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Journal of the Mechanics and Physics of solids
53 (2005) 1128–1162

JOURNAL OF THE
MECHANICS AND
PHYSICS OF SOLIDS

www.elsevier.com/locate/jmps

# High-order asymptotics and perturbation problems for 3D interfacial cracks

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Received 14 June 2004; accepted 30 November 2004

**Abstract**

We present an asymptotic algorithm for analysis of a singularly perturbed problem in a domain containing an interfacial crack. The crack is assumed to be flat and its front, initially straight, is perturbed in the plane containing the crack. The aim of the work is to determine the asymptotic representation of the stress-intensity factors near the edge of the crack. Mathematically, the limit problem is reduced to the analysis of a matrix, $3 \times 3$, Wiener-Hopf problem, and its solution generates the "weight matrix-function" characterised by a special singular solution near the crack edge. The two-term asymptotic representation for the weight function components is required by the asymptotic algorithm, together with two-term asymptotics for stress components associated with the physical fields near the edge of the crack. The particular feature of the solution is the coupling between the normal opening mode (Mode-I), and the shear modes (Mode-II and Mode-III), and the oscillatory behaviour of certain stress components near the crack edge. Explicit asymptotic formulae for the stress-intensity factors are obtained at the edge of a "wavy crack" at an interface.


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doi:10.1016/j.jmps.2004.11.008

## 1. Introduction

The models of straight interfacial cracks in two dimensions, or penny-shaped interfacial cracks are well-developed in the literature. Rice and Sih (1965) evaluated the stress-intensity factors for plane problems of cracks in dissimilar media. Bueckner (1970, 1987) introduced the weight functions to 3D cracks in elastostatics; they are used together with the loads applied to the crack surfaces, such that their weighted integrals over the crack surfaces provide the stress-intensity factors at a chosen point. Three-dimensional interfacial crack problems were studied systematically by Willis (1971a, b, 1972), who analysed the stresses around a crack on the plane interface between two dissimilar anisotropic half-spaces and considered the problems of dissimilar half-spaces bonded together over a circular area in the case of arbitrary loading.

When the front of the crack deviates from the original canonical shape, the physical fields as well as the stress-intensity factors at the crack edge also change. Gao and Rice (1989) studied the in-plane perturbation problems for 3D static cracks in a homogeneous elastic space. Willis and Movchan (1995) found the dynamic weight functions for the 3D opening Mode, and studied the perturbation from straightness of the crack front. Movchan et al. (1998) analysed out-of-plane perturbations of 3D cracks in a homogeneous space. Lazarus and Leblond (1998a, b), studied 3D crack-face weight functions for a semi-infinite interface crack. First, the paper (Lazarus and Leblond, 1998a) discusses the evaluation of the first-order variation of the stress-intensity factors along the crack front arising from some small but otherwise arbitrary in-plane perturbation of the crack front. Second, the work (Lazarus and Leblond, 1998b) presents an application of the earlier result to a certain class of perturbations, which leads to integrodifferential equations for the weight functions. Antipov (1999) has produced an analytical solution of the 3D problem for a semi-infinite half-plane interfacial crack. The associated weight functions were found by quadratures from the analysis of the physical fields.

Our aim is, following Willis and Movchan (1995), to present the Fourier transforms of the weight functions for the interfacial crack problem as singular solutions of a homogeneous boundary value problem and their application to determine the variation of the stress-intensity factors due to a small and smooth in-plane perturbation. In contrast with (Lazarus and Leblond, 1998a, b), our method is of a general type. We shall discuss the factorisation of the $3 \times 3$ matrix of the corresponding Wiener-Hopf problem in (Antipov, 1999), and construct a basis of singular solutions. This problem presents new features in comparison with Willis and Movchan (1995): the oscillatory behaviour of the weight functions and stress components, and a coupling in the stress-intensity factors corresponding to symmetric and skew-symmetric modes. These singular solutions are found *exactly*.

In a way similar to Willis and Movchan (1995), we shall use the reciprocal theorem as a tool that enables us to express the stress-intensity factors as a limit of a convolution of the weight function and the applied loads. The convolution integrals are interpreted in the sense of generalised functions. The main application is in the asymptotics of the stress-intentsity factors, associated with a perturbation of the crack front.

The results of the present paper may be of great help in development of the idea that crack tip instabilities naturally result from a wavy crack through an heterogeneous medium. The roughness observed along a fracture surface may be the direct result of a continuous roughnening of the surface that is driven by small inhomogeneities within the material, as observed by Marder and Fineberg (1999), as well as by loss of symmetry when a crack propagates along an interface between two elastic media. The paper presents a new asymptotic method of study of interfacial cracks with small perturbations of the crack front, and it also leads to an extension to problems of modelling cracks on wavy interface surfaces. Motivated by the earlier work by Willis and Movchan (1995) and Antipov (1999), this paper brings new mathematical results in the analysis of the high-order asymptotics of the weight-functions associated with 3D interfacial cracks.

## 2. Problem description

Before we describe the problem, we need to establish the geometry of the model domains and systems of coordinates, in order to use the factorisation described in Antipov (1999). We will have two different domains: one domain in which the weight function (special singular solution) is defined and one domain in which the displacement field is defined. It is important to remark here that the solution for the limit problem for the physical field is given to us in Antipov (1999).

### *2.1. Model domains and systems of coordinates*

#### *2.1.1. Model domain $\Omega_1$ for the weight function*

Let us consider an elastic isotropic space consisting of two half-spaces

$$\mathbb{R}_3^+ = \{|x|<\infty, 0<y<\infty, |z|<\infty\} \text{ and}$$

$$\mathbb{R}_3^- = \{|x|<\infty, -\infty<y<0, |z|<\infty\},$$

with Poisson's ratios $v_\pm$ and shear moduli $\mu_\pm$, respectively. Along the interface between the media there is a semi-infinite crack $\mathbb{R}_2^+ = \{0<x<\infty, y = \pm 0, -\infty<z<\infty\}$ that is free of tractions

$$\sigma_{yy} = 0, \quad \tau_{xy} = 0, \quad \tau_{yz} = 0.$$

On $\mathbb{R}_2^- = \{-\infty<x<0, y = \pm 0, -\infty<z<\infty\}$ we have ideal contact conditions

$$[\sigma_{yy}] = [\tau_{xy}] = [\tau_{yz}] = 0, \quad [u] = [v] = [w] = 0,$$

where $[f] = f|_{y=+0} - f|_{y=-0}$.

We can see that this problem has the same orientation than the problem in Antipov (1999), but it does not have loads.

#### 2.1.2. Model domain $\Omega_2$ for the physical problem (Problem I)

Let us consider an elastic isotropic space as before, with two half-spaces, $\mathbb{R}_3^{\pm}$. Along the interface between the media there is a semi-infinite crack $\mathbb{R}_2^- = \{-\infty < x < 0, y = \pm 0, -\infty < z < \infty\}$ that is acted on by normal and tangential loads

$$\sigma_{yy} = p_1(x,z), \quad \tau_{xy} = p_2(x,z), \quad \tau_{yz} = p_3(x,z).$$

On $\mathbb{R}_2^+ = \{0 < x < \infty, y = \pm 0, -\infty < z < \infty\}$ we have ideal contact conditions as in the weight function.

#### 2.1.3. Problems description

The displacement, either physical or singular, $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ satisfies the Lamé equations which with no body forces are

$$\mathscr{L}(\mathbf{u}) = \text{grad div } \mathbf{u} + (1 - 2v)\Delta\mathbf{u} = 0, \quad (x,y,z) \in \mathbb{R}_3^{\pm}, \tag{1}$$

where $v$ will be either $v_+$ or $v_-$, depending on the problem we are solving (see Figs. 1 and 2). Let us introduce the Fourier transforms of the stress on the crack faces, and

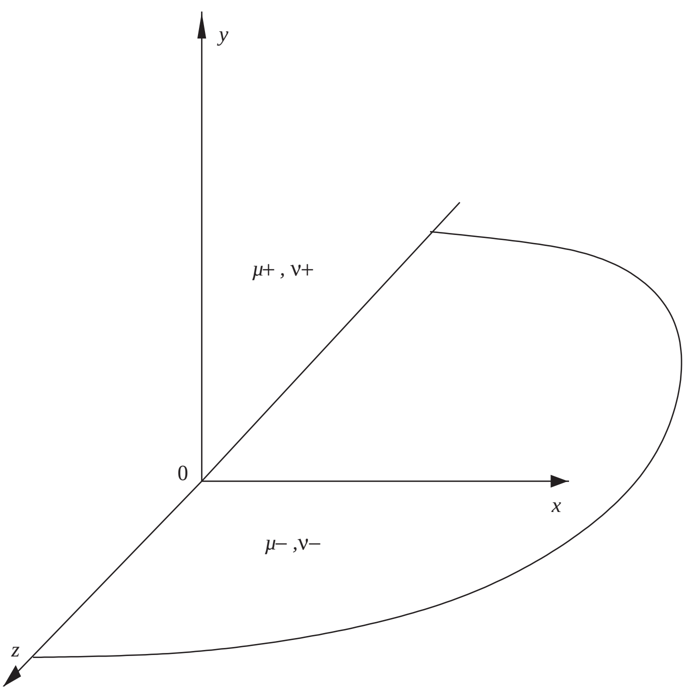


Fig. 1. Domain $\Omega_1$ (weight function).

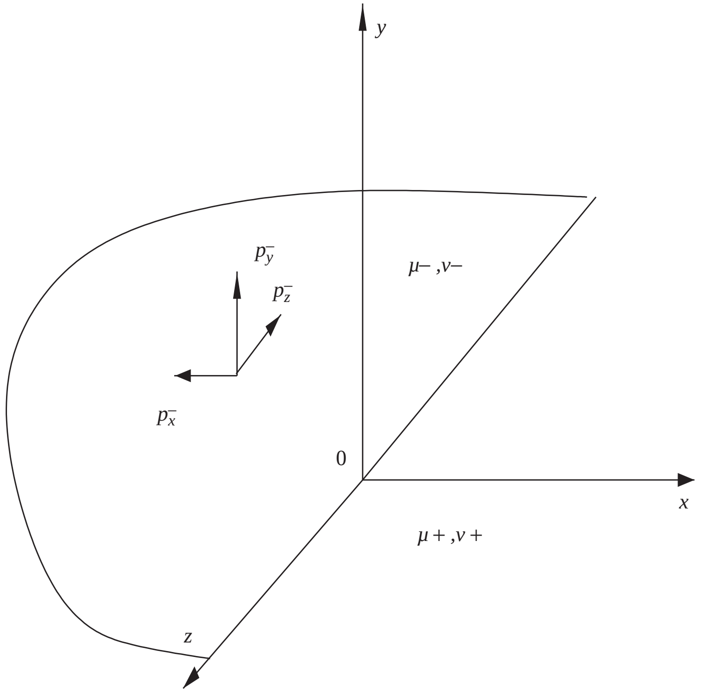


Fig. 2. Domain $\Omega_2$ for problem I.

of the jumps of the displacement on the interface

$$\tilde{\mathbf{p}}(\beta,\lambda)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{p}(x,z)\mathrm{e}^{\mathrm{i}\beta x}\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}x\,\mathrm{d}z,$$

$$[\tilde{\mathbf{v}}](\beta,\lambda)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\mathbf{v}](x,z)\mathrm{e}^{\mathrm{i}\beta x}\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}x\,\mathrm{d}z,$$

where

$$\mathbf{p}(x,z)=\begin{pmatrix}\sigma_{yy}(x,\pm 0,z)\\ \tau_{xy}(x,\pm 0,z)\\ \tau_{yz}(x,\pm 0,z)\end{pmatrix},\quad [\mathbf{v}]=\begin{pmatrix}[v]\\ [w]\\ [u]\end{pmatrix}.$$

Obs. We have to note that if $(x,y,z)\in\Omega_1$ then $(-x,-y,-z)\in\Omega_2$ (reflection of the system of coordinates). This will be used in Section 4, Eq. (25).

### 2.2. Special class of singular solutions and Wiener-Hopf problem

We define now the class of solutions in which we shall define the weight function later (Section 4). We shall use the factorisation in Antipov (1999) (see Appendix A) but our solution is singular and the problem is homogeneous (no stress on crack faces).

**Definition 2.1.** The weight function is defined within the class of singular solutions $U(x,y,z) = \{U_i(x,y,z)\}$, $(i = y,z,x)$ of Eq. (1) with homogeneous boundary conditions as follows:

(a) $U$ satisfies (1);
(b) $[U] = 0$ when $x<0$;
(c) The corresponding vector of stress

$$\Sigma = \begin{pmatrix} 2\mu \dfrac{\partial U_y}{\partial y} + \dfrac{2\mu v}{1-2v} \operatorname{div} U \\ \mu\left(\dfrac{\partial U_z}{\partial y} + \dfrac{\partial U_y}{\partial z}\right) \\ \mu\left(\dfrac{\partial U_x}{\partial y} + \dfrac{\partial U_y}{\partial x}\right) \end{pmatrix}$$

is continuous across the interface and when $y = 0$, $x>0$, then $\Sigma = 0$;
(d) $[U] \to \mathbf{F}(x)x^{-1/2}\delta(z)$ as $x \to 0^+$, $\mathbf{F}(x)$ a bounded vector-function, $\delta(z)$ is the Dirac delta function;
(e) $U$ decays at infinity.

This definition is similar to that in Willis and Movchan (1995, p. 322). However, the bounded vector-function $\mathbf{F}$ in (d) may oscillate, which is different compared to the homogeneous elastic space (see Willis and Movchan, 1995).

