eta,-w)}Y(\eta, -w), \quad \eta \in \mathbb{C}^\pm, \tag{62}$$

solve the factorization problem (57). Here,

$$Y(\eta, w) = \frac{1}{2}\left[I_2 + \frac{1}{w}Q(\eta)\right], \quad Q(\eta) = \begin{pmatrix} l_0(\eta) & l_1(\eta) \\ l_2(\eta) & -l_0(\eta) \end{pmatrix}. \tag{63}$$

The solution to the factorization problem is not unique. We aim to construct a vector-function that satisfies not only the boundary condition (52) but also the symmetry condition (53). Since the winding numbers of the eigenvalues vanish:

$$\frac{1}{2\pi}\left[\arg \lambda_j(t)\right]\big|_L = 0, \quad j = 1, 2, \tag{64}$$

we choose the solution to the problem (61) in the form

$$\chi(\eta, w) = \frac{1}{2\pi i} \int_{\mathfrak{L}} \log \lambda(t, u) dW + \beta(\eta, w) - \beta(-\eta, w), \tag{65}$$

where the branches of the logarithmic functions $\log \lambda_j(\eta) = \log |\lambda_j(\eta)| + i\alpha_j$ are fixed by the conditions $-\pi \le \alpha_j \le \pi$, $j = 1, 2$,

$$\beta(\eta, w) = \sum_{j=1}^{3} \left(\int_{q_{0j}}^{q_{1j}} dW + m_j \oint_{\mathbf{a}_j} dW + n_j \oint_{\mathbf{b}_j} dW \right), \tag{66}$$

m_j and n_j ($j = 1, 2, 3$) are integers to be fixed, $q_{0j} = (\sigma_{0j}, w_{0j}) \in \mathbb{C}_1$, $q_{1j} = (\sigma_{1j}, w_{1j}) \in \mathfrak{R}$, $w_{mj} = w(\sigma_{mj})$, $m = 0, 1$. The points q_{0j} are arbitrary fixed points in the first sheet, while the points q_{1j} may lie in either sheet of the surface and are to be determined,

$$dW(\eta) = \frac{w + u}{2u} \frac{dt}{t - \eta} \tag{67}$$

is the Weierstrass kernel, and $u = w(t)$, $(t, u) \in \mathfrak{L}$.

The contours $\mathbf{a}_j$ and $\mathbf{b}_j$ form a system of canonical cross-sections of the hyperelliptic surface $\mathfrak{R}$. The cross-sections $\mathbf{a}_j$ are closed contours formed by the branch cuts l_1^-, l_1^+ and l_2^+. The positive direction is chosen in the standard way such that the sheet $\mathbb{C}_1$ is on the left. The canonical cross-sections $\mathbf{b}_j$ are chosen to be

$$\mathbf{b}_1 = [-\eta_2, -\eta_1]_{\mathbb{C}_2} \cup [-\eta_1, -\eta_2]_{\mathbb{C}_1}, \tag{68}$$

$$\mathbf{b}_2 = [-\eta_2, -\eta_1]_{\mathbb{C}_2} \cup [-\eta_1, -k]^+_{\mathbb{C}_2} \cup [-k, \eta_1]_{\mathbb{C}_2} \cup [\eta_1, -k]_{\mathbb{C}_1} \cup [-k, -\eta_1]^-_{\mathbb{C}_1} \cup [-\eta_1, -\eta_2]_{\mathbb{C}_1},$$

$$\mathbf{b}_3 = [-\eta_2, -\eta_1]_{\mathbb{C}_2} \cup [-\eta_1, -k]^+_{\mathbb{C}_2} \cup [-k, \eta_1]_{\mathbb{C}_2} \cup [\eta_1, k]^+_{\mathbb{C}_2} \cup [k, \eta_2]_{\mathbb{C}_2}$$
$$\cup [\eta_2, k]_{\mathbb{C}_1} \cup [k, \eta_1]^-_{\mathbb{C}_1} \cup [\eta_1, -k]_{\mathbb{C}_1} \cup [-k, -\eta_1]^-_{\mathbb{C}_1} \cup [-\eta_1, -\eta_2]_{\mathbb{C}_1}.$$

The loop $\mathbf{b}_j$ crosses the contour $\mathbf{a}_j$ from right to left and does not cross the other loops $\mathbf{a}_m$ and $\mathbf{b}_m$ $(m \ne j)$.

