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Method of integral equations for systems of difference equations in diffraction theory

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A system of difference equations of the first order with meromorphic coefficients (not necessary periodic ones) subject to certain conditions of symmetry is analysed. These conditions arise in model problems of diffraction theory. A method for the constructive solution of the system of difference equations is proposed. It consists of three steps. First, it requires factorization of a certain function. Then, scalar integral equations with the same kernel and different right-hand sides should be solved. The kernel of the equations has a fixed singularity, and the solution belongs to a class of functions with a power singularity. Finally, arbitrary constants are fixed from some conditions which guarantee the equivalence of the original system of difference equations and the integral equations. The method is illustrated by solving a model electromagnetic problem of a plane wave diffracted from an anisotropic impedance half-plane at oblique incidence.

Keywords: system of difference equations; Riemann-Hilbert problem; integral equation; electromagnetic diffraction.

1. Introduction

Many problems of applied mechanics (e.g. Atkinson & Craster, 1994; Antipov & Gao, 2000), physics (e.g. Gaudin & Derrida, 1975), the theory of integrable systems (Buslaev & Fedotov, 2001) and diffraction theory (e.g. Maliuzhinets, 1958; Budaev, 1995; Croissille & Lebeau, 1999; Antipov & Silvestrov, 2004b,c) require the solution of scalar first-order difference equations or their systems. The difference equations of diffraction theory subject to certain symmetry conditions, also known as the Maliuzhinets equations, are always solvable in closed form if the equations are scalar. For systems of Maliuzhinets equations, a general procedure is not available. Recently, Antipov & Silvestrov (2004c, 2006) proposed an exact solution for systems of difference equations of the first order:

$$\boldsymbol{\Phi}(\sigma) = G(\sigma)\boldsymbol{\Phi}(\sigma - h) + \mathbf{g}(\sigma), \quad \sigma \in \Omega = \{\mathrm{Re}\, s = \omega\}, \tag{1.1}$$

subject to the symmetry condition

$$\boldsymbol{\Phi}(\omega + \mathrm{i}\tau) = \boldsymbol{\Phi}(\omega - h - \mathrm{i}\tau), \quad -\infty < \tau < \infty. \tag{1.2}$$

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The matrix coefficient $G(\sigma)$ has the following structure:

$$G(\sigma) = a_1(\sigma)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + a_2(\sigma)\begin{pmatrix}f_1(\sigma) & 1\\ f_2(\sigma) & -f_1(\sigma)\end{pmatrix}, \tag{1.3}$$

where $a_1(\sigma)$ and $a_2(\sigma)$ are Hölder functions and $f_1(s)$ and $f_2(s)$ are single-valued meromorphic h-periodic functions. The function $f_1(s)$ has a finite number of poles in the strip $\Pi = \{\omega - h < \operatorname{Re} s < h\}$. The function $f_1^2(s) + f_2(s)$ has a finite number of poles and zeros in the same strip. The key step of the procedure by Antipov and Silvestrov is the solution of the associated scalar Riemann–Hilbert problem on a Riemann surface and a Jacobi inversion problem.

In this paper, we propose a method for the solution of System (1.1, 1.2) when the entries of the matrix $G(s)$ are meromorphic, but the matrix $G(s)$ does not necessarily have the structure (1.3). The main idea is to extend the strip Π and reduce the system to a second-order difference equation for one of the unknown functions, say $\Phi_1(s)$, and to find the second function $\Phi_2(s)$ from System (1.1). In general, this reduction is not equivalent. That is why additional arbitrary constants appear which should be fixed afterwards. A second-order difference equation with T-periodic coefficients when the shift $h = T$ or $h = \frac{1}{2}T$ has been solved in closed form (Antipov & Silvestrov, 2004a,b). For arbitrary T and h, this equation can be converted into an integral equation with a fixed singularity. Different aspects of this equation associated with the diffraction problem for an impedance cone were analysed by Bernard (1997) and Antipov (2002). Lyalinov & Zhu (2003) used the generalized Maliuzhinets functions to analyse the governing second-order difference equation of a problem on E-polarization of a wedge with an impedance sheet on the symmetry plane. In their case, the fixed singularity at the ending point is quenched by the coefficient which vanishes at the end and therefore, the resulting integral equation is of the Fredholm type. A method of the Fredholm integral equation for approximate factorization of the matrix coefficient of a system of Wiener–Hopf equations equivalent to System (1.1, 1.2) was proposed by Daniele (2004).

In this paper, we focus our attention on the general case when the coefficient $a(x)$ in the resulting equation

$$a(x)\psi(x) + \frac{1}{\pi\,\mathrm{i}}\int_0^1 \frac{\psi(t)\mathrm{d}t}{1 - t^2x^2} = f(x), \quad 0 < x < 1, \tag{1.4}$$

has a finite non-zero limit $a(x) \to a_1$, $x \to 1$, $0 < |a_1| < \infty$, and therefore, (1.4) is singular. The dominant equation for (1.4) is the Dixon integral equation (Titschmarsh, 1937). It is equivalent to the convolution equation with the kernel $1\big/\cosh\frac{1}{2}(t-s)$ on a semi-axis solved by Krein (1958) by the Wiener–Hopf method using the Fourier transform. Duduchava (1979) developed the Noether theory for the singular integral equations with fixed singularities. It turns out that for the model diffraction problem considered in this paper, we need to analyse (1.4) in the case $a_1 = \frac{\mathrm{i}}{2}$ (the integral operator is not a Noether one). In the class of Hölder functions on the interval (0, 1) with at most integrable singularities at the ends, depending on the parameter a_1, the equation has either a unique solution (case 1) or two linearly independent solutions (case 2). If $f(x)$ is a Hölder function bounded at the ends, as $x \to 1$, the solution has either a power singularity $\left(a_1 \neq \pm\frac{\mathrm{i}}{2}\right)$: $\psi(x) \sim \mathrm{const}(1-x)^{-\mu}$ $\left(\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ or a power-logarithmic singularity $\left(a_1 = \pm\frac{\mathrm{i}}{2}\right)$: $\psi(x) \sim \mathrm{const}(1-x)^{\mp 1/2}\log(1-x)$.

In Section 2 of this paper, we show (Theorem 2.1) that if the matrix $A(s)$ given by (2.6) is not a zero matrix, then the governing system of difference equations (2.1–2.4) (main problem 2.1) is either uncoupled or the solution does not exist. This work concentrates on the more challenging case $A(s) \equiv 0$. In this case, Problem 2.1 reduces to a scalar difference equation of the second order (auxiliary problem

2.2) in an extended strip. Conditions under which Problems 2.1 and 2.2 are equivalent are summarized in Theorem 2.2.

In Section 3, the second-order difference equation (2.14, 2.15) is converted into an even scalar Riemann–Hilbert-type problem (3.3, 3.4). Theorem 3.1 establishes a link between Problem 2.2 and the integral equation with a fixed singularity (3.26). Its solution by the Mellin transform is derived in Section 4.1 and the solvability conditions are formulated in Theorem 4.1.

In Section 5, the model problem of electromagnetic diffraction of a plane wave from an impedance half-plane at skew incidence is solved. It is shown that the associated matrix $A(s)$ is a zero matrix. For the solution of the problem, two different approaches are proposed. The first one requires solution of a second-order difference equation for the first spectral function $\Phi_1(s)$. The second function $\Phi_2(s)$ is expressed through $\Phi_1(s)$ from the governing system of difference equations by (5.45). Initially, the solution possesses 12 arbitrary constants. They satisfy nine linear non-homogeneous algebraic equations. The other three equations follow from the analysis of the asymptotics of the function $\Phi_2(s)$ at infinity (the relation (5.55)). For practical implementation, we propose another approach that needs the solution of second-order difference equations in the extended strip for both functions $\Phi_1(s)$ and $\Phi_2(s)$. This method gives 24 arbitrary constants. Two of them are found explicitly from the physical conditions. For the other 22 constants, a system of 22 linear algebraic equations is derived. The key step of the procedure of this paper is the solution of the singular integral equation with a fixed singularity. A scheme for its numerical solution based on the method of orthogonal polynomials is proposed at the end of Section 5.

2. System of difference equations of the first order with meromorphic coefficients

Let Π be a strip $\{\omega - h < \operatorname{Re} s < \omega\}$ (ω is real, $h > 0$) in a complex s-plane. Denote the left and the right boundaries of the strip by $\Omega_{-1} = \{s \in \mathbb{C}\colon \operatorname{Re} s = \omega - h\}$ and $\Omega = \{s \in \mathbb{C}\colon \operatorname{Re} s = \omega\}$. Introduce also the left- and the right-hand side strips $\Pi_l = \{s \in \mathbb{C}\colon \omega - h < \operatorname{Re} s < \omega - \frac{1}{2}h\}$ and $\Pi_r = \{s \in \mathbb{C}\colon \omega - \frac{1}{2}h < \operatorname{Re} s < \omega\}$. Analyse the following problem.

PROBLEM 2.1 (Main problem) Find two functions $\Phi_1(s)$ and $\Phi_2(s)$ analytic in the strip Π, continuous up to the boundaries Ω_{-1} and Ω, apart from the simple poles $\alpha_m \in \Pi_l$, $m = 1, 2, \ldots, \nu$, with prescribed non-zero residues

$$\operatorname*{res}_{s=\alpha_m} \Phi_j(s) = r_{jm}, \quad m = 1, 2, \ldots, \nu, \tag{2.1}$$

satisfying the boundary condition

$$\boldsymbol{\Phi}(\sigma) = G(\sigma)\boldsymbol{\Phi}(\sigma - h), \quad \sigma \in \Omega, \tag{2.2}$$

where

$$\boldsymbol{\Phi}(s) = \begin{pmatrix} \Phi_1(s) \\ \Phi_2(s) \end{pmatrix}, \quad G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix}, \tag{2.3}$$

and the symmetry condition

$$\boldsymbol{\Phi}(s) = \boldsymbol{\Phi}(2\omega - h - s), \quad s \in \Pi \cup \Omega \cup \Omega_{-1}. \tag{2.4}$$

The matrix $G(\sigma)$ is not singular on the contour Ω. Its entries are meromorphic functions which have certain limits at infinity: $G_{mj}(\sigma) \sim G_{mj}(\omega \pm \mathrm{i}\infty)$ as $\sigma \to \omega \pm \mathrm{i}\infty$, $m, j = 1, 2$. The functions $\Phi_j(s)$ may have an exponential growth at infinity: $|\Phi_j(s)| \leqslant D_j\, \mathrm{e}^{\pm b_j \operatorname{Im} s}$ as $s \to s_1 \pm \mathrm{i}\infty$, $\omega - h < s_1 < \omega$, b_j are prescribed real numbers and D_j are constants.

REMARK 2.1 The symmetry condition (2.4) for $\omega = 2\pi \pm \pi$ and $h = 4\pi$ appears in two cases of Problem 2.1 associated with the diffraction problem analysed in Section 5. It is a consequence of the symmetry of the double-loop Sommerfeld contour (see Antipov & Silvestrov, 2006).

REMARK 2.2 For simplicity and because of the example considered in Section 5, the system of difference equations (2.2) is taken to be homogeneous. The non-homogeneous system can be treated similarly.

Because of the symmetry condition (2.4), the functions $\Phi_1(s)$ and $\Phi_2(s)$ have simple poles at the points $2\omega - h - \alpha_m \in \Pi_r \subset \Pi$. The matrix $G(s)$ is a meromorphic matrix function, and therefore by the analytical continuation, the boundary condition (2.2) can equivalently be written as a system of difference equations

$$\boldsymbol{\Phi}(s) = G(s)\boldsymbol{\Phi}(s-h), \quad s \in \mathbb{C}, \tag{2.5}$$

with respect to the meromorphic functions $\Phi_1(s)$ and $\Phi_2(s)$. Introduce a 2×2 matrix $A(s)$ with the entries

$$\begin{aligned}
A_{11}(s) &= G_{11}(2\omega - s)G_{11}(s) + G_{12}(2\omega - s)G_{21}(s) - 1,\\
A_{12}(s) &= G_{11}(2\omega - s)G_{12}(s) + G_{12}(2\omega - s)G_{22}(s),\\
A_{21}(s) &= G_{21}(2\omega - s)G_{11}(s) + G_{22}(2\omega - s)G_{21}(s),\\
A_{22}(s) &= G_{21}(2\omega - s)G_{12}(s) + G_{22}(2\omega - s)G_{22}(s) - 1.
\end{aligned} \tag{2.6}$$

THEOREM 2.1 If $A(s) \not\equiv 0$, then either (2.5) is uncoupled or the solution to Problem 2.1 does not exist.