The Fourier transform of the jump $[U]$ shall be written according to (b) as

$$[\tilde{U}] = \int_{-\infty}^{\infty} \int_0^{\infty} [U] e^{i\beta x} e^{i\lambda z}\, dx\, dz. \tag{2}$$

The next result is key to our analysis

**Theorem 2.1** (*Willis, 1971b*). *The relationship between the Fourier transforms of the displacement jump and stress for an interfacial crack problem as defined in Section* 2.1.1 *is*

$$[\tilde{\mathbf{v}}] = \frac{1}{\rho} \mathbf{G}(\beta)\tilde{\mathbf{p}}, \quad \rho = (\lambda^2 + \beta^2)^{1/2}, \tag{3}$$

*where*

$$\tilde{\mathbf{p}}(\beta,\lambda) = \begin{pmatrix} \tilde{\sigma}_{yy}(\beta,\pm 0,\lambda) \\ \tilde{\tau}_{xy}(\beta,\pm 0,\lambda) \\ \tilde{\tau}_{yz}(\beta,\pm 0,\lambda) \end{pmatrix}, \quad [\tilde{\mathbf{v}}](\beta,\lambda) = \begin{pmatrix} [\tilde{v}] \\ [\tilde{w}] \\ [\tilde{u}] \end{pmatrix}.$$

$$\mathbf{G}(\beta) = -\frac{1}{\rho^2}\begin{pmatrix} b\rho^2 & \mathrm{i}d\lambda\rho & \mathrm{i}d\beta\rho \\ -\mathrm{i}d\lambda\rho & b\rho^2 + \mathrm{e}\beta^2 & -\mathrm{e}\beta\lambda \\ -\mathrm{i}d\beta\rho & -\mathrm{e}\beta\lambda & b\rho^2 + \mathrm{e}\lambda^2 \end{pmatrix},$$

$$b = \frac{1-v_+}{\mu_+} + \frac{1-v_-}{\mu_-}, \quad d = \frac{1-2v_+}{2\mu_+} - \frac{1-2v_-}{2\mu_-}, \quad e = \frac{v_+}{\mu_+} + \frac{v_-}{\mu_-}.$$

The single branch $\rho$ is fixed as in Antipov (1999).

We have to observe that $d = 0$ means $(1-2v_+)\mu_- = (1-2v_-)\mu_+$, so the homogeneous space is a particular case, and $\sigma_{yy}$ decouples as we would expect for the opening mode (Mode I) (see Willis and Movchan, 1995).

Considering our orientation, we can use the notations

$$\mathbf{\Phi}^+(\beta, \lambda) = \int_{-\infty}^{\infty}\int_0^{\infty} [U]\mathrm{e}^{\mathrm{i}\beta x}\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}x\,\mathrm{d}z,$$

$$\mathbf{\Phi}^-(\beta, \lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{0} \mathbf{\Sigma}\mathrm{e}^{\mathrm{i}\beta x}\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}x\,\mathrm{d}z,$$

for the Fourier transforms of our singular field $[U]$ and its associated vector of stress $\Sigma$, from (a) to (e). (In our notations, we shall omit the dependance on $\lambda$ in the following sections.) Therefore, using Eq. (3) we obtain the following functional relation:

$$\mathbf{\Phi}^+(\beta) = \frac{1}{\rho}\,\mathbf{G}(\beta)\mathbf{\Phi}^-(\beta). \tag{4}$$

The vector $\mathbf{\Phi}^+(\beta)$ is analytic in the upper half-plane $\mathbb{C}^+ = \{\Im\beta > 0\}$ and $\mathbf{\Phi}^-(\beta)$ is analytic in the lower half-plane $\mathbb{C}^- = \{\Im\beta < 0\}$. So Eq. (4) is a $3 \times 3$ matrix Wiener-Hopf problem. We have to emphasize here that the orientation of the problem is the same as in Antipov (1999), but Eq. (4) is homogeneous and we are searching for a special class of singular solutions (d).

## 3. Singular solutions

Before we proceed to find the solutions to the homogeneous problem with the singularity specified in definition (2.1), we need to factorise the matrix $\mathbf{G}$ (see Appendix A). All the details can be found in Antipov (1999, pp. 1055–1066). The important new features of this problem come from the factorisation. As a result of the factorisation, and considering the special class of solutions, we will find three independent singular solutions which comprise the *matrix-function of singular solutions* $[\mathbf{U}]$.

### 3.1. Fundamental singular solutions

According with definition (2.1), we are seeking solutions with $\beta$-asymptotics

$$\mathbf{\Phi}^+ \sim \mathbf{\Phi}_0^+ \beta^{-1/2}, \quad \beta \to \infty, \quad \beta \in \mathbb{C}^+, \tag{5}$$

with $\mathbf{\Phi}_0^+$ being a vector bounded at infinity. Substituting the representation for the matrix $\mathbf{G}$ (71) we obtain

$$\mathbf{Y}_+(\beta)\mathbf{\Phi}^+(\beta) = \mathbf{Y}_-(\beta)\mathbf{\Phi}^-(\beta), \tag{6}$$

and we can use the notations

$$\mathbf{E}_\pm(\beta) = \mathbf{Y}_\pm(\beta)\mathbf{\Phi}^\pm(\beta). \tag{7}$$

So far, our solution seems to be similar to the one found in Antipov (1999), but we are looking for a certain class of solutions at infinity (5). We can use (see Appendix A) Eq. (72) to see the behaviour of $\mathbf{Y}_+(\beta)$ at infinity. From Eq. (72), and considering the asymptotics in Eqs. (69), (76) and (64) we obtain

$$\mathbf{Y}_+(\beta) \sim \beta^{1/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1 & 0 & c_2 \end{pmatrix}, \quad \beta \to \infty, \;\; \beta \in \mathbb{C}^+, \tag{8}$$

where $c_1, \;\; c_2$ are constants.

It is clear from Eqs. (5), (7) and (8) that

$$\mathbf{E}_+(\beta) \sim \begin{pmatrix} 0 \\ 0 \\ C \end{pmatrix}, \quad \beta \to \infty, \;\; \beta \in \mathbb{C}^+. \tag{9}$$

Thus, to have the desired behaviour at infinity, we need to add a constant, $C_6$ in the entire function found in Antipov (1999), namely

$$\mathbf{E}_+(\beta) = \mathbf{E}_-(\beta) = \begin{pmatrix} \dfrac{C_1}{\beta - \mathrm{i}|\lambda|} + \dfrac{C_2}{\beta + \mathrm{i}|\lambda|} \\ \dfrac{C_3}{\beta + \mathrm{i}|\lambda|} + \dfrac{C_4}{\beta - a} \\ \dfrac{C_5}{\beta - a} + C_6 \end{pmatrix}, \quad \beta \in \mathbb{C}, \tag{10}$$

where $a = -\lambda_R$ if $|\lambda| < b^{-1}|d|^{-1}$ and $a = \mathrm{i}\lambda_R$ for $|\lambda| > b^{-1}|d|^{-1}$. (We have to note here that, since our problem is homogenous, our entire function does not present the elements $H_j^+$, $(j = 1, 2, 3)$ compared to (4.9) in Antipov (1999).) Hence, the solution for the Wiener-Hopf problem is given by

$$\mathbf{\Phi}^+(\beta) = \mathbf{Y}_+^{-1}(\beta)\mathbf{E}_+(\beta), \tag{11}$$

with

$$[\mathbf{Y}_+(\beta)]^{-1} = (\beta + \mathrm{i}|\lambda|)^{-1/2} \frac{1}{d^2} \begin{pmatrix} 0 & \Lambda^+ \Sigma_1^+ & \Lambda^+ \Sigma_2^+ \\ -d^2(b+e)\beta & \lambda \Lambda^+ \Sigma_2^+ & -\dfrac{\lambda}{\rho^2} \Lambda^+ \Sigma_1^+ \\ d^2(b+e)\lambda & \beta \Lambda^+ \Sigma_2^+ & -\dfrac{\beta}{\rho^2} \Lambda^+ \Sigma_1^+ \end{pmatrix}, \quad \beta \in \mathbb{C}^+. \tag{12}$$

In this class of solutions, three out of the six constants $C_i$, $i = 1, \ldots, 6$, are determined imposing analyticity. For this, we also need to study the matrix

$$[\mathbf{Y}_-(\beta)]^{-1} = (\beta - \mathrm{i}|\lambda|)^{1/2} \begin{pmatrix} 0 & -\rho \Lambda^- \sin(\rho B^-) & \Lambda^- \cos(\rho B^-) \\ \beta & \lambda \Lambda^- \cos(\rho B^-) & \dfrac{\lambda}{\rho} \Lambda^- \sin(\rho B^-) \\ -\lambda & \beta \Lambda^- \cos(\rho B^-) & \dfrac{\beta}{\rho} \Lambda^- \sin(\rho B^-) \end{pmatrix}, \quad \beta \in \mathbb{C}^-. \tag{13}$$

In order for the vectors $\mathbf{\Phi}^{\pm}(\beta)$ to be analytic at the points $\pm \mathrm{i}|\lambda|$, studying the behaviour of the matrices (12) and (13) we arrive at

$$\mathbf{\Phi}^+(\beta) = (2\mathrm{i}|\lambda|)^{-1/2} \begin{pmatrix} O(1) \\ -\dfrac{\mathrm{i}\,\mathrm{sgn}\,\lambda}{\beta - \mathrm{i}|\lambda|} L_\lambda + O(1) \\ \dfrac{1}{\beta - \mathrm{i}|\lambda|} L_\lambda + O(1) \end{pmatrix}, \quad \beta \to \mathrm{i}|\lambda|,$$

$$\mathbf{\Phi}^-(\beta) = (-2\mathrm{i}|\lambda|)^{1/2} \begin{pmatrix} O(1) \\ -\dfrac{\mathrm{i}|\lambda|}{\beta + \mathrm{i}|\lambda|} (C_2 + \mathrm{i}\,\mathrm{sgn}\,\lambda M_- C_3) + O(1) \\ -\dfrac{\lambda}{\beta + \mathrm{i}|\lambda|} (C_2 + \mathrm{i}\,\mathrm{sgn}\,\lambda M_- C_3) + O(1) \end{pmatrix}, \quad \beta \to -\mathrm{i}|\lambda|,$$

where

$$L_\lambda = (b+e)\lambda C_1 - \frac{\mathrm{i} M_+}{2d} \left[ \frac{C_5}{\mathrm{i}|\lambda| - a} + C_6 \right],$$

$$M_+ = d_0 (2|\lambda|)^{1/2} \begin{cases} (|\lambda| + \mathrm{i}\lambda_R)^{-1/2}, & |\lambda| < b^{-1}|d|^{-1}, \\ (|\lambda| + \lambda_R)^{-1/2}, & |\lambda| > b^{-1}|d|^{-1}, \end{cases}$$

$$M_- = M_+^{-1}, \quad 0 < \arg(|\lambda| + \mathrm{i}\lambda_R) < \frac{\pi}{2}.$$

Therefore it is necessary and sufficient that

$$L_\lambda = 0, \tag{14}$$

$$C_2 + \mathrm{i}\,\mathrm{sgn}\lambda M_- C_3 = 0. \tag{15}$$

Let us assume first that $|\lambda| < b^{-1}|d|^{-1}$. At the point $a = -\lambda_R$, Eqs. (13), (12) and (10) give

$$\begin{aligned}\mathbf{\Phi}^+(\beta) = {} & \frac{(-\lambda_R + \mathrm{i}|\lambda|)^{-1/2}\Lambda^+}{(\beta+\lambda_R)d^2}\begin{pmatrix} C_4\Sigma_1^+ + C_5\Sigma_2^+ \\ \lambda(C_4\Sigma_2^+ - \rho^{-2}C_5\Sigma_1^+) \\ \beta(C_4\Sigma_2^+ - \rho^{-2}C_5\Sigma_1^+)\end{pmatrix} \\ & + O(1), \quad \beta \to -\lambda_R, \ \beta \in \mathbb{C}^+,\end{aligned}$$

$$\begin{aligned}\mathbf{\Phi}^-(\beta) = {} & \frac{(-\lambda_R - \mathrm{i}|\lambda|)^{1/2}\Lambda^-}{\beta+\lambda_R}\begin{pmatrix} -C_4\rho\sin(\rho B^-) + C_5\cos(\rho B^-) \\ \lambda[C_4\cos(\rho B^-) + \rho^{-1}C_5\sin(\rho B^-)] \\ \beta[C_4\cos(\rho B^-) + \rho^{-1}C_5\sin(\rho B^-)]\end{pmatrix} \\ & + O(1), \quad \beta \to -\lambda_R, \ \beta \in \mathbb{C}^-.\end{aligned}$$

The vectors $\mathbf{\Phi}^\pm$ are analytic at the point $\beta = -\lambda_R$ if the following four conditions are satisfied:

$$\begin{aligned}& C_4\Sigma_1^+ + C_5\Sigma_2^+ = 0, \quad \beta = -\lambda_R, \\ & C_4\Sigma_2^+ - \rho^{-2}C_5\Sigma_1^+ = 0, \quad \beta = -\lambda_R, \\ & \lim_{\beta\to-\lambda_R} \Lambda^-[-C_4\rho\sin(\rho B^-) + C_5\cos(\rho B^-)] = 0, \\ & \lim_{\beta\to-\lambda_R} \Lambda^-[C_4\cos(\rho B^-) + \rho^{-1}C_5\sin(\rho B^-)] = 0,\end{aligned} \tag{16}$$

which give

$$\mathrm{i}C_4 - |d|bC_5 = 0. \tag{17}$$

The same result is obtained for the case $|\lambda| > b^{-1}|d|^{-1}$. We have, from the system

$$\begin{aligned}& 2d(b+e)\lambda C_1 - \mathrm{i}M_+[(\mathrm{i}|\lambda| - a)^{-1}C_5 + C_6] = 0, \\ & C_2 + \mathrm{i}\,\mathrm{sgn}\,\lambda M_- C_3 = 0, \\ & \mathrm{i}C_4 - |d|bC_5 = 0,\end{aligned} \tag{18}$$

with three independent solutions, and we can choose which are the parameters. The three independent solutions give the *matrix-function* $[\mathbf{U}]$ *of singular solutions*.

