Propagation of a Mode-II crack in a viscoelastic medium with different bulk and shear relaxation

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Abstract. A solution for a crack propagating under shear loading in an isotropic viscoelastic medium with different relaxation under volume and shear deformations is presented. The medium is infinite and the semi-infinite crack propagates along the x_1-axis at constant speed V, which may take any value up to the speed of dilatational waves. The requisite Riemann-Hilbert problem for the steady-state case has been solved and the asymptotics of the stress component σ_{12} directly ahead of the crack and at infinity have been obtained.

1. Introduction

It is known [1] that shear cracks can propagate faster than the speed of shear waves, in polymeric materials with a weak plane. The presence of the weak interface is necessary to encourage the crack to run straight rather than deviating towards the direction of maximum tension. In the experiment, it is introduced artificially by bonding together two halves of the specimen. Such planes occur naturally in the Earth's crust and permit the shear faulting associated with earthquakes, some of which have been observed to propagate intersonically [2]. The polymeric materials employed in the experiments display some degree of viscoelastic relaxation. There is thus some direct incentive to study the viscoelastic problem, in addition to the fact that it provides at least an example of the influence of a dissipative process on crack propagation.

The first solution for a crack propagating in a viscoelastic medium was given in [3], for the case of antiplane strain. Subsequent solutions have been produced for plane strain (e.g. [4] for steady-state subsonic propagation, [5] for a case of transient intersonic propagation) but only for the case that the medium has the same relaxation for volumetric and shear deformations. The present work treats the more general case,

* Partial support by Louisiana Board of Regents grant LEQSF(2005-07)-ENH-TR-09 is acknowledged.

together with all speeds up to the speed of high-frequency longitudinal waves.

2. Transient problem formulation

The viscoelastic medium through which the crack propagates is described by constitutive equations

$$\sigma_{ij} = \kappa\delta_{ij} g_1 * de_{kk} + 2\mu g_2 * de'_{ij}, \tag{1}$$

where the symbol $*$ denotes convolution with respect to time, $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk modulus, and $e'_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}$ is the deviatoric, or shear, strain. The relaxation functions $g_1(t)$ and $g_2(t)$ are assumed to be convex and monotone decreasing, with $g_1(0) = g_2(0) = 1$, and tending to positive finite limits $g_1(\infty)$, $g_2(\infty)$ as $t \to \infty$. The medium is assumed to be uniform and infinite and to be loaded in some way that would generate, in the absence of the crack, a stress field $\sigma^A_{ij}(x_1, x_2, t)$. The presence of the crack which occupies, at time t, the surface

$$S(t) = \{\mathbf{x} : -\infty < x_1 < Vt, x_2 = 0, -\infty < x_3 < \infty\}, \tag{2}$$

induces additional stress, strain and displacement fields σ_{ij}, e_{ij}, u_i that satisfy the constitutive relations (1), the equations of motion $\sigma_{ij,j} = \rho\ddot{u}_i$, where ρ is the mass density, together with the boundary conditions

$$\sigma_{i2} = -\sigma^A_{i2} \;\text{ as }\; x_2 = \pm 0, -\infty < x_1 < Vt, \tag{3}$$

and a "radiation condition" that this field is associated with waves outgoing from the crack. The condition of plane strain, $u_3 \equiv 0$, is assumed.

The crack is taken to be subject to pure Mode II loading. Thus, when $x_2 = 0$, only the traction component σ^A_{12} is different from zero. Then, by symmetry, u_2 and σ_{12} are even functions of x_2, while u_1 and σ_{22} are odd functions of x_2. Since it has to be continuous across the plane $x_2 = 0$ ahead of the crack, $u_1(x_1, 0, t) = 0$ ahead of the crack but it has a discontinuity, $[u_1](x_1, t)$, across the crack surface $S(t)$. The stress component σ_{12} is continuous across the plane $x_2 = 0$ for all x_1. The component σ_{22} is zero on the whole plane $x_2 = 0$ because it is an odd function of x_2, continuous across $x_2 = 0$ ahead of the crack, and from the boundary conditions (3) behind the crack, because $\sigma^A_{22} = 0$ there.

A representation for the displacement field in the half-space $x_2 > 0$ follows by relating the displacement to the traction for $x_2 = 0$, through the half-space Green's function $G_{ij}(x_1 - x_1', x_2, x_2', t - t')$:

$$u_i(x_1, x_2, t) = -\int\int G_{i1}(x_1 - x_1', x_2, 0, t - t') * \sigma_{12}(x_1', 0, t')dx_1'dt'. \quad (4)$$

In the sequel, interest will center on the values of u_1 and σ_{12} as the surface $x_2 = 0$ is approached from the side $x_2 > 0$. It is convenient, therefore, to streamline the notation and write $u(x_1, t)$ for $u_1(x_1, +0, t)$, $\sigma(x_1, t)$ for $\sigma_{12}(x_1, +0, t)$ and $G(x_1, t)$ for $G_{11}(x_1, 0, 0, t)$. The relation (4) then gives, when $x_2 = 0$, $u(x_1, t) = -(G * \sigma)(x_1, t)$, the symbol $*$ now representing convolution over the relevant arguments x_1, t. Since the crack extends with speed V, it is helpful to introduce a moving coordinate $x = x_1 - Vt$ and to express the fields u, σ and G as functions of (x, t). Thus, for example, $u(x_1, t) = u(x + Vt, t) = \tilde{u}(x, t)$, and therefore

$$\tilde{u}(x, t) = -(\tilde{G} * \tilde{\sigma})(x, t), \quad (5)$$

the convolution being with respect to the relevant arguments x, t. The function $\tilde{u}$ is zero for $x > 0$; it is helpful to recognize this by appending a subscript so that $\tilde{u}$ becomes $\tilde{u}_-$. Then decompose $\tilde{\sigma}$: $\tilde{\sigma} = \tilde{\sigma}_+ + \tilde{\sigma}_-$, where $\tilde{\sigma}_+$ is the restriction of $\tilde{\sigma}$ to the half-line $x > 0$ and is unknown, whereas $\tilde{\sigma}_-$ is the corresponding restriction to $x < 0$ and is known, from the boundary condition (3).