To analyze the behavior of the function $\chi(\eta, w)$ at infinity, it is convenient to write it in the form

$$\chi(\eta, w) = \chi_1(\eta) + w\chi_2(\eta), \tag{69}$$

where $\chi_1(\eta)$ and $\chi_2(\eta)$ are odd functions given by

$$\chi_1(\eta) = \frac{\eta}{\pi i} \int_0^\infty \frac{\varepsilon_+(t) dt}{t^2 - \eta^2} + \eta \sum_{j=1}^{3} \int_{\sigma_{0j}}^{\sigma_{1j}} \frac{dt}{t^2 - \eta^2}, \tag{70}$$

$$\chi_2(\eta) = \frac{\eta}{\pi i} \int_0^\infty \frac{\varepsilon_-(t) dt}{\sqrt{f(t)}(t^2 - \eta^2)} + \eta \sum_{j=1}^{3} \left(\int_{q_{0j}}^{q_{1j}} \frac{dt}{u(t)(t^2 - \eta^2)} \right. \tag{71}$$

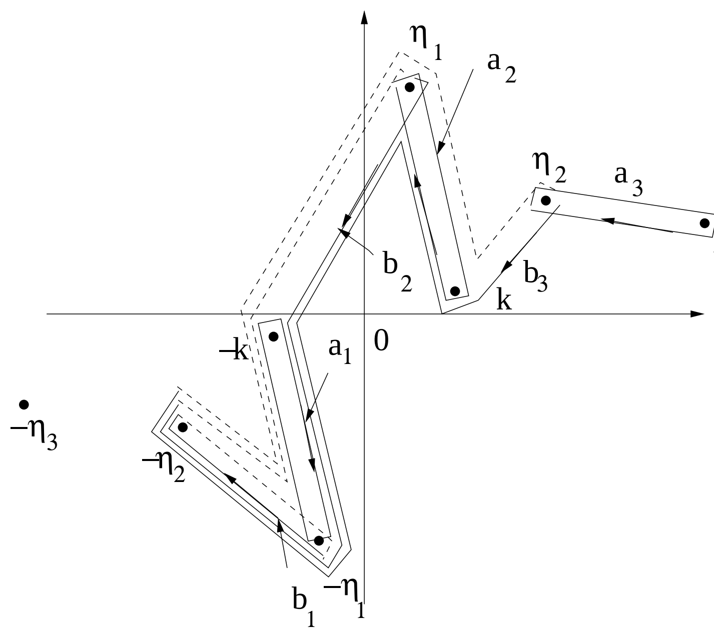

FIGURE 1. Canonical cross-sections

$$+m_j \oint\limits_{\mathbf{a}_j} \frac{dt}{u(t)(t^2-\eta^2)} + n_j \oint\limits_{\mathbf{b}_j} \frac{dt}{u(t)(t^2-\eta^2)} \Bigg),$$

$$\varepsilon_\pm(t) = \frac{1}{2}[\log \lambda_1(t) \pm \log \lambda_2(t)].$$

Since $dW(\eta) = O(\eta^4)$, $(\eta, w) \to (\infty, \infty)_j \in \mathbb{C}_j$, $j = 1, 2$, and $\chi_2(\eta) = O(\eta^{-1})$, $\eta \to \infty$, in general, the function $\chi(\eta, w)$ has an order-3 pole at the two infinite points of the surface, and the factors $X^\pm(\eta)$ have an inadmissible essential singularity at the infinite point in the η-plane. For the function $\exp\{\chi(\eta, w)\}$ being bounded as $(\eta, w) \to (\infty, \infty)_j$, $j = 1, 2$, it is necessary and sufficient that

$$\frac{1}{\pi i} \int_0^\infty \frac{t^{s-1}\varepsilon_-(t)dt}{\sqrt{f(t)}} + \sum_{j=1}^{3} \left(\int_{q_{0j}}^{q_{1j}} \frac{t^{s-1}dt}{u(t)} + m_j A_{sj} + n_j B_{sj} \right) = 0, \quad s = 1, 3, \tag{72}$$

where

$$A_{sj}=\oint_{\mathbf{a}_j}\frac{t^{s-1}dt}{u(t)},\quad B_{sj}=\oint_{\mathbf{b}_j}\frac{t^{s-1}dt}{u(t)} \tag{73}$$

are the A- and B-periods of the abelian integrals

$$\omega_s=\omega_s(\eta,w)=\int_{(-\eta_3,0)}^{(\eta,w)}\frac{t^{s-1}dt}{u(t)},\quad s=1,2,3. \tag{74}$$