Proof. Replacing s by $2\omega - h - s$ in (2.5) and employing the symmetry condition (2.4),

$$\boldsymbol{\Phi}(s) = G(2\omega - h - s)\boldsymbol{\Phi}(s+h). \tag{2.7}$$

By analytical continuation, $\boldsymbol{\Phi}(s+h) = G(s+h)\boldsymbol{\Phi}(s)$ and (2.7) becomes

$$\boldsymbol{\Phi}(s) = G(2\omega - h - s)G(s+h)\boldsymbol{\Phi}(s). \tag{2.8}$$

Therefore, if the vector $\boldsymbol{\Phi}(s)$ solves (2.5) and satisfies the symmetry condition (2.4), then necessarily

$$\begin{aligned}
A_{11}(s+h)\Phi_1(s) + A_{12}(s+h)\Phi_2(s) = 0,\\
A_{21}(s+h)\Phi_1(s) + A_{22}(s+h)\Phi_2(s) = 0,
\end{aligned} \tag{2.9}$$

where the functions $A_{ij}(s)$ are given by (2.6). Clearly, a non-trivial solution to the system of equations (2.9) exists if

$$A_{11}(s)A_{22}(s) - A_{12}(s)A_{21}(s) \equiv 0. \tag{2.10}$$

Assume that the determinant of System (2.9) equals 0 for all s and $A_{11}(s) \neq 0$ and $A_{12}(s) \neq 0$. Then,

$$\Phi_2(s) = -\frac{A_{11}(s+h)}{A_{12}(s+h)}\Phi_1(s), \tag{2.11}$$

and the function $\Phi_1(s)$ solves the difference equation of the first order

$$\begin{aligned}
\Phi_1(s) &= A_0(s)\Phi_1(s-h),\\
A_0(s) &= \left[G_{11}(s) - \tfrac{A_{11}(s)G_{12}(s)}{A_{12}(s)}\right].
\end{aligned} \tag{2.12}$$

The symmetry condition (2.4) applied to (2.11) and (2.12) yields

$$A_{11}(s+h)A_{12}(2\omega-s) = A_{12}(s+h)A_{11}(2\omega-s), \quad A_0(s+h)A_0(2\omega-h-s) = 1. \tag{2.13}$$

If the matrix $G(s)$ fails to meet these conditions, then the solution does not exist. If $A_{12}(s) \neq 0$ and $A_{11}(s) = 0$, then $\Phi_2(s) = 0$, and the function $\Phi_2(s)$ cannot have the non-zero residues at the poles $s = \alpha_m$. The solution does not exist. The other possible cases are analysed similarly. □

It turns out that in the diffraction problem considered in this work the matrix $A(s) \equiv 0$, and (2.5) is coupled. In what follows, we assume that all the entries of the matrix $A(s)$ are equal to zero.

Let $\hat{\Pi} = \{s \in \mathbb{C}\colon \hat{\omega} - 2h < \operatorname{Re} s < \hat{\omega}\}$, where $\omega < \hat{\omega} < \omega + h$. Analyse next the following problem associated with Problem 2.1.

PROBLEM 2.2 (Auxiliary problem) Find a function $\phi(s)$ analytic in the strip $\hat{\Pi}$ continuous up to the boundaries $\hat{\Omega}_{-1} = \{s \in \mathbb{C}\colon \operatorname{Re} s = \hat{\omega} - 2h\}$ and $\hat{\Omega} = \{s \in \mathbb{C}\colon \operatorname{Re} s = \hat{\omega}\}$, apart from the poles $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_{2\hat{\nu}} \in \hat{\Pi}$ ($\hat{\alpha}_{j+\hat{\nu}} = 2\omega - h - \hat{\alpha}_j$, $j = 1, 2, \ldots, \hat{\nu}$), satisfying the boundary condition

$$\phi(\sigma) = b(\sigma)\phi(\sigma - h) + c(\sigma)\phi(\sigma - 2h), \quad \sigma \in \hat{\Omega}, \tag{2.14}$$

and the symmetry condition

$$\phi(s) = \phi(2\omega - h - s). \tag{2.15}$$

On the contour $\hat{\Omega}$, $b(\sigma)$ and $c(\sigma)$ are Hölder-continuous functions and $c(\sigma) \neq 0$. As $\sigma \to \hat{\omega} \pm \mathrm{i}\infty$, the function $b(\sigma)$ is bounded and $c(\sigma) \to c(\hat{\omega} \pm \mathrm{i}\infty)$. As $s \to s_1 \pm \mathrm{i}\infty$, $\hat{\omega} - 2h < s_1 < \hat{\omega}$, the function $\phi(s)$ may grow: $|\phi(s)| \leqslant D_0\, \mathrm{e}^{\pm b_1 \operatorname{Im} s}$, $D_0 =$ constant.

Let $s = \alpha_j^*$ ($j = 1, 2, \ldots, \nu^*$) be the zeros of the function $G_{11}(s)$ in the strip $\omega \leqslant \operatorname{Re} s \leqslant \hat{\omega}$ and $s = \delta_m$ ($m = 1, 2, \ldots, \rho$) be the poles of the function

$$\Lambda(s) = \frac{G_{22}(s)}{G_{12}(s)}\Phi_1(s) + \left[G_{21}(s) - \frac{G_{11}(s)}{G_{12}(s)}G_{22}(s)\right]\Phi_1(s-h) \tag{2.16}$$

in the strip Π. Assume that all the poles α_j^* and δ_m are simple. The following theorem establishes the conditions when the solution to Problem 2.1 can be expressed through the solution to Problem 2.2.

THEOREM 2.2 If the matrix $A(s)$ is a zero matrix, then

$$\Phi_1(s) = \phi(s), \quad \Phi_2(s) = \frac{\phi(s+h) - G_{11}(s+h)\phi(s)}{G_{12}(s+h)}, \tag{2.17}$$

provided the functions $b(s)$ and $c(s)$ are selected as follows:

$$\begin{aligned} b(s) &= G_{11}(s) + \frac{G_{12}(s)G_{22}(s-h)}{G_{12}(s-h)}, \\ c(s) &= G_{12}(s)\left[G_{21}(s-h) - \frac{G_{11}(s-h)}{G_{12}(s-h)}G_{22}(s-h)\right], \end{aligned} \tag{2.18}$$

the set of poles $\hat{\alpha}_m$ ($m = 1, 2, \ldots, 2\hat{\nu}$, $\hat{\nu} = 2\nu + \nu^*$) consists of the points α_m, $2\omega - h - \alpha_m$, $\alpha_m + h$, $2\omega - 2h - \alpha_m$ ($m = 1, 2, \ldots, \nu$), $\alpha_j^* - h$ and $2\omega - \alpha_j^*$ ($j = 1, 2, \ldots, \nu^*$), and the following conditions hold:

$$\operatorname*{res}_{s=\delta_m} \Lambda(s) = 0, \quad m = 1, 2, \ldots, \rho, \tag{2.19}$$

$$\frac{\phi(s+h) - G_{11}(s+h)\phi(s)}{G_{12}(s+h)} = \mathrm{O}(\mathrm{e}^{\pm b_2 \operatorname{Im} s}), \quad s \to s_1 \pm \mathrm{i}\infty, \;\; \omega - h < s_1 < \omega. \tag{2.20}$$

Proof. Eliminating the function $\Phi_2(s-h)$ from System (2.2) gives

$$\Phi_2(s) = \Lambda(s). \tag{2.21}$$

The analytical continuation of (2.16) to the left yields $\Phi_2(s-h) = \Lambda(s-h)$. Now, substitute the function $\Phi_2(s-h)$ into the first equation in System (2.2) to get the second-order difference equation

$$\Phi_1(\sigma) = b(\sigma)\Phi_1(\sigma - h) + c(\sigma)\Phi_1(\sigma - 2h), \quad \sigma \in \hat{\Omega}, \tag{2.22}$$

with the coefficients (2.18) subject to the symmetry condition $\Phi_1(s) = \Phi_1(2\omega - h - s)$. Verify next the symmetry condition for the function $\Phi_2(s)$ given by (2.17). From (2.21) and the symmetry condition for the function $\Phi_1(s)$, it follows

$$\begin{aligned}\Phi_2(2\omega - h - s) = {} & \frac{G_{22}(2\omega - h - s)}{G_{12}(2\omega - h - s)}\Phi_1(s) \\ & + \left[G_{21}(2\omega - h - s) - \frac{G_{11}(2\omega - h - s)}{G_{12}(2\omega - h - s)}G_{22}(2\omega - h - s)\right]\Phi_1(s+h).\end{aligned} \tag{2.23}$$

The analytical continuation of (2.22) to the right recovers the function $\Phi_1(s+h)$:

$$\Phi_1(s+h) = b(s+h)\Phi_1(s) + c(s+h)\Phi_1(s-h). \tag{2.24}$$

By making use of formulas (2.18) and also since all the functions $A_{ij}(s) \equiv 0$ $(i, j = 1, 2)$, we transform the right-hand side of relation (2.23) into

$$\frac{\Phi_1(s+h) - G_{11}(s+h)\Phi_1(s)}{G_{12}(s+h)}, \tag{2.25}$$

that coincides with (2.17) if $\Phi_1(s) = \phi(s)$. Thus, the function $\Phi_2(s)$ meets the symmetry condition. Finally, identify all the poles of the function $\Phi_1(s)$ in the extended strip $\hat{\Pi}$. The function $\Phi_1(s)$ solves Problem 1.1 and that is why it has to have the prescribed poles $\alpha_m \in \Pi$ and $2\omega - h - \alpha_m$ $(m = 1, 2, \ldots, \nu)$. Because of the first equation in (2.2),

$$\Phi_1(s) = \frac{\Phi_1(s+h) - G_{12}(s+h)\Phi_2(s)}{G_{11}(s+h)}, \quad \omega - h \leqslant \operatorname{Re} s \leqslant \hat{\omega} - h. \tag{2.26}$$

Therefore, the function $\Phi_1(s)$ has poles at the points $\alpha_m + h$ $(m = 1, 2, \ldots, \nu)$ and at the zeros $s = \alpha_j^* - h$ $(j = 1, 2, \ldots, \nu^*)$ of the function $G_{11}(s+h)$ in the domain $\omega - h \leqslant \operatorname{Re} s \leqslant \hat{\omega} - h$. The symmetry condition (2.15) yields the other poles $2\omega - 2h - \alpha_m$ $(m = 1, 2, \ldots, \nu)$ and $2\omega - \alpha_j^*$ $(j = 1, 2, \ldots, \nu^*)$. Note that the function (2.16) and therefore the function $\Phi_2(s)$ have ρ simple poles in the strip Π. Conditions (2.19) remove them. The function $G_{12}(s)$ may exponentially decay as $s \to \infty$ and $\operatorname{Re} s$ is finite that causes an unacceptable growth of the function $\Phi_2(s)$. Thus, the function $\phi(s)$ has to meet Condition (2.20). This may bring, say, μ extra conditions. The theorem is proved. □