The next step is to take the two-sided Laplace transform of equation (5), to give

$$\mathcal{L}[\tilde{u}_-] = -\mathcal{L}[\tilde{G}](\mathcal{L}[\tilde{\sigma}_+] + \mathcal{L}[\tilde{\sigma}_-]), \quad (6)$$

where the transform of a function $f(x, t)$ is defined so that

$$\mathcal{L}[f](\zeta, p) = \int\int e^{-(\zeta x + pt)} f(x, t)dxdt, \quad \mathrm{Re}\, p > 0. \quad (7)$$

Thus, the transforms $\mathcal{L}[\tilde{u}_-](\zeta, p)$, $\mathcal{L}[\tilde{\sigma}_-](\zeta, p)$ are analytic in the half-plane $\mathrm{Re}\,\zeta < 0$, whereas $\mathcal{L}[\tilde{\sigma}_+](\zeta, p)$ is analytic in the half-plane $\mathrm{Re}\,\zeta > 0$. In contrast, $\mathcal{L}[\tilde{G}]$ is defined, still for $\mathrm{Re}\, p > 0$ but only on the imaginary axis L in the complex ζ-plane. Make the following definitions

$$F^+(\zeta, p) = \zeta\mathcal{L}[\tilde{u}_-], \quad F^-(\zeta, p) = \mathcal{L}[\tilde{\sigma}_+],$$
$$P^+(\zeta, p) = \mathcal{L}[\tilde{\sigma}_-], \quad K(\zeta, p) = -\zeta\mathcal{L}[\tilde{G}]. \quad (8)$$

and reduce the relation (6) to the Riemann-Hilbert problem

$$F^+(\zeta, p) = K(\zeta, p)F^-(\zeta, p) + K(\zeta, p)P^+(\zeta, p), \quad \zeta \in L, \quad (9)$$

relating the boundary values, as ζ approaches L from their respective domains of analyticity, of the functions F^+ and F^-. The reason for defining F^+ and K in the way given is to ensure that K remains bounded as $|\zeta| \to \infty$ because F^+ is related to a strain and F^- to a stress.

The basic need, at this stage, is to obtain explicitly the transform $\mathcal{L}[\tilde{G}]$. This can be accomplished immediately by noting that

$$\mathcal{L}[\tilde{G}](\zeta, p) = \mathcal{L}[G](\zeta, p - V\zeta), \tag{10}$$

where

$$\mathcal{L}[G](\zeta, p) = \int\int e^{-(\zeta x_1 + pt)} G(x_1, t) dx_1 dt, \quad \mathrm{Re}\, p > 0. \tag{11}$$

The transform $\mathcal{L}[G](\zeta, p)$ is exactly like the corresponding transform of the elastic Green's function, except that the elastic wave speeds a, b of dilatational and shear waves are replaced by their viscoelastic counterparts which are given by

$$a^2(p) = \frac{p}{\rho}\left[\kappa \hat{g}_1(p) + \frac{4}{3}\mu \hat{g}_2(p)\right], \quad b^2(p) = \frac{p}{\rho}\mu \hat{g}_2(p), \tag{12}$$

where

$$\hat{g}_j(p) = \int_0^\infty g_j(t) e^{-pt} dt, \quad j = 1, 2. \tag{13}$$

Explicitly, the function $\mathcal{L}[G](\zeta, p)$ is

$$\mathcal{L}[G](\zeta, p) = \frac{p^2 \beta}{\rho b^4 [4\zeta^2 \alpha\beta + (\beta^2 - \zeta^2)^2]}, \tag{14}$$

where

$$\alpha = (p^2/a^2 - \zeta^2)^{1/2}, \quad \beta = (p^2/b^2 - \zeta^2)^{1/2}. \tag{15}$$

The branches of the square roots are chosen such that $\mathrm{Re}\, \alpha > 0$ and $\mathrm{Re}\, \beta > 0$ for all $\zeta \in L$ ($\mathrm{Re}\, p > 0$), to ensure boundedness of the Green's function as $x_2 \to \infty$.

3. Solution for steady-state loading

The limiting case of steady-state loading is obtained from the general transient case by multiplying by a factor p and then letting $p \to +0$. The relation (9) remains valid, except that the participating functions depend only on ζ; recall, however, that $K(\zeta)$ is obtained by first replacing p in $\mathcal{L}[G](\zeta, p)$ by $p - V\zeta$ before letting $p \to +0$. To specify the solution to the problem, choose the functions $g_1(t)$ and $g_2(t)$ as follows

$$g_j(t) = \frac{1}{1 + f_j}\left[1 + f_j \exp\left(-\frac{1 + f_j}{\tau_j} t\right)\right] H(t), \quad j = 1, 2, \tag{16}$$

which correspond to a standard linear solid with different bulk and shear relaxation. Here $H(t)$ is the Heaviside step function. In this case

$$\hat{g}_j(-V\zeta) = \frac{\zeta - \gamma_j}{\zeta - \gamma_j(1+f_j)}, \quad \gamma_j = \frac{1}{V\tau_j}, \quad j = 1, 2. \tag{17}$$

3.1. Analysis of the function $K(\zeta)$

The coefficient of the Riemann-Hilbert problem $K(\zeta)$ in the case (16) becomes

$$K(\zeta) = -\beta_0(\zeta)[1 + \beta_0^2(\zeta)][\hat{g}_2(-V\zeta)D(\zeta)]^{-1},$$
$$D(\zeta) = 4\alpha_0(\zeta)\beta_0(\zeta) + [\beta_0^2(\zeta) - 1]^2. \tag{18}$$