It will be convenient to add a third equation to the system (72), equation (72) for $s=2$ (it does not affect the properties of the function $\chi_2(\eta)$). This gives

$$\sum_{j=1}^{3}[\omega_s(q_{1j})+m_jA_{sj}+n_jB_{sj}]=\rho_s,\quad s=1,2,3, \tag{75}$$

where

$$\rho_s=\sum_{j=1}^{3}\omega_s(q_{0j})-\frac{1}{\pi i}\int_0^{\infty}\frac{t^{s-1}\varepsilon_-(t)dt}{\sqrt{f(t)}}. \tag{76}$$

To reduce the nonlinear system (75) to the classical genus-3 Jacobi inversion problem, we normalize the basis of abelian integrals. Let A be the matrix whose elements are the A-periods $\{A_{sj}\}$ $(s,j=1,2,3)$ and $A^\circ=\{A^\circ_{sj}\}$ be its inverse, A^{-1}. Denote the normalized (canonical) basis as $\hat\omega=\{\hat\omega_s\}$ $(s=1,2,3)$ with the A- and B-periods

$$\hat A_{sj}=\oint_{\mathbf{a}_j}d\hat\omega_s=\delta_{sj}, \tag{77}$$

$$\hat B_{sj}=\oint_{\mathbf{b}_j}d\hat\omega_s=\sum_{n=1}^{3}A^\circ_{sn}B_{nj}.$$

Here δ_{sj} is the Kronecker symbol. Thus, to eliminate the essential singularity of the solution, one needs to solve the following Jacobi problem:

Find three points $q_{1j}\in\mathfrak{R}$ and six integers m_j and n_j $(j=1,2,3)$ such that

(78)

$$\sum_{j=1}^{3}\hat\omega_s(q_{1j})=e_s-k_s-\sum_{j=1}^{3}n_j\hat B_{sj}-m_i\equiv e_s-k_s\quad\text{(modulo the periods)},\quad s=1,2,3,$$

where $e_s=\hat\rho_s+k_s$,

$$\hat\rho_s=\sum_{j=1}^{3}A^\circ_{sj}\rho_j,\quad s=1,2,3, \tag{79}$$

k_s $(s=1,2,3)$ are the Riemann constants

$$k_s=-1+\frac{s}{2}-\frac{1}{2}\sum_{j=1}^{3}\hat{B}_{sj}. \tag{80}$$

The unknown points q_{1j} $(j=1,2,3)$ coincide with the three zeros of the Riemann Theta function [13], [20]

$$\Theta(q)=\theta(\hat{\omega}_1(q)-e_1,\hat{\omega}_2(q)-e_2,\hat{\omega}_3(q)-e_3)=\sum_{t_1,t_2,t_3=-\infty}^{\infty}\hat{\theta}_{\mathbf{t}}(q), \tag{81}$$

where

$$\hat{\theta}_{\mathbf{t}}(q)=\exp\left\{\pi i\sum_{r=1}^{3}\sum_{s=1}^{3}\hat{B}_{rs}t_rt_s+2\pi i\sum_{s=1}^{3}t_s[\hat{\omega}_s(q)-e_s]\right\},\quad \mathbf{t}=(t_1,t_2,t_3). \tag{82}$$

