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Subsonic semi-infinite crack with a finite friction zone in a bimaterial

Y.A. Antipov *

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

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ABSTRACT

Propagation of a semi-infinite crack along the interface between an elastic half-plane and a rigid half-plane is analyzed. The crack advances at constant subsonic speed. It is assumed that, ahead of the crack, there is a finite segment where the conditions of Coulomb friction law are satisfied. The contact zone of unknown a priori length propagates with the same speed as the crack. The problem reduces to a vector Riemann–Hilbert problem with a piece-wise constant matrix coefficient discontinuous at three points, 0, 1, and ∞. The problem is solved exactly in terms of Kummer's solutions of the associated hypergeometric differential equation. Numerical results are reported for the length of the contact friction zone, the stress singularity factor, the normal displacement u_2, and the dynamic energy release rate G. It is found that in the case of frictionless contact for both the sub-Rayleigh and super-Rayleigh regimes, G is positive and the stress intensity factor K_{II} does not vanish. In the sub-Rayleigh case, the normal displacement is positive everywhere in the opening zone. In the super-Rayleigh regime, there is a small neighborhood of the ending point of the open zone where the normal displacement is negative.

1. Introduction

Modeling of a mode-I, II crack propagating at constant speed along the interface has been attracted many researchers (Freund, 1990; Broberg, 1999). The models, static and dynamic, which assume the direct transition from the free boundary condition to the condition of ideal contact lead to an oscillating singularity which causes the non-physical self-penetration of the crack banks (Williams, 1959; England, 1965). In static contact mechanics an analogue of this problem is the study of the ideal contact of a punch with a half-plane. The non-physical singularity has been removed by Falkovich (1945) by considering a frictionless contact zone and Galin (1945) by introducing a finite Coulomb friction contact zone. Comninou (1977) introduced a friction contact zone in static fracture mechanics. The model problem was reduced to a singular integral equation and solved numerically. By the method of the vector Riemann–Hilbert problem, Antipov (1995) solved analytically the static model problem for an interfacial finite crack with two Coulomb friction zones.

The dynamic steady-state model problem for a semi-infinite crack propagating along an interface with a finite friction contact zone ahead leads to a Riemann–Hilbert problem. If the crack propagates at intersonic speed, then the model reduces to a scalar Riemann–Hilbert problem and admits a closed-form solution (Huang et al., 1998). In the subsonic case (the sub-Rayleigh and super-Rayleigh regimes), the Riemann–Hilbert problem is a vector problem whose coefficient is a piece-wise constant matrix discontinuous at 0, 1, and ∞. Assuming that in the contact zone there was no friction, Simonov

* Tel.: +1 225 578 1567.
E-mail address: antipov@math.lsu.edu

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(1984) analyzed a semi-infinite crack propagating at sub-Rayleigh speed along the interface between two elastic half-planes. The matrix coefficient was factorized exactly by the method of conformal mappings.

To the knowledge of the author, the dynamic problem for a semi-infinite interfacial subsonically propagating crack with a finite friction zone ahead has not been solved in the literature. The main idea of the method to be proposed is to construct a partial solution to the Riemann–Hilbert problem in terms of Kummer's solutions to the hypergeometric differential equation with unknown parameters and use it to factorize the matrix coefficient. The parameters are recovered from the boundary condition of the Riemann–Hilbert problem such that the solution is within the class prescribed. This method was designed for some specific problems of filtration and the theory of conformal mappings by Polubarinova-Kochina (1962). Tsitskishvili (1977) carried out a method for the Riemann–Hilbert problem with a piece-wise constant matrix coefficient based on the theory of functions of matrices. Khvoshchinskaya (1986) developed the method further to make it practical not only for the case when the number of discontinuity points, n, is three, but also for $n>3$. The method of factorization based on the theory of the hypergeometric equation was also applied by Moiseyev and Popov (1994) for a mode-III crack orthogonal to two interfaces of a composite and by Craster and Obnosov (2006) to analyze a model problem for a doubly periodic composite.

In Section 2, we formulate the 2-d-problem of a semi-infinite crack with a finite friction zone ahead propagating at constant subsonic speed. We show that the problem is equivalent to a Riemann–Hilbert problem for a pair of functions. The matrix coefficient of the Riemann–Hilbert problem has three points of discontinuity. In Section 3, we derive a particular solution to the Riemann–Hilbert problem in terms of Kummer's solutions to a certain hypergeometric equation. Then we use this solution to factorize the matrix coefficient and solve the original Riemann–Hilbert problem. It turns out that the factorization formulas are different for the sub-Rayleigh and super-Rayleigh cases. First, in Section 4, we analyze the sub-Rayleigh regime. We study the asymptotics of the solution, fix the unknown parameters, find the length of the contact zone and derive the stress singularity factor, the normal displacement, and the dynamic energy release rate. In Section 5, we study the super-Rayleigh case. Numerical results show that in the super-Rayleigh case the normal displacement is positive everywhere in the opening zone except at a small neighborhood of the ending point of the open zone. Based on the numerical results we propose to model a semi-infinite interfacial crack propagating at constant super-Rayleigh speed by splitting it into three parts, a leading finite mode-II crack, an intermediate cohesive zone and a semi-infinite trailing mode-I,II crack.

This reminisces the "mother–daughter" mechanism proposed by Abraham and Gao (2000) and Gao et al. (2001) for modeling crack propagation along a weak interface between the same materials. They used atomic simulation and showed that a mode-II shear dominated crack could accelerate to the Rayleigh speed and then nucleate an intersonic finite crack accelerated up to the longitudinal wave speed. Another model of transition from subsonic to intersonic propagation of a crack in a homogeneous plane was studied by Geubelle and Kubair (2001). To couple the normal and shear modes, the authors used a quasi-linear rate independent cohesive failure model. By using the method of boundary integral equations, they found that in some cases, a secondary cohesive zone appears ahead of the main crack and then quickly coalesces with the primary cohesive zone. In other cases, the transition occurs through a rapid smooth acceleration of the main cohesive zone.

2. Formulation

A semi-infinite crack which occupies, at time t, the surface

$$S(t) = \{\mathbf{x} : -\infty < x_1 < Vt + b,\ x_2 = 0,\ \ -\infty < x_3 < +\infty\} \tag{2.1}$$

propagates along an interface $\{-\infty < x_1 < +\infty, x_2 = 0, -\infty < x_3 < +\infty\}$ between an elastic solid ($x_2 > 0$) bonded to a rigid substrate ($x_2 < 0$) (Fig. 1). The speed V is constant and $V < c_s$, where c_s is the shear wave speed for the elastic solid of

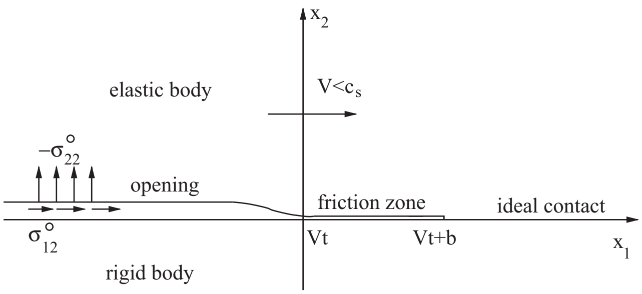

Fig. 1. The geometry of the problem.

density ρ_+ with Lame's constants λ_+ and μ_+. The banks of the crack are subjected to plane-strain loading

$$\sigma_{i2}|_{x_2=0} = \begin{cases} (-1)^{i-1}\sigma^o_{i2}(x_1 - Vt), & -b_1 < x_1 - Vt < -b_2, \\ 0, & x_1 - Vt < -b_1 \text{ or } -b_2 < x_1 - Vt < 0, \end{cases} \tag{2.2}$$

where b_1 and b_2 are prescribed positive parameters ($b_1 > b_2$), and $\sigma^o_{i2}(x)$ ($i = 1, 2$) are prescribed Hölder continuous non-negative functions in the segment $x \in [-b_1, -b_2]$. Along a finite zone $0 < x_1 - Vt < b$ whose length b is unknown a priori, the crack banks contact according to Coulomb law of dry friction,

$$\sigma_{12} = f\sigma_{22}, \quad u_2 = 0, \quad 0 < x_1 - Vt < b, \quad x_2 = 0, \tag{2.3}$$

where $f = f_k$ is the kinematic friction coefficient.

Since the crack and the loading move with the same constant speed V, the problem is steady-state, and it is helpful to introduce a moving dimensionless coordinate $\xi = (x_1 - Vt)/b$. In the new coordinates (ξ, x_2), the boundary conditions become

$$\sigma_{12} = \tau_0(\xi), \quad \sigma_{22} = -\sigma_0(\xi), \quad -\infty < \xi < 0, \quad x_2 = 0,$$

$$\sigma_{12} - f\sigma_{22} = 0, \quad u_2 = 0, \ 0 < \xi < 1, \quad x_2 = 0,$$

$$u_1 = u_2 = 0, \ 1 < \xi < \infty, \quad x_2 = 0, \tag{2.4}$$

where

$$\tau_0(\xi) = \begin{cases} \sigma^o_{12}(b\xi), & -1/l < \xi < -h/l, \\ 0, & \xi < -1/l \text{ or } -h/l < \xi < 0, \end{cases}$$

$$\sigma_0(\xi) = \begin{cases} \sigma^o_{22}(b\xi), & -1/l < \xi < -h/l, \\ 0, & \xi < -1/l \text{ or } -h/l < \xi < 0, \end{cases} \quad l = \frac{b}{b_1}, \quad h = \frac{b_2}{b_1} \in (0, 1). \tag{2.5}$$

The displacement and stress fields are expressible through the displacement potentials $\phi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ which satisfy the wave equations (Freund, 1990)

$$c_l^2 \Delta\phi = \phi_{tt}, \quad c_s^2 \Delta\psi = \psi_{tt}, \quad -\infty < x_1 < +\infty, \ x_2 > 0, \ t > 0, \tag{2.6}$$

where c_l and c_s are the longitudinal and shear wave speeds, respectively,

$$c_l = \sqrt{\frac{\lambda_+ + 2\mu_+}{\rho_+}} = \sqrt{\frac{E_+(1 - \nu_+)}{\rho_+(1 + \nu_+)(1 - 2\nu_+)}},$$

$$c_s = \sqrt{\frac{\mu_+}{\rho_+}} = \sqrt{\frac{E_+}{2\rho_+(1 + \nu_+)}}, \tag{2.7}$$

E_+ is the Young modulus, and ν_+ is the Poisson's ratio.