The functions $\alpha_0 = \zeta^{-1}\alpha$ and $\beta_0 = \zeta^{-1}\beta$ in (18) have four and two branch points, respectively:

$$\alpha_0(\zeta) = -i\eta_l \frac{(\zeta - a_{11})^{1/2}(\zeta - a_{12})^{1/2}}{(\zeta - a_{21})^{1/2}(\zeta - a_{22})^{1/2}}, \quad \eta_l = |d_l|^{1/2},$$
$$\beta_0(\zeta) = -i\eta_s(\zeta - a_s)^{1/2}(\zeta - \gamma_2)^{-1/2}, \quad \eta_s = |d_s|^{1/2}, \tag{19}$$

For the function α_0, the branch points are

$$a_{mj} = \tfrac{1}{2}[r_m + (-1)^{j-1}\sqrt{r_m^2 - 4t_m}], \quad m, j = 1, 2, \tag{20}$$

where

$$r_1 = d_l^{-1}\left\{V^2 c_l^{-2}[\gamma_1(1+f_1) + \gamma_2(1+f_2)] - r_2\right\},$$
$$t_1 = d_l^{-1}\left[V^2 c_l^{-2}\gamma_1\gamma_2(1+f_1)(1+f_2) - t_2\right],$$
$$r_2 = (\delta+1)^{-1}\{\delta[\gamma_1(1+f_1) + \gamma_2] + \gamma_1 + \gamma_2(1+f_2)\},$$
$$t_2 = \gamma_1\gamma_2(\delta+1)^{-1}[\delta(1+f_1) + 1 + f_2],$$
$$\delta = \frac{4\mu}{3\kappa}, \quad c_s = \sqrt{\frac{\mu}{\rho}}, \quad c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad d_l = \frac{V^2}{c_l^2} - 1. \tag{21}$$

The branch points of the function β_0 are $\zeta = a_s$ and $\zeta = \gamma_2$, where

$$a_s = \gamma_2 d_s^{-1}[V^2 c_s^{-2}(1+f_2) - 1], \quad d_s = V^2 c_s^{-2} - 1. \tag{22}$$

It can be proved that for all speeds $V \in (0, c_l)$ the branch points a_{11}, a_{21}, and a_{22} are in the half-plane $\mathcal{D}^- = \{\zeta : \mathrm{Re}\,\zeta > 0\}$. As for a_{12}, its position depends on the sign of the quantity $V - c_l\delta_f$, where

$$\delta_f = \sqrt{(1+\delta)^{-1}[(1+f_1)^{-1} + \delta(1+f_2)^{-1}]}. \tag{23}$$

If $V < \delta_f c_l$, then $a_{12} \in \mathcal{D}^-$. Otherwise $a_{12} \in \mathcal{D}^+= \{\zeta : \operatorname{Re}\zeta < 0\}$. Physically, $\delta_f c_l$ is the low-frequency limit of the phase velocity of longitudinal waves, calculated from the relaxed moduli $\kappa/(1+f_1)$, $\mu/(1+f_2)$.

The single branch $\alpha = \zeta\alpha_0$ has to meet the condition $\operatorname{Re}\alpha(\zeta) > 0$ for $\zeta \in L$. In the case $0 < V < \delta_f c_l$, it can be achieved by cutting the complex plane along the lines joining the branch points and passing through the infinite point. Let a^* be the branch point such that $\operatorname{Re} a^* = \min\{\operatorname{Re} a_{mj}\}$, $m, j = 1, 2$. Fix the arguments by

$$0 \le \arg(\zeta - a_{mj}) < 2\pi, \quad a_{mj} \neq a^*, \quad -\pi < \arg(\zeta - a^*) \le \pi; \qquad (24)$$

then $\operatorname{Re}\alpha(\zeta) > 0$, $\zeta \in L$. At $\zeta = 0$ and at infinity, the function α is discontinuous:

$$\alpha_0(\zeta) \sim \mp i\mu_l, \quad \zeta \to \pm i0, \quad \alpha_0(\zeta) \sim \mp i\eta_l, \quad \zeta \to \pm i\infty, \qquad (25)$$

where $\mu_l = \sqrt{|V^2(c_l\delta_f)^{-2} - 1|}$.

Suppose now that $\delta_f c_l < V < c_l$. Then $a_{12} \in \mathcal{D}^+$, and the other three branch points are in $\mathcal{D}^-$. To achieve $\operatorname{Re}\alpha(\zeta) > 0$, $\zeta \in L$, fix a single branch by the inequalities $-\pi < \arg(\zeta - a_{12}) \le \pi$, and for the other points by $0 \le \arg(\zeta - a_{mj}) < 2\pi$. The function $\alpha_0(\zeta)$ is continuous at the point $\zeta = 0$: $\alpha_0(\zeta) \sim -\mu_l$, $\zeta \to \pm i0$, and, as before, it is discontinuous at infinity: $\alpha_0(\zeta) \sim \mp i\eta_l$, $\zeta \to \pm i\infty$.

Analyze now the function $\beta(\zeta)$. In the case $0 < V < (1+f_2)^{-1/2}c_s$, both branch points belong to $\mathcal{D}^-$ and $a_s > \gamma_2$. Fix a single branch of the function (19) by

$$-\pi < \arg(\zeta - \gamma_2) \le \pi, \quad 0 \le \arg(\zeta - a_s) < 2\pi. \qquad (26)$$

Then for the chosen branch, $\operatorname{Re}\beta > 0$, $\zeta \in L$. At zero the branch is discontinuous: $\beta_0(\zeta) \sim \mp i\mu_s$, $\zeta \to \pm i0$, where $\mu_s = \sqrt{|V^2 c_s^{-2}(1+f_2) - 1|}$. In the next case, $(1+f_2)^{-1/2}c_s < V < c_s$, $a_s \in \mathcal{D}^+$, $d_s^{1/2} = -i\eta_s$, and the single branch of $\beta_0(\zeta)$ satisfies the requirement $\operatorname{Re}\beta(\zeta) > 0$, and it is continuous at zero:

$$\beta_0(\zeta) = -i\eta_s(\zeta - a_s)^{1/2}(\zeta - \gamma_2)^{-1/2} \sim -\mu_s, \quad \zeta \to \pm i0. \qquad (27)$$

Here $-\pi < \arg(\zeta - a_s) \le \pi$ and $0 \le \arg(\zeta - \gamma_2) < 2\pi$.