Since the matrix $\operatorname{Im}\hat{B}$ is positive definite, the series (81) converges exponentially. The affixes of the points q_{11},q_{12}, and q_{13} can be expressed through the roots of the cubic equation [7]

$$(\varepsilon_1^3-3\varepsilon_1\varepsilon_2+2\varepsilon_3)\sigma^3+3(\varepsilon_2-\varepsilon_1^2)\sigma^2+6\varepsilon_1\sigma-6=0, \tag{83}$$

where the coefficients of the equation are

$$\varepsilon_s=2\sum_{j=1}^{3}\sum_{r=1}^{3}A^\circ_{jr}\int_{\gamma_j}\frac{\tau^{r-s-1}d\tau}{f^{1/2}(\tau)}-\sum_{m=1}^{2}\underset{q=0_m\in\mathbb{C}_m}{\operatorname{res}}\frac{\eta^{-s}\Theta'(q)}{\Theta(q)}, \tag{84}$$

$\gamma_1=[-k,-\eta_1]^+$, $\gamma_2=[\eta_1,k]^+$, $\gamma_3=[\eta_2,\eta_3]^+$, $0_m=(0,(-1)^{m-1}f^{1/2}(0))\in\mathbb{C}_m$, and without loss of generality, at the two zero points of the Riemann surface the Theta function $\Theta(q)$ does not vanish. By computing the three roots of the cubic equations we identify three pairs of points, $(\sigma_{1s},\sqrt{f(\sigma_{1s})})$ and $(\sigma_{1s},-\sqrt{f(\sigma_{1j})})$ $(s=1,2,3)$. However, one and only one point, $q_{1s}=(\sigma_{1s},(-1)^{m-1}\sqrt{f(\sigma_{1s})})\in\mathbb{C}_m$, is a zero of the Theta function $\Theta(q)$. To complete the solution of the Jacobi inversion problem, one needs to identify the integers m_s and n_s. The integers n_s solve the system of linear algebraic equations

$$\sum_{j=1}^{3}n_j\operatorname{Im}(\hat{B}_{sj})=\operatorname{Im}v_s,\quad s=1,2,3, \tag{85}$$

and the integers m_s are defined explicitly through the integers n_s by

$$m_s=\operatorname{Re}v_s-\sum_{j=1}^{3}n_j\operatorname{Re}(\hat{B}_{sj}). \tag{86}$$

Here,

$$v_s = e_s - k_s - \sum_{j=1}^{3} \hat{\omega}_s(q_j). \tag{87}$$

4.2. Symmetry condition. We have found the general form of the vectors $\hat{\mathbf{\Omega}}_0^{\pm}(\eta)$ satisfying the boundary condition (52). It is

$$\hat{\mathbf{\Omega}}_0^{+}(\eta) = \frac{1}{2\nu} X^{+}(\eta)\mathbf{r}(\eta), \quad \eta \in \mathbb{C}^{+}, \tag{88}$$

$$\hat{\mathbf{\Omega}}_0^{-}(\eta) = X^{-}(\eta)\mathbf{r}(\eta), \quad \eta \in \mathbb{C}^{-},$$

where $\mathbf{r}(\eta)$ is a rational vector-function to be determined. In general, this solution does not satisfy the symmetry condition (53). To meet this condition, we reconstruct the vector $\mathbf{r}(\eta)$. First, we rewrite the factors (62) in terms of functions defined in the complex η-plane, not in the surface $\mathfrak{R}$, as

$$X^{\pm}(\eta) = e^{\chi_1^{\pm}(\eta)} \left[\cosh(f^{1/2}(\eta)\chi_2^{\pm}(\eta)) I_2 + \frac{1}{f^{1/2}(\eta)} \sinh(f^{1/2}(\eta)\chi_2^{\pm}(\eta)) Q(\eta) \right], \tag{89}$$

where $\eta \in \mathbb{C}^{\pm}$, and $I_2 = \operatorname{diag}\{1, 1\}$. The functions $\chi_1^{\pm}$ and $\chi_2^{\pm}$ possess the symmetry property

$$\chi_j^{-}(-\eta) = -\chi_j^{+}(\eta), \quad j = 1, 2, \tag{90}$$

and the integral (70) can be computed explicitly

$$\chi_1^{\pm}(\eta) = \frac{1}{2} \log \prod_{j=1}^{3} \frac{(\eta + \sigma_{0j})(\eta - \sigma_{1j})}{(\eta - \sigma_{0j})(\eta + \sigma_{1j})} \pm \frac{1}{2} \log \frac{\eta \pm \gamma}{\eta \pm \gamma_0}, \quad \eta \in \mathbb{C}^{\pm}. \tag{91}$$