In new coordinates, (ξ, η_l) and (ξ, η_s),

$$\eta_l = \frac{a_l x_2}{b}, \quad \eta_s = \frac{a_s x_2}{b}, \quad a_l = \sqrt{1 - \frac{V^2}{c_l^2}}, \quad a_s = \sqrt{1 - \frac{V^2}{c_s^2}}, \tag{2.8}$$

the functions

$$\phi_0(\xi, \eta_l) = \phi\left(b\xi + Vt, \frac{b\eta_l}{a_l}\right), \quad \psi_0(\xi, \eta_s) = \psi\left(b\xi + Vt, \frac{b\eta_s}{a_s}\right) \tag{2.9}$$

are harmonic and therefore can be represented as the real and the imaginary parts of two functions, Φ and Ψ, respectively,

$$\phi_0(\xi, \eta_l) = \operatorname{Re}\Phi(\zeta_l), \quad \psi_0(\xi, \eta_s) = \operatorname{Im}\Psi(\zeta_s), \quad \zeta_l = \xi + i\eta_l, \quad \zeta_s = \xi + i\eta_s, \tag{2.10}$$

analytic in the upper half-plane $\mathbb{C}^+$. To rewrite the boundary conditions (2.4) in terms of the complex potentials Φ and Ψ, we express the displacements u_1 and u_2 and the stresses σ_{12} and σ_{22} in terms of these functions,

$$2bu_1 = \Phi'(\zeta_l) + \overline{\Phi'(\zeta_l)} + a_s[\Psi'(\zeta_s) + \overline{\Psi'(\zeta_s)}],$$

$$2bu_2 = ia_l[\Phi'(\zeta_l) - \overline{\Phi'(\zeta_l)}] + i[\Psi'(\zeta_s) - \overline{\Psi'(\zeta_s)}],$$

$$\frac{2b^2}{\mu_+}\sigma_{12} = 2ia_l[\Phi''(\zeta_l) - \overline{\Phi''(\zeta_l)}] + ia[\Psi''(\zeta_s) - \overline{\Psi''(\zeta_s)}],$$

$$\frac{2b^2}{\mu_+}\sigma_{22} = -a[\Phi''(\zeta_l) + \overline{\Phi''(\zeta_l)}] - 2a_s[\Psi''(\zeta_s) + \overline{\Psi''(\zeta_s)}], \tag{2.11}$$

where $a = 2 - V^2/c_s^2$.

In what follows we aim to reduce the physical problem (2.4), (2.6) to a vector Riemann–Hilbert problem with a piece-wise constant 2×2 matrix coefficient. The opening crack boundary conditions read

$$\Omega^+(\xi) = \Omega^-(\xi) + \frac{b^2}{\mu_+}\begin{pmatrix} \tau_0(\xi) \\ -\sigma_0(\xi) \end{pmatrix}, \quad -\infty<\xi<0, \tag{2.12}$$

where $\Omega^\pm(\xi) = \Omega(\xi \pm i0)$, and the vector $\Omega(\zeta)$ is defined by

$$\Omega(\zeta) = \begin{pmatrix} ia_l & ia/2 \\ -a/2 & -a_s \end{pmatrix}\begin{pmatrix} \Phi''(\zeta_l) \\ \Psi''(\zeta_s) \end{pmatrix}, \quad \zeta = \xi + ix_2/b \in \mathbb{C}^+,$$

$$\Omega(\zeta) = \begin{pmatrix} ia_l & ia/2 \\ a/2 & a_s \end{pmatrix}\begin{pmatrix} \overline{\Phi''}(\zeta_l) \\ \overline{\Psi''}(\zeta_s) \end{pmatrix}, \quad \zeta = \xi - ix_2/b \in \mathbb{C}^-. \tag{2.13}$$

The vector-function $\Omega(\zeta)$ is piece-wise analytic in the plane $\mathbb{C}$ with the line of discontinuity $\{-\infty<\xi<+\infty\}$.

In dealing with the ideal contact boundary conditions in (2.4) write them first in terms of the tangential derivatives of the displacements, $\partial u_1/\partial\xi = \partial u_2/\partial\xi = 0$, $1<\xi<\infty$. Then from (2.11),

$$\begin{pmatrix} 1 & a_s \\ ia_l & i \end{pmatrix}\begin{pmatrix} \Phi''(\xi) \\ \Psi''(\xi) \end{pmatrix} = \begin{pmatrix} -1 & -a_s \\ ia_l & i \end{pmatrix}\begin{pmatrix} \overline{\Phi''}(\xi) \\ \overline{\Psi''}(\xi) \end{pmatrix}, \quad 1<\xi<+\infty, \tag{2.14}$$

which in turn implies

$$\Omega^+(\xi) = A_1\Omega^-(\xi), \quad 1<\xi<+\infty, \tag{2.15}$$

where

$$A_1 = \begin{pmatrix} d_1 & ia_ld_2 \\ -ia_sd_2 & d_1 \end{pmatrix}, \quad d_1 = \frac{\Delta_1^+}{\Delta_1^-},$$

$$d_2 = -\frac{2}{\Delta_1^-}(2-a)(a-2a_sa_l), \quad \Delta_1^\pm = (a-2a_sa_l)^2 \pm (a-2)^2a_sa_l. \tag{2.16}$$

Consider now the friction boundary condition, $\partial u_2/\partial\xi = 0$, $\sigma_{12} - f\sigma_{22} = 0$, $0<\xi<1$. Using formulas (2.11) we may find

$$\begin{pmatrix} 2ia_l + af & ia + 2a_sf \\ ia_l & i \end{pmatrix}\begin{pmatrix} \Phi''(\xi) \\ \Psi''(\xi) \end{pmatrix} = \begin{pmatrix} 2ia_l - af & ia - 2a_sf \\ ia_l & i \end{pmatrix}\begin{pmatrix} \overline{\Phi''}(\xi) \\ \overline{\Psi''}(\xi) \end{pmatrix}. \tag{2.17}$$

By inverting the matrix in the left-hand side and exploiting the relations (2.13) we obtain the boundary condition for the one-sided limits $\Omega^+(\xi)$ and $\Omega^-(\xi)$ of the vector-function $\Omega(\zeta)$ in $0<\xi<1$,

$$\Omega^+(\xi) = A_0\Omega^-(\xi), \quad 0<\xi<1, \tag{2.18}$$

where

$$A_0 = \frac{1}{\Delta_0}\begin{pmatrix} \Delta_0 & 2ifa_l(a-2) \\ 0 & f(a-2a_la_s) + ia_l(a-2) \end{pmatrix},$$

$$\Delta_0 = ia_l(2-a) + f(a-2a_la_s). \tag{2.19}$$

Thus, the model problem on propagating a semi-infinite crack along the interface between an elastic and rigid half-planes with a finite friction zone ahead is equivalent to the following vector Riemann–Hilbert problem:

$$\Omega^+(\xi) = A(\xi)\Omega^-(\xi) + g(\xi), \quad -\infty<\xi<+\infty, \tag{2.20}$$

where

$$A(\xi) = \begin{cases} I, & -\infty<\xi<0, \\ A_0, & 0<\xi<1, \\ A_1, & 1<\xi<+\infty, \end{cases} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$g(\xi)=\frac{b^2}{\mu_+}\begin{pmatrix}\tau_0(\xi)\\ -\sigma_0(\xi)\end{pmatrix},\quad -\infty<\xi<0,\quad g(\xi)=0,\ \ 0<\xi<+\infty. \tag{2.21}$$

3. Solution of the Riemann–Hilbert problem

The key step in the solution of the inhomogeneous Riemann–Hilbert problem (2.20) is factorization of the matrix $A(\xi)$,

$$A(\xi)=X^+(\xi)[X^-(\xi)]^{-1},\qquad -\infty<\xi<+\infty, \tag{3.1}$$

where $X^{\pm}(\xi)$ are the boundary values $\zeta=\xi\pm i0$ of a piece-wise analytic 2×2 matrix $X(\zeta)$. The matrix $A(\xi)$ has three points of discontinuity, 0, 1, and ∞, and therefore it can be factorized in closed form in terms of hypergeometric functions (Golubev, 1950; Khvoshchinskaya, 1986).

First, we diagonalize the matrices A_0 and A_1,

$$A_j=T_j\Lambda_jT_j^{-1},\quad j=0,1, \tag{3.2}$$

where

$$\Lambda_j=\begin{pmatrix}\lambda_1^{(j)} & 0\\ 0 & \lambda_2^{(j)}\end{pmatrix},\quad T_0=\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix},\quad T_1=\begin{pmatrix}\sqrt{a_l} & \sqrt{a_l}\\ -i\sqrt{a_s} & i\sqrt{a_s}\end{pmatrix},$$

$$\lambda_1^{(0)}=1,\quad \lambda_2^{(0)}=\frac{ia_l(a-2)+f(a-2a_la_s)}{\Delta_0},$$

$$\lambda_m^{(1)}=d_1+(-1)^{m-1}\sqrt{a_sa_l}d_2,\quad m=1,2. \tag{3.3}$$

Next, we find a partial solution to the homogeneous vector Riemann–Hilbert problem,

$$\omega^+(\xi)=A(\xi)\omega^-(\xi),\qquad -\infty<\xi<+\infty. \tag{3.4}$$

This solution will later be used for the reconstruction of the matrix $X(\zeta)$.