Finally, if $c_s < V < c_l$, then a_s returns to the domain $\mathcal{D}^-$. In contrast to the previous case, $d_s^{1/2}$ is real: $d_s^{1/2} = \eta_s$. Let $0 \le \arg(\zeta - a_s) < 2\pi$ and $0 \le \arg(\zeta - \gamma_2) < 2\pi$. The branch of β_0 in (19) can be written as $\beta_0 = -\eta_s(\zeta - a_s)^{1/2}(\zeta - \gamma_2)^{-1/2}$. It is continuous at zero: $\beta_0(\zeta) \sim -\mu_s$, $\zeta \to \pm i0$ and $\operatorname{Re}\beta(\zeta) > 0$, $\zeta \in L$.

Obviously, at infinity, the function $\beta_0(\zeta)$ is discontinuous for $V \in (0, c_s)$, $\beta_0(\zeta) \sim \mp i\eta_s$, $\zeta \to \pm i\infty$, and it is continuous in the transonic case $c_s < V < c_l$: $\beta_0(\zeta) \sim -\eta_s$, $\zeta \to \pm i\infty$.

To factorize the function $K(\zeta)$ it is crucial to describe its behavior at zero and infinity. This function is always discontinuous at infinity:

$$K(\zeta) \sim \begin{cases} \mp i\omega, & 0 < V < c_s \\ \omega_\pm, & c_s < V < c_l \end{cases} \quad \text{for} \quad \zeta \to \pm i\infty. \tag{28}$$

Here

$$\omega = \frac{\eta_s(1-\eta_s^2)}{4\eta_s\eta_l - (\eta_s^2+1)^2}, \quad \omega_\pm = \frac{\eta_s(1+\eta_s^2)}{\pm 4i\eta_s\eta_l + (\eta_s^2-1)^2}. \tag{29}$$

As for the point $\zeta = 0$, for some regimes the function $K(\zeta)$ is discontinuous and for the others it is continuous: $K(\zeta) \sim \mp i\nu_0, \quad \zeta \to \pm i0$, for lower speeds $0 < V < (1+f_2)^{-1/2}c_s$, and $K(\zeta) \sim \nu_\pm, \quad \zeta \to \pm i0$ for either $(1+f_2)^{-1/2}c_s < V < \delta_f c_l < c_s$, or $(1+f_2)^{-1/2}c_s < V < c_s < \delta_f c_l$, or $c_s < V < \delta_f c_l$. Notice that $\delta_f c_l > (1+f_2)^{-1/2}c_s$. Here

$$\nu_0 = \frac{\mu_s(1-\mu_s^2)(1+f_2)}{4\mu_s\mu_l - (\mu_s^2+1)^2}, \quad \nu_\pm = \frac{\mu_s(1+\mu_s^2)(1+f_2)}{\pm 4i\mu_s\mu_l + (\mu_s^2-1)^2}. \tag{30}$$

Otherwise, if $\delta_f c_l < V < c_s$, $\delta_f c_l < c_s < V < c_l$, or $c_s < \delta_f c_l < V < c_l$, the function $K(\zeta)$ is continuous at $\zeta = 0$: $K(\zeta) \sim \nu_1$, $\zeta \to \pm i0$, where

$$\nu_1 = \mu_s(1+\mu_s^2)(1+f_2)[4\mu_s\mu_l + (\mu_s^2-1)^2]^{-1}. \tag{31}$$

3.2. Factorization of the function $K(\zeta)$

Sub-Rayleigh regime. To remove the discontinuity of the function $K(\zeta)$ at infinity, split the function $K(\zeta)$ into two factors:

$$K(\zeta) = -\omega \tan \pi(\zeta - \delta_0) K_0(\zeta). \tag{32}$$

Factorize the first factor in terms of the Γ-function:

$$-\omega \tan \pi(\zeta - \delta_0) = T^+(\zeta)[T^-(\zeta)]^{-1},$$

$$T^+(\zeta) = \frac{\omega\Gamma(\frac{1}{2} + \delta_0 - \zeta)}{\Gamma(\delta_0 - \zeta)} \sim \omega(-\zeta)^{1/2}, \quad \zeta \to \infty, \quad \zeta \in \mathcal{D}^+,$$

$$T^-(\zeta) = \frac{\Gamma(1 - \delta_0 + \zeta)}{\Gamma(\frac{1}{2} - \delta_0 + \zeta)} \sim \zeta^{1/2}, \quad \zeta \to \infty, \quad \zeta \in \mathcal{D}^-. \tag{33}$$

Here δ_0 is any fixed real number such that $0 < \delta_0 < \frac{1}{2}$. The second factor $K_0(\zeta)$ is continuous at infinity: $K_0(\zeta) = 1 + O(\zeta^{-1})$, $\zeta \to \infty$.