We are now ready to recover the rational vector-function $\mathbf{r}(\eta)$. Because of the property (90) we have

$$X^{-}(-\eta) = e^{-\chi_1^{+}(\eta)} \left[\cosh(f^{1/2}(\eta)\chi_2^{+}(\eta)) I_2 - \frac{1}{f^{1/2}(\eta)} \sinh(f^{1/2}(\eta)\chi_2^{\pm}(\eta)) Q(-\eta) \right], \tag{92}$$

where $\eta \in \mathbb{C}^{+}$. Rewrite the symmetry condition (53) as

$$\mathbf{r}(\eta) = \tilde{S}(\eta)\mathbf{r}(-\eta), \tag{93}$$

where

$$\tilde{S}(\eta) = -\frac{\eta - \gamma}{4\nu\gamma(\eta - \gamma_0)(\eta + \gamma)} \prod_{j=1}^{3} \frac{(\eta - \sigma_{0j})(\eta + \sigma_{1j})}{(\eta + \sigma_{0j})(\eta - \sigma_{1j})} \times \begin{pmatrix} -(\gamma^2 - 4\nu^2\xi^2) & -4\nu^2\xi^2 \\ 2(\gamma^2 + 2\nu^2\xi^2) & \gamma^2 - 4\nu^2\xi^2 \end{pmatrix}. \tag{94}$$

Recall that the functions $\Omega_j^+(\eta)$ $(j = 3, 4)$ have simple zeros at the point $\eta = \gamma \in \mathbb{C}^+$ and simple poles at the point $\eta = \gamma_0 \in \mathbb{C}^+$. Therefore, the points $\eta = \gamma$ and $\eta = \gamma_0$ are a simple zero and a simple pole, respectively, of the rational vector $\mathbf{r}(\eta)$. Analysis of the function (91) and the singular integral (71) shows that the matrix $X(\eta)$ has simple zeros at the points $\eta = \sigma_{1j}$ and $\eta = -\sigma_{0j}$. This means that the components of the vector $\mathbf{r}(\eta)$, the functions $r_1(\eta)$ and $r_2(\eta)$, have simple poles at these points. At the geometric optics poles of the vector-functions $\mathbf{\Phi}^\pm(\eta)$, the vector $\mathbf{r}(\eta)$ cannot have poles, otherwise the symmetry condition (93) fails. As $\eta \to \infty$, the vectors $\hat{\mathbf{\Omega}}^\pm(\eta)$ grow as in (38) $(\alpha = -1)$. Having established all the properties of the rational vector $\mathbf{r}(\eta)$ we can write its most general form

$$\mathbf{r}(\eta) = \frac{(\eta-\gamma)\mathbf{p}(\eta)}{(\eta-\gamma_0)\prod_{j=1}^3(\eta+\sigma_{0j})(\eta-\sigma_{1j})} \tag{95}$$

where $\mathbf{p}(\eta) = (p_1(\eta), p_2(\eta))^\top$, and $p_1(\eta)$ and $p_2(\eta)$ are degree-8 polynomials

$$p_1(\eta) = \sum_{j=0}^{8} a_j\eta^j, \quad p_2(\eta) = \sum_{j=0}^{8} b_j\eta^j \tag{96}$$

with the coefficients to be determined from the symmetry condition (93) that reduces to

$$\mathbf{p}(\eta) = S^\circ(\eta)\mathbf{p}(-\eta), \tag{97}$$

where $S^\circ(\eta)$ is a 2×2 matrix. It satisfies the identity $S^\circ(\eta) = [S^\circ(-\eta)]^{-1}$ and is given by

$$S^\circ(\eta) = \frac{1}{4\nu\gamma(\eta+\gamma_0)}\begin{pmatrix} \gamma^2 - 4\nu^2\xi^2 & 4\nu^2\xi^2 \\ -2(\gamma^2+2\nu^2\xi^2) & -(\gamma^2-4\nu^2\xi^2)\end{pmatrix}. \tag{98}$$