**Lemma 3.1.** *A fundamental system of solutions for* $\mathbf{\Phi}^+(\beta)$ *is given by*

$$\mathbf{\Phi}^+(\beta) = \{\mathbf{\Phi}_1^+(\beta), \mathbf{\Phi}_2^+(\beta), \mathbf{\Phi}_3^+(\beta)\}, \tag{19}$$

*where*

$$\mathbf{\Phi}_k^+(\beta) = \begin{pmatrix} \Phi_{1k}^+ \\ \Phi_{2k}^+ \\ \Phi_{3k}^+ \end{pmatrix}, \quad k = 1, 2, 3,$$

*and*

$$\mathbf{\Phi}_1^+ = (\beta + \mathrm{i}|\lambda|)^{-1/2} \begin{pmatrix} -2\mathrm{i}dM_-(b+e)\lambda\Lambda^+\Sigma_2^+ \\ -d^2(b+e)\beta(\beta - \mathrm{i}|\lambda|)^{-1} + 2\mathrm{i}dM_-(b+e)\dfrac{\lambda^2}{\rho^2}\Lambda^+\Sigma_1^+ \\ \lambda\left[d^2(b+e)(\beta - \mathrm{i}|\lambda|)^{-1} + 2\mathrm{i}dM_-(b+e)\dfrac{\beta}{\rho^2}\Lambda^+\Sigma_1^+\right] \end{pmatrix}, \tag{20}$$

$$\mathbf{\Phi}_2^+ = (\beta + \mathrm{i}|\lambda|)^{-3/2} \begin{pmatrix} \Lambda^+\Sigma_1^+ \\ d^2\mathrm{i}\,\mathrm{sgn}\,\lambda M_-(b+e)\beta + \lambda\Lambda^+\Sigma_2^+ \\ -\mathrm{i}d^2(b+e)|\lambda|M_- + \beta\Lambda^+\Sigma_2^+ \end{pmatrix}, \tag{21}$$

$$\mathbf{\Phi}_3^+ = (\beta + \mathrm{i}|\lambda|)^{-1/2} \begin{pmatrix} \Lambda^+[-\mathrm{i}|d|b(\beta-a)^{-1}\Sigma_1^+ + ((\beta-a)^{-1} - (\mathrm{i}|\lambda| - a)^{-1})\Sigma_2^+], \\ \lambda\Lambda^+\left[-\mathrm{i}|d|b(\beta-a)^{-1}\Sigma_2^+ - ((\beta-a)^{-1} - (\mathrm{i}|\lambda| - a)^{-1})\dfrac{\Sigma_1^+}{\rho^2}\right] \\ \beta\Lambda^+\left[-\mathrm{i}|d|b(\beta-a)^{-1}\Sigma_2^+ - ((\beta-a)^{-1} - (\mathrm{i}|\lambda| - a)^{-1})\dfrac{\Sigma_1^+}{\rho^2}\right] \end{pmatrix}. \tag{22}$$

**Proof.** Assume first $C_3 = C_5 = 0$ and $C_1 = d^2$. Then the first basis-vector becomes $\mathbf{\Phi}_1^+(\beta) = \{\Phi_{j1}^+(\beta)\}$, $j = 1, 2, 3$, where

$$\mathbf{\Phi}_1^+(\beta) = \mathbf{Y}_+^{-1}(\beta)\mathbf{E}_+(\beta),$$

$$\mathbf{E}_+(\beta) = d^2 \begin{pmatrix} (\beta - \mathrm{i}|\lambda|)^{-1} \\ 0 \\ -2\mathrm{i}d(b+e)\lambda M_- \end{pmatrix},$$

and the components $\Phi_{j1}^+(\beta)$ are given in Eq. (20).

Putting next $C_1 = C_5 = 0$ and $C_3 = d^2$ we have

$$\mathbf{E}_+(\beta) = \frac{d^2}{\beta + \mathrm{i}|\lambda|} \begin{pmatrix} -\mathrm{i}\,\mathrm{sgn}\,\lambda M_- \\ 1 \\ 0 \end{pmatrix},$$

and therefore the corresponding vector $\mathbf{\Phi}_2^+(\beta) = \{\Phi_{j2}^+(\beta)\}$ is given in Eq. (21). Finally, if we take $C_1 = C_3 = 0$, $C_5 = d^2$, then

$$\mathbf{E}_+(\beta) = d^2 \begin{pmatrix} 0 \\ -\mathrm{i}|d|b(\beta - a)^{-1} \\ (\beta - a)^{-1} - (\mathrm{i}|\lambda| - a)^{-1} \end{pmatrix},$$

and the components $\Phi_{j3}^+(\beta)$, $j = 1, 2, 3$ of the third basis-vector $\mathbf{\Phi}_3^+$ are given in Eq. (22). $\square$

## 4. Leading-order asymptotics. The weight function

Similar to Willis and Movchan (1995), the reciprocal theorem is used in our analysis. It will allow us to express the stress-intesity factors as a limit of a convolution (in the sense of generalised functions) of the weight function and the applied loads.

### 4.1. Integral identity

**Theorem 4.1** (*Willis and Movchan, 1995*). *If* $U(x, y, z) = (U_y, U_x, U_z)^{\mathrm{t}}$ *satisfies* (1) *in the domain* $\Omega_1$ *and* $\mathbf{u}(x, y, z) = (u_y, u_x, u_z)^{\mathrm{t}}$ *does satisfy* (1) *in* $\Omega_2$, *then*

$$\lim_{x \to 0} \{[U]^{\mathrm{t}} * \boldsymbol{\sigma}_y - \Sigma_y^{\mathrm{t}} * [\mathbf{u}]\}(x) = 0, \tag{23}$$

*where* $\boldsymbol{\sigma}_y(\mathbf{u}) = (\sigma_{yy}(\mathbf{u}), \sigma_{xy}(\mathbf{u}), \sigma_{yz}(\mathbf{u}))^{\mathrm{t}}$ *is the vector of stresses for* $\mathbf{u}$ *and* $\Sigma_y(U) = (\Sigma_{yy}(U), \Sigma_{xy}(U), \Sigma_{yz}(U))^{\mathrm{t}}$ *is the vector of stresses for* $U$.

**Proof.** For this purpose, we adapt the method of Willis and Movchan (1995) to our problem. If $\mathbf{u}$ satisfies Eq. (1), we can write $\mathscr{L}^{\pm}(\mathbf{u}) = 0$, where $\mathscr{L}^{\pm}$ is the elliptic self-adjoint operator

$$\mathscr{L}^{\pm}(\mathbf{u}) = \text{grad div } \mathbf{u} + (1 - 2\nu_{\pm})\Delta\mathbf{u}.$$

Integrating by parts, where $\Omega$ encloses a hemispherical domain in the half-space $y > 0$, whose plane boundary is $y = +0$, and whose radius $R$ is allowed to go to infinity,

$$\int_{\Omega} (\mathbf{v} \cdot \mathscr{L}^{\pm}\mathbf{u} - \mathbf{u} \cdot \mathscr{L}^{\pm}\mathbf{v})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_{(y=+0)} (\mathbf{v} \cdot \mathscr{T}_n^{\pm}\mathbf{u} - \mathbf{u} \cdot \mathscr{T}_n^{\pm}\mathbf{v})\, \mathrm{d}S = 0, \tag{24}$$

where $\mathscr{T}_n^{\pm}$ is the functional of the boundary conditions $\mathscr{T}_n^{\pm}\mathbf{u} = \sigma_{ij}^{\pm} n_j \mathrm{e}^{\mathrm{i}}$, with

$$\sigma_{ij}^{\pm} = \mu_{\pm}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \frac{2\mu_{\pm}\nu_{\pm}}{1 - 2\nu_{\pm}}\delta_{ij} \operatorname{div} \mathbf{u}, \quad \{e^i\} \text{ orthonormal basis of } \mathbb{R}^3.$$

We can emphasise the fact that if $v_i$ satisfies Eq. (1) in the domain $\Omega_2$, then

$$U_i(x - x', y, z - z') = -v_i(x' - x, -y, z' - z) \tag{25}$$

satisfies Eq. (1) in the domain $\Omega_1$. Therefore, we can write

$$\int_{(y=+0)} \{U^{\mathrm{t}}(x'-x,-y,z'-z)\boldsymbol{\sigma}_y(x,y,z) + \Sigma_y^{\mathrm{t}}(x'-x,-y,z'-z)\mathbf{u}(x,y,z)\}\,\mathrm{d}S = 0,$$

and thus,

$$\lim_{x'\to 0} \{U^{\mathrm{t}}(y=-0) * \boldsymbol{\sigma}_y(y=+0) + \Sigma_y^{\mathrm{t}}(y=-0) * \mathbf{u}(y=+0)\}(x') = 0.$$

Here, the convolution $*$ is taken over $x$ and $z$ in the sense of generalised functions. A similar equation can be obtained by integrating over a hemisphere in $y<0$. The designations $+0$ and $-0$ are interchanged. If one representation is subtracted from the other, and $\mathbf{v}$ is constructed like $\mathbf{u}$ so that its associated stresses are continuous across $y=0$, while $\mathbf{v}$ may have a discontinuity when $x'>0$, then

$$\lim_{x'\to 0} \{[U]^{\mathrm{t}} * \boldsymbol{\sigma}_y - \Sigma_y^{\mathrm{t}} * [\mathbf{u}]\}(x') = 0. \qquad \square \tag{26}$$

It is important to remark here, that when we have the matrix-function $[\mathbf{U}]$, whose columns are independent singular solutions in the system of reference $(y,x,z)$, the identity (23) becomes

$$\lim_{x\to 0} \{[\mathbf{U}]^{\mathrm{t}} * \boldsymbol{\sigma}_y - \boldsymbol{\Sigma}_y^{\mathrm{t}} * [\mathbf{u}]\}(x) = 0. \tag{27}$$

From definition (2.1) we can deduce that if we evaluate the convolution for $x>0$,

$$\lim_{x\to 0} \{\boldsymbol{\Sigma}_y^{\mathrm{t}} * [\mathbf{u}]\}(x) = 0,$$

as $\boldsymbol{\Sigma}_y^{\mathrm{t}}$ and $[\mathbf{u}]$ are both "$-$" functions.

Therefore, we have

$$\lim_{x\to 0} \{[\mathbf{U}]^{\mathrm{t}} * \boldsymbol{\sigma}_y\}(x) = 0.$$

We can express the stress $\boldsymbol{\sigma}_y(x,z)$ as a sum of two fields:

(i) Ahead of the physical crack, when $x>0$, $\boldsymbol{\sigma}(x,z)$
(ii) On the faces of the crack, when $x<0$, we have the loads $\mathbf{p}(x,z)$.

Thus, the reciprocal theorem becomes for the singular matrix-function

$$\lim_{x\to 0} \{[\mathbf{U}]^{\mathrm{t}} * \boldsymbol{\sigma} - \boldsymbol{\Sigma}_y^{\mathrm{t}} * [\mathbf{u}]\}(x) = -\lim_{x\to 0} \{[\mathbf{U}]^{\mathrm{t}} * \mathbf{p}\}(x). \tag{28}$$

### *4.2. 2D interfacial crack. Illustrative example*

We want to illustrate the use of the integral identity in the evaluation of the stress-intensity factors in a well-known 2D interfacial crack problem, studied in Rice and Sih (1965) and Hutchinson et al. (1987). In particular, the cancellation of oscillatory terms becomes apparent, which will be used in the normalisation of the weight function. The formulae for the stress-intensity factors are well-known for this case, and one can easily follow the illustrative derivation based on the integral identity.

In this problem the $xy$ plane is divided in two half-planes by an interfacial crack $x<0$. The upper (lower) half-plane $y>0$ ($y<0$) presents elastic constants $\nu_+$, $\mu_+$ ($\nu_-$,$\mu_-$). The bimaterial constant $\varepsilon$ is defined as in Eq. (69).

Ahead of the crack tip ($x>0$), the stress is given by

$$\sigma_{yy} + i\tau_{xy} = \mathscr{K}(2\pi)^{-1/2}x^{-1/2+\mathrm{i}\varepsilon},$$

where $\mathscr{K} = K_I + iK_{II}$ is the complex stress-intensity factor and $\mathscr{K} = \sqrt{\pi}\cosh(\pi\varepsilon)(k_I + ik_{II})$, where $\sqrt{\pi}$ is standard in converting the lower case $k$'s of Rice and Sih (1965) to $K$'s in Hutchinson et al. (1987), and $\cosh(\pi\varepsilon)$ is included so the magnitude of the *traction* vector is given by $(\sigma_{yy}^2 + \tau_{xy}^2)^{1/2} = |\mathscr{K}|/\sqrt{2\pi r}$, where $r$ is the distance between the crack tip and the point where the point forces are applied on the crack edges. Therefore

$$\begin{pmatrix} \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \frac{x^{-1/2}}{\sqrt{2\pi}} N(x) \begin{pmatrix} K_I \\ K_{II} \end{pmatrix},$$

with $N(x) = \begin{pmatrix} \cos(\varepsilon\log x) & -\sin(\varepsilon\log x) \\ \sin(\varepsilon\log x) & \cos(\varepsilon\log x) \end{pmatrix}$.

The displacement jumps behind the tip are given by

$$[u_y + \mathrm{i}u_x] = 2\,\frac{\{(1-\nu_+)/\mu_+ + (1-\nu_-)/\mu_-\}}{(1+2\mathrm{i}\varepsilon)\cosh(\pi\varepsilon)}\,\mathscr{K}(2\pi)^{-1/2}r^{1/2+\mathrm{i}\varepsilon},$$

where $[f] = f(-r,0^+) - f(-r,0^-)$.