3. Riemann–Hilbert-type problem

To solve Problem 2.2, we map it to a boundary-value problem on a plane. The conformal mapping

$$z = -\mathrm{i}\tan\frac{\pi}{2h}(s-\hat{\omega}), \quad s = \hat{\omega} + \frac{\mathrm{i}h}{\pi}\log\frac{1+z}{1-z} \tag{3.1}$$

transforms the extended s-strip $\hat{\Pi}$ onto a z-plane cut along the segment $[-1, 1]$. A single branch of the logarithmic function is fixed such that the function $\log[(1+z)(1-z)^{-1}]$ is real on the upper side of the cut $[-1, 1]^+$. The mapping $z = z(s)$ transforms the left and the right boundaries of the strip, the contours $\hat{\Omega}_{-1}$ and $\hat{\Omega}$, into the lower and the upper banks of the cut, respectively. This means that the image of a point $\sigma \in \hat{\Omega}$ is a point $t^+ \in [-1, 1]^+$ and a point $\sigma \in \hat{\Omega}_{-1}$ is mapped into a point $t^- \in [-1, 1]^-$. Choose the parameter $\hat{\omega} \in (\omega, \omega + h)$ such that $z = -t^-$ as $s = 2\omega - h - \sigma$ and $\sigma \in \hat{\Omega}$. This gives $\hat{\omega} = \omega + \frac{1}{2}h$. Then, for any point $s \in \hat{\Pi}$, the point $2\omega - h - s \in \hat{\Pi}$ is mapped into a point $-z$ of the z-plane. Also note that if a point $\sigma \in \Omega_0 = \{s \in \hat{\Pi} : \mathrm{Re}\, s = \hat{\omega} - h\}$, then its image is the point $z = t^{-1}$, $t \in (-1, 1)$. Let now

$$F(z) = \phi(s(z)), \quad s \in \hat{\Pi},$$
$$B(t) = b(\sigma), \quad C(t) = c(\sigma), \quad \sigma \in \hat{\Omega}, \quad \sigma = \hat{\omega} + \frac{\mathrm{i}h}{\pi}\log\frac{1+t}{1-t}. \tag{3.2}$$

Then, the difference equation (2.14) becomes

$$F^+(t) = C(t)F^-(t) + B(t)F\left(\frac{1}{t}\right), \quad t \in (-1, 1). \tag{3.3}$$

The condition of symmetry (2.15) requires the function $F(z)$ to be even:

$$F(z) = F(-z). \tag{3.4}$$

Now since $F^+(t) = F^-(-t)$, the boundary values of the function $F(z)$ have to satisfy the condition

$$F^+(t) = \frac{F^-(t)}{C(-t)} - \frac{B(-t)}{C(-t)}F\left(-\frac{1}{t}\right), \quad t \in (-1, 1), \tag{3.5}$$

which follows from (3.3). Comparison of Conditions (3.3) and (3.5) gives the two necessary conditions

$$C(t)C(-t) = 1, \quad B(t) + C(t)B(-t) = 0 \tag{3.6}$$

for an even solution $F(z)$ to exist. Consider next the Cauchy integral

$$X_0(z) = \exp\left\{\frac{1}{2\pi\,\mathrm{i}}\int_{-1}^{1}\frac{\log C(t)\mathrm{d}t}{t-z}\right\}, \tag{3.7}$$

where a single branch of the logarithmic function is fixed by the condition $\log C(0) = \mathrm{i}\gamma$, $-\pi < \gamma \leqslant \pi$ ($|C(0)| = 1$). From (3.6), it follows

$$\arg C(t) + \arg C(-t) = 2\pi n, \tag{3.8}$$

where n is an integer. Particularly, for $t = 0$, $\arg C(0) = \gamma = \pi n$. Thus, $n = 0$ or $n = 1$. Introduce the real number

$$\lambda = \frac{1}{2\pi}[\arg C(t)]_{t=0}^{t=1} = \frac{1}{2\pi}[\arg c(\sigma)]_{\sigma=\hat{\omega}}^{\sigma=\hat{\omega}+\mathrm{i}\infty}. \tag{3.9}$$

Then, $\arg C(t)$ at the ending points can be represented in the form

$$\arg C(\pm 1) = \gamma \pm 2\pi\lambda. \tag{3.10}$$

Therefore, near the ends, the Cauchy integral (3.7) is described by

$$X_0(z) \sim D_\pm (z \mp 1)^{\lambda \pm n/2 \mp \mathrm{i}\mu_\pm}, \quad z \to \pm 1, \quad D_\pm = \text{constant}, \tag{3.11}$$

where

$$\mu_\pm = \frac{1}{2\pi} \log |C(\pm 1)| = \frac{1}{2\pi} \log |c(\hat{\omega} \pm \mathrm{i}\infty)|. \tag{3.12}$$

Because of the properties (3.6) and (3.8), the function $X_0(z)$ reduces to

$$X_0(z) = \left(\frac{z}{z+1}\right)^n \exp\left\{\frac{1}{\pi\,\mathrm{i}}\int_0^1 \frac{\log C(t) t\,\mathrm{d}t}{t^2 - z^2}\right\}. \tag{3.13}$$

This function solves the factorization problem $C(t) = X_0^+(t)[X_0^-(t)]^{-1}$, $-1 < t < 1$. To use this factorization for solving the boundary-value problem (3.3), one needs to make sure that the factors $X_0^\pm$ have the required asymptotics at the ends. The class of solutions for $\phi(s)$ is fixed by the condition at the infinite points: $\phi(s) = \mathrm{O}(\mathrm{e}^{\pm b_1 \operatorname{Im} s})$, $\operatorname{Im} s \to \pm\infty$, $\operatorname{Re} s$ is finite. Therefore,

$$F(z) = \mathrm{O}((1 \mp z)^{-hb_1/\pi}), \quad z \to \pm 1. \tag{3.14}$$

Let now $\lambda_0 = \lambda + hb_1/\pi$ and $\kappa = \left[\lambda_0 + \frac{1}{2}n\right]$ $(n = 0, 1)$, where $[a]$ is the entire part of a number a. Then, the new function

$$X(z) = (z+1)^n (z^2 - 1)^{-\kappa} X_0(z) = z^n (z^2 - 1)^{-\kappa} \exp\left\{\frac{1}{\pi\,\mathrm{i}}\int_0^1 \frac{\log C(t) t\,\mathrm{d}t}{t^2 - z^2}\right\} \tag{3.15}$$

also factorizes the function $C(t)$: $C(t) = X^+(t)[X^-(t)]^{-1}$, $-1 < t < 1$. At the ends of the interval $[-1, 1]$,

$$X(z) \sim D'_\pm (z \mp 1)^{\lambda + n/2 - \kappa \mp \mathrm{i}\mu_\pm}, \quad z \to \pm 1, \quad D'_\pm = \text{constant}. \tag{3.16}$$

From (3.15), it follows that if $\arg C(0) = \pi$, then the factorization is odd, and if $\arg C(0) = 0$, the function $X(z)$ is even. According to the choice of the class of the coefficients $G_{ij}(s)$ $(i, j = 1, 2)$, the function $B(t)$ is bounded at the ends $t = \pm 1$ and the function $B(t)F(t^{-1})[X^+(t)]^{-1}$ may have an integrable singularity

$$\left|\frac{B(t)F(1/t)}{X^+(t)}\right| \leqslant \hat{D}|t \mp 1|^{\kappa - \lambda_0 - n/2}, \quad t \to \pm 1, \quad \hat{D} = \text{constant}. \tag{3.17}$$

Introduce next the Cauchy integral

$$\Psi(z) = \frac{1}{2\pi\,\mathrm{i}} \int_{-1}^{1} \frac{B(t)F(1/t)\mathrm{d}t}{X^{+}(t)(t-z)}. \tag{3.18}$$

Because of the relations

$$\frac{1}{X^{+}(-t)} = (-1)^n \frac{C(t)}{X^{+}(t)}, \quad B(-t) = -\frac{B(t)}{C(t)}, \quad F\left(-\frac{1}{t}\right) = F\left(\frac{1}{t}\right), \tag{3.19}$$

the integral (3.18) can be transformed into the form

$$\Psi(z) = \frac{1}{\pi\,\mathrm{i}} \int_{0}^{1} \frac{B(t)F(1/t)t^{1-n}z^n\,\mathrm{d}t}{X^{+}(t)(t^2-z^2)}. \tag{3.20}$$

By the generalized Liouville's theorem, we have

$$F(z) = X(z)[\Psi(z) + R(z) + Q(z)], \tag{3.21}$$

where $R(z)$ is a rational function

$$R(z) = \sum_{j=1}^{\hat{\nu}} \frac{z^n E_j}{z^2 - \zeta_j^2}, \quad \zeta_j = -\mathrm{i}\tan\frac{\pi}{2h}(\hat{\alpha}_j - \hat{\omega}), \tag{3.22}$$

E_j $(j = 1, 2, \ldots, \hat{\nu})$ are arbitrary constants. If $n = 0$, then $Q(z)$ is an even polynomial of degree 2κ with $\kappa + 1$ arbitrary coefficients and $Q(z) \equiv 0$ in the case $\kappa < 0$. If $n = 1$, then $Q(z)$ is an odd polynomial of degree $2\kappa - 1$ with κ arbitrary coefficients and $Q(z) \equiv 0$ in the case $\kappa \leqslant 0$. Analysis of formula (3.21) shows that regardless of whether $n = 0$ or $n = 1$ the function $F(z)$ is even.

The representation of the solution (3.21) involves the values of function $F(t)$ for $t \in (1, \infty)$ which are unknown. Note that the left- and the right-hand sides of (3.21) are analytic in $\mathbb{C} \setminus [-1, 1]$. Therefore, it is sufficient for the equation to be held on any curve in the z-plane cut along the segment $[-1, 1]$. Choose that curve as the contour $(1, \infty)$. Then, (3.21) is satisfied everywhere in $\mathbb{C} \setminus [-1, 1]$. This means that Problem 2.2 is equivalent to the integral equation

$$F\left(\frac{1}{x}\right) = X\left(\frac{1}{x}\right)\left[\frac{1}{\pi\,\mathrm{i}} \int_{0}^{1} \frac{B(t)F(1/t)t^{1-n}x^{-n}\,\mathrm{d}t}{X^{+}(t)(t^2 - x^{-2})} + R\left(\frac{1}{x}\right) + Q\left(\frac{1}{x}\right)\right], \quad 0 < x < 1, \tag{3.23}$$

and the following theorem has been proved.

Let $H^{\nu,\mu}[\alpha, \beta]$ be the space of Hölder functions $\varphi(x)$ on the interval (α, β) which may have power singularities at the ends:

$$|\varphi(x)| \leqslant d_0|x-\alpha|^{\nu}, \;\; x \to \alpha; \quad |\varphi(x)| \leqslant d_1|x-\beta|^{\mu}, \;\; x \to \beta, \tag{3.24}$$

where ν and μ are finite real numbers and d_0 and d_1 are constants.

THEOREM 3.1 If the function

$$\psi(x) = \frac{B(x)}{X^{+}(x)} F\left(\frac{1}{x}\right) \in H^{0,\kappa-\lambda_0-n/2}[0, 1] \tag{3.25}$$

solves the integral equation

$$a(x)\psi(x) + \frac{1}{\pi\,\mathrm{i}}\int_0^1 \frac{\psi(t)x^{2-n}t^{1-n}\,\mathrm{d}t}{1-t^2x^2} = f(x), \quad 0 < x < 1, \tag{3.26}$$

where

$$a(x) = \frac{X^+(x)}{B(x)X(1/x)}, \quad f(x) = R\left(\frac{1}{x}\right) + Q\left(\frac{1}{x}\right), \tag{3.27}$$

then the even function $F(z) = \phi(s(z))$ given by (3.21) solves Problem 2.2.

4. Integral equation with a fixed singularity

Consider the integral equation (3.26) for the case $\kappa - \lambda_0 - n/2 \in (-1, 1)$, i.e. in the space of Hölder functions $\psi(x)$ on the open interval $(0, 1)$, bounded at the point $x = 0$ and admitting an integrable singularity at the end $x = 1$. This space is the one of interest for the physical problem analysed in Section 5. The procedure should be changed accordingly if the parameter $\kappa - \lambda_0 - n/2 \in (m-1, m+1)$, $m \neq 0$. The function $f(x)$ is chosen to be a Hölder function on $(0, 1)$ bounded when $x \to 0$ and $x \to 1$ as it is in (3.26).

If $a(x) \to \infty$, $x \to 1$, then (3.26) is a Fredholm integral equation. This case appears in the diffraction problem considered by Lyalinov & Zhu (2003).