3.1. Partial solution to the homogeneous vector Riemann–Hilbert problem

Consider the hypergeometric equation

$$\zeta(1-\zeta)w''+[\gamma-(\alpha+\beta+1)\zeta]w'-\alpha\beta w=0,\quad w=w(\zeta), \tag{3.5}$$

where the parameters of the equations are to be determined. Eq. (3.5) has two linearly independent solutions,

$$\chi_1(\zeta)=F(\alpha,\beta,\gamma;\zeta),$$

$$\chi_2(\zeta)=(-\zeta)^{1-\gamma}F(1+\alpha-\gamma,1+\beta-\gamma,2-\gamma;\zeta),\qquad -\pi<\arg(-\zeta)<\pi. \tag{3.6}$$

These solutions can analytically be continued from the interior of the unit disk into its exterior by the relation (Bateman and Erdelyi, 1953)

$$\begin{pmatrix}\chi_1(\xi)\\ \chi_2(\xi)\end{pmatrix}=B\begin{pmatrix}\varphi_1(\xi)\\ \varphi_2(\xi)\end{pmatrix},\quad B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}, \tag{3.7}$$

where the elements of the transition matrix B are given by

$$b_{11}=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)},\quad b_{12}=\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)},$$

$$b_{21}=\frac{\Gamma(\beta-\alpha)\Gamma(2-\gamma)}{\Gamma(1-\alpha)\Gamma(1+\beta-\gamma)},\quad b_{22}=\frac{\Gamma(\alpha-\beta)\Gamma(2-\gamma)}{\Gamma(1-\beta)\Gamma(1+\alpha-\gamma)}, \tag{3.8}$$

and $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$ are two out of 24 Kummer's solutions

$$\varphi_1(\zeta)=(-\zeta)^{-\alpha}F(\alpha,1+\alpha-\gamma,1+\alpha-\beta;\zeta^{-1}),$$

$$\varphi_2(\zeta)=(-\zeta)^{-\beta}F(\beta,1+\beta-\gamma,1+\beta-\alpha;\zeta^{-1}),\qquad -\pi<\arg(-\zeta)<\pi. \tag{3.9}$$

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We seek a partial solution to the problem (3.4) in the form

$$\omega(\zeta) = T_0(-\zeta)^{-\sigma}(1-\zeta)^{-\rho}\mathrm{diag}(p_1, p_2)\chi(\zeta), \quad |\zeta|<1,$$

$$\omega(\zeta) = T_1(-\zeta)^{-\sigma}(1-\zeta)^{-\rho}\mathrm{diag}(q_1, q_2)\varphi(\zeta), \quad |\zeta|>1, \tag{3.10}$$

where $-\pi<\arg(-\zeta)<\pi$, $-\pi<\arg(1-\zeta)<\pi$, σ, ρ, p_1, p_2, q_1, and q_2 are some parameters to be fixed, $\chi=(\chi_1,\chi_2)^\top$, $\varphi=(\varphi_1,\varphi_2)^\top$, and the matrices T_0 and T_1 are given in (3.3). The second formula in (3.10) continues analytically the first one from the unit disc into its exterior, $|\zeta|>1$.

Because the problem is homogeneous, without loss of generality, one of the parameters p_j and q_j ($j=1,2$) say p_1, can be fixed. Let $p_1=1$. After this we still have eight parameters to be determined. At the same time we need to make sure that the boundary values $\omega^+(\xi)$ and $\omega^-(\xi)$ of the function $\omega(\zeta)$ satisfy the condition (3.4).

Let first $-\infty<\xi<0$. A branch of the function $(-\zeta)^\mu$, where μ is one of the parameters $-\alpha$, $-\beta$, $1-\gamma$, and $-\sigma$, is fixed by the condition $-\pi<\arg(-\zeta)<\pi$. It is a single valued function in the ζ-plane cut along the positive semi-axis, $\{\mathrm{Im}\,\zeta=0,\ 0<\mathrm{Re}\,\zeta<+\infty\}$. Similarly, the function $(1-\zeta)^{-\rho}$, $-\pi<\arg(1-\zeta)<\pi$, is single valued in the ζ-plane cut along the semi-infinite segment, $\{\mathrm{Im}\,\zeta=0,\ 1<\mathrm{Re}\,\zeta<+\infty\}$. Thus, from Eq. (3.10) for $-\infty<\xi<0$, we have

$$\mathrm{diag}(1, p_2)B = T\,\mathrm{diag}(q_1, q_2), \tag{3.11}$$

where

$$T = T_0^{-1}T_1 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{a_l}+if\sqrt{a_s} & \sqrt{a_l}-if\sqrt{a_s} \\ -i\sqrt{a_s} & i\sqrt{a_s} \end{pmatrix}. \tag{3.12}$$

We derive from the matrix equation (3.11) immediately

$$q_1 = \frac{b_{11}}{t_{11}}, \quad q_2 = \frac{b_{12}}{t_{12}}, \quad p_2 = \frac{b_{11}t_{21}}{t_{11}b_{21}}, \tag{3.13}$$

and an extra equation for the parameters α, β, and γ

$$t_{11}t_{22}\sin\pi\alpha\sin\pi(\gamma-\beta) = t_{12}t_{21}\sin\pi\beta\sin\pi(\gamma-\alpha). \tag{3.14}$$

Consider now the case $0<\xi<1$. Then $A(\xi)=A_0=T_0\Lambda_0T_0^{-1}$, and from the boundary condition (3.4) and the first relation in (3.10) we obtain

$$(-\xi)_+^{-\sigma}(1-\xi)_+^{-\rho}\begin{pmatrix} \chi_1^+(\xi) \\ p_2\chi_2^+(\xi) \end{pmatrix} = \begin{pmatrix} \lambda_1^{(0)} & 0 \\ 0 & \lambda_2^{(0)} \end{pmatrix}(-\xi)_-^{-\sigma}(1-\xi)_-^{-\rho}\begin{pmatrix} \chi_1^-(\xi) \\ p_2\chi_2^-(\xi) \end{pmatrix}, \tag{3.15}$$

where $(-\xi)_\pm^{-\sigma}$ and $(1-\xi)_\pm^{-\rho}$ are the boundary values as $\zeta=\xi\pm i0$ of the functions $(-\zeta)^{-\sigma}$ and $(1-\zeta)^{-\rho}$, respectively. Evidently, for $0<\xi<1$,

$$(-\xi)_+^{-\sigma} = e^{2\pi i\sigma}(-\xi)_-^{-\sigma}, \quad (1-\xi)_+^{-\rho} = (1-\xi)_-^{-\rho},$$

$$\chi_1^+(\xi) = \chi_1^-(\xi), \quad \chi_2^+(\xi) = e^{2\pi i(\gamma-1)}\chi_2^-(\xi). \tag{3.16}$$

Therefore, in order to match the values of the vectors $\omega^+(\xi)$ and $A_0\omega^-(\xi)$ on the segment $(0,1)$, we require

$$e^{2\pi i\sigma} = 1, \quad e^{2\pi i(\gamma-1)} = \lambda_2^{(0)}. \tag{3.17}$$

Choose $\sigma=0$ and notice that since

$$\lambda_2^{(0)} = \frac{f(a-2a_sa_l)+ia_l(a-2)}{f(a-2a_sa_l)-ia_l(a-2)} \tag{3.18}$$

we have $|\lambda_2^{(0)}|=1$, and the parameter γ is real. Introduce a new parameter

$$q = \frac{1}{\pi}\tan^{-1}\frac{(2-a)a_l}{(a-2a_sa_l)f}. \tag{3.19}$$

Since $0<V<c_s<c_l$ we obtain $a>2a_sa_l$, and therefore $q\in(0,\frac{1}{2})$ for $f>0$, and $q=\frac{1}{2}$ when the contact is frictionless, $f=0$. In view of the second equation in (3.17) we can determine the parameter γ up to an arbitrary integer k_γ,

$$\gamma = 1-q+k_\gamma. \tag{3.20}$$

Finally, consider the case $1<\xi<+\infty$. Because of the chosen value of σ the boundary condition for $\xi>1$ becomes

$$(1-\xi)_+^{-\rho}\begin{pmatrix} q_1\varphi_1^+(\xi) \\ q_2\varphi_2^+(\xi) \end{pmatrix} = \begin{pmatrix} \lambda_1^{(1)} & 0 \\ 0 & \lambda_2^{(1)} \end{pmatrix}(1-\xi)_-^{-\rho}\begin{pmatrix} q_1\varphi_1^-(\xi) \\ q_2\varphi_2^-(\xi) \end{pmatrix}. \tag{3.21}$$

From here we find two equations for the parameters α, β, and ρ

$$e^{2\pi i(\alpha+\rho)} = \lambda_1^{(1)}, \quad e^{2\pi i(\beta+\rho)} = \lambda_2^{(1)}. \tag{3.22}$$