The *weight function* can be defined as the stress-intensity factors when the loadings are point forces, say $P + \mathrm{i}Q$. We have expressions for $k_i$ from Rice and Sih (1965) at the crack tip, namely

$$k_1(0) = \frac{1}{\pi}\left(\frac{2}{r}\right)^{1/2}\{P\cos(\varepsilon\log r) + Q\sin(\varepsilon\log r)\},$$

$$k_2(0) = \frac{1}{\pi}\left(\frac{2}{r}\right)^{1/2}\{Q\cos(\varepsilon\log r) - P\sin(\varepsilon\log r)\}.$$

Therefore,

$$k_1 + \mathrm{i}k_2 = \frac{1}{\pi}\left(\frac{2}{r}\right)^{1/2}\{P + \mathrm{i}Q\}r^{-\mathrm{i}\varepsilon}.$$

With a distribution of stresses $\sigma_{yy} = -p(x)$, $\tau_{xy} = -q(x)$ on the crack faces we obtain

$$\begin{pmatrix} K_I \\ K_{II} \end{pmatrix} = -\sqrt{\frac{2}{\pi}}\cosh(\pi\varepsilon)\int_{-\infty}^{0}(-x)^{-1/2}M(x)\begin{pmatrix} p(x) \\ q(x) \end{pmatrix}\mathrm{d}x,$$

$M(-x) = \begin{pmatrix} \cos(\varepsilon\log(-x)) & \sin(\varepsilon\log(-x)) \\ -\sin(\varepsilon\log(-x)) & \cos(\varepsilon\log(-x)) \end{pmatrix}$, and we can write

$$\begin{pmatrix} K_I \\ K_{II} \end{pmatrix} = \int_{-\infty}^{\infty}[U](\xi - x)P(x)\,\mathrm{d}x = [U]*P(\xi), \quad \xi \to 0, \tag{29}$$

and the weight function

$$[U](x) = -\sqrt{\frac{2}{\pi}}\cosh(\pi\varepsilon)x^{-1/2}M(x),$$

where $x>0$. We show that Eq. (29) can also be obtained by applying Betti's identity (23) which in terms of Fourier transforms becomes

$$[\tilde{U}]\tilde{\sigma} - \tilde{\Sigma}[\tilde{u}] = -[\tilde{U}]\tilde{P}, \quad \beta \to \infty,$$

with $[\tilde{U}] = -\sqrt{\frac{2}{\pi}}\cosh(\pi\varepsilon)\int_0^\infty e^{i\beta x}x^{-1/2}M(x)\,dx$. We can write

$$\cos(\varepsilon\log x) = \frac{1}{2}(x^{i\varepsilon} + x^{-i\varepsilon}),$$
$$\sin(\varepsilon\log x) = \frac{1}{2i}(x^{i\varepsilon} - x^{-i\varepsilon}).$$

Using the fact from Gradshtein and Ryzhik (1980)

$$\int_0^\infty x^{k-1}e^{i\beta x}\,dx = e^{i\pi k/2}\Gamma(k)\beta^{-k}, \quad -\frac{\pi}{2} < \arg\beta < \frac{3\pi}{2}, \tag{30}$$

we obtain

$$[\tilde{U}] = -\sqrt{\frac{2}{\pi}}\cosh(\pi\varepsilon)\begin{pmatrix} \frac{1}{2}(a_1\beta^{-1/2-i\varepsilon} + a_2\beta^{-1/2+i\varepsilon}) & \frac{-i}{2}(a_1\beta^{-1/2-i\varepsilon} - a_2\beta^{-1/2+i\varepsilon}) \\ \frac{-i}{2}(a_2\beta^{-1/2+i\varepsilon} - a_1\beta^{-1/2-i\varepsilon}) & \frac{1}{2}(a_1\beta^{-1/2-i\varepsilon} + a_2\beta^{-1/2+i\varepsilon}) \end{pmatrix},$$

$$\tilde{\sigma} = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} \frac{1}{2}(a_1\mathscr{K}\beta^{-1/2-i\varepsilon} + a_2\overline{\mathscr{K}}\beta^{-1/2+i\varepsilon}) \\ -\frac{i}{2}(a_1\mathscr{K}\beta^{-1/2-i\varepsilon} - a_2\overline{\mathscr{K}}\beta^{-1/2+i\varepsilon}) \end{pmatrix},$$

where $a_1 = \Gamma(1/2 + i\varepsilon)e^{i\pi/4}/e_0$, $a_2 = \Gamma(1/2 - i\varepsilon)e^{i\pi/4}e_0$, and $\overline{(\cdot)}$ denotes the complex conjugate.

Since

$$\Gamma(1/2 + i\varepsilon)\Gamma(1/2 - i\varepsilon) = \frac{\pi}{\cosh(\pi\varepsilon)},$$

we derive

$$[\tilde{U}]\tilde{\sigma} \sim -i\begin{pmatrix} K_I \\ K_{II} \end{pmatrix}\frac{1}{\beta + 0i}, \quad \beta \to \infty.$$

Similarly, we obtain

$$\tilde{\Sigma}[\tilde{u}] \sim -i\begin{pmatrix} K_I \\ K_{II} \end{pmatrix}\frac{1}{\beta - 0i}, \quad \beta \to \infty,$$

so the identity (28) thus yields

$$-\mathrm{i}\begin{pmatrix} K_I \\ K_{II} \end{pmatrix}\left\{\frac{1}{\beta+0\mathrm{i}}-\frac{1}{\beta-0\mathrm{i}}\right\} = -2\pi\delta(\beta)\begin{pmatrix} K_I \\ K_{II} \end{pmatrix} = -[\tilde{U}]\tilde{P}, \quad \beta\to\infty,$$

where $\delta(\beta)$ is the Dirac $\delta$-function, and finally, taking the inverse Fourier transform,

$$\begin{pmatrix} K_I \\ K_{II} \end{pmatrix} = [U]*P(\xi), \quad \xi\to 0.$$

### 4.3. Leading order asymptotics for the singular field

**Theorem 4.2.** *The Fourier transform of the matrix-function of singular solutions* **[U]** *whose columns are defined in Eq.* (2) *has leading order asymptotics*

$$[\tilde{\mathbf{U}}]^{\mathrm{t}}(\beta) \sim \begin{pmatrix} \psi_1 c^+ & \psi_1 s^+ & -d^2(b+\mathrm{e}) \\ \psi_2 s^+ & -\psi_2 c^+ & \mathrm{i}\alpha(\lambda) \\ \psi_3 c^+ & \psi_3 s^+ & -|d|b\alpha(\lambda) \end{pmatrix}\beta^{-1/2}, \quad \beta\to\infty,\ \beta\in\mathbb{C}^+, \tag{31}$$

*with* $\psi_1 = 2\mathrm{i}d_0 d^3 M_-(b+\mathrm{e})b\lambda$, $\psi_2 = d^2 d_0 b$, $\psi_3 = d_0 d^2 b(\mathrm{i}|\lambda|-a)^{-1}$, $\alpha(\lambda) = (b+\mathrm{e})d^2 \operatorname{sgn}\lambda M_-$ *and* $c^+$, $s^+$ *as in Eq.* (70).

**Proof.** To derive two-term-asymptotics of the vectors $\mathbf{\Phi}_j^+(\beta)$ at infinity we use formulae (64), (68) and (76). Then, for the function $\Phi_{11}^+(\beta)$, we obtain

$$\Phi_{11}^+(\beta) \sim \beta^{-1/2}\left(1-\frac{\mathrm{i}|\lambda|}{2\beta}\right) \times\left[-2\mathrm{i}dM_-(b+\mathrm{e})\lambda d_0\left\{-bd^2c^+ + \beta^{-1}\left(dis^+ - \frac{\mathrm{i}}{2}\,\lambda_j bd^2c^+\right)\right\}\right].$$

In the same manner, similar formulae as $\beta\to\infty$, $\beta\in\mathbb{C}^+$, can be written for the other eight functions $\Phi_{jk}(\beta)$. Taking the leading order term, $O(\beta^{-1/2})$ we obtain

$$\Phi_{jk}^+(\beta) \sim f_{jk}^0(\beta), \quad j,k=1,2,3, \tag{32}$$

with

$$\begin{aligned} f_{11}^0 &= \psi_1 c^+\beta^{-1/2}, \\ f_{21}^0 &= -(b+\mathrm{e})d^2\beta^{-1/2}, \\ f_{31}^0 &= \psi_1 s^+\beta^{-1/2}, \end{aligned} \tag{33}$$

where $\psi_1 = 2\mathrm{i}d_0 d^3 M_-(b+\mathrm{e})b\lambda$,

$$\begin{aligned} f_{12}^0 &= \psi_2 s^+\beta^{-1/2}, \\ f_{22}^0 &= \mathrm{i}\alpha(\lambda)\beta^{-1/2}, \\ f_{32}^0 &= -\psi_2 c^+\beta^{-1/2}, \end{aligned} \tag{34}$$

$\psi_2 = d^2 d_0 b,\ \alpha(\lambda) = (b+e)d^2 \mathrm{sgn}\,\lambda M_-.$

$$\begin{aligned} f^0_{13} &= (\psi_3^- s^+ + \psi_3 c^+)\beta^{-1/2},\\ f^0_{23} &= 0,\\ f^0_{33} &= (\psi_3 s^+ - \psi_3^- c^+)\beta^{-1/2}, \end{aligned} \tag{35}$$

$\psi_3^- = -\mathrm{i}d_0|d|d^2b^2, \quad \psi_3 = d_0 d^2 b(\mathrm{i}|\lambda| - a)^{-1}.$

We can write the leading order asymptotics for $[\tilde{\mathbf{U}}](\beta)$, in the system of coordinates $(y, x, z)$ as

$$[\tilde{\mathbf{U}}](\beta) \sim \mathbf{U}(\beta)\beta^{-1/2}, \quad \beta \to \infty,\ \beta \in \mathbb{C}^+, \tag{36}$$

with

$$\mathbf{U}(\beta) = \begin{pmatrix} \psi_1 c^+ & \psi_2 s^+ & \psi_3^- s^+ + \psi_3 c^+ \\ \psi_1 s^+ & -\psi_2 c^+ & \psi_3 s^+ - \psi_3^- c^+ \\ -d^2(b+e) & \mathrm{i}\alpha(\lambda) & 0 \end{pmatrix}. \tag{37}$$

Denoting the second and third column of $\mathbf{U}(\beta)$ $c2$ and $c3$, respectively, $c3 - (\psi_3^-/\psi_2)c2$ gives Eq. (31). $\square$

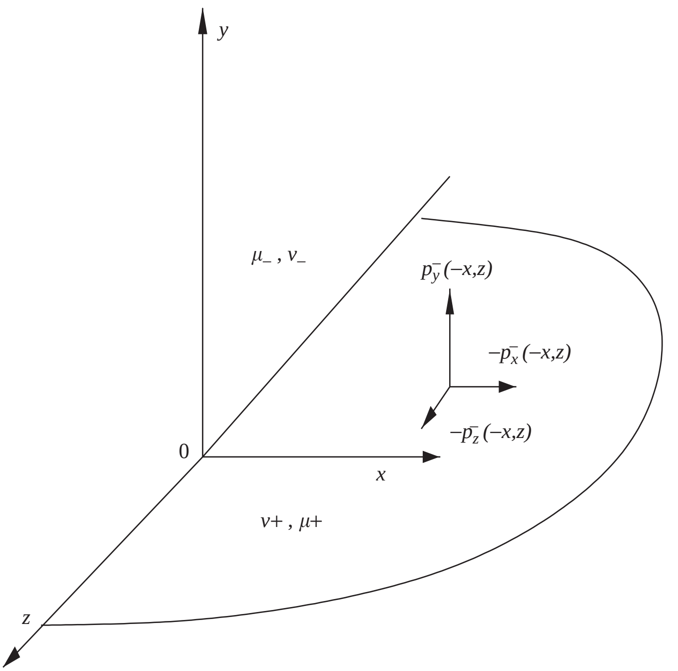


Fig. 3. Domain $\Omega_3$ (problem II).

### 4.4. Leading order asymptotics for the physical stress field

The stress field $\boldsymbol{\sigma}$ is associated with the physical displacement $\mathbf{u}$ that satisfies the non-homogenous problem described in Fig. 2, problem I, on the domain $\Omega_2$. The leading asymptotics for stress were found in Antipov (1999), with a configuration similar to Fig. 1 but with loads. If we consider a rotation about the $y$-axis through an angle $\pi$ ($\mathbf{R}_{y,\pi} = \mathrm{diag}(1,-1,-1)$) of the domain $\Omega_2$ in Fig. 2, we can see in Fig. 3 (problem II), the situation in comparison with Antipov (1999). The main difference is the sign of some components of the vector of stress, and the fact that the elastic constants are interchanged in domain $\Omega_3$.

**Lemma 4.1.** *The Fourier transform of stress in the physical problem described in Fig.* 2 (*Problem I*) *can be obtained from the physical problem in* Antipov (1999) (*Problem II after interchanging the elastic constants*) *in the following manner*: *If we denote* $\mathbf{\Phi}_I^+(\beta)$, $\mathbf{\Phi}_{II}^-(\beta)$ *the Fourier transform of the physical stress in problems I and II*, *respectively*, *then*

$$\mathbf{\Phi}_I^+(\beta) = \mathbf{R}_{y,\pi}\mathbf{\Phi}_{II}^-(-\beta), \quad \beta \in \mathbb{C}^+, \tag{38}$$

*where* $\mathbf{R}_{y,\pi} = diag(1,-1,-1)$.

**Proof.** Let $\mathbf{\Phi}_I^+ = \int_\infty^\infty \mathrm{e}^{\lambda z}\,\mathrm{d}z \int_0^\infty \begin{pmatrix} \sigma_{yy}(x,0,z) \\ \tau_{yz}(x,0,z) \\ \tau_{xy}(x,0,z) \end{pmatrix} \mathrm{e}^{\mathrm{i}\beta x}\,\mathrm{d}x$. We can see from Fig. 3 that

$$\mathbf{\Phi}_{II}^-(\beta) = \mathbf{R}_{y,\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \int_{-\infty}^{0} \begin{pmatrix} \sigma_{yy}(-x,0,z) \\ \tau_{yz}(-x,0,z) \\ \tau_{xy}(-x,0,z) \end{pmatrix} \mathrm{e}^{\mathrm{i}\beta x}\,\mathrm{d}x,$$

is the Fourier transform of the traction vector

$$\mathbf{R}_{y,\pi} \begin{pmatrix} \sigma_{yy}(-x,0,z) \\ \tau_{yz}(-x,0,z) \\ \tau_{xy}(-x,0,z) \end{pmatrix},$$

with $\mathbf{R}_{y,\pi} = \mathrm{diag}(1,-1,-1)$. Therefore

$$\mathbf{\Phi}_{II}^-(-\beta) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \int_{0}^{\infty} \begin{pmatrix} \sigma_{yy}(x,0,z) \\ -\tau_{yz}(x,0,z) \\ -\tau_{xy}(x,0,z) \end{pmatrix} \mathrm{e}^{\mathrm{i}\beta x}\,\mathrm{d}x,$$

is an analytic function in $\mathbb{C}^+$, and $\mathbf{\Phi}_{II}^-(-\beta) = \mathbf{\Phi}_I^+(\beta)$. And finally,

$$\mathbf{\Phi}_{II}^-(\beta) = \mathbf{\Phi}_I^+(-\beta) = \mathbf{R}_{y,\pi}\mathbf{\Phi}^-(\beta; -d), \quad \beta \in \mathbb{C}^-,$$

where $\mathbf{\Phi}^-(\beta)$ is defined in Antipov (1999), (2.6) and

$$\mathbf{\Phi}_I^+(\beta) = \mathbf{R}_{y,\pi}\mathbf{\Phi}^-(-\beta; -d), \quad \beta \in \mathbb{C}^+. \qquad \square$$

**Theorem 4.3.** *The Fourier transform of the physical stress has leading order asymptotics*

$$\tilde{\boldsymbol{\sigma}}(\beta) \sim \frac{1}{2}\begin{pmatrix} \dfrac{e_0}{c_1}\tilde{K}(\lambda)(-\beta)^{-\mathrm{i}\varepsilon} + \dfrac{1}{e_0 c_2}\bar{\tilde{K}}(\lambda)(-\beta)^{\mathrm{i}\varepsilon} \\ \mathrm{i}\left[\dfrac{e_0}{c_1}\tilde{K}(\lambda)(-\beta)^{-\mathrm{i}\varepsilon} - \dfrac{1}{e_0 c_2}\bar{\tilde{K}}(\lambda)(-\beta)^{\mathrm{i}\varepsilon}\right] \\ (1-\mathrm{i})\tilde{K}_{III}(\lambda) \end{pmatrix}(-\beta)^{-1/2}, \quad \beta \to \infty, \quad \beta \in \mathbb{C}^+, \tag{39}$$

*with* $\tilde{K}(\lambda) = \tilde{K}_I(\lambda) - \mathrm{i}\tilde{K}_{II}(\lambda)$, $c_1 = \frac{(1+\mathrm{i})\sqrt{\pi}}{2\Gamma(1/2+\mathrm{i}\varepsilon)}$, $c_2 = \frac{(1+\mathrm{i})\sqrt{\pi}}{2\Gamma(1/2-\mathrm{i}\varepsilon)}$, $e_0$ as in Eq. (70).