Let $a(x) \to a_1$, $x \to 1$, $0 < |a_1| < \infty$. In this case, the equation can be regularized by the Carleman–Vekua procedure (Gakhov, 1966). Rewrite (3.26) in the form

$$a_1\psi(x) + \frac{1}{2\pi\,\mathrm{i}}\int_0^1 \frac{\psi(t)\mathrm{d}t}{2-t-x} + T\psi(x) = f(x), \quad 0 < x < 1, \tag{4.1}$$

where

$$\begin{aligned} T\psi(x) &= \frac{1}{2\pi\,\mathrm{i}}\int_0^1 V(x,t)\psi(t)\mathrm{d}t + [a(x) - a_1]\psi(x), \\ V(x,t) &= \frac{2x^{2-n}t^{1-n}}{1-t^2x^2} - \frac{1}{2-t-x}. \end{aligned} \tag{4.2}$$

Clearly, T is a Fredholm operator. Indeed, as $t \to 1$, $V(1,t) \sim n - \frac{1}{2}$ and $V(x,1) \sim n - \frac{3}{2}$ when $x \to 1$. Making the substitution $t = 1 - \tau$, $x = 1 - \xi$ transforms (4.1) to

$$a_1\tilde{\psi}(\xi) + \frac{1}{2\pi\,\mathrm{i}}\int_0^1 \frac{\tilde{\psi}(\tau)\mathrm{d}\tau}{\tau+\xi} = \tilde{f}(\xi), \quad 0 < \xi < 1, \tag{4.3}$$

where $\tilde{\psi}(\xi) = \psi(1-\xi)$ and $\tilde{f}(\xi) = f(1-\xi) - T\psi(1-\xi)$. This equation can be solved exactly by the Wiener–Hopf method by using the Mellin transform (see also the solution by Krein (1958) obtained by the Fourier transform in a different form). Extend the equation for the whole semi-axis $0 < \xi < \infty$:

$$a_1\tilde{\psi}^-(\xi) + \frac{1}{2\pi\,\mathrm{i}}\int_0^\infty \frac{\tilde{\psi}^-(\tau)}{1+\xi/\tau}\frac{\mathrm{d}\tau}{\tau} = \tilde{f}^-(\xi) + \tilde{\psi}^+(\xi), \quad 0 < \xi < \infty, \tag{4.4}$$

where the functions $\tilde{\psi}^-(x)$ and $\tilde{f}^-(x)$ coincide with $\tilde{\psi}(x)$ and $\tilde{f}(x)$ for $0 \leqslant x \leqslant 1$ and $\operatorname{supp}\{\tilde{\psi}^-, \tilde{f}^-\} \subset [0, 1]$. The function $\tilde{\psi}^+(x)$ is unknown on the interval $[1, \infty)$ and $\operatorname{supp} \tilde{\psi}^+ \subset [1, \infty)$. Apply next the Mellin transform and map the integral equation into the Riemann–Hilbert problem

$$K(p)\tilde{\Psi}^-(p) = \tilde{F}^-(p) + \tilde{\Psi}^+(p), \quad \Gamma\colon \operatorname{Re} p = \varepsilon \in (\max\{\varepsilon_0, 0\}, 1), \tag{4.5}$$

where $\tilde{\Psi}^\pm(p)$ and $\tilde{F}^-(p)$ are the Mellin transforms of the functions $\tilde{\psi}^\pm(\xi)$ and $\tilde{f}^-(\xi)$, respectively, defined by

$$H(p) = \int_0^\infty h(\xi)\xi^{p-1}\,\mathrm{d}\xi, \tag{4.6}$$

ε_0 is the real part of the zero of the function

$$K(p) = a_1 - \frac{\mathrm{i}}{2\sin\pi p} \tag{4.7}$$

lying in the half-plane $\operatorname{Re} p < 1$ and whose real part is the closest one to $p = 1$ among all its zeros in this half-plane.

Introduce next a parameter q:

$$a_1 = \frac{\mathrm{i}}{2\sin\pi q}, \quad q \neq 0, \quad -\frac{1}{2} \leqslant \operatorname{Re} q \leqslant \frac{1}{2}, \quad -\infty < \operatorname{Im} q < \infty. \tag{4.8}$$

Consider two cases:

1. $-\frac{1}{2} \leqslant \operatorname{Re} q \leqslant 0$, $-\infty < \operatorname{Im} q < \infty$ ($q = 0$ is excluded). Then, the coefficient of the Riemann–Hilbert problem (4.5) can be factorized in terms of the Γ-function as follows:

$$K(p) = \frac{K^+(p)}{K^-(p)}, \quad p \in \Gamma, \tag{4.9}$$

where

$$\begin{aligned} K^+(p) &= \frac{a_1\Gamma(1-p/2)\Gamma(1/2-p/2)}{\Gamma(1+q/2-p/2)\Gamma(1/2-q/2-p/2)} \sim a_1, \quad p \to \infty, \quad p \in \mathcal{D}^+, \\ K^-(p) &= \frac{\Gamma(p/2-q/2)\Gamma(1/2+q/2+p/2)}{\Gamma(p/2)\Gamma(1/2+p/2)} \sim 1, \quad p \to \infty, \quad p \in \mathcal{D}^-, \end{aligned} \tag{4.10}$$

where $\mathcal{D}^+ = \{\operatorname{Re} p < \varepsilon\}$ and $\mathcal{D}^- = \{\operatorname{Re} p > \varepsilon\}$. Since the functions $\tilde{\psi}^\pm(\xi)$ are bounded at $\xi = 1$, the Mellin transforms $\tilde{\Psi}^\pm(p)$ vanish at infinity: $\tilde{\Psi}^\pm(p) = \mathrm{O}(p^{-1})$, $p \to \infty$, $p \in \mathcal{D}^\pm$. By the Liouville's theorem, the solution to the Riemann–Hilbert problem (4.5) is unique:

$$\begin{aligned} \tilde{\Psi}^\pm(p) &= -K^\pm(p)\Omega^\pm(p), \quad p \in \mathcal{D}^\pm, \\ \Omega^\pm(p) &= \frac{1}{2\pi\,\mathrm{i}}\int_\Gamma \frac{\tilde{F}^-(p_0)\,\mathrm{d}p_0}{K^+(p_0)(p_0-p)}, \quad p \in \mathcal{D}^\pm. \end{aligned} \tag{4.11}$$

2. $0 < \operatorname{Re} q \leqslant \frac{1}{2}$, $-\infty < \operatorname{Im} q < \infty$. Then, the factors $K^+(p)$ and $K^-(p)$ in (4.9) become

$$\begin{aligned} K^+(p) &= -\frac{a_1 \Gamma(1-p/2)\Gamma(1/2-p/2)}{\Gamma(1+q/2-p/2)\Gamma(3/2-q/2-p/2)} \sim \frac{a_1}{p}, \quad p \to \infty, \quad p \in \mathcal{D}^+, \\ K^-(p) &= \frac{\Gamma(p/2-q/2)\Gamma(-1/2+q/2+p/2)}{\Gamma(p/2)\Gamma(1/2+p/2)} \sim \frac{1}{p}, \quad p \to \infty, \quad p \in \mathcal{D}^-. \end{aligned} \tag{4.12}$$

The solution to the Riemann–Hilbert problem is not unique in this case:

$$\tilde{\Psi}^\pm(p) = K^\pm(p)[M - \Omega^\pm(p)], \quad p \in \mathcal{D}^\pm, \tag{4.13}$$

where M is an arbitrary constant. The solution to the integral equation (4.3) is recovered by the inverse Mellin transform

$$\tilde{\psi}(\xi) = \frac{1}{2\pi\,\mathrm{i}} \int_\Gamma \tilde{\Psi}^-(p)\xi^{-p}\,\mathrm{d}p. \tag{4.14}$$

Formulas (4.11) and (4.13) indicate that in the prescribed class the integral equation (3.26) is uniquely solvable in case 1 and has two linearly independent solutions in case 2.

To find the asymptotics of the solution to (4.1) at the point $x = 1$, analyse the behaviour of the function $\tilde{\psi}(\xi)$ at the point $\xi = 0$. The use of the analytical continuation yields

$$\tilde{\psi}(\xi) = \frac{1}{2\pi\,\mathrm{i}} \int_\Gamma \{K^+(p)[M - \Omega^+(p)] + \tilde{F}^-(p)\} \frac{\sin \pi p}{a_1 \sin \pi p - \mathrm{i}/2} \xi^{-p}\,\mathrm{d}p \tag{4.15}$$

($M = 0$ in case 1). If $a_1 \neq \pm\frac{\mathrm{i}}{2}$, by the Cauchy theorem,

$$\tilde{\psi}(\xi) \sim E_0 \xi^{-\mu}, \quad \xi \to 0, \tag{4.16}$$

where $\mu = \frac{1}{\pi}\sin^{-1} \mathrm{i}/(2a_1)$. In the case $a_1 = \pm\frac{\mathrm{i}}{2}$, the kernel $K(p)$ has a zero $p = \mp\frac{1}{2}$ of the second multiplicity and therefore,

$$\tilde{\psi}(\xi) \sim E_0 \xi^{\mp 1/2} + E_1 \xi^{\mp 1/2} \log \xi, \quad \xi \to 0, \ a_1 = \pm\frac{\mathrm{i}}{2}, \tag{4.17}$$

where E_0 and E_1 are certain constants. This asymptotics can directly be verified from the analysis of the integral equation (4.1). Let $a_1 = \frac{\mathrm{i}}{2}$. Assume that

$$\psi(x) \sim E'(1-x)^\alpha + E''(1-x)^\alpha \log(1-x), \quad x \to 1-0, \ -1 < \alpha < 1, \tag{4.18}$$

where E' and E'' are constants. Next, evaluate the singular integrals

$$\begin{aligned} V(\xi;\alpha) &= \int_0^1 \frac{\tau^\alpha\,\mathrm{d}\tau}{\tau+\xi} = -\frac{\pi\xi^\alpha}{\sin \pi\alpha} + \sum_{j=0}^\infty \frac{(-\xi)^j}{\alpha - j}, \quad 0 < \xi < 1, \\ \int_0^1 \frac{\tau^\alpha \log\tau\,\mathrm{d}\tau}{\tau+\xi} &= \lim_{\beta\to 0} \frac{V(\xi;\alpha+\beta) - V(\xi;\alpha)}{\beta} \\ &= -\frac{\pi\xi^\alpha}{\sin\pi\alpha}(\log\xi - \pi\cot\pi\alpha) - \sum_{j=0}^\infty \frac{(-\xi)^j}{(\alpha-j)^2}, \quad 0 < \xi < 1. \end{aligned} \tag{4.19}$$

Since $a(x) \sim \frac{\mathrm{i}}{2}$, $x \to 1-0$, by using the above integrals, it is obtained from (4.3) that

$$E'\xi^{\alpha}\left(1+\frac{1}{\sin\pi\alpha}-\frac{\pi\cot\pi\alpha}{\sin\pi\alpha}\right)+E''\xi^{\alpha}\log\xi\left(1+\frac{1}{\sin\pi\alpha}\right)$$
$$= \mathrm{O}(1), \quad \xi\to+0 \ \ (x\to 1-0). \tag{4.20}$$

Clearly, these relations are satisfied if the parameter $\alpha \in (-1, 1)$ is chosen to be $\alpha = -\frac{1}{2}$.

Finally note that for the second case, when $a_1 = \frac{\mathrm{i}}{2}$ $\left(q = \frac{1}{2}\right)$, the solution is unique in the class $H^{-1/2,-1/2}[-1, 1]$. For the solution to be within this class, the arbitrary constant M must be fixed by the condition

$$M = \Omega^{+}\left(\frac{1}{2}\right) - \frac{\tilde{F}^{-}\left(\frac{1}{2}\right)}{K^{+}\left(\frac{1}{2}\right)}, \tag{4.21}$$

where $\tilde{F}^{-}(p)$ is bounded at the point $p = \frac{1}{2}$ since it is analytic in the half-plane $\mathrm{Re}\, p > 0$.