We have used here the relation $(1-\xi)_+^{-\rho} = e^{2\pi i\rho}(1-\xi)_-^{-\rho}$, $1<\xi<+\infty$, and formulas (3.9) for the functions φ_1 and φ_2. To solve Eqs. (3.22) for the parameters α and β, analyze the sign of the eigenvalues

$$\lambda_j^{(1)} = \frac{[(a-2a_s a_l)+(-1)^j(2-a)\sqrt{a_s a_l}]^2}{(1-a_s a_l)D}, \quad j=1,2, \tag{3.23}$$

of the matrix A_1. Here

$$D = a^2 - 4a_s a_l. \tag{3.24}$$

It can directly be verified that the eigenvalues are expressed through each other by

$$\lambda_2^{(1)} = \frac{1}{\lambda_1^{(1)}}. \tag{3.25}$$

We next observe that $\operatorname{sign}\lambda_j^{(1)} = \operatorname{sign} D$ and determine the range for the speed V when the eigenvalues are negative. Notice that

$$D = \left(2-\frac{V}{c_s^2}\right)^2 - 4\sqrt{1-\frac{V}{c_s^2}}\sqrt{1-\frac{V}{c_l^2}}. \tag{3.26}$$

Introduce new parameters

$$r = \frac{1-2\nu_+}{2(1-\nu_+)} \in \left(0,\frac{1}{2}\right), \quad s = \frac{V^2}{c_s^2} \in (0,1). \tag{3.27}$$

Then $V^2/c_l^2 = sr$ and

$$D = (2-s)^2 - 4\sqrt{(1-sr)(1-s)}. \tag{3.28}$$

We now wish to find $s \in (0,1)$ when $D<0$ or, equivalently, when the cubic polynomial

$$H(s) = s^3 - 8s^2 + \kappa_0 s - \kappa_0 + 8 \tag{3.29}$$

is negative. Here $\kappa_0 = 8(2-\nu_+)/(1-\nu_+) \in (16,24)$. Since $H(0) = 8-\kappa_0<0$, $H(1)=1$, and $H'(s)>0$ for all $s\in(0,1)$ we conclude that in the interval $(0,1)$, there is one and only one value say $s_* = V^2/c_s^2$, such that $H(s_*)=0$. This zero is simple and given by

$$s_* = \frac{1}{3}(8 - R_+ - R_-), \tag{3.30}$$

where

$$R_\pm = \left(\frac{45\kappa_0}{2} - 404 \pm \frac{3\sqrt{3R}}{2}\right)^{1/3},$$

$$R = -14656 + 2768\kappa_0 - 181\kappa_0^2 + 4\kappa_0^3. \tag{3.31}$$

The corresponding value of $V = c_s\sqrt{s_*}$ is the Rayleigh speed, $V = c_R$, for the elastic solid with the characteristic speeds c_s and c_l.

To sum up, we have shown that for the sub-Rayleigh regime both of the eigenvalues, $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$, are negative. In the super-Rayleigh case, $c_R<V<c_s$, they are positive.

Let first $0<V<c_R$. The super-Rayleigh regime will be analyzed in Section 5. From (3.22) we find

$$\alpha = -\frac{1}{2} - \rho - i\varepsilon + k_\alpha, \quad \beta = -\frac{1}{2} - \rho + i\varepsilon + k_\beta, \tag{3.32}$$

where k_α and k_β are arbitrary integers and

$$\varepsilon = \frac{\ln|\lambda_1^{(1)}|}{2\pi} = -\frac{\ln|\lambda_2^{(1)}|}{2\pi}. \tag{3.33}$$

The last undetermined parameter, ρ, can be recovered from Eq. (3.14) which reads

$$2i\sqrt{a_s a_l}\cos\pi(\alpha+\beta-\gamma) = t_{11}t_{22}\cos\pi(\alpha-\beta+\gamma) - t_{21}t_{12}\cos\pi(\beta-\alpha+\gamma). \tag{3.34}$$

By using the expressions (3.32) and (3.20) we can transform Eq. (3.34) into the form

$$\cos\pi(q-2\rho) = S, \tag{3.35}$$

where

$$S=-\cos\pi q\cosh 2\pi\varepsilon-f\sqrt{\frac{a_s}{a_l}}\sin\pi q\sinh 2\pi\varepsilon. \tag{3.36}$$

In order to utilize the relation (3.35), we have to simplify the expression for S. Since $\lambda_1^{(1)}\lambda_2^{(1)}=1$ and $|\lambda_j^{(1)}|=-\lambda_j^{(1)}, j=1,2$, we have

$$\cosh 2\pi\varepsilon=-d_1>0,\quad \sinh 2\pi\varepsilon=-\sqrt{a_sa_l}d_2<0. \tag{3.37}$$

Combining formulas (3.19), (3.37), (2.16), and (3.36) we obtain

$$S=\cos\pi q\left(d_1+\frac{(2-a)a_sa_ld_2}{a-2a_sa_l}\right)=\cos\pi q. \tag{3.38}$$

Therefore, Eq. (3.34) is equivalent to the following one:

$$\cos\pi(q-2\rho)=\cos\pi q. \tag{3.39}$$

This equation has two families of solutions

$$\text{(I) } \rho=-k_\rho' \quad \text{and} \quad \text{(II) } \rho=q-k_\rho,\quad 0<V<c_R, \tag{3.40}$$

where k_ρ' and k_ρ are integers. Since we seek a partial solution we may consider the second family only. Choose $k_\rho=0$ and therefore $\rho=q$. It will later be verified that this choice leads to an integrable singularity of the solution at the point $\zeta=1$. Thus, for the sub-Rayleigh regime the following parameters are fixed: $p_1=1$, $\sigma=0$, and $\rho=q$. At this stage, the parameters α β, and γ are determined up to arbitrary integers k_α, k_β, and k_γ, respectively, by (3.32) and (3.20). The parameters p_2, q_1, and q_2 are expressed in terms of α, β, and γ by (3.13).

3.2. Factorization of the matrix $A(\xi)$

We seek a matrix $X(\zeta)$ which possesses the following properties:

(i) its elements, $X_{mj}(\zeta)$ $(m,j=1,2)$, are piece-wise analytic functions in the cut complex plane $\hat{\mathbb{C}}=\mathbb{C}\backslash\{0<\xi<+\infty\}$, Hölder-continuous up to the contour $\{0<\xi<+\infty\}$ except for the points $\xi=0$ and 1, where they may have integrable singularities,
(ii) $\det X(\zeta)$ is finite and non-zero everywhere in the region $\hat{\mathbb{C}}$ except at most for the three singular points $0, 1$, and ∞, and
(iii) its boundary values $X^+(\xi)$ and $X^-(\xi)$ satisfy the boundary condition

$$X^+(\xi)=A(\xi)X^-(\xi),\quad \xi\in(-\infty,0)\cup(0,1)\cup(1,+\infty). \tag{3.41}$$

Introduce functions

$$\hat{\chi}_{j1}(\zeta)=(1-\zeta)^{-q}\chi_j(\zeta),\quad \hat{\chi}_{j2}(\zeta)=D_\zeta\hat{\chi}_{j1}(\zeta),$$

$$\hat{\varphi}_{j1}(\zeta)=(1-\zeta)^{-q}\varphi_j(\zeta),\quad \hat{\varphi}_{j2}(\zeta)=D_\zeta\hat{\varphi}_{j1}(\zeta),\quad j=1,2, \tag{3.42}$$

where D_ζ is the following differential operator:

$$D_\zeta=-\zeta\frac{d}{d\zeta}+\frac{q}{1-\zeta}. \tag{3.43}$$

Evidently, the matrix

$$X(\zeta)=T_0\,\mathrm{diag}(1,p_2)\begin{pmatrix}\hat{\chi}_{11}(\zeta) & \hat{\chi}_{12}(\zeta)\\ \hat{\chi}_{21}(\zeta) & \hat{\chi}_{22}(\zeta)\end{pmatrix},\quad |\zeta|<1,$$

$$X(\zeta)=T_1\,\mathrm{diag}(q_1,q_2)\begin{pmatrix}\hat{\varphi}_{11}(\zeta) & \hat{\varphi}_{12}(\zeta)\\ \hat{\varphi}_{21}(\zeta) & \hat{\varphi}_{22}(\zeta)\end{pmatrix},\quad |\zeta|>1, \tag{3.44}$$

is continuous through the line $-\infty<\xi<0$ and satisfies the boundary condition (3.41) for $0<\xi<1$. By using analytical continuation of the functions $\hat{\chi}_{mj}(\zeta)$ into the exterior of the unit disk we can show that it satisfies the boundary condition (3.41) for $1<\xi<+\infty$ as well.