**Proof.** From the previous lemma, we can see that in order to find the leading and higher order terms for the physical stress, we need to make some adjustments to the solution in Antipov (1999). (See Appendix B for details, Eq. (82).) We need to find first $\mathbf{\Phi}^-(-\beta; -d)$; from relations (70) we can see that

$$\begin{aligned} c^-(-\beta; -d) &= \frac{1}{2}\left(\frac{e_0}{D}(-\beta)^{-\mathrm{i}\varepsilon} + \frac{D}{e_0}(-\beta)^{\mathrm{i}\varepsilon}\right), \\ s^-(-\beta; -d) &= \frac{1}{2\mathrm{i}}\left(\frac{e_0}{D}(-\beta)^{-\mathrm{i}\varepsilon} - \frac{D}{e_0}(-\beta)^{\mathrm{i}\varepsilon}\right), \quad \beta \in \mathbb{C}^+. \end{aligned} \tag{40}$$

Applying now $\mathbf{R}_{y,\pi}$ to the leading order term of $\mathbf{\Phi}^-(-\beta; -d)$ from Eq. (82) we obtain

$$\mathbf{\Phi}^+(\beta) = \begin{pmatrix} N_1 s^-(-\beta; -d) + N_2 c^-(-\beta; -d) \\ -N_3 \\ N_1 c^-(-\beta; -d) - N_2 s^-(-\beta; -d) \end{pmatrix}(-\beta)^{-1/2} + o(\beta^{-1/2}\mathbf{1}),$$
$$\beta \to \infty, \quad \beta \in \mathbb{C}^+, \tag{41}$$

$\mathbf{1} = (1,1,1)^{\mathrm{t}}$ and $N_i$ are defined as in Antipov (1999) (see Eq. (81) in Appendix B).

We next write $\tilde{\sigma}(\beta) = (\tilde{\sigma}_{yy}, \tilde{\tau}_{xy}, \tilde{\tau}_{yz})^{\mathrm{t}}$

$$\tilde{\boldsymbol{\sigma}}(\beta) \sim \begin{pmatrix} \dfrac{e_0}{2D}(-\mathrm{i}N_1 + N_2)(-\beta)^{-1/2-\mathrm{i}\varepsilon} + \dfrac{D}{2e_0}(\mathrm{i}N_1 + N_2)(-\beta)^{-1/2+\mathrm{i}\varepsilon} \\ \mathrm{i}\left[\dfrac{e_0}{2D}(-\mathrm{i}N_1 + N_2)(-\beta)^{-1/2-\mathrm{i}\varepsilon} - \dfrac{D}{2e_0}(\mathrm{i}N_1 + N_2)(-\beta)^{-1/2+\mathrm{i}\varepsilon}\right] \\ -N_3(-\beta)^{-1/2} \end{pmatrix},$$
$$\beta \to \infty, \quad \beta \in \mathbb{C}^+. \tag{42}$$

The exact representations for $\tilde{K}_I(\lambda)$, $\tilde{K}_{II}(\lambda)$ and $\tilde{K}_{III}(\lambda)$, can be obtained by taking the Fourier transform of

$$\begin{pmatrix} \sigma_{yy} \\ -\tau_{xy} \\ -\tau_{yz} \end{pmatrix} (-x, 0, z; -d),$$

where

$$\begin{pmatrix} \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \end{pmatrix} (x, 0, z; d)$$

can be found in Antipov (1999, 5.9), and comparing with the expression $\tilde{\sigma}$ above. In this way we relate $\tilde{K}_i$, $N_j$ and $D$ (This shall be used when finding the higher-order asymptotics in Appendix C.)

$$\begin{aligned} &\tilde{K}(\lambda) = \frac{c_1}{D}(-\mathrm{i}N_1 + N_2), \\ &\overline{\tilde{K}}(\lambda) = c_2 D(\mathrm{i}N_1 + N_2), \\ &\tilde{K}_{III}(\lambda) = -(1+\mathrm{i})N_3, \end{aligned} \tag{43}$$

with $c_1 = (1+\mathrm{i})\sqrt{\pi}/2\Gamma(1/2 + \mathrm{i}\varepsilon)$, $c_2 = (1+\mathrm{i})\sqrt{\pi}/2\Gamma(1/2 - \mathrm{i}\varepsilon)$ from Eq. (30), and we can write the leading order asymptotics for $\tilde{\sigma}(\beta)$ from these. $\square$

### *4.5. Normalisation. The weight matrix-function*

The solutions found in Eqs. (33)–(35) are linearly independent, but they do not necessarily give the stress-intensity factors when performing a convolution with the loads. For that reason, we need to normalise the weight matrix-function.

**Theorem 4.4.** *The weight matrix-function for the interfacial crack problem is given by*

$$[\tilde{\mathscr{U}}] = \mathbf{C}^{-1}[\tilde{\mathbf{U}}]^{\mathrm{t}}, \tag{44}$$

*where* $\mathbf{C} = -\mathbf{C}_0$, $[\tilde{\mathrm{U}}]^{\mathrm{t}}$, $\mathbf{C}_0$ *as in Eqs.* (47) *and* (31).

**Proof.** Since we have found previously the expressions for the leading terms of $[\tilde{\mathbf{U}}]$ and $\tilde{\sigma}$, we can use the convolution theorem for Fourier transforms to find large-$\beta$ asymptotics of the Fourier transforms involved in Eq. (28).

Thus, considering the single branch fixed as in (Antipov, 1999),

$$(\beta + 0\mathrm{i})^{-1/2}(-\beta - 0\mathrm{i})^{-1/2} = \mathrm{i}(\beta + 0\mathrm{i})^{-1},$$

we can express $\tilde{\sigma}$ as

$$\tilde{\boldsymbol{\sigma}}(\beta) \sim \frac{\mathrm{i}}{2}\begin{pmatrix} \dfrac{1}{e_0 c_1}\tilde{K}(\lambda)(\beta+0\mathrm{i})^{-\mathrm{i}\varepsilon} + \dfrac{e_0}{c_2}\tilde{\bar{K}}(\lambda)(\beta+0\mathrm{i})^{\mathrm{i}\varepsilon} \\ \mathrm{i}\left[\dfrac{1}{e_0 c_1}\tilde{K}(\lambda)(\beta+0\mathrm{i})^{-\mathrm{i}\varepsilon} - \dfrac{e_0}{c_2}\tilde{\bar{K}}(\lambda)(\beta+0\mathrm{i})^{\mathrm{i}\varepsilon}\right] \\ (1-\mathrm{i})\tilde{K}_{III}(\lambda) \end{pmatrix}(\beta+0\mathrm{i})^{-1/2}, \quad \beta \to \infty, \tag{45}$$

and multiplying $[\tilde{\mathbf{U}}](\beta)$ and $\tilde{\sigma}(\beta)$,

$$[\tilde{\mathbf{U}}]^{\mathrm{t}}\tilde{\boldsymbol{\sigma}} \sim \mathbf{C}_0\tilde{\mathbf{K}}(\lambda)\mathrm{i}(\beta+0\mathrm{i})^{-1}, \quad \beta \to \infty, \tag{46}$$

where $\tilde{\mathbf{K}}(\lambda) = \begin{pmatrix} \tilde{K}_I(\lambda) \\ \tilde{K}_{II}(\lambda) \\ \tilde{K}_{III}(\lambda) \end{pmatrix}$,

$$\mathbf{C}_0 = \frac{1}{2c_1c_2D}\begin{pmatrix} \psi_1 D^+ & -\mathrm{i}\psi_1 D^- & d^2(b+e)(\mathrm{i}-1)c_1c_2D \\ -\mathrm{i}\psi_2 D^- & -\psi_2 D^+ & (\mathrm{i}+1)\alpha(\lambda)c_1c_2D \\ \psi_3 D^+ & -\mathrm{i}\psi_3 D^- & |d|b(\mathrm{i}-1)\alpha(\lambda)c_1c_2D \end{pmatrix}, \tag{47}$$

$D^+ = c_2D^2 + c_1$, $D^- = c_2D^2 - c_1$.

Similarly,

$$\tilde{\Sigma}[\tilde{u}] \sim \mathbf{C}_0\tilde{\mathbf{K}}(\lambda)\mathrm{i}(\beta-0\mathrm{i})^{-1}, \quad \beta \to \infty.$$

And therefore,

$$\mathrm{i}\left\{\frac{1}{\beta+0\mathrm{i}} - \frac{1}{\beta+0\mathrm{i}}\right\}\tilde{\mathbf{K}}(\lambda) = 2\pi\delta(\beta)\tilde{\mathbf{K}}(\lambda) = \{-\mathbf{C}_0^{-1}[\tilde{\mathbf{U}}]^{\mathrm{t}}\}\tilde{\mathbf{p}}, \quad \beta \to \infty, \tag{48}$$

and taking inverse Fourier transform we obtain

$$\mathbf{K}(z) = \lim_{x\to 0}[\mathscr{U}] * \tilde{\mathbf{p}}. \qquad \square$$

We have to remark here the oscillatory behaviour of the weight matrix-function from Eq. (31), in contrast with the homogeneous case (see Willis and Movchan, 1995).

## 5. Asymptotics of higher order of $[\tilde{\mathscr{U}}]$ and $\tilde{\boldsymbol{\sigma}}$

We have obtained formulae for the weight function up to the power $\beta^{-1/2}$. Using relations (70) we can write

$$[\tilde{\mathscr{U}}] = \mathscr{U}(\beta+0\mathrm{i})^{-1/2} + \sum_{\pm}\mathscr{U}^{\pm}(\beta+0\mathrm{i})^{-1/2\pm\mathrm{i}\varepsilon}, \tag{49}$$

with

$$\mathscr{U} = \mathbf{C}^{-1}U,\ \mathscr{U}^{\pm} = \mathbf{C}^{-1}U^{\pm},\ U = \begin{pmatrix} 0 & 0 & -d^2(b+e) \\ 0 & 0 & \mathrm{i}\alpha(\lambda) \\ 0 & 0 & -|d|b\alpha(\lambda) \end{pmatrix},$$

$$U^{+} = \frac{e_0 D}{2}\begin{pmatrix} \psi_1 & -\mathrm{i}\psi_1 & 0 \\ -\mathrm{i}\psi_2 & -\psi_2 & 0 \\ \psi_3 & -\mathrm{i}\psi_3 & 0 \end{pmatrix},$$

$$U^{-} = \frac{1}{2e_0 D}\begin{pmatrix} \psi_1 & \mathrm{i}\psi_1 & 0 \\ \mathrm{i}\psi_2 & -\psi_2 & 0 \\ \psi_3 & \mathrm{i}\psi_3 & 0 \end{pmatrix}.$$

We need to obtain the next term in the asymptotic expansion (equivalent to the function $Q$ in Willis and Movchan (1995) (see Appendix C)), i.e., Eq. (32) should be modified at infinity as

$$\mathbf{\Phi}^{+}_{jk}(\beta) \sim f^{0}_{jk}(\beta) + f^{1}_{jk}(\beta) + O(\beta^{-3/2}), \quad \beta \to \infty,\ \beta \in \mathbb{C}^{+},\ j,k = 1,2,3, \tag{50}$$

where $f^{1}_{jk}(\beta)$ are defined in Appendix B. We can write the two-term asymptotics for large $\beta$:

$$\begin{aligned}[\tilde{\mathscr{U}}] \sim{}& \mathscr{U}(\beta+0\mathrm{i})^{-1/2} + \sum_{\pm} \mathscr{U}^{\pm}(\beta+0\mathrm{i})^{-1/2\pm\mathrm{i}\varepsilon} \\ &+ \mathscr{Q}(\beta+0\mathrm{i})^{-3/2} + \sum_{\pm} \mathscr{Q}^{\pm}(\beta+0\mathrm{i})^{-3/2\pm\mathrm{i}\varepsilon},\end{aligned} \tag{51}$$

with $\mathscr{Q} = \mathbf{C}^{-1}(Q_{jk})$, $\mathscr{Q}^{\pm} = \mathbf{C}^{-1}(Q^{\pm}_{jk})$, and $Q^{\pm}_{jk}$, $Q_{jk}$ defined in Appendix C. We have also obtained formulae for the physical stress up to the power $\beta^{-1/2}$. Using Eq. (39) we can write at infinity

$$\tilde{\mathscr{K}} = \mathscr{K}(\beta+0\mathrm{i})^{-1/2} + \sum_{\pm} \mathscr{K}^{\pm}(\beta+0\mathrm{i})^{-1/2\pm\mathrm{i}\varepsilon}, \tag{52}$$

with

$$\mathscr{K} = \frac{1+\mathrm{i}}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tilde{K}_{III},\ \mathscr{K}^{+} = \frac{e_0}{2c_2}\begin{pmatrix} \mathrm{i} \\ 1 \\ 0 \end{pmatrix} \tilde{\bar{K}},\ \mathscr{K}^{-} = \frac{1}{2e_0 c_1}\begin{pmatrix} \mathrm{i} \\ -1 \\ 0 \end{pmatrix} \tilde{K}.$$

Expression (43) gives the higher asymptotics for stress at infinity (see Appendix C)

$$\mathbf{A} \sim A(\beta+0\mathrm{i})^{-3/2} + \sum_{\pm} A^{\pm}(\beta+0\mathrm{i})^{-3/2\pm\mathrm{i}\varepsilon} + O(\beta^{-3/2}), \quad \beta \to \infty, \tag{53}$$

with $A$, $A^{+}$ and $A^{-}$ defined in Appendix C (84).