Let now $a(x) \to 0$ as $x \to 1$. Analysis of the Cauchy integral as $x \to 1$ shows that the asymptotics of the left- and right-hand sides as $x \to 1$ are not compatible. Thus, in the space $H^{0,\mu}(0, 1)$ $(-1 < \mu < 1)$, the solution does not exist. This result also follows from the fact that if $a_1 = 0$, then the dominant integral equation (4.3) is not solvable in the class of Hölder functions with integrable singularities at the ending points.

Let us now summarize the derived results.

THEOREM 4.1 If $a(x) \to \infty$, $x \to 1$, then (3.26) is a Fredholm integral equation.

If $a(x) \to 0$, $x \to 1$, then the solution of (3.26) does not exist.

Let $a(x) \to a_1$, $x \to 1$, $0 < |a_1| < \infty$ and $a_1 = \mathrm{i}(2\sin\pi q)^{-1}$. Then, the dominant integral equation (4.3) in the space $H^{0,\mu}(0, 1)$ $(-1 < \mu < 1)$ has a unique solution (4.11, 4.14) if $-\frac{1}{2} \leqslant \mathrm{Re}\, q \leqslant 0$, $-\infty < \mathrm{Im}\, q < \infty$ $(q \neq 0)$ (case 1). If $0 < \mathrm{Re}\, q \leqslant \frac{1}{2}$, $-\infty < \mathrm{Im}\, q < \infty$ (case 2), then the general solution to (3.26) has an arbitrary constant and the solution is given by (4.13) and (4.14).

If $a_1 \neq \pm\frac{\mathrm{i}}{2}$, then the solution has a power singularity at $x = 1$:

$$\psi(x) \sim E(1-x)^{-\mu}, \quad x \to 1, \ \ \mu = \frac{1}{\pi}\sin^{-1}\frac{\mathrm{i}}{2a_1}, \tag{4.22}$$

with $0 \leqslant \mathrm{Re}\,\mu < \frac{1}{2}$ in case 1 and $-\frac{1}{2} < \mathrm{Re}\,\mu < 0$ in case 2.

If $a_1 = \mp\frac{\mathrm{i}}{2}$, then the solution has a power-logarithmic singularity at $x = 1$:

$$\psi(x) \sim E'(1-x)^{\pm 1/2} + E''(1-x)^{\pm 1/2}\log(1-x), \quad x \to 1, \tag{4.23}$$

with '+' in case 1 $\left(a_1 = -\frac{\mathrm{i}}{2}\right)$ and '−' in case 2 $\left(a_1 = \frac{\mathrm{i}}{2}\right)$.

5. Electromagnetic diffraction from an anisotropic impedance half-plane

5.1 *Formulation*

The technique proposed will be illustrated by analysing scattering of an electromagnetic wave at skew incidence from an anisotropic impedance half-plane $S = \{0 < \rho < \infty, \theta = \pm\pi \mp 0, -\infty < z < +\infty\}$. The surface impedance parameters are η_1^{+} and η_2^{+} on the upper sides, $\theta = \pi - 0$, and η_1^{-} and η_2^{-} on the lower side, $\theta = -\pi + 0$. In general, they are different and could be complex numbers. Let the incoming plane wave be described by

$$\begin{aligned} E_z^i &= e_1\, \mathrm{e}^{\mathrm{i}k\rho \sin\beta \cos(\theta-\theta_0) - \mathrm{i}kz\cos\beta}, \\ Z_0 H_z^i &= e_2\, \mathrm{e}^{\mathrm{i}k\rho \sin\beta \cos(\theta-\theta_0) - \mathrm{i}kz\cos\beta}, \end{aligned} \tag{5.1}$$

where (ρ, θ, z) are cylindrical coordinates, k is the wave number ($\mathrm{Im}(k) \leqslant 0$), Z_0 is the intrinsic impedance of free space, $\beta \in (0, \pi)$ is the angle of incidence ($0 < \beta < \pi/2$), the angle $\theta_0 \in (-\pi, \pi)\backslash\{0\}$ defines the direction of incidence, e_1 and e_2 are prescribed parameters and a time factor $\mathrm{e}^{\mathrm{i}\omega t}$ is suppressed. The two components of the electric and magnetic field $V_1 = E_z$ and $V_2 = Z_0 H_z$ solve the Helmholtz equation

$$(\nabla^2 + k^2)V_j = 0, \quad (\rho, \theta, z) \in \mathbb{R}^3 \setminus S, \tag{5.2}$$

and are coupled by the following boundary conditions (Senior, 1978):

$$\frac{1}{\rho}\frac{\partial V_j}{\partial \theta} + (-1)^j \cos\beta \frac{\partial V_{3-j}}{\partial \rho} \pm \mathrm{i}k\hat{\eta}_j^{\pm} \sin^2\beta V_j = 0, \quad \theta = \pm\pi \mp 0, \quad j = 1, 2, \tag{5.3}$$

where $\hat{\eta}_1^{\pm} = 1/\eta_1^{\pm}$ and $\hat{\eta}_2^{\pm} = \eta_2^{\pm}$.

The total field $\mathbf{V} = (V_1, V_2)$ is represented in the form of the Sommerfeld integral (Maliuzhinets, 1958):

$$\mathbf{V}(\rho, \theta, z) = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\,\mathrm{i}} \int_\gamma \mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos s} \boldsymbol{\mathcal{S}}(s + \theta)\mathrm{d}s. \tag{5.4}$$

Here, γ is the Sommerfeld contour that consists of two loops symmetric with respect to the origin. The asymptotes for the branches are the lines $s = \frac{3}{2}\pi$ and $s = -\frac{1}{2}\pi$ for the upper loop and the lines $s = \frac{1}{2}\pi$ and $s = -\frac{3}{2}\pi$ for the lower loop. The spectral vector function $\boldsymbol{\mathcal{S}}(s) = (\mathcal{S}_1(s), \mathcal{S}_2(s))$ is analytic everywhere in the strip $-\pi \leqslant \mathrm{Re}\, s \leqslant \pi$ apart from the point $s = \theta_0$, where its components have a simple pole with the residues defined by the incident field (5.1):

$$\operatorname*{res}_{s=\theta_0} \mathcal{S}_j(s) = e_j, \quad j = 1, 2. \tag{5.5}$$

At the infinite points $s = x \pm \mathrm{i}\infty$ ($|x| < \infty$), the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ are bounded. In a vicinity of the point $r = 0$, the energy is assumed to be finite (the electromagnetic field satisfies the Meixner condition). The boundary conditions (5.3) can be written in terms of the spectral functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ as the following system of Maliuzhinets difference equations:

$$\begin{aligned} &(\sin s \pm \hat{\eta}_j^{\pm} \sin\beta)\mathcal{S}_j(s \pm \pi) + (-1)^j \cos s \cos\beta \mathcal{S}_{3-j}(s \pm \pi) \\ &\quad = (-\sin s \pm \hat{\eta}_j^{\pm} \sin\beta)\mathcal{S}_j(-s \pm \pi) + (-1)^j \cos s \cos\beta \mathcal{S}_{3-j}(-s \pm \pi), \quad j = 1, 2. \end{aligned} \tag{5.6}$$

Following Antipov & Silvestrov (2006), introduce the following four functions:

$$\Phi_j^{\pm}(s) = (-\sin s \pm \hat{\eta}_j^{\pm} \sin\beta)\mathcal{S}_j(s) - (-1)^j \cos s \cos\beta \mathcal{S}_{3-j}(s), \quad j = 1, 2. \tag{5.7}$$

The spectral functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ have to be invariant with respect to the transformation

$$\mathcal{T}\colon (\theta, \theta_0, \beta, \eta_1^{\pm}, \eta_2^{\pm}) \to (-\theta, -\theta_0, \pi - \beta, \eta_1^{\mp}, \eta_2^{\mp}). \tag{5.8}$$

This property can be achieved if the functions $\mathcal{S}_1(s)$ and $\mathcal{S}_2(s)$ are represented as

$$\mathcal{S}_j(s) = \frac{1}{2}[\mathcal{S}_j^+(s) + \mathcal{S}_j^-(s)], \quad j = 1, 2, \tag{5.9}$$

where

$$\mathcal{S}_j^\pm(s) = \frac{1}{\Gamma_s(\mp\hat{\eta}_1^\pm, \mp\hat{\eta}_2^\pm)}[(-\sin s \pm \hat{\eta}_{3-j}^\pm \sin\beta)\Phi_j^\pm(s) + (-1)^j \cos s \cos\beta \Phi_{3-j}^\pm(s)], \quad j = 1, 2, \tag{5.10}$$

and

$$\Gamma_s(a, b) = (\sin s + a \sin\beta)(\sin s + b \sin\beta) + \cos^2 s \cos^2\beta. \tag{5.11}$$

The difference equations (5.6) reduce to two cases of Problem 2.1 for the strips $\Pi^\pm$ described by

$$\Pi^+ = \{s \in \mathbb{C} \mid -\pi < \operatorname{Re} s < 3\pi\}, \quad \Pi^- = \{s \in \mathbb{C} \mid -3\pi < \operatorname{Re} s < \pi\}. \tag{5.12}$$

We shall refer to these cases as Problems 2.1$^\pm$. The functions $\Phi_j^\pm(s)$ $(j = 1, 2)$ are analytic in the strips $\Pi^\pm$ apart from the geometrical optics poles at the points $s = \theta_0$ and $s = \pm 2\pi - \theta_0$. They are continuous up to the boundary of the strips $\Pi^\pm$ and satisfy the boundary conditions

$$\begin{aligned}
&\Phi_1^\pm(\sigma) = G_{11}(\sigma)\Phi_1^\pm(\sigma - 4\pi) + G_{12}^\pm(\sigma)\Phi_2^\pm(\sigma - 4\pi),\\
&\Phi_2^\pm(\sigma) = G_{21}^\pm(\sigma)\Phi_1^\pm(\sigma - 4\pi) + G_{22}(\sigma)\Phi_2^\pm(\sigma - 4\pi),\\
&\sigma \in \Omega^\pm = \{s \in \mathbb{C} \mid \operatorname{Re} s = 2\pi \pm \pi\}
\end{aligned} \tag{5.13}$$

and the conditions of symmetry

$$\Phi_j^\pm(s) = \Phi_j^\pm(\pm 2\pi - s), \quad s \in \Pi^\pm. \tag{5.14}$$

The coefficients of System (5.14) are given by

$$\begin{aligned}
G_{11}(\sigma) &= \frac{\Gamma_\sigma(1/\eta_1^-, -\eta_2^+)\Gamma_\sigma(1/\eta_1^+, -\eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma \sin^2 2\beta}{D(\sigma)},\\
G_{22}(\sigma) &= \frac{\Gamma_\sigma(-1/\eta_1^-, \eta_2^+)\Gamma_\sigma(-1/\eta_1^+, \eta_2^-) + \eta_2\eta_1^{-1}\cos^2\sigma \sin^2 2\beta}{D(\sigma)},\\
G_{12}^\pm(\sigma) &= \mp\frac{\eta_0^\mp \sin\beta \sin 2\beta \sin 2\sigma}{\eta_1 D(\sigma)}, \quad G_{21}^\pm(\sigma) = \mp\frac{\eta_0^\pm \eta_2 \sin\beta \sin 2\beta \sin 2\sigma}{D(\sigma)},
\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}
&D(\sigma) = \Gamma_\sigma(-1/\eta_1^+, -\eta_2^+)\Gamma_\sigma(-1/\eta_1^-, -\eta_2^-),\\
&\frac{1}{\eta_1} = \frac{1}{2}\left(\frac{1}{\eta_1^+} + \frac{1}{\eta_1^-}\right), \quad \eta_2 = \frac{\eta_2^+ + \eta_2^-}{2}, \quad \eta_0^+ = \eta_2^+ - \frac{1}{\eta_1^-}, \quad \eta_0^- = \eta_2^- - \frac{1}{\eta_1^+}.
\end{aligned} \tag{5.16}$$