In order to check the property (ii) of the matrix $X(\zeta)$ for $|\zeta|<1$, rewrite the first expression for $X(\zeta)$ in formula (3.44) as

$$X(\zeta)=\begin{pmatrix}\hat{\chi}_{11}(\zeta)+fp_2\hat{\chi}_{21}(\zeta) & \hat{\chi}_{12}(\zeta)+fp_2\hat{\chi}_{22}(\zeta)\\ p_2\hat{\chi}_{21}(\zeta) & p_2\hat{\chi}_{22}(\zeta)\end{pmatrix} \tag{3.45}$$

and find its determinant. Simple calculations show that

$$\det X(\zeta) = -p_2\zeta(1-\zeta)^{-2q}W(\chi_1,\chi_2), \tag{3.46}$$

where $W(\chi_1,\chi_2)=\chi_1\chi_2' - \chi_1'\chi_2$ is the Wronskian of the system of functions $\{\chi_1,\chi_2\}$. Prove next that

$$W(\chi_1,\chi_2) = (\gamma-1)(-\zeta)^{-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-1}. \tag{3.47}$$

Note first that as $\zeta\to 0$,

$$\chi_1(\zeta) = F(\alpha,\beta,\gamma;\zeta)\sim 1,$$

$$\chi_2(\zeta) = (-\zeta)^{1-\gamma}F(1+\alpha-\gamma,1+\beta-\gamma,2-\gamma;\zeta)\sim(-\zeta)^{1-\gamma},$$

$$\chi_1'(\zeta) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;\zeta)\sim\frac{\alpha\beta}{\gamma},$$

$$\begin{aligned}\chi_2'(\zeta) &= (\gamma-1)(-\zeta)^{-\gamma}F(1+\alpha-\gamma,1+\beta-\gamma,2-\gamma;\zeta)\\ &\quad+(-\zeta)^{1-\gamma}\frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{2-\gamma}F(2+\alpha-\gamma,2+\beta-\gamma,3-\gamma;\zeta)\sim(\gamma-1)(-\zeta)^{-\gamma},\end{aligned} \tag{3.48}$$

and therefore $W(\chi_1,\chi_2)\sim(\gamma-1)(-\zeta)^{-\gamma}$, $\zeta\to 0$. This implies

$$\frac{W(\chi_1,\chi_2)}{(-\zeta)^{-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-1}}\sim\gamma-1, \quad \zeta\to 0. \tag{3.49}$$

On the other hand, for any $\zeta\in\{\zeta:|\zeta|<1\}$,

$$\frac{d}{d\zeta}\frac{W(\chi_1,\chi_2)}{(-\zeta)^{-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-1}} = 0. \tag{3.50}$$

Indeed, by direct differentiation,

$$\frac{d}{d\zeta}[(-\zeta)^{\gamma}(1-\zeta)^{\alpha+\beta+1-\gamma}(\chi_1\chi_2' - \chi_1'\chi_2)] = (-\zeta)^{\gamma-1}(1-\zeta)^{\alpha+\beta-\gamma}(-\chi_1\mathcal{L}[\chi_2]+\chi_2\mathcal{L}[\chi_1]), \tag{3.51}$$

where

$$\mathcal{L}[\chi_j] = \zeta(1-\zeta)\chi_j'' + [\gamma-(\alpha+\beta+1)\zeta]\chi_j' - \alpha\beta\chi_j, \quad j=1,2. \tag{3.52}$$

Since $\chi_1(\zeta)$ and $\chi_2(\zeta)$ are two linearly independent solutions to the hypergeometric equation (3.5), the relation (3.50) becomes evident. From (3.49) and (3.50), formula (3.47) follows for $|\zeta|<1$ and therefore

$$\det X(\zeta) = p_2(\gamma-1)(-\zeta)^{1-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-2q-1}, \quad |\zeta|<1. \tag{3.53}$$

Combining (3.45) and (3.53) we now have the inverse matrix for $|\zeta|<1$

$$[X(\zeta)]^{-1} = \frac{(-\zeta)^{\gamma-1}(1-\zeta)^{\alpha+\beta-\gamma+2q+1}}{(\gamma-1)p_2}\begin{pmatrix} p_2\hat\chi_{22}(\zeta) & -\hat\chi_{12}(\zeta)-fp_2\hat\chi_{22}(\zeta)\\ -p_2\hat\chi_{21}(\zeta) & \hat\chi_{11}(\zeta)+fp_2\hat\chi_{21}(\zeta)\end{pmatrix}, \quad |\zeta|<1. \tag{3.54}$$

Similarly, we can determine the determinant and the inverse matrix in the case $|\zeta|>1$. From the second relation in (3.44) we obtain

$$X(\zeta) = \begin{pmatrix}\sqrt{a_l}[q_1\hat\varphi_{11}(\zeta)+q_2\hat\varphi_{21}(\zeta)] & \sqrt{a_l}[q_1\hat\varphi_{12}(\zeta)+q_2\hat\varphi_{22}(\zeta)]\\ i\sqrt{a_s}[-q_1\hat\varphi_{11}(\zeta)+q_2\hat\varphi_{21}(\zeta)] & i\sqrt{a_s}[-q_1\hat\varphi_{12}(\zeta)+q_2\hat\varphi_{22}(\zeta)]\end{pmatrix}, \quad |\zeta|>1. \tag{3.55}$$

It is a matter of simple algebra to show that

$$\det X(\zeta) = -2i\sqrt{a_s a_l}q_1q_2\zeta(1-\zeta)^{-2q}W(\varphi_1,\varphi_2), \quad |\zeta|>1, \tag{3.56}$$

where $W(\varphi_1,\varphi_2)=\varphi_1\varphi_2' - \varphi_1'\varphi_2$. Analysis of the Wronskian $W(\varphi_1,\varphi_2)$ at infinity shows

$$\frac{W(\varphi_1,\varphi_2)}{(-\zeta)^{-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-1}}\sim\beta-\alpha, \quad \zeta\to\infty. \tag{3.57}$$

On the other hand, for any $\zeta\in\{\zeta:|\zeta|>1\}$,

$$\frac{d}{d\zeta}\frac{W(\varphi_1,\varphi_2)}{(-\zeta)^{-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-1}} = 0, \tag{3.58}$$

and therefore, for any finite ζ such that $|\zeta|>1$,

$$W(\varphi_1,\varphi_2)=(\beta-\alpha)(-\zeta)^{-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-1}. \tag{3.59}$$

This defines the determinant and the inverse of the matrix $X(\zeta)$ for $|\zeta|>1$,

$$\det X(\zeta)=2i\sqrt{a_s a_l}q_1q_2(\beta-\alpha)(-\zeta)^{1-\gamma}(1-\zeta)^{\gamma-\alpha-\beta-2q-1}, \quad |\zeta|>1,$$

$$[X(\zeta)]^{-1}=\frac{(-\zeta)^{\gamma-1}(1-\zeta)^{\alpha+\beta-\gamma+2q+1}}{2i\sqrt{a_s a_l}q_1q_2(\beta-\alpha)}\times\begin{pmatrix} i\sqrt{a_s}[-q_1\hat{\varphi}_{12}(\zeta)+q_2\hat{\varphi}_{22}(\zeta)] & -\sqrt{a_l}[q_1\hat{\varphi}_{12}(\zeta)+q_2\hat{\varphi}_{22}(\zeta)] \\ -i\sqrt{a_s}[-q_1\hat{\varphi}_{11}(\zeta)+q_2\hat{\varphi}_{21}(\zeta)] & \sqrt{a_l}[q_1\hat{\varphi}_{11}(\zeta)+q_2\hat{\varphi}_{21}(\zeta)]\end{pmatrix}, \quad |\zeta|>1. \tag{3.60}$$

Thus, we have proved that the matrix $X(\zeta)$ is non-singular everywhere in the complex plane except for the points 0, 1, and ∞.

3.3. *Definition of the parameters*

At this point, the integers k_α, k_β and k_γ are arbitrary. In order to fix the parameter k_γ, determine the asymptotics of the matrix $X(\zeta)$ at the point $\zeta=0$. From the definition of the functions $\hat{\chi}_{mj}(\zeta)$,

$$\hat{\chi}_{11}(\zeta)\sim 1+c_{11}\zeta,$$

$$\hat{\chi}_{12}(\zeta)\sim q+c_{12}\zeta,$$

$$\hat{\chi}_{21}(\zeta)\sim(-\zeta)^{1-\gamma}+c_{21}(-\zeta)^{2-\gamma},$$

$$\hat{\chi}_{22}(\zeta)\sim(q+\gamma-1)(-\zeta)^{1-\gamma}+c_{22}(-\zeta)^{2-\gamma}, \quad \zeta\to 0, \tag{3.61}$$

where c_{mj} $(m,j=1,2)$ are non-zero constants. From (3.20), $1-\gamma=q-k_\gamma$, $q\in(0,\frac{1}{2}]$. The elements of the matrix $X(\zeta)$ may have at most an integrable singularity at the point $\zeta=0$. Consequently $k_\gamma=1$ and $\gamma=2-q$. Then, at the point $\zeta=0$, the matrix $X(\zeta)$ has the following asymptotic representation:

$$X(\zeta)=p_2(-\zeta)^{q-1}\begin{pmatrix} f & f \\ 1 & 1\end{pmatrix}+\begin{pmatrix} 1 & q \\ 0 & 0\end{pmatrix}+O(\zeta^q), \quad \zeta\to 0. \tag{3.62}$$

The elements of the inverse matrix are bounded at the point $\zeta=0$,

$$[X(\zeta)]^{-1}=\frac{1}{\gamma-1}\begin{pmatrix} 1 & -f \\ -1 & f\end{pmatrix}+\frac{(-\zeta)^{1-q}}{p_2(\gamma-1)}\begin{pmatrix} 0 & -q \\ 0 & 1\end{pmatrix}+O(\zeta), \quad \zeta\to 0. \tag{3.63}$$

Fix next the integers k_α and k_β. For this we need the asymptotics of the matrix $X(\zeta)$ as $\zeta\to\infty$. Since

$$\hat{\varphi}_{11}(\zeta)\sim(-\zeta)^{-\alpha-q}, \quad \hat{\varphi}_{21}(\zeta)\sim(-\zeta)^{-\beta-q},$$

$$\hat{\varphi}_{12}(\zeta)\sim(\alpha+q)(-\zeta)^{-\alpha-q}, \quad \hat{\varphi}_{22}(\zeta)\sim(\beta+q)(-\zeta)^{-\beta-q}, \quad \zeta\to\infty, \tag{3.64}$$

we have

$$X(\zeta)\sim q_1(-\zeta)^{-\alpha-q}\begin{pmatrix} \sqrt{a_l} & \sqrt{a_l}(q+\alpha) \\ -i\sqrt{a_s} & -i\sqrt{a_s}(q+\alpha)\end{pmatrix}+q_2(-\zeta)^{-\beta-q}\begin{pmatrix} \sqrt{a_l} & \sqrt{a_l}(q+\beta) \\ i\sqrt{a_s} & i\sqrt{a_s}(q+\beta)\end{pmatrix}, \quad \zeta\to\infty. \tag{3.65}$$