Case $0 < V < (1+f_2)^{-1}c_s$. Then the function $K_0(\zeta)$ is discontinuous at zero: $K_0(\zeta) \sim \mp i\nu_0\omega^{-1}\cot\pi\delta_0$, $\zeta \to \pm i0$. To remove this discontinuity, introduce the function

$$K_1(\zeta) = -\tan\pi\delta_1 K_0(\zeta)\cot\pi(\zeta^{-1}-\delta_1) \tag{34}$$

with a parameter δ_1 to be described later. Then $K_1(\zeta) = 1 + O(\zeta^{-1})$, $\zeta \to \infty$, and it is continuous at zero:

$$K_1(\zeta) \sim -\nu_0\omega^{-1}\tan\pi\delta_1\cot\pi\delta_0, \quad \zeta \to \pm i0. \tag{35}$$

It is directly verified that the parameter ω introduced in (29), is positive if $0 < V < c_R$, and $\omega < 0$ for $c_R < V < c_s$, where c_R is the speed of Rayleigh waves. It can be shown that $\nu_0 > 0$ for lower speeds: $0 < V < c_{Rf}$, and $\nu_0 < 0$ if $c_{Rf} < V < (1+f_2)^{-1/2}c_s$. Here $c_{Rf} = \sqrt{w_f(1+f_2)^{-1}}c_s$, and w_f is the only root in the interval $0 < w < 1$, of the cubic equation

$$w^3 - 8w^2 + (24-16r)w + 16r - 16 = 0, \quad r = c_s^2[(1+f_2)c_l^2\delta_f^2]^{-1}. \tag{36}$$

Physically, c_{Rf} is the speed of low-frequency Rayleigh waves, calculated like c_R but from the relaxed elastic moduli.

Factorize now the function $K_1(\zeta)$. If $0 < V < c_{Rf}$, then take $\delta_1 \in (-\frac{1}{2}, 0)$. From (35) it follows that $K_1(\zeta)$ is continuous and positive at $\zeta = 0$. At infinity $K_1(\zeta) = 1+O(\zeta^{-1})$, and the increment of the argument of $K_1(\zeta)$, $[\arg K_1(\zeta)]_L$, as ζ traces the imaginary axis from $-i\infty$ to $i\infty$ is equal to zero (this is verified numerically for different sets of the parameters of the problem). Therefore, $K_1(\eta) = X_1^+(\eta)[X_1^-(\eta)]^{-1}$, $\eta \in L$, where $X_1^\pm(\eta)$ are the limiting values from the left and from the right hand-side on the contour L of the function

$$X_1(\zeta) = \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\log K_1(\eta)d\eta}{\eta-\zeta}\right\}, \quad -\pi < \arg K_1(\eta) < \pi. \tag{37}$$

To complete the factorization of the function $K(\zeta)$, we also need the following representation

$$-\tan\pi(\zeta^{-1}-\delta_1)\cot\pi\delta_1 = T_1^+(\zeta)[T_1^-(\zeta)]^{-1}, \tag{38}$$

with

$$T_1^+(\zeta) = -\frac{\Gamma(\frac{1}{2}+\delta_1-\frac{1}{\zeta})}{\tan\pi\delta_1\Gamma(1+\delta_1-\frac{1}{\zeta})}, \quad T_1^-(\zeta) = \frac{\Gamma(-\delta_1+\frac{1}{\zeta})}{\Gamma(\frac{1}{2}-\delta_1+\frac{1}{\zeta})}. \tag{39}$$

where $-\frac{1}{2} < \delta_1 < 0$. The factorization $K(\eta) = X^+(\eta)[X^-(\eta)]^{-1}$, $\eta \in L$, of the function $K(\zeta)$ is given by

$$X^\pm(\eta) = T^\pm(\eta)T_1^\pm(\eta)X_1^\pm(\eta), \quad \eta \in L. \tag{40}$$

Case $c_{Rf} \leq V < (1+f_2)^{-1/2}c_s$. The parameter ν_0 is negative, and we choose $\delta_1 \in (0, \frac{1}{2})$. Then

$$K_1(\pm i0) = -\frac{\nu_0 \tan \pi\delta_1}{\omega \tan \pi\delta_0} > 0, \quad [\arg K_1(\zeta)]_L = 0, \tag{41}$$

and formulas (39) should be replaced by $(0 < \delta_1 < \frac{1}{2})$

$$T_1^+(\zeta) = \frac{\Gamma(\frac{1}{2} + \delta_1 - \frac{1}{\zeta})}{\tan \pi\delta_1 \Gamma(\delta_1 - \frac{1}{\zeta})}, \quad T_1^-(\zeta) = \frac{\Gamma(1 - \delta_1 + \frac{1}{\zeta})}{\Gamma(\frac{1}{2} - \delta_1 + \frac{1}{\zeta})}. \tag{42}$$

Case $(1+f_2)^{-1}c_s \leq V \leq \min\{\delta_f c_l, c_R\}$. The function $K_0(\zeta)$ is still discontinuous at zero:

$$K_0(\zeta) \sim \nu_\pm \omega^{-1} \cot \pi\delta_0, \quad \zeta \to \pm i0. \tag{43}$$

Therefore, to remove this discontinuity we split the function $K_0(\zeta)$ into two factors

$$K_0(\zeta) = \frac{\tan \pi\omega_1}{-\tan \frac{\pi}{\zeta} + \tan \pi\omega_1} K_1(\zeta), \tag{44}$$

with

$$\omega_1 = \frac{1}{\pi} \tan^{-1} \frac{(\mu_s^2 - 1)^2}{4\mu_s\mu_l} \in [0, \tfrac{1}{2}]. \tag{45}$$

Note that $\omega_1 = 0$ when $V = \sqrt{2/(1+f_2)}c_s$, and $\omega_1 = \frac{1}{2}$ for either $V = \delta_f c_l$, or $V = (1+f_2)^{-1/2}c_s$. Then, the new function $K_1(\zeta)$ is continuous at zero:

$$K_1(\zeta) \sim \frac{\mu_s(\mu_s^2+1)(f_2+1)}{\omega(\mu_s^2-1)^2 \tan \pi\delta_0} > 0, \quad \zeta \to \pm i0, \tag{46}$$

and at infinity: $K_1(\zeta) = 1 + O(\zeta^{-1})$, $\zeta \to \pm i\infty$. As before $[\arg K_1(\zeta)]_L = 0$, and the function $K_1(\zeta)$ is factorized by the Cauchy integral (37). The factorization (40) of the coefficient of the Riemann-Hilbert problem (9) can still be used if we take the functions $T_1^\pm(\zeta)$ as follows

$$T_1^+(\zeta) = \frac{\sin \pi\omega_1 \Gamma(\omega_1 - \frac{1}{\zeta})}{\Gamma(\frac{1}{2} - \frac{1}{\zeta})}, \quad T_1^-(\zeta) = \frac{\Gamma(\frac{1}{2} + \frac{1}{\zeta})}{\Gamma(1 - \omega_1 + \frac{1}{\zeta})}. \tag{47}$$