At the ends of the strip, i.e. as $s \to \infty$ ($\operatorname{Re} s$ is finite), $\Phi_j^\pm(s) = \mathrm{O}(\mathrm{e}^{|s|})$ $(j = 1, 2)$. In the derivation of the formula for the function $D(\sigma)$ in (5.16), the following identity was used:

$$\Gamma_s\left(-\frac{1}{\eta_1^-}, \eta_2^+\right)\Gamma_s\left(\frac{1}{\eta_1^+}, -\eta_2^-\right) + \frac{\eta_2}{\eta_1}\cos^2 \sin^2 2\beta = \Gamma_s\left(-\frac{1}{\eta_1^-}, -\eta_2^-\right)\Gamma_s\left(\frac{1}{\eta_1^+}, \eta_2^+\right). \tag{5.17}$$

The functions $\mathcal{S}_j^{\pm}(s)$ have to be analytic everywhere in the strip $-\pi \leqslant \operatorname{Re} s \leqslant \pi$ apart from the point $s = \theta_0$, where they have a simple pole. To remove the poles $\xi_m^{\pm}$ ($m = 1, 2, 3, 4$) of the right-hand side in (5.10), it is required that

$$(-\sin\xi_m^{\pm} \pm \eta_2^{\pm} \sin\beta)\Phi_1^{\pm}(\xi_m^{\pm}) - \cos\xi_m^{\pm} \cos\beta\Phi_2^{\pm}(\xi_m^{\pm}) = 0, \quad m = 1, 2, 3, 4. \tag{5.18}$$

The points $\xi_m^{\pm}$ are those four zeros

$$2\pi m - \mathrm{i}\log(\mathrm{i}M_\nu^{\pm} \pm \sqrt{1 - (M_\nu^{\pm})^2}), \quad m = 0, \pm 1, \ldots, \quad \nu = 1, 2, \tag{5.19}$$

of the functions $\Gamma_s(\mp 1/\eta_1^{\pm}, \mp\eta_2^{\pm})$ which lie in the strip $-\pi < \operatorname{Re} s < \pi$. Here,

$$\begin{aligned} M_\nu^{\pm} &= \pm\frac{1}{2\eta_1^{\pm}\sin\beta}[1 + \eta_1^{\pm}\eta_2^{\pm} + (-1)^\nu\sqrt{h_\pm}], \\ h_\pm &= (1 - \eta_1^{\pm}\eta_2^{\pm})^2 + 4\eta_1^{\pm}(\eta_2^{\pm} - \eta_1^{\pm})\cos^2\beta, \end{aligned} \tag{5.20}$$

and $\sqrt{h_\pm}$ and $\log(\mathrm{i}M_\nu^{\pm} \pm \sqrt{1 - (M_\nu^{\pm})^2})$ are fixed branches of the square root and the logarithmic functions, respectively.

5.2 *Reduction to an integral equation*

Show first that for Problems 2.1$^{\pm}$, the matrix A given by (2.6) ($\omega = \omega^{\pm} = 2\pi \pm \pi$) is a zero matrix. Let $G^{\pm}(s)$ be the matrix whose entries are the coefficients of System (5.14). It can be directly verified that

$$\det G^{\pm}(s) = G_{11}(s)G_{22}(s) - G_{12}^{\pm}(s)G_{21}^{\pm}(s) = \frac{D(-s)}{D(s)} \tag{5.21}$$

and also

$$\begin{aligned} G_{11}(-s) &= \frac{D(s)}{D(-s)}G_{22}(s), \quad G_{22}(-s) = \frac{D(s)}{D(-s)}G_{11}(s), \\ G_{12}^{\pm}(-s) &= -\frac{D(s)}{D(-s)}G_{12}^{\pm}(s), \quad G_{21}^{\pm}(-s) = -\frac{D(s)}{D(-s)}G_{21}^{\pm}(s). \end{aligned} \tag{5.22}$$

The identities $A_{ij} \equiv 0$ ($i, j = 1, 2$) follow immediately from (2.6), (5.21) and (5.22). Therefore, System (5.14) cannot be decoupled. According to Theorem 2.2, since the functions (5.16) are 2π-periodic functions and $h = 4\pi$, the coefficients $b(s)$ and $c(s)$ of the second-order difference equations

$$\phi^{\pm}(\sigma) = b(\sigma)\phi^{\pm}(\sigma - 4\pi) + c(\sigma)\phi^{\pm}(\sigma - 8\pi), \quad \sigma \in \hat{\Omega}^{\pm}, \tag{5.23}$$

are given by

$$b(s) = G_{11}(s) + G_{22}(s), \quad c(s) = -\frac{D(-s)}{D(s)}. \tag{5.24}$$

For Problems 2.1$^{\pm}$, $\hat{\omega}^{\pm} = 4\pi \pm \pi$, and the symmetry condition (2.15) becomes

$$\phi^{\pm}(s) = \phi^{\pm}(\pm 2\pi - s). \tag{5.25}$$

Specify now all possible poles of the functions $\phi^+(s)$ and $\phi^-(s)$ in the extended strips

$$\hat{\Pi}^+ = \{s \in \mathbb{C}\colon -3\pi < \operatorname{Re} s < 5\pi\}, \quad \hat{\Pi}^- = \{s \in \mathbb{C}\colon -5\pi < \operatorname{Re} s < 3\pi\}. \tag{5.26}$$

Because of the geometric optics pole and the symmetry condition, in the strips $\Pi^\pm$, the functions $\phi^\pm(s)$ have simple poles at the points θ_0, $2\pi - \theta_0$, $-2\pi - \theta_0$ and $\pm 4\pi + \theta_0$. From the analytical continuation,

$$\begin{aligned} \Phi_1^\pm(s) &= G_{11}(s)\Phi_1^\pm(s-4\pi) + G_{12}^\pm(s)\Phi_2^\pm(s-4\pi), \quad 2\pi \pm \pi < \operatorname{Re} s < 4\pi \pm \pi, \\ \Phi_1^\pm(s) &= G_{11}(-s)\Phi_1^\pm(s+4\pi) + G_{12}^\pm(-s)\Phi_2^\pm(s+4\pi), \quad -4\pi \pm \pi < \operatorname{Re} s < -2\pi \pm \pi, \end{aligned} \tag{5.27}$$

it follows that the functions $\phi^\pm(s)$ may have additional poles in the domains $\hat{\Pi}^\pm \backslash \Pi^\pm$. These poles are $\pm\xi_j^+ \pm 4\pi$, $\mp\xi_j^- \pm 4\pi$, $\mp\xi_j^+ \mp 2\pi$ and $\pm\xi_j^- \mp 2\pi$ $(j = 1, 2, 3, 4)$, where $\xi_j^\pm$ are the four simple zeros of the functions $\Gamma_s(\mp 1/\eta_1^\pm, \mp\eta_2^\pm)$ which lie in the strip $-\pi < \operatorname{Re} s < \pi$ and are determined in (5.19) and (5.20). Note that the symmetry condition (5.25) does not bring extra poles. Thus, both functions $\phi^+(s)$ and $\phi^-(s)$ have $2\hat{\nu} = 20$ simple poles in the strips $\hat{\Pi}^+$ and $\hat{\Pi}^-$, respectively.

We now implement the procedure of Section 3. Transform the difference equation (2.14) into the Riemann–Hilbert-type problem (3.3). The conformal mapping (3.1) for Problem 2.1$^\pm$ becomes

$$z = z^\pm(s) = \mathrm{i}\cot\frac{s \mp \pi}{8}, \quad s = s^\pm(z) = 4\pi \pm \pi + 4\,\mathrm{i}\log\frac{1+z}{1-z}. \tag{5.28}$$

It is directly verified that the functions $b(\sigma)$ and $c(\sigma)$ have the properties

$$c(\sigma) = -\frac{D(\bar{\sigma})}{D(\sigma)}, \quad c(2\pi - \sigma) = c(\bar{\sigma}), \quad b(2\pi - \sigma) = b(\bar{\sigma}), \quad \sigma \in \hat{\Omega}^\pm. \tag{5.29}$$

Therefore, by using the identity

$$G_{11}(\bar{\sigma}) + G_{22}(\bar{\sigma}) = \frac{D(\sigma)}{D(\bar{\sigma})}[G_{11}(\sigma) + G_{22}(\sigma)], \quad \sigma \in \hat{\Omega}^\pm, \tag{5.30}$$

we establish the following:

$$\begin{gathered} C(t)C(-t) = c(\sigma)c(\bar{\sigma}) = 1, \quad t = z(\sigma), \ \ \sigma \in \hat{\Omega}^\pm, \\ B(t) + C(t)B(-t) = 1. \end{gathered} \tag{5.31}$$

Thus, the necessary conditions (3.6) for the existence of an even solution $F_\pm(z) = \phi^\pm(s^\pm(z))$, $s^\pm \in \hat{\Pi}^\pm$, are satisfied.

Factorize now the function $C(t)$, $-1 < t < 1$. As $\sigma = 4\pi \pm \pi + \mathrm{i}\sigma_2 \in \hat{\Omega}^\pm$ and $\sigma_2 \to \pm\infty$, $c(\sigma) \sim -1$. As $\sigma \to 4\pi \pm \pi$, $c(\sigma) \sim -1$. So, $C(\pm 1) = -1$ and $C(0) = -1$. Fix a branch of the logarithmic function $\log C(t)$ by the condition $\log C(0) = \pi\,\mathrm{i}$. Then, the integer n introduced in (3.8) equals 1. It has been numerically verified for different sets of the parameters of the problem that

$$[\arg c(\sigma)]_{\sigma=3\pi}^{\sigma=3\pi+\mathrm{i}\infty} = [\arg c(\sigma)]_{\sigma=5\pi}^{\sigma=5\pi+\mathrm{i}\infty} = -4\pi. \tag{5.32}$$

Therefore,

$$\log C(\pm 1) = \pi\,\mathrm{i} \pm 2\pi\,\mathrm{i}\lambda, \quad \lambda = -2. \tag{5.33}$$

In the case under consideration, $h = 4\pi$, $b_1 = 1$, $n = 1$, $\lambda_0 = 2$ and $\kappa = 2$. Thus, the function $X(z)$ becomes

$$X(z) = z(z^2-1)^{-2}\exp\left\{\frac{1}{\pi\,\mathrm{i}}\int_0^1 \frac{\log C(t)t\,\mathrm{d}t}{t^2-z^2}\right\}. \tag{5.34}$$

The even function $F_\pm(z)$ has 20 simple poles at the images of the poles of the functions $\phi^\pm(s)$ in the strips $\hat{\Pi}^\pm$. For the function $F_+(z)$, these poles are $z = \pm z_0^+$, $\pm 1/z_0^+$, $\pm\delta_j$ and $\pm\gamma_j$ $(j = 1, 2, 3, 4)$. The poles of the function $F_-(z)$ are the following points: $z = \pm z_0^-$, $\pm 1/z_0^-$, $\pm\delta_j$ and $\pm\gamma_j$ $(j = 1, 2, 3, 4)$. Here,

$$z_0^\pm = \mathrm{i}\cot\frac{\theta_0 \mp \pi}{8}, \quad \delta_j = \mathrm{i}\tan\frac{\xi_j^+ - \pi}{8}, \quad \gamma_j = \mathrm{i}\tan\frac{\xi_j^- + \pi}{8}, \quad j = 1, 2, 3, 4. \tag{5.35}$$

Thus, the functions $F_\pm(z)$ admit the following representations:

$$F_\pm(z) = X(z)[\Psi_\pm(z) + R_\pm(z) + Q_\pm(z)]. \tag{5.36}$$

The polynomials $Q_\pm(z)$ and the rational functions $R_\pm(z)$ are given by

$$\begin{aligned} Q_\pm(z) &= d_0^\pm z + d_1^\pm z^3, \\ R_\pm(z) &= \frac{d_2^\pm z}{z^2-(z_0^\pm)^2} + \frac{d_3^\pm z}{z^2-(z_0^\pm)^{-2}} + \sum_{j=1}^4 \frac{d_{j+3}^\pm z}{z^2-\delta_j^2} + \sum_{j=1}^4 \frac{d_{j+7}^\pm z}{z^2-\gamma_j^2}, \end{aligned} \tag{5.37}$$

where $d_j^\pm$ $(j = 0, 1, \ldots, 11)$ are constants to be determined. From (3.18) and (3.20), the functions $\Psi_\pm(z)$ are expressed through the solution to the integral equations

$$a(x)\psi^\pm(x) + \frac{x}{\pi\,\mathrm{i}}\int_0^1 \frac{\psi^\pm(t)\mathrm{d}t}{1-t^2x^2} = f^\pm(x), \quad 0 < x < 1, \tag{5.38}$$

as follows:

$$\Psi_\pm(z) = \frac{1}{2\pi\,\mathrm{i}}\int_{-1}^1 \frac{\psi^\pm(t)\mathrm{d}t}{t-z} = \frac{z}{\pi\,\mathrm{i}}\int_0^1 \frac{\psi^\pm(t)\mathrm{d}t}{t^2-z^2}. \tag{5.39}$$

Here, the function $a(x)$ is the same as in (3.27) and

$$\psi^\pm(x) = \frac{B(x)}{X^+(x)}F_\pm\left(\frac{1}{x}\right), \quad f^\pm(x) = R_\pm\left(\frac{1}{x}\right) + Q_\pm\left(\frac{1}{x}\right). \tag{5.40}$$

Both functions $f^+(x)$ and $f^-(x)$ possess 12 arbitrary constants.