Since the elements of the matrix $X(\zeta)$ have to be Hölder continuous functions at $\zeta=\infty$ we require $-1<\mathrm{Re}(-q-\alpha)<0$ and $-1<\mathrm{Re}(-q-\beta)<0$. From (3.32) we obtain $k_\alpha=k_\beta=1$ and therefore

$$\alpha=\frac{1}{2}-i\varepsilon-q, \quad \beta=\frac{1}{2}+i\varepsilon-q. \tag{3.66}$$

Finally, show that the elements of the matrix $X(\zeta)$ have an integrable singularity at the point $\zeta=1$. Using the analytical continuation of the hypergeometric function into a neighborhood of the point $\zeta=1$ (Bateman and Erdelyi, 1953),

$$F(\alpha,\beta,\gamma;\zeta)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-\zeta)$$
$$+(1-\zeta)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-\zeta), \quad |\arg(1-\zeta)|<\pi, \tag{3.67}$$

formulas (3.9) and (3.42) and also since $\sigma=0$, $\rho=q$, $\alpha=\frac{1}{2}-i\varepsilon-q$, $\beta=\overline{\alpha}$, and $\gamma=2-q$, we have the following asymptotic relations:

$$\hat{\chi}_{11}(\zeta)=\frac{\Gamma(1+q)\Gamma(2-q)}{\Gamma(\frac{3}{2}+i\varepsilon)\Gamma(\frac{3}{2}-i\varepsilon)}(1-\zeta)^{-q}+O\{(1-\zeta)^{1-q}\},$$

$$\hat{\chi}_{12}(\zeta)=\frac{\Gamma(q)\Gamma(2-q)(q-\frac{1}{4}-\varepsilon^2)}{\Gamma(\frac{3}{2}+i\varepsilon)\Gamma(\frac{3}{2}-i\varepsilon)}(1-\zeta)^{-q}+O(1),\quad \zeta\to 1. \tag{3.68}$$

To write down the asymptotics at $\zeta=1$ of the other two functions, $\hat{\chi}_{21}(\zeta)$ and $\hat{\chi}_{22}(\zeta)$, for the chosen branch of $(-\zeta)^{-\gamma}$ we compute

$$(-\zeta)^{-\gamma}|_{\zeta=1+i0}=e^{-i\pi q}. \tag{3.69}$$

Similarly to the relations (3.68) we derive

$$\hat{\chi}_{21}(\zeta)\sim-\frac{e^{-i\pi q}\Gamma(1+q)\Gamma(q)}{\Gamma(\frac{1}{2}+i\varepsilon+q)\Gamma(\frac{1}{2}-i\varepsilon+q)}(1-\zeta)^{-q}+O\{(1-\zeta)^{1-q}\},$$

$$\hat{\chi}_{22}(\zeta)\sim-\frac{e^{-i\pi q}\Gamma^2(q)(q-\frac{1}{4}-\varepsilon^2)}{\Gamma(\frac{1}{2}+i\varepsilon+q)\Gamma(\frac{1}{2}-i\varepsilon+q)}(1-\zeta)^{-q}+O(1),\quad \zeta\to 1. \tag{3.70}$$

Finally, we need to specify the parameters p_2, q_1, and q_2. We have already fixed the other parameters in the representation (3.6), (3.9), and (3.44). They are given by

$$\sigma=0,\quad \rho=q,\quad \alpha=\frac{1}{2}-q-i\varepsilon,\quad \beta=\frac{1}{2}-q+i\varepsilon,\quad \gamma=2-q,\quad p_1=1. \tag{3.71}$$

In accordance with the relations (3.13) and (3.71) we obtain

$$p_2=\frac{i\sqrt{a_s}\Gamma(2-q)\Gamma(\frac{1}{2}+q+i\varepsilon)}{(\sqrt{a_l}+if\sqrt{a_s})(\varepsilon^2+\frac{1}{4})\Gamma(q)\Gamma(\frac{1}{2}-q+i\varepsilon)},$$

$$q_1=\frac{\Gamma(2-q)\Gamma(2i\varepsilon)}{(\sqrt{a_l}+if\sqrt{a_s})\Gamma(\frac{3}{2}+i\varepsilon)\Gamma(\frac{1}{2}-q+i\varepsilon)},\quad q_2=\overline{q_1}. \tag{3.72}$$

If now we turn to (3.14) we find that $p_2=\overline{p_2}$, that is the parameter p_2 is real.

3.4. Inhomogeneous Riemann–Hilbert problem

By replacing the matrix coefficient $A(\xi)$ in (2.20) by $X^+(\xi)[X^-(\xi)]^{-1}$ and introducing a new piece-wise analytic vector

$$\Upsilon(\zeta)=\frac{1}{2\pi i}\int_{-1/l}^{-h/l}[X^+(\xi)]^{-1}\frac{g(\xi)\,d\xi}{\xi-\zeta}, \tag{3.73}$$

we can rewrite the boundary condition (2.20) of the Riemann–Hilbert problem as follows:

$$[X^+(\xi)]^{-1}\Omega^+(\xi)-\Upsilon^+(\xi)=[X^-(\xi)]^{-1}\Omega^-(\xi)-\Upsilon^-(\xi),\quad -\infty<\xi<+\infty, \tag{3.74}$$

where $\Upsilon^{\pm}(\xi)=\Upsilon(\xi\pm i0)$, the boundary values $\Upsilon^+(\xi)$ and $\Upsilon^-(\xi)$, $\xi\in(-\infty,+\infty)$, are not the same in the segment $[-1/l,-h/l]$ and coincide otherwise. At infinity, the inverse matrix $[X(\zeta)]^{-1}$ has the following asymptotics:

$$[X(\zeta)]^{-1}\sim(-\zeta)^{1/2+i\varepsilon}E_0+(-\zeta)^{1/2-i\varepsilon}\overline{E_0},\quad \zeta\to\infty, \tag{3.75}$$

where

$$E_0=\frac{1}{4\varepsilon q_2}\begin{pmatrix}\frac{i}{\sqrt{a_l}}\left(\frac{1}{2}-i\varepsilon\right) & \frac{1}{\sqrt{a_s}}(\frac{1}{2}-i\varepsilon)\\ -\frac{i}{\sqrt{a_l}} & -\frac{1}{\sqrt{a_s}}\end{pmatrix}. \tag{3.76}$$

By the Liouville theorem, the Riemann–Hilbert problem (2.20) has a unique solution given by

$$\Omega(\zeta)=X(\zeta)\Upsilon(\zeta). \tag{3.77}$$

4. Analysis of the solution: the sub-Rayleigh regime

In this section we analyze the asymptotics of the solution at the points $\xi=0,1$, and $+\infty$ in the case $0<V<c_R$, define the length of the friction zone, the stress singularity factor at the point $\zeta=1$, and the dynamic energy release rate. For this we

shall evidently need the stresses and the tangential derivatives of the displacements,

$$\sigma_{11} = \frac{2\mu_+}{b^2}\mathrm{Re}\left[\left(a_l^2 - a_s^2 + \frac{a}{2}\right)\Phi''(\zeta_l) + a_s\Psi''(\zeta_s)\right],$$

$$\sigma_{12} = -\frac{\mu_+}{b^2}\mathrm{Im}[2a_l\Phi''(\zeta_l) + a\Psi''(\zeta_s)],$$

$$\sigma_{22} = -\frac{\mu_+}{b^2}\mathrm{Re}[a\Phi''(\zeta_l) + 2a_s\Psi''(\zeta_s)],$$

$$\frac{\partial u_1}{\partial\xi} = \frac{1}{b}\mathrm{Re}[\Phi''(\zeta_l) + a_s\Psi''(\zeta_s)],$$

$$\frac{\partial u_2}{\partial\xi} = -\frac{1}{b}\mathrm{Im}[a_l\Phi''(\zeta_l) + \Psi''(\zeta_s)], \tag{4.1}$$

where

$$\begin{pmatrix}\Phi''(\zeta_l)\\ \Psi''(\zeta_s)\end{pmatrix} = \frac{2}{iD}\begin{pmatrix}-2a_s\Omega_1(\zeta) - ia\Omega_2(\zeta)\\ a\Omega_1(\zeta) + 2ia_l\Omega_2(\zeta)\end{pmatrix}. \tag{4.2}$$

4.1. Asymptotics of the solution at $+\infty$

It follows from (3.77) and (3.65) that the vector-function $\Omega(\zeta)$ decays at infinity,

$$\Omega(\zeta) \sim N^\infty(-\zeta)^{-q-\alpha-1} - \overline{N^\infty}(-\zeta)^{-q-\beta-1}, \quad \zeta\to\infty, \tag{4.3}$$

where $q+\alpha+1 = \frac{3}{2} - i\varepsilon$, $q+\beta+1 = \frac{3}{2} + i\varepsilon$, $N^\infty = (N_1^\infty, N_2^\infty)^\top$ is a constant vector,

$$N^\infty = -q_1\begin{pmatrix} i\sqrt{a_l}\\ \sqrt{a_s}\end{pmatrix}[\Upsilon_1^\infty + (q+\alpha)\Upsilon_2^\infty],$$

$$\Upsilon^\infty = \begin{pmatrix}\Upsilon_1^\infty\\ \Upsilon_2^\infty\end{pmatrix} = \frac{1}{2\pi}\int_{-1/l}^{-h/l}[X(\xi)]^{-1}g(\xi)\,d\xi, \quad \mathrm{Im}\,\Upsilon^\infty = 0. \tag{4.4}$$

In view of (4.3), (4.1), and (4.2) we can derive the asymptotics of the tractions σ_{12} and σ_{22} as $\xi\to+\infty$,

$$\sigma_{m2} \sim \xi^{-3/2}(e_i'\cos\varepsilon\ln\xi + e_i''\sin\varepsilon\ln\xi), \quad \xi\to+\infty,\ m=1,2, \tag{4.5}$$

where e_m' and e_m'' are non-zero real constants. The tangential derivatives of the displacements when $x_2 = 0^+$ have the same asymptotics as $\xi\to-\infty$,

$$\frac{\partial u_m}{\partial\zeta} \sim (-\xi)^{-3/2}(\hat{e}_i'\cos\varepsilon\ln\xi + \hat{e}_i''\sin\varepsilon\ln\xi), \quad \zeta = \xi + i0,\ \xi\to-\infty,\ m=1,2, \tag{4.6}$$

where $\hat{e}_m'$ and $\hat{e}_m''$ are non-zero real constants.