Case $\delta_f c_l < V < c_R$. The factorization of the function $K(\zeta)$ can be constructed in an easier way. This is because of the continuity of the function $K_0(\zeta)$ at $\zeta = 0$:

$$K_0(\zeta) = -\frac{K(\zeta)}{\omega \tan \pi(\zeta - \delta_0)} \sim \frac{\nu_1}{\omega \tan \pi\delta_0} > 0, \quad \zeta \to \pm i0. \tag{48}$$

Since $[\arg K_0(\zeta)]_L = 0$, one can use the formulas of factorisation (37), (40) if $T_1^{\pm}$ and K_1 are replaced by 1 and K_0, respectively.

Super-Rayleigh regime. Factorize the coefficient of the Riemann-Hilbert problem for the speeds exceeding the Rayleigh speed c_R. As before, the function $K(\zeta)$ is discontinuous at infinity. What is different from the sub-Rayleigh regime is the sign of ω: $\omega < 0$ for $c_R < V < c_s$. Split the function $K(\zeta)$ as follows $K(\zeta) = T^+(\zeta)[T^-(\zeta)]^{-1}K_0(\zeta)$, where $\delta_0 \in (0, \frac{1}{2})$ and

$$T^+(\zeta) = -\frac{\omega\Gamma(\frac{1}{2} - \delta_0 - \zeta)}{\Gamma(1 - \delta_0 - \zeta)}, \quad T^-(\zeta) = \frac{\Gamma(\delta_0 + \zeta)}{\Gamma(\frac{1}{2} + \delta_0 + \zeta)}. \tag{49}$$

As for the function $K_0(\zeta)$, it may or may not be discontinuous at zero.

Case $\max\{(1 + f_2)^{-1}c_s, c_R\} < V < \min\{\delta_f c_l, c_s\}$. The function $K_0(\zeta)$ is discontinuous at zero. To factorize $K_0(\zeta)$, split it first into two factors

$$K_0(\zeta) = \frac{\tan \pi\omega_1 K_1(\zeta)}{-\tan\frac{\pi}{\zeta} + \tan \pi\omega_1}. \tag{50}$$

Then, at infinity, $K_1(\zeta) = 1 + O(\zeta^{-1})$. It is continuous and positive at zero, and $[\arg K_1(\zeta)]_L = 0$. Therefore, the factorization of the function $K(\zeta)$ is given by formulas (40), (37), (47), (45) and (49).

Case $\max\{c_R, \delta_f c_l\} < V < c_s$. The function $K_0(\zeta)$ is continuous and positive at $\zeta = 0$. At infinity, $K_0(\zeta) = 1+O(\zeta^{-1})$, and $[\arg K_0(\zeta)]_L = 0$. Therefore its factorization $K_0(\zeta) = X^+(\eta)[X^-(\eta)]^{-1}$, $\eta \in L$, is defined by (40), (37), where $K_1(\eta)$ should be replaced by $K_0(\eta)$, and $T_1^{\pm} = 1$. For the functions $T^{\pm}(\eta)$, formulas (49) need to be taken.

Transonic regime. For $c_s < V < c_l$, the function $K(\zeta)$ is discontinuous at infinity: $K(\zeta) \sim \omega_{\pm}$, $\zeta \to \pm i\infty$. By following [5] represent it in the form $K(\zeta) = T^+(\zeta)[T^-(\zeta)]^{-1}K_0(\zeta)$, where

$$T^+(\zeta) = \frac{\omega_* \cos \pi\omega_0 \Gamma(1 - \omega_0 - \zeta)}{\Gamma(\frac{1}{2} - \zeta)}, \quad T^-(\zeta) = \frac{\Gamma(\frac{1}{2} + \zeta)}{\Gamma(\omega_0 + \zeta)}, \tag{51}$$

with

$$\omega_* = \frac{\eta_s^2 + 1}{4\eta_l}, \quad \omega_0 = \frac{1}{\pi}\tan^{-1}\frac{(2 - V^2/c_s^2)^2}{4\eta_s\eta_l}, \quad 0 \le \omega_0 < \tfrac{1}{2}. \tag{52}$$

Case $c_s < V < \delta_f c_l$. The function $K_0(\zeta)$ is discontinuous at zero and it factorized by

$$K_0(\eta) = T_1^+(\eta)X_1^+(\eta)[T_1^-(\eta)X_1^-(\eta)]^{-1}, \quad \eta \in L, \tag{53}$$

with the function $T_1^{\pm}(\eta)$, $X_1^{\pm}(\eta)$ given by (47) and (37), (44). The function $K_1(\zeta)$ is continuous everywhere on the real axis and $[\arg K_1(\zeta)]_L =$

0. Making use of formulas (40) and (51), completes the splitting of the coefficient of the Riemann-Hilbert problem.

Case $\delta_f c_l < V < c_l$. This is the last possible case. Now, the function $K_0(\zeta)$ is continuous and positive at $\zeta = 0$: $K_0(\zeta) \sim \nu_1 \omega_*^{-1} \tan \pi\omega_0 > 0$, $\zeta \to \pm i0$. Since $[\arg K_0(\eta)]_L = 0$ we may use the factorization given by (40) with $T_1^\pm = 1$, $T^\pm(\eta)$ defined in (51). Replacing K_1 by K_0 in (37) gives the formula for the function $X_1(\zeta)$.