Analyse the function $a(x)$ as $x \to 1$. Since

$$B(x) \sim 2, \quad X(x^{-1})[X^+(x)]^{-1} \sim -\mathrm{i}, \quad x \to 1-0, \tag{5.41}$$

the function $a(x)$ has a finite non-zero limit: $a(x) \sim \frac{\mathrm{i}}{2}$ as $x \to 1-0$. Therefore, the singular equations (5.38) have two linearly independent solutions. As $x \to 1$, the functions $\psi^\pm(x)$ have a power-logarithmic singularity:

$$\psi^\pm(x) \sim E'_\pm(1-x)^{-1/2} + E''_\pm(1-x)^{-1/2}\log(1-x), \quad x \to 1-0, \tag{5.42}$$

where $E'_\pm$ and $E''_\pm$ are constants. On the other hand, analysis of the exact solution (Antipov & Silvestrov, 2006) shows that the functions $\Phi_j(s)$ $(j = 1, 2)$ have the exponential growth at infinity: $|\Phi_j(s)| \leqslant c_j\, \mathrm{e}^{|s|}$, $s \to \infty$, $\operatorname{Re} s$ is finite and c_j are constants, not polynomials. Therefore, the function $\psi^\pm(x)$ cannot have the power-logarithmic singularity and (5.38) should be solved in the class $H^{-1/2,-1/2}[-1, 1]$. In this class, the solution is unique and the arbitrary constants are fixed as in (4.21).

5.3 *Functions* $\Phi_2^\pm(s)$

To find the electromagnetic field, one needs to recover the four functions $\Phi_1^\pm(s)$ and $\Phi_2^\pm(s)$. All these functions have to be analytic everywhere in the strip $-\pi \leqslant \operatorname{Re} s \leqslant \pi$ apart from the geometrical optics pole $s = \theta_0$. From (5.5) and (5.7), they have to meet the four conditions

$$\operatorname*{res}_{s=\theta_0} \Phi_j^\pm(s) = (-\sin\theta_0 \pm \hat{\eta}_j^\pm \sin\beta)e_j - (-1)^j \cos\theta_0 \cos\beta e_{3-j}, \quad j = 1, 2. \tag{5.43}$$

In addition, they must satisfy the eight conditions (5.18). At infinity, as $\operatorname{Im} s \to \infty$ and $\operatorname{Re} s$ is finite, these functions may grow as $\mathrm{e}^{|s|}$. The functions $\Phi_1^\pm(s)$ have been expressed through the functions $F_\pm(z)$ by

$$\Phi_1^\pm(s) = F_\pm(z^\pm(s)), \quad s \in \hat{\Pi}^\pm, \; z^\pm(s) = \mathrm{i}\cot\frac{s \mp \pi}{8}. \tag{5.44}$$

Define now the functions $\Phi_2^\pm(s)$. From the system of difference equations (5.14) and by using (5.22),

$$\begin{aligned} \Phi_2^+(s) &= -\frac{\eta_1 D(s)}{\eta_0^- \sin\beta \sin 2\beta \sin 2s}[\Phi_1^+(s+4\pi) - G_{11}(s)\Phi_1^+(s)], \quad -\pi \leqslant \operatorname{Re} s \leqslant \pi, \\ \Phi_2^-(s) &= -\frac{\eta_1 D(-s)}{\eta_0^+ \sin\beta \sin 2\beta \sin 2s}[\Phi_1^-(s-4\pi) - G_{11}(-s)\Phi_1^-(s)], \quad -\pi \leqslant \operatorname{Re} s \leqslant \pi. \end{aligned} \tag{5.45}$$

Because of the function $\sin 2s$, both functions $\Phi_2^+(s)$ and $\Phi_2^-(s)$ may have inadmissible poles at the points $0, \pm\frac{1}{2}\pi$ and $\pm\pi$. These poles will be removed if

$$\Phi_1^\pm(s \pm 4\pi) - G_{11}(\pm s)\Phi_1^\pm(s) = 0, \quad s = 0, \frac{1}{2}\pi, -\frac{1}{2}\pi, \pi, -\pi. \tag{5.46}$$

Since $b(s)$ and $c(s)$ are meromorphic functions, (5.23) can be analytically continued from the contour $\operatorname{Re} s = 5\pi$ into the complex plane by

$$\phi^+(s+4\pi) = b(s)\phi^+(s) + c(s)\phi^+(s-4\pi). \tag{5.47}$$

For $s = \pi$, because $b(\pi) = 2$ and $c(\pi) = -1$, (5.47) yields $\phi^+(5\pi) = 2\phi^+(\pi) - \phi^+(-3\pi)$. By the symmetry condition (5.14), $\phi^+(-3\pi) = \phi^+(5\pi)$, and therefore, since $G_{11}(\pi) = 1$, Condition (5.46) for $s = \pi$ is satisfied automatically. The relation (5.46) for $s = -\pi$ reads $\phi^+(-\pi) = \phi^+(3\pi)$, and this follows from the symmetry condition (5.14). Thus, the points $s = \pi, -\pi$ are removable singularities of the functions $\Phi_2^\pm(s)$. For $s = 0$, $s = \frac{1}{2}\pi$ and $s = -\frac{1}{2}$, the relations (5.46) yield six additional conditions for the constants $d_j^\pm$ $(j = 0, 1, \ldots, 11)$:

$$\begin{gathered} \phi^\pm(0) = \phi^\pm(\pm 4\pi), \\ \phi^+\left(\pm\frac{\pi}{2}\right) = \tau_*^{\pm 1}\phi^+\left(4\pi \pm \frac{\pi}{2}\right), \quad \phi^-\left(\pm\frac{\pi}{2}\right) = \tau_*^{\mp 1}\phi^-\left(-4\pi \pm \frac{\pi}{2}\right), \end{gathered} \tag{5.48}$$

where

$$\tau_* = \frac{(\eta_1^- - \sin\beta)(\eta_1^+ - \sin\beta)}{(\eta_1^- + \sin\beta)(\eta_1^+ + \sin\beta)}. \tag{5.49}$$

The functions $\Phi_2^\pm(s)$ may grow as $s \to \infty$ ($\mathrm{Re}\, s$ is finite), but not faster than $\mathrm{e}^{|s|}$. From (5.45), it is clear that, in general, the right-hand side grows as $\mathrm{e}^{|3s|}$. The functions $\Phi_2^\pm(s)$ will have the required asymptotics at infinity if and only if

$$\Phi_1^\pm(s \pm 4\pi) - G_{11}(\pm s)\Phi_1^\pm(s) = \mathrm{O}(\mathrm{e}^{-|s|}), \quad |s| \to \infty, \ \ s \in \hat{\Pi}^\pm, \tag{5.50}$$

or equivalently, by letting $\mathrm{Re}\, s = \pm\pi$,

$$\begin{aligned} F_+^+(x) - G_{11}(\sigma)F_+(x^{-1}) &= \mathrm{O}\{(1-x)^4\}, \quad x \to 1-0, \ \ \sigma \to 5\pi + \mathrm{i}\infty, \\ F_-^-(x) - G_{11}(-\sigma)F_-(x^{-1}) &= \mathrm{O}\{(1-x)^4\}, \quad x \to 1-0, \ \ \sigma \to -5\pi + \mathrm{i}\infty, \end{aligned} \tag{5.51}$$

By the Sokhotski–Plemelj formulas,

$$\Psi_\pm^\pm(x) = \pm\frac{\psi^\pm(x)}{2\pi\,\mathrm{i}} + \frac{1}{2\pi\,\mathrm{i}}\int_{-1}^{1}\frac{\psi^\pm(t)\mathrm{d}t}{t-x}, \quad x \in (-1,1). \tag{5.52}$$

Next, using the formulas

$$F_+\left(\frac{1}{x}\right) = \frac{X^+(x)}{B(x)}\psi^+(x), \quad F_-\left(\frac{1}{x}\right) = -\frac{X^-(x)}{B(-x)}\psi^-(x), \quad 0 < x < 1, \tag{5.53}$$

and the asymptotics of the functions $X^\pm(x)$ as $x \to 1$,

$$X^\pm(x) \sim E_\pm(x \mp 1)^{-7/2}, \quad x \to \pm 1, \ \ E_\pm = \text{constant}, \tag{5.54}$$

the relations (5.51) reduce to

$$\begin{aligned} &\frac{1}{2\pi\,\mathrm{i}}\int_{-1}^{1}\frac{\psi^\pm(t)\mathrm{d}t}{t-x} \pm \psi^\pm(x)\left[-\frac{1}{2} + \frac{G_{11}(\pm\sigma)}{B(\pm x)}\right] + R_\pm(x) + Q_\pm(x) \\ &\quad = \mathrm{O}\{(1-x)^{15/2}\}, \quad x \to 1-0, \ \ \sigma \to \pm 5\pi + \mathrm{i}\infty. \end{aligned} \tag{5.55}$$

To satisfy these conditions, there are three free constants for each function, $\Phi_2^+(s)$ and $\Phi_2^-(s)$.

5.4 *Definition of arbitrary constants*

It is possible to avoid the analysis of the asymptotics of the left-hand side in (5.55) and fix all the arbitrary constants. To do this, we reduce System (5.14) to two separate difference equations of the second order:

$$\Phi_m^\pm(\sigma) = b(\sigma)\Phi_m^\pm(\sigma - 4\pi) + c(\sigma)\Phi_m^\pm(\sigma - 8\pi), \quad \sigma \in \hat{\Omega}^\pm, \ \ m = 1, 2, \tag{5.56}$$

with the coefficients given by (5.24). The solution of the equations must meet the symmetry condition $\Phi_m^\pm(s) = \Phi_m^\pm(\pm 2\pi - s)$. Clearly, if the functions $\Phi_m^\pm(s)$ solve System (5.14), then they solve the difference equations (5.56). In general, the inverse statement is not correct: not each solution of (5.56) solves System (5.14). The general solutions to (5.56) are defined by

$$\Phi_m^{\pm}(s) = \sum_{j=0}^{11} d_{mj}^{\pm} \tau_j^{\pm}(s), \quad s \in \hat{\Pi}^{\pm}, \ \ m = 1, 2, \tag{5.57}$$

where $d_{mj}^{\pm}$ are arbitrary constants,

$$\begin{gathered}\tau_j^{\pm}(s) = X(z)\left[f_j^{\pm}(z) + \frac{z}{\pi \,\mathrm{i}} \int_0^1 \frac{\psi_j^{\pm}(t)\mathrm{d}t}{t^2 - z^2} \right], \quad s \in \hat{\Pi}^{\pm}, \ \ j = 0, 1, \dots, 11, \\ f_0^{\pm}(z) = z, \quad f_1^{\pm}(z) = z^3, \quad f_2^{\pm}(z) = \frac{z}{z^2 - (z_0^{\pm})^2}, \quad f_3^{\pm}(z) = \frac{z}{z^2 - (z_0^{\pm})^{-2}}, \\ f_{l+3}(z) = \frac{z}{z^2 - \delta_l^2}, \quad f_{l+7}(z) = \frac{z}{z^2 - \gamma_l^2} \quad (l = 1, 2, 3, 4),\end{gathered} \tag{5.58}$$

and the functions $\psi_j^{\pm}(x) \in H^{0,-1/2}[0, 1]$ ($j = 0, 1, \dots, 11$) are the unique solutions of the integral equations

$$a(x)\psi_j^{\pm}(x) + \frac{x}{\pi \,\mathrm{i}} \int_0^1 \frac{\psi_j^{\pm}(t)\mathrm{d}t}{1 - t^2 x^2} = f_j^{\pm}\left(\frac{1}{x}\right), \quad 0 < x < 1, \ \ j = 0, 1, \dots, 11. \tag{5.59}$$