4.2. Length of the friction zone

Show next that the length of the friction zone cannot be arbitrary and has to be chosen such that the traction σ_{22} is bounded at the point $\zeta = 0$. In general, for an arbitrary positive value of the parameter l, the traction σ_{22} and the function $\partial u_2/\partial\xi$ have integrable singularities as $\zeta = \xi$ and $\xi\to 0^+$ and $\xi\to 0^-$, respectively. Indeed, from (3.62), we find

$$\Phi''(\zeta_l) \sim \frac{4p_2(\Upsilon_1^0 + \Upsilon_2^0)}{D}\left(a_s f + \frac{ia}{2}\right)(-\zeta)^{q-1},$$

$$\Psi''(\zeta_s) \sim -\frac{4p_2(\Upsilon_1^0 + \Upsilon_2^0)}{D}\left(\frac{af}{2} + ia_l\right)(-\zeta)^{q-1}, \quad \zeta\to 0, \tag{4.7}$$

where

$$\Upsilon^0 = \begin{pmatrix}\Upsilon_1^0\\ \Upsilon_2^0\end{pmatrix} = \frac{1}{2\pi}\int_{-1/l}^{-h/l}[X(\xi)]^{-1}g(\xi)\frac{d\xi}{\xi}, \quad \mathrm{Im}\,\Upsilon^0 = 0. \tag{4.8}$$

For the chosen branch, $(-\zeta)^{q-1} = -e^{-i\pi q}\xi^{q-1}$, $\zeta = \xi + i0$, $\xi > 0$. Therefore, from (4.1),

$$\sigma_{22} \sim \frac{2\mu_+ p_2 \sin\pi q}{b^2}(\Upsilon_1^0 + \Upsilon_2^0)\xi^{q-1}, \quad \xi\to 0^+. \tag{4.9}$$

Similarly, for the function $\partial u_2/\partial\xi$,

$$\frac{\partial u_2}{\partial\xi}\sim\frac{2p_2a_lV^2}{bDc_s^2}(\Upsilon_1^0+\Upsilon_2^0)(-\xi)^{q-1},\quad \xi\to 0^-. \tag{4.10}$$

Clearly, the traction σ_{22} has to be negative for $0<\xi<1$. On the other hand, for ξ negative and close to the point $\xi=0$, the tangential derivative $\partial u_2/\partial\xi$ has to be negative as well. However, the analysis of formulas (4.9) and (4.10) shows that the functions σ_{22} and $\partial u_2/\partial\xi$ have different signs as $\xi\to 0^+$ and $\xi\to 0^-$, respectively,

$$\lim_{\xi\to 0^+}\sigma_{22}\xi^{1-q}\left[\lim_{\xi\to 0^-}\frac{\partial u_2}{\partial\xi}(-\xi)^{1-q}\right]^{-1}=\frac{\mu_+c_s^2D\sin\pi q}{ba_lV^2}<0,\quad V<c_R. \tag{4.11}$$

Hence, both functions, σ_{22} and $\partial u_2/\partial\xi$, have to be bounded at the point $\zeta=0$. Because of the asymptotic relations (4.10) and (4.11) this occurs if

$$\Upsilon_1^0+\Upsilon_2^0=0. \tag{4.12}$$

Then the vector $\Omega(\zeta)$ and the functions $\Phi''(\zeta)$ and $\Psi''(\zeta)$ are bounded at the point $\zeta=0$,

$$\Omega(\zeta)=-i\begin{pmatrix}\Upsilon_1^0+q\Upsilon_1^0\\ 0\end{pmatrix}+O(\zeta^q),\quad \zeta\to 0,$$

$$\begin{pmatrix}\Phi''(\zeta_l)\\ \Psi''(\zeta_s)\end{pmatrix}=\frac{4}{D}\begin{pmatrix}a_s\\ -\dfrac{a}{2}\end{pmatrix}(\Upsilon_1^0+q\Upsilon_2^0)+O(\zeta^q),\quad \zeta\to 0. \tag{4.13}$$

On substituting formula (4.13) into (4.1) we find that the tractions σ_{12} and σ_{22} and the function $\partial u_2/\partial\xi$ vanish at the point $\xi=0$ while the function $\partial u_1/\partial\xi$ is bounded and does not equal zero,

$$\sigma_{12}=O(\xi^q),\quad \sigma_{22}=O(\xi^q),\quad \frac{\partial u_2}{\partial\xi}=O(\xi^q),\quad \xi\to 0,$$

$$\frac{\partial u_1}{\partial\xi}=\frac{2a_s(2-a)}{bD}(\Upsilon_1^0+q\Upsilon_2^0)+O(\xi^q),\quad \xi\to 0. \tag{4.14}$$

We next transform the condition (4.12) in a form more suitable for computations. In view of (3.54) and (3.60), Eq. (4.12) becomes

$$\int_{-1/l}^{-h/l}(-\xi)^{-q}P(\xi)\,d\xi=0, \tag{4.15}$$

where

$$P(\xi)=p_2[\chi_2^o(\xi)-\chi_2(\xi)]\tau_0(\xi)-\{\chi_1(\xi)-\chi_1^o(\xi)+fp_2[\chi_2(\xi)-\chi_2^o(\xi)]\}\sigma_0(\xi) \tag{4.16}$$

for $-1<\xi<0$, and

$$\begin{aligned}P(\xi)=&\,i\sqrt{a_s}[-q_1\varphi_1^o(\xi)+q_2\varphi_2^o(\xi)+q_1\varphi_1(\xi)-q_2\varphi_2(\xi)]\tau_0(\xi)\\ &-\sqrt{a_l}[-q_1\varphi_1^o(\xi)-q_2\varphi_2^o(\xi)+q_1\varphi_1(\xi)+q_2\varphi_2(\xi)]\sigma_0(\xi)\end{aligned} \tag{4.17}$$

for $\xi<-1$. Here the functions $\chi_j(\zeta)$ and $\varphi_j(\zeta)$ are given by (3.6) and (3.9) and

$$\chi_j^o(\zeta)=-\zeta\chi_j'(\zeta)+q\chi_j(\zeta),$$

$$\varphi_j^o(\zeta)=-\zeta\varphi_j'(\zeta)+q\varphi_j(\zeta),\quad j=1,2,$$

$$\varphi_1'(\zeta)=\alpha(-\zeta)^{-\alpha-1}F\left(\alpha,1+\alpha-\gamma,1+\alpha-\beta,\frac{1}{\zeta}\right)-(-\zeta)^{-\alpha-2}\frac{\alpha(1+\alpha-\gamma)}{1+\alpha-\beta}F\left(1+\alpha,2+\alpha-\gamma,2+\alpha-\beta,\frac{1}{\zeta}\right). \tag{4.18}$$

The derivative $\varphi_2'(\zeta)$ coincides with $\varphi_1'(\zeta)$ if α and β are interchanged, and the derivatives $\chi_j'(\zeta)$ are given by (3.48). The condition (4.15) is a transcendental equation with respect to the dimensionless parameter $l=b/b_1$ which ultimately defines the length of the friction zone b. For computations of the functions $\chi_j(\xi)$ and $\chi_j^o(\xi)$ for $\xi\in(-1,-\frac{1}{2})$ and the functions $\varphi_j(\xi)$ and $\varphi_j^o(\xi)$ for $\xi\in(-2,-1)$, it is convenient to use first the formula

$$F(\alpha,\beta,\gamma,\xi)=(1-\xi)^{-\alpha}F\left(\alpha,\gamma-\beta,\gamma,\frac{\xi}{\xi-1}\right) \tag{4.19}$$

and then (3.67).