## 6. Perturbation of the crack front and asymptotics for the stress-intensity factors

We are going to adapt the method in Willis and Movchan (1995) relating a small deviation from straightness of the crack front to the associated perturbation of the stress intensity factors. The problem is exactly the same as the physical problem posed in Section 2, but the crack occupies the region

$$x < \delta\phi(z), \quad -\infty < z < \infty, \quad y = 0. \tag{54}$$

The function $\phi(z)$ is bounded and smooth, and we will find a solution for small $\delta$. The displacement and stress fields associated with the perturbed crack are represented as $\mathbf{u} + \Delta\mathbf{u}$ and $\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}$, where $\mathbf{u}$ and $\boldsymbol{\sigma}$ are the fields that give the solution when $\delta = 0$.

**Theorem 6.1.** *The equation relating the stress intensity factors corresponding to the limit problem and the perturbed problem* (54) *is given as follows*:

$$\mathbf{C}\tilde{\Delta\mathbf{K}} = -\mathrm{i}\delta\tilde{\phi}(\mathbf{B}\tilde{\mathbf{K}} + \mathscr{B}), \tag{55}$$

*with* $\mathbf{B} = (2c_1c_2D)^{-1}(b_{jk}),\ j,k = 1,2,3,$ $\mathbf{C}$ *as in Eq.* (47), $\lambda_m$, $m = 1,2$ *as in Eq.* (64),

$$\tilde{\mathbf{K}} = \begin{pmatrix} \tilde{K}_I \\ \tilde{K}_{II} \\ \tilde{K}_{III} \end{pmatrix}, \quad \tilde{\Delta\mathbf{K}} = \begin{pmatrix} \tilde{\Delta K}_I \\ \tilde{\Delta K}_{II} \\ \tilde{\Delta K}_{III} \end{pmatrix},$$

$$b_{11} = c_{11} + \frac{1}{2}(\lambda_m - |\lambda|)\psi_1 D^+$$

$$b_{21} = c_{21} - \frac{\mathrm{i}}{2}(\lambda_m - |\lambda|)\psi_2 D^-,$$

$$b_{31} = c_{31} + \frac{1}{2}(\lambda_m - |\lambda|)\psi_3 D^+,$$

$$c_{j1} = 2D(e_0c_1Q_{j2}^- - \tfrac{c_2}{e_0}Q_{j2}^+),$$

$$b_{12} = c_{12} - \frac{\mathrm{i}}{2}(\lambda_m - |\lambda|)\psi_1 D^-,$$

$$b_{22} = c_{22} - \frac{1}{2}(\lambda_m - |\lambda|)\psi_2 D^+,$$

$$b_{32} = c_{32} - \frac{\mathrm{i}}{2}(\lambda_m - |\lambda|)\psi_3 D^-,$$

$$c_{j2} = 2\mathrm{i}D(\tfrac{c_2}{e_0}Q_{j2}^+ + e_0c_1Q_{j2}^-),$$

$$\begin{aligned} b_{13} &= c_1c_2D(1+\mathrm{i})(Q_{13} - \tfrac{\mathrm{i}}{2}|\lambda|d^2(b+e)), \\ b_{23} &= c_1c_2D(1+\mathrm{i})(Q_{23} - \tfrac{1}{2}|\lambda|\alpha(\lambda)), \\ b_{33} &= c_1c_2D(1+\mathrm{i})(Q_{33} - \tfrac{\mathrm{i}}{2}\lambda|d|b\alpha(\lambda)), \end{aligned} \qquad \mathscr{B} = -\mathrm{i}N_4\begin{pmatrix} \psi_1 \\ 0 \\ \psi_3 \end{pmatrix}.$$

**Proof.** The integral identity (26) applies exactly both to original and to the perturbed fields, and by subtraction, in terms of Fourier transforms and considering the weight function $[\mathscr{U}]$,

$$[\tilde{\mathscr{U}}]\tilde{\Delta\sigma} - \tilde{\boldsymbol{\Sigma}}[\tilde{\Delta\mathbf{u}}] = 0. \tag{56}$$

We have to remark that the perturbation of the crack front, in terms of Fourier transforms $\mathscr{F}_\beta[f(x-\delta\phi);\beta] = \int_{-\infty}^{\infty} e^{i\beta x}[f(x-\delta\phi)]\,dx = e^{i\delta\phi\beta}\mathscr{F}_\beta[f(x);\beta]$, and since $\delta$ is small we can write the Fourier transform of the perturbed field $\mathscr{F}_{\beta\lambda}$ as $(1+i\delta\tilde{\phi}\beta)\tilde{f}(\beta)$.

From the section above, we have found the asymptotics for $[\tilde{\mathscr{U}}](\beta)$ and $\tilde{\boldsymbol{\sigma}}(\beta)$ in Eqs. (51)–(53). Thus, at infinity,

$$\begin{aligned}\{\boldsymbol{\sigma} \tilde{+} \Delta\boldsymbol{\sigma}\} \sim{}& (1+i\delta\tilde{\phi}\beta)(\mathscr{K}+\Delta\mathscr{K})(\beta+0i)^{-1/2} + (1+i\delta\tilde{\phi}\beta)(\mathscr{K}^{\pm}\\ &+\Delta\mathscr{K}^{\pm})(\beta+0i)^{-1/2\pm i\varepsilon} + (1+i\delta\tilde{\phi}\beta)(A+\Delta A)(\beta+0i)^{-3/2}\\ &+(1+i\delta\tilde{\phi}\beta)(A^{\pm}+\Delta A^{\pm})(\beta+0i)^{-3/2\pm i\varepsilon},\end{aligned} \tag{57}$$

with

$$\Delta\mathscr{K} = \frac{1+i}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}\Delta\tilde{K},\ \Delta\mathscr{K}^{+} = \frac{e_0}{2c_2}\begin{pmatrix}i\\1\\0\end{pmatrix}\Delta\tilde{\bar{K}},\ \Delta\mathscr{K}^{-} = \frac{1}{2e_0c_1}\begin{pmatrix}i\\-1\\0\end{pmatrix}\Delta\tilde{K},$$

and therefore

$$\begin{aligned}\{\tilde{\Delta\boldsymbol{\sigma}}\} \sim{}& \frac{\Delta\mathscr{K}+i\delta\beta\tilde{\phi}\mathscr{K}}{(\beta+0i)^{1/2}} + \frac{\Delta\mathscr{K}^{+}+i\delta\beta\tilde{\phi}\mathscr{K}^{+}}{(\beta+0i)^{1/2-i\varepsilon}} + \frac{\Delta\mathscr{K}^{-}+i\delta\beta\tilde{\phi}\mathscr{K}^{-}}{(\beta+0i)^{1/2+i\varepsilon}}\\ &+\frac{\Delta A+i\delta\beta\tilde{\phi}A}{(\beta+0i)^{3/2}} + \frac{\Delta A^{+}+i\delta\beta\tilde{\phi}A^{+}}{(\beta+0i)^{3/2-i\varepsilon}} + \frac{\Delta A^{-}+i\delta\beta\tilde{\phi}A^{-}}{(\beta+0i)^{3/2+i\varepsilon}},\quad \beta\to\infty.\end{aligned} \tag{58}$$

Here we can see the need for higher asymptotics as we want to collect the coefficients in $(\beta+0i)^{-1}$. And thus, at infinity,

$$\begin{aligned}[\tilde{\mathscr{U}}]\tilde{\Delta\boldsymbol{\sigma}} \sim{}& [\mathscr{U}\Delta\mathscr{K} + \mathscr{U}^{+}\Delta\mathscr{K}^{-} + \mathscr{U}^{-}\Delta\mathscr{K}^{+} + i\delta\tilde{\phi}\{\mathscr{U}A + \mathscr{U}^{+}A^{-} + \mathscr{U}^{-}A^{+}\\ &+\mathscr{Q}\mathscr{K} + \mathscr{Q}^{+}\mathscr{K}^{-} + \mathscr{Q}^{-}\mathscr{K}^{+}\}]\,(\beta+0i)^{-1},\quad \beta\to\infty.\end{aligned} \tag{59}$$

The identity (48) requires that the coefficients of $1/(\beta+0i)$ should be zero.

We have to remark here that given the normalisation in Eq. (48), we can see that in the above expression, $\mathbf{C}^{-1}$ is a common factor, so in what follows, we can use the matrix $[\tilde{\mathbf{U}}]^{t}$, so we have the following equation to solve:

$$\begin{aligned}&U\Delta\mathscr{K} + U^{+}\Delta\mathscr{K}^{-} + U^{-}\Delta\mathscr{K}^{+} + i\delta\tilde{\phi}\{UA + U^{+}A^{-} + U^{-}A^{+} + Q\mathscr{K} + Q^{+}\mathscr{K}^{-}\\ &\quad + Q^{-}\mathscr{K}^{+}\} = 0.\end{aligned} \tag{60}$$

After multiplying the matrices, we obtain Eq. (55). □

The above asymptotic analysis suggests that all three modes are coupled for the case of a semi-infinite wavy crack at an interface. In contrary, for a semi-infinite wavy crack in a homogeneous space, Mode I decouples from Modes II and III, to first-order approximation (see Willis and Movchan, 1995).

Hence, a quasi-static interfacial crack with a wavy front may also have a tendency to depart from the plane of the interface (out-of-plane deflection) and this would result in crack tip instabilities. Ramanathan and Fisher (1997) used the calculations by Willis and Movchan (1995), to find that in Mode I fracture weak hetereogeneity of the medium could lead to a non-decaying unstable mode that propagates along the crack front. In the same manner, our calculations could be used in an analogous quasi-static interfacial analysis. Analysis of rough interface surfaces and propagation of cracks along such surfaces is a challenging open problem in theoretical modelling. The method developed in this paper enables one to make a good progress in this direction by obtaining the accurate asymptotics of the stress-intensity factors, with a limited additional technical input.

## 7. Conclusions

We have studied a singularly perturbed problem and obtained the asymptotics for the stress-intensity factors associated with the perturbation of the crack front. We assumed first to have an interfacial crack with a straight front, and then introduced an in-plane perturbation. We have developed a rigorous mathematical procedure that treats such a perturbation problem for a general type of loads and a wide range of perturbation functions. The work (Lazarus and Leblond, 1998a) did not contain the mathematical analysis of the full problem; on the other hand, it gave a comprehensive study based on the physical properties of elastic fields near the edge of the interfacial crack. Our method is "general", since we have looked for the complete solutions of the elasticity problem using the Wiener-Hopf method to find the weight matrix-function. Explicit representations of Fourier transforms of the weight functions are obtained as a result of our study.

Our definition of the weight functions in Section 4 was established in terms of generalised functions, and we followed the algorithm in Willis and Movchan (1995), considering the weight function as a singular solution of a homogeneous problem. The definition in Lazarus and Leblond (1998a) comes from physics, *the crack-face weight function* $h_{pi}(x, z; z')$ $(p = I, II, III; i = x, y, z)$ *is the pth stress-intensity factor generated at the point* $z'$ *of the crack front by unit point forces exerted on the points* $(x, y = 0^{\pm}, z)$ *of the crack faces in the direction* $\pm\mathbf{e}_i$, and we showed through the 2D example in Section 4.2 that indeed both definitions are equivalent.

The method in Lazarus and Leblond (1998a, b) demands the *bimaterial constant* $\varepsilon$ being small. Otherwise, the solution involves a set of unknown coefficients.

In our method, as in Willis and Movchan (1995), the reciprocal identity enables us (see Section 4) to express the stress-intensity factors as limit of convolutions (in the sense of generalised functions) of the weight matrix-function and the loads.

In order to apply the asymptotic algorithm, we have constructed high-order asymptotic representations of the weight matrix-function and physical stress near the crack edge. These data, together with the high-order asymptotics of the physical components of stress, are used in the explicit representation of perturbation of the stress-intensity factors, written in terms of the Fourier transforms. These formulae show the coupling phenomena between stress-intensity factors of different modes as well as the oscillatory behaviour of the weight functions in a neighbourhood of the crack edge.

Needless to say, the crack instabilities can be modelled very efficiently by considering a fault in a lattice, as illustrated in the theoretical papers by Marder and Gross (1995), Kulakhmetova et al. (1984) and the monograph by Slepyan (2002). One essential feature of the papers cited is that they involve functional equations of the Wiener-Hopf type written in terms of discrete Fourier transforms of displacement and stress on the fault plane. Direct numerical analysis of cracks in a 3D lattice would involve substantial computational challenges. However, if formulated in terms of the Fourier transforms, the problem is readily reducible to a system of functional equations of the Wiener-Hopf type, and the perturbation algorithm, developed in the present paper, can be used with limited number of technical modifications. Construction of dynamic weight functions for interfacial cracks is still a challenge, but nevertheless our algorithm is applicable, without major changes, to dynamic cracks propagating along the interface of slightly dissimilar media when the difference between the elastic moduli of upper and lower parts of the elastic solid is relatively small.

## Acknowledgements

J.P. Bercial-Velez acknowledges the support of an EPSRC studentship and the authors would like to thank the referees for their valuable comments.