Each function $\Phi_m^{\pm}(s)$ ($m = 1, 2$) has 12 arbitrary constants (in total, 48 constants). The four constants $d_{12}^{\pm}$ and $d_{22}^{\pm}$ can be defined explicitly from the physical conditions (5.43):

$$d_{m2}^{\pm} = \mathrm{i}[(z_0^{\pm})^2 - 1][4X(z_0^{\pm})]^{-1}[(-\sin\theta_0 \pm \hat{\eta}_m^{\pm} \sin\beta)e_m - (-1)^m \cos\theta_0 \cos\beta e_{3-m}], \ \ m = 1, 2. \tag{5.60}$$

Equations (5.18) yield the following eight conditions for the functions $\Phi_1^{\pm}(s)$ and $\Phi_2^{\pm}(s)$:

$$\sum_{j=0}^{11} [(-\sin\xi_m^{\pm} \pm \eta_2^{\pm} \sin\beta) d_{1j}^{\pm} \tau_j^{\pm}(\xi_m^{\pm}) - \cos\xi_m^{\pm} \cos\beta d_{2j}^{\pm} \tau_j^{\pm}(\xi_m^{\pm})] = 0, \quad m = 1, 2, 3, 4. \tag{5.61}$$

For each function $\Phi_m^{\pm}(s)$ ($m = 1, 2$), the difference between the number of the arbitrary constants and the number of the conditions is equal to nine. The solutions to the difference equations (5.56) solve the original system of difference equations (5.14) if the number of constants and the conditions is the same. The other 36 conditions come from the original system of difference equations (5.14). The general solutions of (5.56) have to satisfy System (5.14) at some fixed points. Clearly, because of the relations

$$\begin{gathered}\Phi_{3-m}^{+}(s) = \frac{1}{G_{m3-m}^{+}(s)}[\Phi_m^{+}(s + 4\pi) - G_{mm}(s)\Phi_m^{+}(s)], \quad -\pi \leqslant \operatorname{Re} s \leqslant \pi, \\ \Phi_{3-m}^{-}(s) = \frac{1}{G_{m3-m}^{-}(-s)}[\Phi_m^{-}(s - 4\pi) - G_{mm}(-s)\Phi_m^{-}(s)], \quad -\pi \leqslant \operatorname{Re} s \leqslant \pi, \ \ m = 1, 2,\end{gathered} \tag{5.62}$$

it is convenient to take the points $s = 0$, $s = \frac{1}{2}\pi$ and $s = -\frac{1}{2}\pi$. This choice explicitly shows that apart from the geometrical optics pole at the point $s = \theta_0$, the right-hand sides of (5.62) are free of poles everywhere in the strip $-\pi \leqslant \operatorname{Re} s \leqslant \pi$. The six conditions for the $+$ functions and the six equations for the $-$ functions can be written as

$$\sum_{j=0}^{11}[\tau_j^{\pm}(s \pm 4\pi) - G_{mm}(\pm s)\tau_j^{\pm}(s)]d_{mj}^{\pm} = 0, \quad s = 0, \frac{\pi}{2}, -\frac{\pi}{2}, \quad m = 1, 2. \tag{5.63}$$

The other points, say, $s_l^{\pm}$ ($l = 1, 2, \ldots, 6$) are arbitrary fixed points which meet the following conditions:

- $\operatorname{Re} s_j^{\pm} \in (-\pi, \pi)$,
- $s_j^{\pm}$ are invariant with respect to the transformation (5.8): $\mathcal{T} s_j^{+} = s_j^{-}$,
- the two 22×22 matrices of the $+$ and $-$ systems of algebraic equations for the constants d_{mj}^{+} and d_{mj}^{-} ($m = 1, 2,\ j = 0, 1, \ldots, 11,\ j \neq 2$), (5.61), (5.63) and

$$\sum_{j=0}^{11}[\tau_j^{\pm}(s_l^{\pm} \pm 4\pi) - G_{mm}(\pm s_l^{\pm})\tau_j^{\pm}(s_l^{\pm}) - G_{m3-m}^{\pm}(\pm s_l^{\pm})\tau_j^{\pm}(s_l^{\pm})]d_{mj}^{\pm} = 0,$$
$$l = 1, 2, \ldots, 6, \quad m = 1, 2, \tag{5.64}$$

  are non-singular.

5.5 *Numerical scheme for the integral equation*

It has been shown that the solutions of the integral equations (5.38) are even, bounded at the point $x = 0$ and have the square-root singularity at the point $x = 1$. The functions $\psi^{\pm}(x)$ admit the following expansion in terms of the Chebyshev polynomials:

$$\psi^{\pm}(x) = \frac{\psi_*^{\pm}(x)}{\sqrt{1 - x^2}}, \quad \psi_*^{\pm}(x) = \sum_{m=0}^{\infty} \Psi_m^{\pm} T_{2m}(x), \quad -1 < x < 1. \tag{5.65}$$

The series in (5.65) converge to $\psi_*^{\pm}(x)$ in the space $L_\rho^2(-1, 1)$ of even functions with the norm

$$\|\psi_*(x)\| = \left[\int_{-1}^{1} \psi_*^2(x)\rho(x)\mathrm{d}x\right]^{1/2}, \quad \rho(x) = (1 - x^2)^{-1/2}. \tag{5.66}$$

Substituting (5.65) into the integral equations (5.38) yields

$$\sum_{m=0}^{\infty}\left[\frac{T_{2m}(x)}{\sqrt{1 - x^2}} + \frac{\tau_m(x)}{\mathrm{i}a(x)}\right]\Psi_m^{\pm} = \frac{f^{\pm}(x)}{a(x)}, \quad 0 < x < 1, \tag{5.67}$$

where $\tau_m(x)$ is the singular integral

$$\tau_m(x) = \frac{x}{\pi}\int_0^1 \frac{T_{2m}(t)\mathrm{d}t}{\sqrt{1 - t^2}(1 - t^2x^2)}. \tag{5.68}$$

The functions $\tau_m(x)$ can be represented in terms of the Mellin convolution-type integral

$$\tau_m(x) = \int_0^\infty h_1\left(\frac{1}{xt}\right) h_2(t)\frac{\mathrm{d}t}{t} = \frac{1}{2\pi\,\mathrm{i}}\int_\Gamma H_1(p)H_2(p)x^p\,\mathrm{d}p. \tag{5.69}$$

Here,

$$h_1(t) = \frac{1}{\pi(t-1/t)}, \quad h_2(t) = \begin{cases} (1-t^2)^{-1/2}T_{2m}(t), & 0 < t < 1, \\ 0, & t > 1, \end{cases} \tag{5.70}$$

$H_1(p)$ and $H_2(p)$ are the Mellin transforms of the functions $h_1(t)$ and $h_2(t)$ (Gradshteyn & Ryzhik, 2000, formula 7.346):

$$H_j(p) = \int_0^\infty h_j(t)t^{p-1}\,\mathrm{d}t, \quad j = 1, 2,$$

$$H_1(p) = \frac{1}{2}\tan\frac{\pi p}{2}, \quad -1 < \mathrm{Re}\,p < 1,$$

$$H_2(p) = \frac{\pi\,\Gamma(p)}{2^p\Gamma\left(\frac{p}{2}+\frac{1}{2}+m\right)\Gamma\left(\frac{p}{2}+\frac{1}{2}-m\right)}, \quad \mathrm{Re}\,p > 0, \tag{5.71}$$

and the contour $\Gamma = \{\mathrm{Re}\,p = \gamma\}$, $0 < \gamma < 1$. By the Cauchy theorem, the integral (5.69) can be expanded as follows:

$$\tau_m(x) = \left(\frac{x}{2}\right)^{2m+1} F\left(m+\frac{1}{2}, m+1; 2m+1; x^2\right). \tag{5.72}$$

The use of the orthogonality property of the Chebyshev polynomials converts (5.67) into infinite systems of linear algebraic equations with respect to the coefficients $\Psi_m^\pm$:

$$\Psi_n^\pm + \sum_{m=0}^\infty a_{nm}\Psi_m^\pm = f_n^\pm, \quad n = 0, 1, \ldots. \tag{5.73}$$

Here,

$$a_{nm} = \frac{1}{\mathrm{i}\sigma_n}\int_{-1}^1 \frac{\tau_m(x)T_{2n}(x)\mathrm{d}x}{a(x)},$$

$$f_n^\pm = \frac{1}{\sigma_n}\int_{-1}^1 \frac{f^\pm(x)T_{2n}(x)\mathrm{d}x}{a(x)}, \quad \sigma_0 = \frac{\pi}{2}, \quad \sigma_n = \pi, \quad n = 1, 2, \ldots \tag{5.74}$$

(the functions $\tau(x)/a(x)$ and $f^\pm(x)/a(x)$ are even). Analytical expansion of the Gauss function in (5.72) shows that $\tau_m(x) \sim \frac{1}{2\sqrt{2}}(1-x)^{-1/2}$, $x \to 1$. For the coefficients a_{nm}, the Gauss quadratures formulas can be used.

6. Conclusions

This paper has proposed a method of integral equations for a system of difference equations with meromorphic coefficients subject to symmetry conditions for the unknown functions. It has been shown how the system of two difference equations of the first order can be reduced to a second-order scalar difference equation. The unknown function is analytic everywhere in an extended strip apart from a finite number of poles. By mapping the strip into a complex plane with a cut, the difference equation converts into an even scalar Riemann–Hilbert-type problem. Its solution has been found in terms of the solution of associated integral equations on the interval $(0, 1)$ with a fixed singularity at the point $x = 1$. If the coefficient of the equation $a(x)$ has a finite non-zero limit a_1 at the point $x = 1$, then in the class of Hölder functions admitting integrable singularities at the ends, depending on the parameter a_1, the integral equation is either uniquely solvable or has two linearly independent solutions. Moreover, for two exceptional values of the parameter a_1, the solution has a power-logarithmic singularity. This method has been applied to find an efficient approximate solution to the model problem of diffraction theory that concerns scattering of an electromagnetic plane wave at oblique incidence from an anisotropic impedance half-plane. It has been shown that in the physical class of solutions and for a known right-hand side, the integral equation has a unique solution. For its numerical solution, it has been proposed to expand the solution in terms of the Chebyshev polynomials and reduce the equation to an infinite system of linear algebraic equations. The same diffraction problem has recently been solved by Antipov & Silvestrov (2006). The solution was found in closed form. The numerical implementation of the method of Riemann surfaces requires the solution of the Jacobi inversion problem on a hyperelliptic surface of genus 3 (or, equivalently, a certain cubic equation with complex coefficients) and two separate linear systems of 15 algebraic equations to fix 30 free constants. This paper has derived another solution that does not use the theory of Riemann surfaces and bypasses the solution of the Jacobi inversion problem. To obtain numerical results, one needs to solve an integral equation (for these purposes, we propose to use the method of orthogonal polynomials). To fix 44 free constants, we have two linear algebraic systems of 22 equations each.

The method proposed does not derive a closed-form solution. However, the technique developed can be applied not only for a half-plane (a closed-form solution for this problem is now available) but also for more general problems (e.g. the problem on scattering of an electromagnetic wave as skew incidence from an anisotropic impedance wedge).

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