It is convenient to rewrite the vector equation (97) as

$$\hat{p}_1(\eta) = 0, \quad \hat{p}_2(\eta) = 0, \tag{99}$$

where

$$\hat{p}_1(\eta) = \sum_{j=0}^{8}\eta^j\{[4\nu\gamma(\eta+\gamma_0) - (-1)^j(\gamma^2-4\nu^2\xi^2]a_j - 4(-1)^j\nu^2\xi^2 b_j\} \tag{100}$$

$$\hat{p}_2(\eta) = \sum_{j=0}^{8}\eta^j\{[4\nu\gamma(\eta+\gamma_0) + (-1)^j(\gamma^2-4\nu^2\xi^2]b_j + 2(-1)^j(\gamma^2+2\nu^2\xi^2)a_j\}$$

By equating the coefficients of the degree-10 polynomial $\hat{p}_1(\eta)$ to zero we obtain 11 equations. Only 10 of them are linearly independent. They express the coefficients b_j through a_j $(j = 0, 1, \ldots, 8)$ as

$$b_8 = a_8, \tag{101}$$

$$b_7 = a_7 - \nu_1 a_8,$$

$$b_6 = a_6 + \nu_1 a_7 + \nu_- a_8,$$
$$b_5 = a_5 - \nu_1 a_6 - \nu_+ a_7 - \nu_1 k^2 a_8,$$
$$b_4 = a_4 + \nu_1 a_5 + \nu_- a_6 + \nu_1 k^2 a_7 + \nu_- k^2 a_8,$$
$$b_3 = a_3 - \nu_1 a_4 - \nu_+ a_5 - \nu_1 k^2 a_6 - \nu_+ k^2 a_7 - \nu_1 k^4 a_8,$$
$$b_2 = a_2 + \nu_1 a_3 + \nu_- a_4 + \nu_1 k^2 a_5 + \nu_- k^2 a_6 + \nu_1 k^4 a_7 + \nu_- k^4 a_8,$$
$$b_1 = a_1 - \nu_1 a_2 - \nu_+ a_3 - \nu_1 k^2 a_4 - \nu_+ k^2 a_5 - \nu_1 k^4 a_6 - \nu_+ k^4 a_7 - \nu_1 k^6 a_8,$$
$$b_0 = a_0 + \nu_1 a_1 + \nu_- a_2 + \nu_1 k^2 a_3 + \nu_- k^2 a_4 + \nu_1 k^4 a_5 + \nu_- k^4 a_6 + \nu_1 k^6 a_7 + \nu_- k^6 a_8,$$

and a_0 through a_j $(j = 1, 2, \ldots, 8)$ as

(102) $$a_0 = -\frac{\nu_+}{\nu_1} a_1 - k^2 a_2 - \frac{\nu_+}{\nu_1} k^2 a_3 - k^4 a_4 - \frac{\nu_+}{\nu_1} k^4 a_5 - k^6 a_6 - \frac{\nu_+}{\nu_1} k^6 a_7 - k^8 a_8.$$

Here,

(103) $$\nu_1 = \frac{\gamma}{\nu}, \quad \nu_\pm = \frac{\gamma(4\gamma_0 \nu \pm \gamma)}{4\nu^2}.$$

The polynomial $\hat{p}_2(\eta)$ identically vanishes provided the coefficients a_0 and b_j $(j = 0, 1, \ldots, 8)$ are determined by (101) and (102). Thus, the rational vector-function $\mathbf{r}(\eta)$ has eight free parameters $a_1, a_2, \ldots, a_8$. In total, there are 13 parameters, C_j $(j = 1, 2, 3)$, D_j $(j = 1, 2)$ and a_j $(j = 1, 2, \ldots, 8)$, to be determined.

Analysis of the function (91) and the singular integral (71) shows that the matrix $X(\eta)$ has simple poles at the points $\eta = -\sigma_{1j}$ and $\eta = \sigma_{0j}$ $(j = 1, 2, 3)$

(104) $$X(\eta) \sim \frac{\chi_*(\sigma_j)}{\eta - \sigma_j} X_*(\sigma_j), \quad \eta \to \sigma_j, \quad j = 1, 2, \ldots, 6,$$

where $\sigma_j = -\sigma_{1j}$ and $\sigma_{j+3} = \sigma_{0j}$ $(j = 1, 2, 3)$, $\chi_*(\eta)$ is a function bounded at the points σ_j and having a certain nonzero limit at these points, and $X_*(\eta)$ is a 2×2 rank-1 matrix