3.3. Solution to the Riemann-Hilbert problem

Analysis of the asymptotics of the factors $X^\pm(\zeta)$ shows that $X^\pm(\zeta) = O(\zeta^\varkappa)$, $\zeta \to \infty$, $\zeta \in \mathcal{D}^\pm$, where $\varkappa = \frac{1}{2}$ if $0 < V < c_R$, $\varkappa = -\frac{1}{2}$ if $c_R < V < c_s$, and $\varkappa = \frac{1}{2} - \omega_0$ if $c_s < V < c_l$. The functions $\tau_+(x)$ and $u'_-(x)$ may have integrable singularities at $x = 0$. The abelian theorem implies $F^\pm(\zeta) = O(\zeta^\epsilon)$, $\zeta \to \infty$, $\zeta \in \mathcal{D}^\pm$ with $\epsilon \in (-1, 0)$. Therefore, by the Liouville theorem, the solution is unique in the sub-Rayleigh and transonic cases

$$F^\pm(\zeta) = X^\pm(\zeta)\Psi^\pm(\zeta), \quad \zeta \in \mathcal{D}^\pm, \tag{54}$$

and it is not unique for the super-Rayleigh regime:

$$F^\pm(\zeta) = X^\pm(\zeta)[C + \Psi^\pm(\zeta)], \quad \zeta \in \mathcal{D}^\pm, \tag{55}$$

where C is an arbitrary constant. In (54) and (55),

$$\Psi^\pm(\zeta) = \frac{1}{2\pi i} \int_L \frac{P^+(\eta)}{X^-(\eta)} \frac{d\eta}{\eta - \zeta}, \quad \zeta \in \mathcal{D}^\pm. \tag{56}$$

4. Asymptotics of the shear traction

The super-Rayleigh regime (in which the solution is not unique) is not, in fact, attainable physically and is not discussed in this section. For all other speed ranges (up to the speed of longitudinal waves), the solution (54), (56) for F^- is equivalent, in real space, to the relation

$$\tilde{\sigma}_+(x) = (W_+)^{-1} * \{W_+ * \tilde{\sigma}_-\}_+. \tag{57}$$

Here, $W_+(x)$ has Laplace transform $[X^-(\zeta)]^{-1}$ and $(W_+)^{-1}(x)$ has Laplace transform $X^-(\zeta)$, so that W_+, $(W_+)^{-1}$ are inverses with respect to the operation of convolution. The symbol $\{.\}_+(x)$ represents the restriction of the function enclosed to positive values of x.

In the sub-Rayleigh regime, the traction component $\tilde{\sigma}_+(x)$ has the square root singularity at the tip of the crack $\tilde{\sigma}_+(x) \sim K_{II}(2\pi x)^{-1/2}$,

$x \to +0$. Here K_{II} is the stress-intensity factor defined by the abelian theorem from the asymptotics as $\zeta \to \infty$ of the representation (54) for $F^-(\zeta)$: $K_{II} = -\sqrt{2}\Psi_*\Lambda_s$, where

$$\Psi_* = \frac{1}{2\pi i}\int_L \frac{P^+(\eta)}{X^-(\eta)}d\eta, \tag{58}$$

$$\Lambda_s = \begin{cases} \Gamma(-\delta_1)[\Gamma(1/2-\delta_1)]^{-1}, & 0 < V < c_{Rf} \\ \Gamma(1-\delta_1)[\Gamma(1/2-\delta_1)]^{-1}, & c_{Rf} < V < (1+f_2)^{-1/2}c_s \\ \sqrt{\pi}[\Gamma(1-\omega_1)]^{-1}, & (1+f_2)^{-1/2}c_s < V < \min\{\delta_f c_l, c_R\} \\ 1, & \delta_f c_l < V < c_R \end{cases}. \tag{59}$$

In line with the real-space relation (57),

$$\Psi_* = \int_{-\infty}^0 W_+(-x')\tilde{\sigma}_-(x')\,dx'. \tag{60}$$

Thus, it is appropriate to regard W_+ as an un-normalized “weight function” which provides the stress singularity factor as a weighted integral of the applied loading $\tilde{\sigma}_-$. The function $W_+(x)$ decays exponentially as $x \to +\infty$ whenever $X^-(\zeta)$ can be defined by analytic continuation to the half-plane $\operatorname{Re}\zeta > -\lambda$ for some real $\lambda > 0$. This can be done if the kernel function $K(\zeta)$ is analytic in a strip that includes $-\lambda < \operatorname{Re}\zeta < 0$. Typically, λ increases with crack speed, as first found in [3] for antiplane loading. This has the effect that any given pattern of loading generates a stress singularity factor that reduces as the crack speed increases, thus tending to stabilize the propagation of the crack. The decay as $x \to \infty$ of $W_+(x)$ is intimately connected with the decay as $x \to \infty$ of the stress $\tilde{\sigma}_+(x)$.

To derive the asymptotics of the shear traction as $y = 0$, $x \to \infty$, apply the Jordan lemma. It yields $\tilde{\sigma}_+(x) = \sigma_+^{(\alpha)}(x) + \sigma_+^{(\beta)}(x)$, $x > 0$, where

$$\sigma_+^{(\alpha)}(x) = -\frac{4c_s^2}{\pi i V^2}\int_{-\lambda_a}^{-\infty+i0} \alpha_0(\zeta)\hat{g}_2^2(-V\zeta)F^+(\zeta)e^{\zeta x}d\zeta, \quad 0 < V < c_l,$$

$$\sigma_+^{(\beta)}(x) = -\frac{c_s^2}{\pi i V^2}\int_{-\lambda_b}^{-\infty+i0} \frac{[\beta_0^2(\zeta)-1]^2}{\beta_0(\zeta)}\hat{g}_2^2(-V\zeta)F^+(\zeta)e^{\zeta x}d\zeta, \quad 0 < V < c_R,$$

$$\sigma_+^{(\beta)}(x) = 0, \quad c_s < V < c_l,$$

$$-\lambda_a = \begin{cases} 0, & 0 < V < \delta_f c_l \\ a_{12}, & \delta_f c_l < V < c_l \end{cases},$$

$$-\lambda_b = \begin{cases} 0, & 0 < V < (1+f_2)^{-1/2} c_s \\ a_s, & (1+f_2)^{-1/2} c_s < V < c_R \end{cases}, \quad \lambda_a \geq 0, \quad \lambda_b \geq 0. \quad (61)$$