## Appendix A. Factorisation of the matrix $\boldsymbol{\rho}^{-1}\mathbf{G}(\boldsymbol{\beta})$

In order to find the solution of problem (4), the matrix $\rho^{-1}\mathbf{G}(\beta)$ has to be split. The factorisation described here is similar to Antipov (1999). However, in Antipov (1999) the factorisation was obtained with the restriction $d>0$, whereas here we allow $d$ to be negative. An important step in Antipov (1999) is to reduce this $3\times 3$ problem to a $2\times 2$, in order to adapt Khrapkov's procedure (see Khrapkov, 1971). This is achieved by representing $\mathbf{G}$ as a block diagonal matrix

$$\mathbf{G}(\beta) = -\frac{1}{\rho^2}\begin{pmatrix} 0 & \mathrm{i}/d & -b \\ -(b+e)\beta & -b\lambda & -\mathrm{i}\lambda/(d\rho^2) \\ (b+e)\lambda & -b\beta & -\mathrm{i}\beta/(d\rho^2) \end{pmatrix}\mathbf{G}_1(\beta)\begin{pmatrix} 0 & \beta/\rho^2 & -\lambda/\rho^2 \\ 0 & \lambda/\rho^2 & \beta/\rho^2 \\ 1 & 0 & 0 \end{pmatrix},$$

where $\mathbf{G}_1(\beta) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2(\beta) \end{pmatrix}$, and

$$\mathbf{G}_2(\beta) = \frac{d^2\rho(b^2\rho+1)}{\delta_0}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\mathrm{i}db(d^2\rho+1)}{\delta_0}\begin{pmatrix} 0 & 1 \\ -\rho^2 & 0 \end{pmatrix},$$

$\delta_0 = b^2 d^2 \rho^2 - 1$. The $2 \times 2$ sub-matrix $\mathbf{G}_2$, can be now factorised after the following steps:

(i) *Factorisation of* $\Delta^{1/2}(\beta)$

$\Delta(\beta) = \det \mathbf{G}_2 = d^2(b^2 - d^2)\rho^2/\delta_0$. The function $\Delta^{1/2}(\beta)$ can be factorised as follows:

$$\Delta^{1/2}(\beta) = \frac{\Lambda^+(\beta)}{\Lambda^-(\beta)}, \quad \beta \in \Gamma. \tag{61}$$

$\Gamma$ is a contour that splits $\mathbb{C}$ in two connected components. The factors $\Lambda^+$, $\Lambda^-$ must be analytically continued into the upper and lower half-planes, and on $\Gamma$ satisfy Eq. (61). There are two possibilities which bring different contours:

(a) When $|\lambda| < (b|d|)^{-1}$, the function $\delta_0$ has two real roots: $\beta = \lambda_R$ and $\beta = -\lambda_R$, with $\lambda_R = \sqrt{|d^{-2}b^{-2} - \lambda^2|}$. We cut the $\beta$-plane by a semi-infinite section $(\lambda_R, \infty)$ of the real axis and demand that $0 < \arg(\beta - \lambda_R) < 2\pi$. Then, the function

$$\Lambda^+(\beta) = \frac{d_0(\beta + \mathrm{i}|\lambda|)^{1/2}}{(\beta - \lambda_R)^{1/2}}, \tag{62}$$

is analytic in the plane cut by the lines $\{\lambda_R < \Re\beta < \infty, \Im\beta = \pm 0\}$ and $\{\Re\beta = \pm 0, -\infty < \Im\beta < -|\lambda|\}$, where $d_0 = (1 - d^2/b^2)^{1/4}$. Similarly, the function

$$\Lambda^-(\beta) = \frac{(\beta + \lambda_R)^{1/2}}{d_0(\beta - \mathrm{i}|\lambda|)^{1/2}}, \quad -\pi < \arg(\beta + \lambda_R) < \pi, \tag{63}$$

is analytic in the $\beta$-plane if we cut it by the lines $\{-\infty < \Re\beta < -\lambda_R, \Im\beta = \pm 0\}$ and $\{\Re\beta = \pm 0, |\lambda| < \Im\beta < \infty\}$. The contour $\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ (see Fig. 4), where

$$\begin{aligned}
\Gamma_- &= \{\beta = \tau - \mathrm{i}0,\ -\infty < \tau < \lambda_R\},\\
\Gamma_0 &= \{\beta = \tau,\ -\lambda_R < \tau < \lambda_R\},\\
\Gamma_+ &= \{\beta = \tau + \mathrm{i}0,\ \lambda_R < \tau < \infty\},
\end{aligned}$$

in this case avoids the poles in the real axis, so the upper half plane $\mathbb{C}^+$ contains $-\lambda_R$, and the lower half-plane $\mathbb{C}^-$ contains $\lambda_R$.

(b) When $|\lambda| > (b|d|)^{-1}$, the function $\delta_0$ has two complex conjugate roots: $\beta = \pm\mathrm{i}\lambda_R$, with $\lambda_R$ as before. The function $\Delta^{1/2}(\beta)$ admits decomposition (61), with

$$\begin{aligned}
\Lambda^+(\beta) &= \frac{d_0(\beta + \mathrm{i}|\lambda|)^{1/2}}{(\beta + \mathrm{i}\lambda_R)^{1/2}}, \quad -\frac{\pi}{2} < \arg(\beta + \mathrm{i}\lambda_R) < \frac{3\pi}{2},\\
\Lambda^-(\beta) &= \frac{(\beta - \mathrm{i}\lambda_R)^{1/2}}{d_0(\beta - \mathrm{i}|\lambda|)^{1/2}}, \quad -\frac{3\pi}{2} < \arg(\beta - \mathrm{i}\lambda_R) < \frac{\pi}{2}.
\end{aligned}$$

The contour $\Gamma$ in this case is the real axis (see Fig. 5).

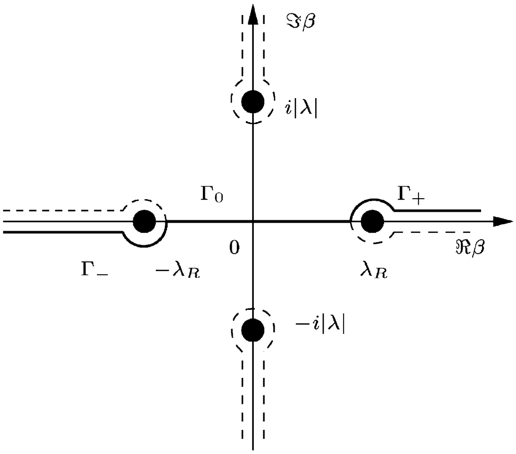


Fig. 4. Branch cuts and the contours $\Gamma_-$, $\Gamma_0$, $\Gamma_+$, for $|\lambda|<b^{-1}|d|^{-1}$.

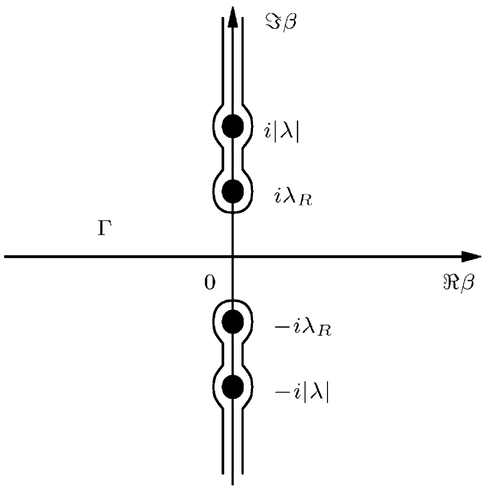


Fig. 5. Branch cuts and the contour $\Gamma$ for $|\lambda|>b^{-1}|d|^{-1}$.

It is convenient to write the asymptotics of the functions $\Lambda^{\pm}(\beta)$

$$\Lambda^{+}(\beta) \sim d_0\left(1+\frac{i\lambda_j}{2}\,\beta^{-1}\right)+O(\beta^{-2}), \quad \beta\to\infty,\ \beta\in\mathbb{C}^{+},$$

$$\Lambda^{-}(\beta) \sim \frac{1}{d_0}\left(1+\frac{i\lambda_j}{2}\,\beta^{-1}\right)+O(\beta^{-2}), \quad \beta\to\infty,\ \beta\in\mathbb{C}^{-}, \tag{64}$$

with $\lambda_1=|\lambda|-i\lambda_R$ for $|\lambda|<(b|d|)^{-1}$ and $\lambda_2=|\lambda|-\lambda_R$ for $|\lambda|>(b|d|)^{-1}$.

(ii) *Khrapkov's factorisation and the functions* $B^{\pm}(\beta)$
Let

$$\Xi(\beta; d) = \frac{1}{2}\log\frac{\Lambda_1(\beta)}{\Lambda_2(\beta)} = \frac{1}{2}\log\left(\frac{b+d}{b-d}\,\frac{bd\rho+1}{bd\rho-1}\right),$$

$\Lambda_i$ are the eigenvalues of the matrix $\mathbf{G}_2$. Clearly, the function $\Xi(\beta; d)$ is odd with respect to the parameter $d$, i.e. $\Xi(\beta; d) = -\Xi(\beta; -d)$. Once the branches of this function are fixed (see (3.9), (3.10) in Antipov (1999)), following Khrapkov's procedure (Khrapkov, 1971), the factorisation of $\mathbf{G}_1$ is constructed by the formulae

$$\mathbf{G}_1(\beta) = \mathbf{X}_1^{+}(\beta)[\mathbf{X}_1^{-}(\beta)]^{-1}, \quad \beta \in \Gamma,$$

$$\mathbf{X}_1^{\pm}(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda^{\pm}(\beta)\cos\{\rho B^{\pm}(\beta)\} & \rho^{-1}\Lambda^{\pm}(\beta)\sin\{\rho B^{\pm}(\beta)\} \\ 0 & -\rho\Lambda^{\pm}(\beta)\sin\{\rho B^{\pm}(\beta)\} & \Lambda^{\pm}(\beta)\cos\{\rho B^{\pm}(\beta)\} \end{pmatrix}, \quad \beta \in \mathbb{C}^{\pm} \cup \Gamma.$$

The functions $B^{\pm}$, involved in the oscillatory behaviour, are determined by the equation

$$B^{+}(\beta) - B^{-}(\beta) = \frac{1}{\mathrm{i}\rho}\,\Xi(\beta), \quad \beta \in \Gamma.$$

The solution of this problem is given by

$$B^{\pm}(\beta; d) = \frac{d_1}{2\mathrm{i}}\,\Psi^{\pm}(\beta) + B_1^{\pm}(\beta; d), \quad \beta \in \mathbb{C}^{\pm}, \tag{65}$$

$$\Psi^{\pm}(\beta) = \frac{\mathrm{i}}{\pi\rho}\log\frac{\beta+\rho}{\pm\mathrm{i}|\lambda|}, \quad -\pi \leqslant \arg\frac{\beta+\rho}{\pm\mathrm{i}|\lambda|} \leqslant \pi, \tag{66}$$

$$B_1^{\pm}(\beta; d) = -\frac{1}{4\pi}\int_{\Gamma}\log\frac{bd(t^2+\lambda^2)^{1/2}+1}{bd(t^2+\lambda^2)^{1/2}-1}\,\frac{\mathrm{d}t}{(t^2+\lambda^2)^{1/2}(t-\beta)}, \quad \beta \in \mathbb{C}^{\pm}, \tag{67}$$

with $d_1 = \log(b+d)/(b-d)$, $B^{\pm}(\beta; d) = -B^{\pm}(\beta; -d)$ and $B_1^{\pm}(\beta; d) = -B_1^{\pm}(\beta; -d)$. After the examination of the behaviour of $B^{\pm}(\beta)$ as $\beta \to \infty$, we have the asymptotics

$$B^{\pm}(\beta) = \frac{d_1}{2\pi\beta}\log\beta + \frac{1}{\beta}\left(\frac{d_1}{2\pi}\log\frac{2}{\pm\mathrm{i}|\lambda|} + B_0\right) + O\left(\frac{1}{\beta^3}\right), \quad \beta \to \infty,\ \beta \in \mathbb{C}^{\pm}, \tag{68}$$

where $B_0$ is the quantity

$$B_0 = \frac{1}{2\pi}\int_0^{\infty}\log\frac{bd(t^2+\lambda^2)^{1/2}+1}{bd(t^2+\lambda^2)^{1/2}-1}\,\frac{\mathrm{d}t}{(t^2+\lambda^2)^{1/2}}, \quad \arg\frac{bd\rho(t)+1}{bd\rho(t)-1} = -\pi\ \mathrm{sgn}\ d,$$

and $B_0(d) = -B_0(-d)$. It is convenient for the future to have the following notations:

$$c^{\pm}(\beta) = \cos\left(\varepsilon\log\beta \mp \frac{\pi\varepsilon}{2}\,\mathrm{i} + \varepsilon\log\frac{2}{|\lambda|} + B_0\right),$$

$$s^{\pm}(\beta) = \sin\left(\varepsilon\log\beta \mp \frac{\pi\varepsilon}{2}\,\mathrm{i} + \varepsilon\log\frac{2}{|\lambda|} + B_0\right),$$

$$\varepsilon = \frac{1}{2\pi}\log\frac{b+d}{b-d} = \frac{1}{2\pi}\log\left(\frac{\mu_+ + \mu_-(3-4\nu_+)}{\mu_- + \mu_+(3-4\nu_-)}\right). \tag{69}$$

$\varepsilon$ is known in the literature as the *bimaterial* constant.

The functions $c^{\pm}, s^{\pm}$ can be rewritten in the form

$$c^{\pm} = \frac{1}{2}\,(e_0^{\pm 1} D\beta^{\mathrm{i}\varepsilon} + e_0^{\mp 1} D^{-1}\beta^{-\mathrm{i}\varepsilon}),$$

$$s^{\pm} = \frac{1}{2\mathrm{i}}\,(e_0^{\pm 1} D\beta^{\mathrm{i}\varepsilon} - e_0^{\mp 1} D^{-1}\beta^{-\mathrm{i}\varepsilon}), \tag{70}$$

$e_0 = \left(\frac{b+d}{b-d}\right)^{1/4}$, $D = \exp\{\mathrm{i}(\varepsilon\log\frac{2}{|\lambda|} + B_0)\}$, $e_0(d) = 1/e_0(-d)$ and $D(d) = 1/D(-d)$.