(105) $$X_*(\eta) = \left(\begin{array}{cc} 1 - \frac{l_0(\eta)}{\sqrt{f(\eta)}} & -\frac{l_1(\eta)}{\sqrt{f(\eta)}} \\ -\frac{l_2(\eta)}{\sqrt{f(\eta)}} & 1 + \frac{l_0(\eta)}{\sqrt{f(\eta)}} \end{array} \right).$$

These six poles are removable singularities of the vectors $\hat{\Omega}^\pm(\eta)$ if and only if the following six conditions are satisfied:

(106) $$\left[\sqrt{f(\sigma_j)} - l_0(\sigma_j)\right] p_1(\sigma_j) - l_1(\sigma_j) p_2(\sigma_j) = 0, \quad j = 1, 2, \ldots, 6.$$

Since $\mathbf{r}(\gamma) = 0$, the functions $\Omega_3^+(\eta)$ and $\Omega_4^+(\eta)$ satisfy the condition (41) automatically. The conditions (42) - (44) bring five additional equations for the unknown constants. One extra equation follows from the physical conditions (12) - (14) in Theorem 1. Finally, we verify the asymptotics of the functions $\Phi_j^\pm(\eta)$ at infinity and derive the last, the 13th, equation. Since

(107) $$\cosh(f^{1/2}(\eta)\chi_2(\eta)) \sim 1,$$

$$\sinh(f^{1/2}(\eta)\chi_2(\eta)) \sim \frac{s_0}{\eta}, \quad \eta \to \infty, \quad s_0 = \text{const},$$

the matrix $X^+(\eta)$ admits the following asymptotic expansion:

$$X^+(\eta) \sim \begin{pmatrix} 1 - \frac{16\nu^4 s_0}{\nu_0} & \frac{16\nu^4 s_0}{\nu_0} \\ -\frac{16\nu^4 s_0}{\nu_0} & 1 + \frac{16\nu^4 s_0}{\nu_0} \end{pmatrix} - \frac{8\gamma\nu^2 s_0}{\nu_0 \eta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta \to \infty, \tag{108}$$

and therefore

$$\hat{\mathbf{\Omega}}^+(\eta) \sim \frac{a_8 \eta^2}{2\nu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\eta}{2\nu} \begin{pmatrix} d_0 \\ d_0 - \frac{\gamma}{\nu} a_8 \end{pmatrix}, \quad \eta \to \infty, \tag{109}$$

where d_0 is a constant. Thus, $\Omega_3^+(\eta) - \Omega_4^+(\eta) \sim (2\nu^2)^{-1}\gamma a_8 \eta$, $\eta \to \infty$. As $\eta \to \infty$, the functions $\Phi_j^\pm(\eta)$ $(j = 1, 2, 4)$ vanish, $\Phi_j^\pm(\eta) = O(\eta^{-1})$, and the functions $\Phi_3^\pm(\eta)$ are bounded if and only if the condition (40) is satisfied. It ultimately gives

$$C_3 = -a_8. \tag{110}$$

This equation excludes the constant C_3 from the representations (50) of the functions $\Omega_1(\eta)$ and $\Omega_2(\eta)$. The definition of the 12 constants C_j $(j = 1, 2)$, D_j $(j = 1, 2)$ and a_j $(j = 1, 2, \ldots, 8)$ from the 12 linear equations (12) - (14), (42) - (44) and (106) completes the solution of the Riemann-Hilbert problem (15).

5. Solution of the discontinuous boundary-value problem

We have established that the solution to the vector Riemann-Hilbert problem (15), the functions $\Phi_j^\pm(\eta)$ $(j = 1, 2, 3, 4)$, have the following asymptotics at infinity:

$$\Phi_j^\pm(\eta) = O(\eta^{-1}), \quad \Phi_3^\pm(\eta) = O(1), \quad \eta \to \infty, \quad \eta \in \mathbb{C}^\pm. \tag{111}$$

Since

$$\int_{-\infty}^{\infty} H_y(0^-, y) e^{i\eta y} dy = \tilde{H}_{y+}(0^-, \eta) + \tilde{H}_{y-}(0^-, \eta), \tag{112}$$