It becomes evident that the *exponential* decay of the traction $\tilde{\sigma}_+(x)$ as $x \to +\infty$ is observed if $\delta_f c_l < V < c_R$ for the sub-Rayleigh regime and for $\delta_f c_l < V < c_l$ for the transonic regime. Indeed, in the former case, putting $\zeta = -\zeta_1 - \lambda_a$, $\zeta = -\zeta_1 - \lambda_b$, respectively, transforms (61) into

$$\tilde{\sigma}_+(x) = e^{-\lambda_a x} T_1(x) + e^{-\lambda_b x} T_2(x), \quad x > 0,$$

$$T_1(x) = \frac{4c_s^2}{\pi i V^2} \int_0^\infty \alpha_0(-\zeta - \lambda_a + i0) \hat{g}_2^2(V(\zeta + \lambda_a)) F^+(-\zeta - \lambda_a) e^{-\zeta x} d\zeta,$$

$$T_2(x) = \frac{c_s^2}{\pi i V^2} \int_0^\infty \frac{[\beta_0^2(-\zeta - \lambda_b) - 1]^2}{\beta_0(-\zeta - \lambda_b + i0)} \hat{g}_2^2(V(\zeta + \lambda_b)) F^+(-\zeta - \lambda_b) e^{-\zeta x} d\zeta. \quad (62)$$

For the transonic case, when $\delta_f c_l < V < c_l$, this formula simplifies to

$$\tilde{\sigma}_+(x) = e^{-\lambda_a x} T_1(x), \quad x > 0. \quad (63)$$

Applying the abelian theorem for the case $\delta_f c_l < V < c_R$ gives

$$\tilde{\sigma}_+(x) \sim T' e^{-\lambda_a x} x^{-3/2}, \quad x \to \infty \quad \text{if} \quad a_{12} > a_s \quad (\lambda_a < \lambda_b), \quad (64)$$

$$\tilde{\sigma}_+(x) \sim T'' e^{-\lambda_b x} x^{-1/2}, \quad x \to \infty \quad \text{if} \quad a_{12} < a_s \quad (\lambda_a > \lambda_b), \quad (65)$$

where

$$T' = -\frac{2c_s^2 \eta_l}{\sqrt{\pi} V^2} \left[\frac{a_{11} - a_{12}}{(a_{21} - a_{12})(a_{22} - a_{12})} \right]^{1/2} \hat{g}_2^2(-V a_{12}) F^+(a_{12}),$$

$$T'' = \frac{c_s^2}{\sqrt{\pi} V^2 \eta_s} [\beta_0^2(a_s) - 1]^2 (\gamma_2 - a_s)^{1/2} \hat{g}_2^2(-V a_s) F^+(a_s). \quad (66)$$

In the transonic case for the speed $V \in (\delta_f c_l, c_l)$, the decay of the traction is also exponential:

$$\tilde{\sigma}_+(x) \sim T' e^{-\lambda_a x} x^{-3/2}, \quad x \to \infty, \quad (67)$$

where the constant T' is the same as that in (66).

For lower speeds in the sub-Rayleigh and transonic regimes, the decay of the shear traction is *algebraic*. To find the asymptotics of the shear traction $\tilde{\sigma}_+(x)$ as $x \to \infty$, we use the tauberian theorem for the Fourier transform. In the subsonic case, for the range of speeds $(0, c_{Rf})$,

$$\tilde{\sigma}_+(x) \sim -\frac{\Psi^-(0) X_1^-(0) \Gamma(1 - \delta_0)}{2\sqrt{\pi} \Gamma(\frac{1}{2} - \delta_0)} x^{-3/2}, \quad x \to \infty. \quad (68)$$

For $c_{Rf} < V < \min\{(1+f_2)^{-1/2}c_s, c_R\}$, it is found that

$$\tilde{\sigma}_+(x) \sim \frac{\Psi^-(0)X_1^-(0)\Gamma(1-\delta_0)}{\sqrt{\pi}\Gamma(\frac{1}{2}-\delta_0)}x^{-1/2}, \quad x \to \infty. \tag{69}$$

Finally, if $(1+f_2)^{-1/2}c_s < V < \min\{\delta_f c_l, c_R\}$, then the function $\tilde{\sigma}_+(x)$ decays at infinity as

$$\tilde{\sigma}_+(x) \sim \frac{\Psi^-(0)X_1^-(0)\Gamma(1-\delta_0)}{\Gamma(\omega_1-\frac{1}{2})\Gamma(\frac{1}{2}-\delta_0)}x^{-3/2+\omega_1}, \quad x \to \infty. \tag{70}$$

For the lower speeds in the transonic regime $c_s < V < \delta_f c_l$, the decay is also algebraic

$$\tilde{\sigma}_+(x) \sim \frac{\Psi^-(0)X_1^-(0)\sqrt{\pi}}{\Gamma(\omega_1-\frac{1}{2})\Gamma(\omega_0)}x^{-3/2+\omega_1}, \quad x \to \infty. \tag{71}$$

Notice that formulas (64), (68), (70), and (71) are derived for the case when $\tilde{\sigma}'_-(x) \in L(-\infty, 0)$ and also, as $x \to -\infty$, $\tilde{\sigma}_-(x) = o(x^{-3/2})$ for (64) and (68) and $\tilde{\sigma}_-(x) = o(x^{-3/2+\omega_1})$ for (70) and (71). Otherwise, the principal term in the asymptotics of the traction $\tilde{\sigma}_+(x)$ depends on the asymptotics as $x \to -\infty$ of the applied loading $\sigma_{12}^A(x, 0)$, $x = x_1 - Vt$.

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