Therefore, from the split of $\mathbf{G}_1(\beta)$ and $\rho = (\beta + \mathrm{i}|\lambda|)^{1/2}(\beta - \mathrm{i}|\lambda|)^{1/2}$, we arrive to this important result

**Theorem A.1** (*Antipov, 1999*). *If* $d \neq 0$ *then a Wiener-Hopf factorisation of the matrix* $\rho^{-1}\mathbf{G}(\beta)$ *in Eq.* (3), *is given by*

$$\rho^{-1}\mathbf{G}(t) = [\mathbf{Y}_+(t)]^{-1}\mathbf{Y}_-(t), \quad t \in \Gamma, \tag{71}$$

*with*

$$\mathbf{Y}_+(\beta) = (\beta + \mathrm{i}|\lambda|)^{1/2}\begin{pmatrix} 0 & -\dfrac{\beta}{(b+e)\rho^2} & \dfrac{\lambda}{(b+e)\rho^2} \\ \dfrac{\Sigma_1^+}{\delta_0\Lambda^+} & \dfrac{\lambda\Sigma_2^+}{\delta_0\Lambda^+} & \dfrac{\beta\Sigma_2^+}{\delta_0\Lambda^+} \\ \dfrac{\rho^2\Sigma_2^+}{\delta_0\Lambda^+} & -\dfrac{\lambda\Sigma_1^+}{\delta_0\Lambda^+} & -\dfrac{\beta\Sigma_1^+}{\delta_0\Lambda^+} \end{pmatrix}, \tag{72}$$

$$\mathbf{Y}_-(\beta) = (\beta - \mathrm{i}|\lambda|)^{-1/2}\begin{pmatrix} 0 & \dfrac{\beta}{\rho^2} & -\dfrac{\lambda}{\rho^2} \\ -\dfrac{\sin(\rho B^-)}{\rho\Lambda^-} & \dfrac{\lambda\cos(\rho B^-)}{\rho^2\Lambda^-} & \dfrac{\beta\cos(\rho B^-)}{\rho^2\Lambda^-} \\ \dfrac{\cos(\rho B^-)}{\Lambda^-} & \dfrac{\lambda\sin(\rho B^-)}{\rho\Lambda^-} & \dfrac{\beta\sin(\rho B^-)}{\rho\Lambda^-} \end{pmatrix}, \tag{73}$$

*and*

$$\Sigma_1^+ = \mathrm{i}d[\cos(\rho B^+) - \mathrm{i}bd\rho\sin(\rho B^+)], \tag{74}$$

$$\Sigma_2^+ = \mathrm{i}d\left[\frac{1}{\rho}\sin(\rho B^+) + \mathrm{i}bd\cos(\rho B^+)\right], \tag{75}$$

*which have the following asymptotics at infinity*:

$$\Sigma_1^+ \sim d^2 bs^+\beta + \mathrm{i}dc^+ + O(\beta^{-1}),$$
$$\Sigma_2^+ \sim -bd^2c^+ + \mathrm{i}ds^+\beta^{-1} + O(\beta^{-2}). \tag{76}$$

## Appendix B. Physical stress

Here we present a summary of the calculations in Antipov (1999) to obtain $\Phi^-(\beta)$, as we have to find higher order asymptotics of $\Phi_I^+(\beta)$ for our procedure. The crack is assumed to be as in Fig. 1, but with loads $p_{\mathrm{i}}^{\pm}$. After Fourier transform, and considering the orientation of the physical problem we have the Wiener-Hopf equation

$$\mathbf{\Phi}^+(\beta) = \frac{1}{\rho}\,\mathbf{G}[\mathbf{\Phi}^-(\beta) + \mathbf{F}^+(\beta)], \tag{77}$$

with

$$\mathbf{F}^+ = \int_{-\infty}^{\infty}\int_0^{\infty} \mathbf{p}(x,z)\mathrm{e}^{\mathrm{i}\beta x}\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}x\,\mathrm{d}z,$$

which leads to

$$\mathbf{Y}_+(t)\mathbf{\Phi}^+(t) - \mathbf{H}^+(t) = \mathbf{Y}_-(t)\mathbf{\Phi}^-(t) - \mathbf{H}^-(t), \quad t \in \Gamma,$$

with

$$\mathbf{H}^{\pm} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\mathbf{Y}_-(t)\mathbf{F}^+(t)}{t-\beta}\,\mathrm{d}t, \quad \beta \in \mathbb{C}^{\pm},$$

and the contour $\Gamma$ defined as before. From Eqs. (7) and (13), considering the class of solutions (finite energy) the entire function $\mathbf{E}_-(\beta)$ has the same form as in Antipov (1999),

$$\mathbf{E}_-(\beta) = \begin{pmatrix} \dfrac{C_1}{\beta - \mathrm{i}|\lambda|} + \dfrac{C_2}{\beta + \mathrm{i}|\lambda|} + H_1^-(\beta) \\ \dfrac{C_3}{\beta + \mathrm{i}|\lambda|} + \dfrac{C_4}{\beta - a} + H_2^-(\beta) \\ \dfrac{C_5}{\beta - a} + H_3^-(\beta) \end{pmatrix}, \quad \beta \in \mathbb{C}^-. \tag{78}$$

Hence, the solution of the Wiener-Hopf problem is given by

$$\mathbf{\Phi}^-(\beta) = \mathbf{Y}_-(\beta)\mathbf{E}_-(\beta), \tag{79}$$

with $\mathbf{Y}_-(\beta)$ as in Eq. (73). At infinity, we have the following asymptotics:

$$H_j^-(\beta) \sim H_j^0\beta^{-1} + H_j^1\beta^{-2},$$

$$\mathbf{E}_-(\beta) \sim \begin{pmatrix} (C_1 + C_2 + H_1^0)\beta^{-1} + ((C_1 - C_2)\mathrm{i}|\lambda| + H_1^1)\beta^{-2} \\ (C_3 + C_4 + H_2^0)\beta^{-1} + (C_4 a - C_3\mathrm{i}|\lambda| + H_2^1)\beta^{-2} \\ (C_5 + H_3^0)\beta^{-1} + (C_5 a + H_3^1)\beta^{-2} \end{pmatrix},$$

and

$$\mathbf{Y}_{-}(\beta) \sim \beta^{1/2}\left(1 - \frac{\mathrm{i}|\lambda|}{2\beta}\right)\frac{1}{d_0 b}\begin{pmatrix} 0 & -\beta\lambda^{-}s^{-} & \lambda^{-}c^{-} \\ \beta & \lambda\lambda^{-}c^{-} & \dfrac{\lambda}{\beta}\lambda^{-}s^{-} \\ -\lambda & \beta\lambda^{-}c^{-} & \lambda^{-}s^{-} \end{pmatrix},$$

$$\lambda^{-} = \left(1 + \frac{\mathrm{i}\lambda_j}{2\beta}\right), \ \ \beta \in \mathbb{C}^{-}$$

In order to have a physical stress, relations (4.15) in Antipov (1999) hold, namely

$$C_1 + C_2 + H_1^0 = 0,$$
$$C_3 + C_4 + H_2^0 = 0, \tag{80}$$

and using the notations

$$N_1 = \frac{1}{d_0}\,(\mathrm{i}|\lambda|C_3 - aC_4 - H_2^1),$$
$$N_2 = \frac{1}{d_0}\,(C_5 + H_3^0),$$
$$N_3 = \mathrm{i}|\lambda|(C_1 - C_2) + H_1^1,$$
$$N_4 = \frac{1}{d_0}\,(aC_5 + H_3^1), \tag{81}$$

we can write, as $\beta \to \infty, \ \beta \in \mathbb{C}^{-}$,

$$\Phi_1^{-} \sim \beta^{1/2}\left(1 - \frac{\mathrm{i}|\lambda|}{2\beta}\right)\left[s^{-}\left(1 + \frac{\mathrm{i}\lambda_j}{2\beta}\right)N_1\beta^{-1} + c^{-}\left(1 + \frac{\mathrm{i}\lambda_j}{2\beta}\right)(N_2\beta^{-1} + N_4\beta^{-2})\right],$$
$$\Phi_2^{-} \sim \beta^{1/2}\left(1 - \frac{\mathrm{i}|\lambda|}{2\beta}\right)\left[\beta^{-1}N_3 - \lambda c^{-}\left(1 + \frac{\mathrm{i}\lambda_j}{2\beta}\right)N_1\beta^{-2} + \lambda s^{-}\left(1 + \frac{\mathrm{i}\lambda_j}{2\beta}\right)N_2\beta^{-2}\right],$$

$$\Phi_3^{-} \sim \beta^{1/2}\left(1 - \frac{\mathrm{i}|\lambda|}{2\beta}\right)\left[-\lambda N_3\beta^{-2} - c^{-}\left(1 + \frac{\mathrm{i}\lambda_j}{2\beta}\right)N_1\beta^{-1} + s^{-}\left(1 + \frac{\mathrm{i}\lambda_j}{2\beta}\right)(N_2\beta^{-1} + N_4\beta^{-2})\right]. \tag{82}$$

## Appendix C. Higher order asymptotics

We need to obtain the next term in the asymptotic expansions in Eq. (50), from Eqs. (20)–(22).

$$f_{jk}^1 = Q_{jk}\beta^{-3/2} + Q_{jk}^{+}e_0 D\beta^{-3/2+\mathrm{i}\varepsilon} + Q_{jk}^{-}e_0^{-1}D^{-1}\beta^{-3/2-\mathrm{i}\varepsilon}, \quad j,k = 1,2,3.$$

Since we are finding the higher order asymptotics of $[\mathbf{U}]^{\mathrm{t}}$, we have to use the same system of reference $(y,x,z)$ (i.e., the second and third row have to be interchanged) and obviously $Q_{mn}$, $Q^{\pm}_{mn}$ are obtained after transposition.

$$Q^{\pm}_{11} = d^2M_-(b+e)\lambda d_0\left(\mp \mathrm{i} + (|\lambda| - \lambda_j)\,\frac{db}{2}\right), \quad Q_{11} = Q_{21} = Q_{31} = 0,$$

$$Q_{13} = -\frac{\mathrm{i}}{2}|\lambda| d^2(b+e), \quad Q^{+}_{13} = d^3M_-b(b+e)\lambda^2 d_0, \quad Q^{-}_{13} = -Q^{+}_{13},$$

$$Q_{12} = d^2(b+e)\lambda, \quad Q^{+}_{12} = -\mathrm{i}Q^{+}_{11}, \quad Q^{-}_{12} = \mathrm{i}Q^{-}_{11},$$

$$Q^{\pm}_{21} = \frac{1}{2}\,d_0 d\left(\pm\frac{1}{2}\,(\lambda_j - 3|\lambda|)db + \mathrm{i}\right),$$

$$Q_{23} = \frac{3}{2}\,\lambda d^2(b+e)M_-, \quad Q^{+}_{23} = Q^{-}_{23} = -\frac{\lambda}{2}\,bd^2 d_0,$$

$$Q_{22} = -\mathrm{i}d^2|\lambda|(b+e)M_-, \quad Q^{+}_{22} = -\mathrm{i}Q^{+}_{21}, \quad Q^{-}_{22} = \mathrm{i}Q^{-}_{21},$$

$$Q^{\pm}_{31} = \pm\frac{1}{2}\,d_0|d|\left[d^2b^2\left(\frac{\mathrm{i}}{2}\,(|\lambda| - \lambda_j) - a\right) - (\mathrm{i}|\lambda| - a)^{-1}\left(1 \pm \frac{\mathrm{i}}{2}\,(|\lambda| - \lambda_j)b|d|\right)\right],$$

$$Q^{\pm}_{33} = \frac{1}{2}\,\mathrm{i}d_0 bd^2\lambda(b|d| \mp (\mathrm{i}|\lambda| - a)^{-1}),$$

$$Q^{+}_{32} = -\mathrm{i}Q^{+}_{31}, \quad Q^{-}_{32} = \mathrm{i}Q^{-}_{31}, \quad Q_{33} = Q_{32} = 0.$$

In order to find the elements of $\mathbf{A} = (A_1, A_2, A_3)^{\mathrm{t}}$, in Eq. (53), we need to find first the terms of order $O(\beta^{-3/2})$ in Eq. (82) as follows:

$$\begin{aligned}
\mathscr{A}_1 &= \beta^{-3/2}\left[\frac{\mathrm{i}}{2}\,(\lambda_j - |\lambda|)N_1 s^- + \left(\frac{\mathrm{i}}{2}\,(\lambda_j - |\lambda|)N_2 + N_4\right)c^-\right],\\
\mathscr{A}_2 &= \beta^{-3/2}\left[-\frac{\mathrm{i}}{2}\,|\lambda|N_3 - \lambda c^- N_1 + \lambda s^- N_2\right],\\
\mathscr{A}_3 &= \beta^{-3/2}\left[-\lambda N_3 + \left(\frac{\mathrm{i}}{2}\,(\lambda_j - |\lambda|)N_2 + N_4\right)s^- - \frac{\mathrm{i}}{2}\,(\lambda_j - |\lambda|)N_1 c^-\right].
\end{aligned}$$

We can find $\mathbf{A} = \mathbf{R}_{y,\pi}\mathscr{A}(-\beta; -d)$ from Lemma 4.1, using Eq. (40) and $\sigma = (\sigma_{yy}, \tau_{xy}, \tau_{yz})^{\mathrm{t}}$,

$$\begin{aligned}
A_1 \sim \frac{-\mathrm{i}}{2}\Bigg[&\frac{1}{e_0 D}\left(\frac{1}{2}\,(\lambda_j - |\lambda|)(\mathrm{i}N_2 + N_1) + N_4\right)\beta^{-\mathrm{i}\varepsilon} + \\
&+ e_0 D\left(\frac{1}{2}\,(\lambda_j - |\lambda|)(\mathrm{i}N_2 - N_1) + N_4\right)\beta^{\mathrm{i}\varepsilon}\Bigg]\beta^{-3/2}
\end{aligned}$$

$$A_2 \sim \frac{\mathrm{i}}{2}\left[-2\lambda N_3 + \frac{1}{e_0 D}\left(\frac{1}{2}(\lambda_j - |\lambda|)(-\mathrm{i}N_1 + N_2) - \mathrm{i}N_4\right)\beta^{-\mathrm{i}\varepsilon} + e_0 D\left(\frac{1}{2}(\lambda_j - |\lambda|)(-\mathrm{i}N_1 - N_2) + \mathrm{i}N_4\right)\beta^{\mathrm{i}\varepsilon}\right]\beta^{-3/2}$$

$$A_3 \sim \frac{\mathrm{i}}{2}\left[-\mathrm{i}|\lambda|N_3 + -\lambda\frac{1}{e_0 D}(N_1 + \mathrm{i}N_2)\beta^{-\mathrm{i}\varepsilon} - \lambda e_0 D(N_1 - \mathrm{i}N_2)\beta^{\mathrm{i}\varepsilon}\right]\beta^{-3/2}, \quad \beta \to \infty. \tag{83}$$

From these, and using relations (40) and (43) we obtain the higher asymptotics for stress at infinity

$$\mathbf{A} \sim A\beta^{-3/2} + A^{+}\beta^{-3/2+\mathrm{i}\varepsilon} + A^{-}\beta^{-3/2-\mathrm{i}\varepsilon}, \tag{84}$$

with

$$\mathbf{A} = \frac{1+\mathrm{i}}{2}\begin{pmatrix} 0 \\ \lambda \\ \frac{1}{2}\,\mathrm{i}|\lambda| \end{pmatrix}\tilde{K}_{III},$$

$$\mathbf{A}^{+} = \frac{e_0}{2c_2}\begin{pmatrix} \frac{1}{2}(\lambda_j - |\lambda|)\tilde{\bar{K}} - \mathrm{i}c_2 D N_4 \\ -\frac{\mathrm{i}}{2}(\lambda_j - |\lambda|)\tilde{\bar{K}} - c_2 D N_4 \\ -\lambda\tilde{\bar{K}} \end{pmatrix},$$

$$\mathbf{A}^{-} = \frac{1}{2c_1 e_0}\begin{pmatrix} \frac{1}{2}(\lambda_j - |\lambda|)\tilde{K} - \frac{\mathrm{i}c_1 N_4}{D} \\ \frac{\mathrm{i}}{2}(\lambda_j - |\lambda|)\tilde{K} + \frac{c_1 N_4}{D} \\ \lambda\tilde{K} \end{pmatrix},$$

where $N_4$ is defined in Eq. (81).

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