$\tilde{H}_{y+}(0^-, \eta) = i(kZ)^{-1}\Phi_2^+(\eta)$ and $\tilde{H}_{y-}(0^-, \eta) = i(kZ)^{-1}\Phi_3^-(\eta)$, the asymptotics (111) implies that the function $H_y(0^-, y)$ can be represented as

$$H_y(0^-, y) = H_y^\circ(0^-, y) + C\delta(y), \tag{113}$$

where $C = \text{const}$, $\delta(y)$ is the generalized δ-function, and the function $H_y^\circ(0^-, y)$ is bounded as $y \to 0$. Also,

$$\int_0^{\infty} H_y^\circ(0^-, y) e^{i\eta y} dy = \tilde{H}_{y+}^\circ(0^-, \eta) = O(\eta^{-1}), \quad \eta \to \infty, \quad \eta \in \mathbb{C}^+, \tag{114}$$

$$\int_{-\infty}^{0} H_y^\circ(0^-,y)e^{i\eta y}dy = \tilde{H}_{y-}^\circ(0^-,\eta) = O(\eta^{-1}), \quad \eta\to\infty, \quad \eta\in\mathbb{C}^-.$$

This leads to the relations

$$\tilde{H}_{y+}(0^-,\eta) = \tilde{H}_{y+}^\circ(0^-,\eta) = \frac{i}{kZ}\Phi_2^+(\eta) = O(\eta^{-1}), \quad \eta\to\infty, \quad \eta\in\mathbb{C}^+, \tag{115}$$

$$\tilde{H}_{y-}(0^-,\eta) = C + \tilde{H}_{y-}^\circ(0^-,\eta) = \frac{i}{kZ}\Phi_3^-(\eta) = C + O(\eta^{-1}), \quad \eta\to\infty, \quad \eta\in\mathbb{C}^-.$$

Finally, we write down the solution to the discontinuous boundary value problem for the Helmholtz equation (5), (6). We have

$$E_z(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}_z(x,\eta)e^{-i\eta y}d\eta, \tag{116}$$

where

$$\tilde{E}_z(x,\eta) = A(\eta)e^{-\xi x} - \frac{1}{2\xi}\int_0^\infty F(x_1,\eta)e^{-\xi|x-x_1|}dx_1, \tag{117}$$

$$A(\eta) = \tilde{E}_{z+}(0^+,\eta) - \frac{i\eta}{2\xi}\hat{E}_{z+}(i\xi,0^+) + \frac{ikZ}{2\xi}\hat{H}_{x+}(i\xi,0^+),$$
$$F(x,\eta) = -i\eta E_z(x,0^+) + ikZH_x(x,0^+).$$

By using the relations (8) and (10) we express the functions $A(\eta)$, $E_z(x,0^+)$ and $H_x(x,0^+)$ through the solution of the Riemann-Hilbert problem (15). The function $A(\eta)$ is given by

$$A(\eta) = -\frac{\gamma - i\nu\xi}{2\gamma\xi}\Phi_1^+(\eta) - \frac{i\nu}{2\gamma}\Phi_2^+(\eta), \tag{118}$$

while the other two functions are the inverse Laplace transforms

$$E_z(x,0^+) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\hat{E}_z(i\xi,0^+)e^{-\xi x}d\xi, \tag{119}$$

$$H_x(x,0^+) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\hat{H}_x(i\xi,0^+)e^{-\xi x}d\xi,$$

and

$$\hat{E}_z(i\xi,0^+) = -\frac{i}{2\eta\gamma}\{(\gamma+i\nu\xi)[\Phi_1^+(\eta)+\Phi_1^-(\eta)] - i\nu\xi[\Phi_2^+(\eta)+\Phi_2^-(\eta)]\}, \tag{120}$$

$$\hat{H}_x(i\xi,0^+) = \frac{i}{2\gamma kZ}\{(\gamma+i\nu\xi)[\Phi_1^+(\eta)-\Phi_1^-(\eta)] - i\nu\xi[\Phi_2^+(\eta)-\Phi_2^-(\eta)]\}.$$

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Yuri A. Antipov E-MAIL: antipov@math.lsu.edu
ADDRESS: *Department of Mathematics, Louisiana State University, LA 70803 Baton Rouge, United States*