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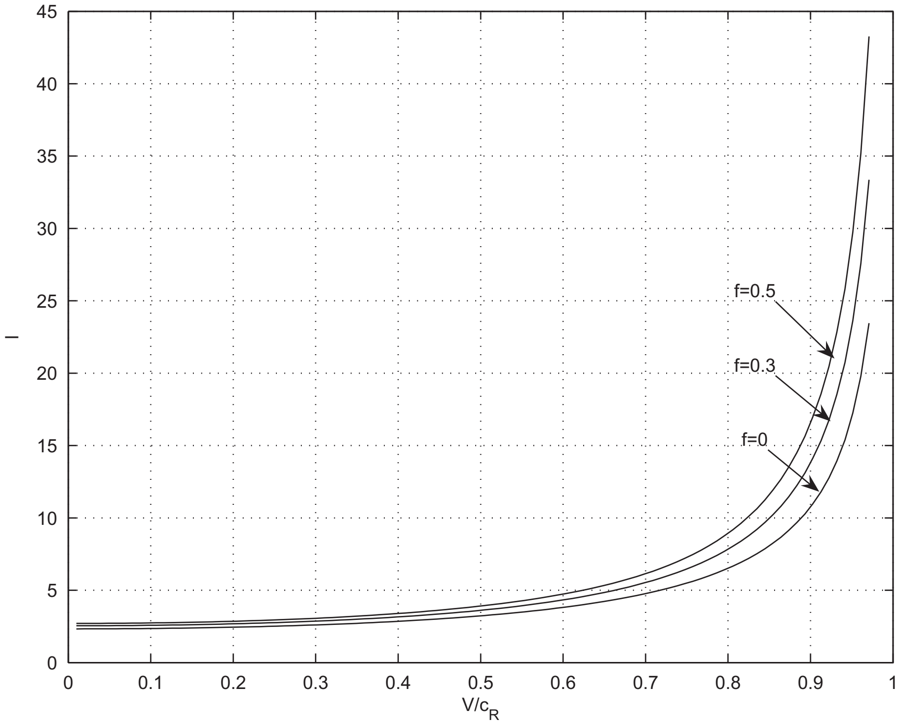

Fig. 2. The dimensionless length of the friction zone $l = b/b_1$ vs. the parameter $V/c_R < 1$ for the friction coefficients $f = 0$, 0.3, and 0.5 when $\nu_+ = 0.3$, and the loading has the form (4.20).

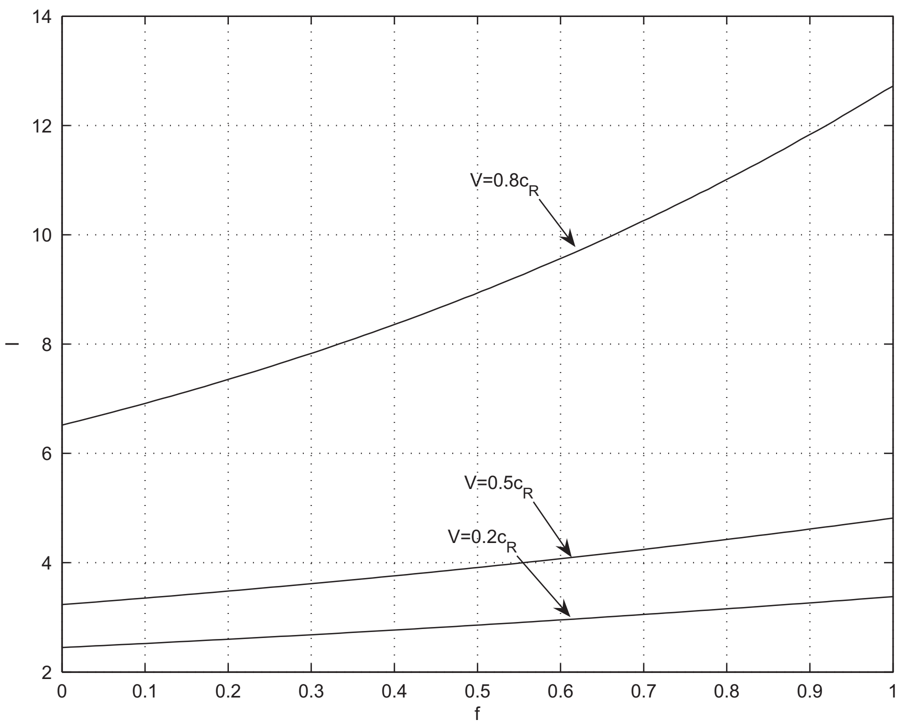

Fig. 3. The dimensionless length of the friction zone $l = b/b_1$ vs. the friction coefficient f for the speeds $V = 0.2c_R$, $0.5c_R$, and $0.8c_R$ when $\nu_+ = 0.3$ and the loading has the form (4.20).

Computations are implemented for the loading

$$\sigma_{j2}=\begin{cases}(-1)^{j-1}\sigma^o_{j2}, & -1/l<\xi<-h/l,\\ 0, & \xi<-1/l \text{ or } -h/l<\xi<0,\end{cases}$$

$$\sigma^o_{j2}=\text{const},\quad j=1,2,\quad \frac{\sigma^o_{22}}{\sigma^o_{12}}=0.1,\quad h=\frac{b_2}{b_1}=0.5, \tag{4.20}$$

Poisson's ratio $\nu_+=0.3$, and the friction coefficients $f=0$, 0.3, and 0.5. The results show that the length of the contact zone $b=lb_1$ increases when the speed increases and tends to infinity as $V\to c_R$ (Fig. 2). Fig. 3 shows the effect on the contact zone length of changing the friction coefficient. It turns out that the length increases as the friction coefficient increases.

4.3. Stress singularity factor

We wish now to analyze the tractions as $\xi\to 1^{\pm}$. Derive first the asymptotics of the matrix $X(\zeta)$ as $\zeta\to 1$. According to formulas (3.45), (3.68), and (3.70), the matrix $X(\zeta)$ has an integrable singularity at this point,

$$X(\zeta)\sim\Gamma(q)(1-\zeta)^{-q}\begin{pmatrix}\tau_1 q & \tau_1(q-\frac{1}{4}-\varepsilon^2)\\ \tau_2 q & \tau_2(q-\frac{1}{4}-\varepsilon^2)\end{pmatrix},\quad \zeta\to 1, \tag{4.21}$$

where

$$\tau_1=\frac{\Gamma(2-q)}{|\Gamma(\frac{3}{2}+i\varepsilon)|^2}+f\tau_2,\quad \tau_2=-\frac{p_2e^{-i\pi q}\Gamma(q)}{|\Gamma(\frac{1}{2}+i\varepsilon+q)|^2}. \tag{4.22}$$

Notice that we can also use the analytic continuation (3.55) of the matrix $X(\zeta)$ into the exterior $\zeta>1$ of the unit disc. By exploiting formula (3.67) we continue the elements of the matrix (3.55) in a neighborhood of the point $\zeta=1$,

$$X(\zeta)\sim ie^{-i\pi q}\Gamma(q)(1-\zeta)^{-q}\begin{pmatrix}\sqrt{a_l}q\tau_+ & \sqrt{a_l}(q-\frac{1}{4}-\varepsilon^2)\tau_+\\ i\sqrt{a_s}q\tau_- & i\sqrt{a_s}(q-\frac{1}{4}-\varepsilon^2)\tau_-\end{pmatrix},\quad \zeta\to 1. \tag{4.23}$$

where

$$\tau_\pm=\pm\frac{q_1\Gamma(1-2i\varepsilon)e^{\pi\varepsilon}}{\Gamma(\frac{1}{2}-i\varepsilon+q)\Gamma(\frac{3}{2}-i\varepsilon)}+\frac{q_2\Gamma(1+2i\varepsilon)e^{-\pi\varepsilon}}{\Gamma(\frac{1}{2}+i\varepsilon+q)\Gamma(\frac{3}{2}+i\varepsilon)}. \tag{4.24}$$

It can directly be shown that the matrices in the right-hand sides of formulas (4.21) and (4.23) coincide. By using the asymptotic formula (4.21) and the relations (3.77) and (4.2) we find

$$\begin{pmatrix}\Phi''(\zeta_l)\\ \Psi''(\zeta_s)\end{pmatrix}\sim-\frac{4\Upsilon_0(1-\zeta)^{-q}}{D}\times\left[\frac{\Gamma(2-q)}{|\Gamma(\frac{3}{2}+i\varepsilon)|^2}\begin{pmatrix}-a_s\\ \dfrac{a}{2}\end{pmatrix}+\frac{p_2\Gamma(q)e^{-i\pi q}}{|\Gamma(\frac{1}{2}+i\varepsilon+q)|^2}\begin{pmatrix}\dfrac{ia}{2}+fa_s\\ -ia_l-\dfrac{fa}{2}\end{pmatrix}\right],\quad \zeta\to 1, \tag{4.25}$$

where

$$\Upsilon_0=\Gamma(q)[q\Upsilon^1_1+(q-\tfrac{1}{4}-\varepsilon^2)\Upsilon^1_2],$$

$$\Upsilon^1=\begin{pmatrix}\Upsilon^1_1\\ \Upsilon^1_2\end{pmatrix}=\frac{1}{2\pi}\int_{-1/l}^{-h/l}[X(\xi)]^{-1}g(\xi)\frac{d\xi}{\xi-1},\quad \operatorname{Im}\Upsilon^1=0. \tag{4.26}$$

Proceeding as before, we obtain that the traction σ_{12} has an integrable singularity as $\zeta=\xi\to 1^{\pm}$ while the traction σ_{22} is unbounded as $\xi\to 1^-$ and bounded as $\xi\to 1^+$,

$$\sigma_{22}\sim\text{const},\quad \sigma_{12}\sim\frac{2\mu_+\Upsilon_0\Gamma(2-q)\sin\pi q}{b^2|\Gamma(\frac{3}{2}+i\varepsilon)|^2}(\xi-1)^{-q},\quad \xi\to 1^+,$$

$$\sigma_{22}\sim\frac{2\mu_+p_2\Gamma(q)\Upsilon_0\sin\pi q}{b^2|\Gamma(\frac{1}{2}+i\varepsilon+q)|^2}(1-\xi)^{-q},\quad \sigma_{12}=f\sigma_{22},\quad \xi\to 1^-. \tag{4.27}$$

These formulas are consistent with those for the traction components in the static case (Antipov, 1995). We emphasize that the normal component σ_{22} of the traction has an integrable singularity as x_1 approaches the tip $x_1=Vt+b$ along the slip

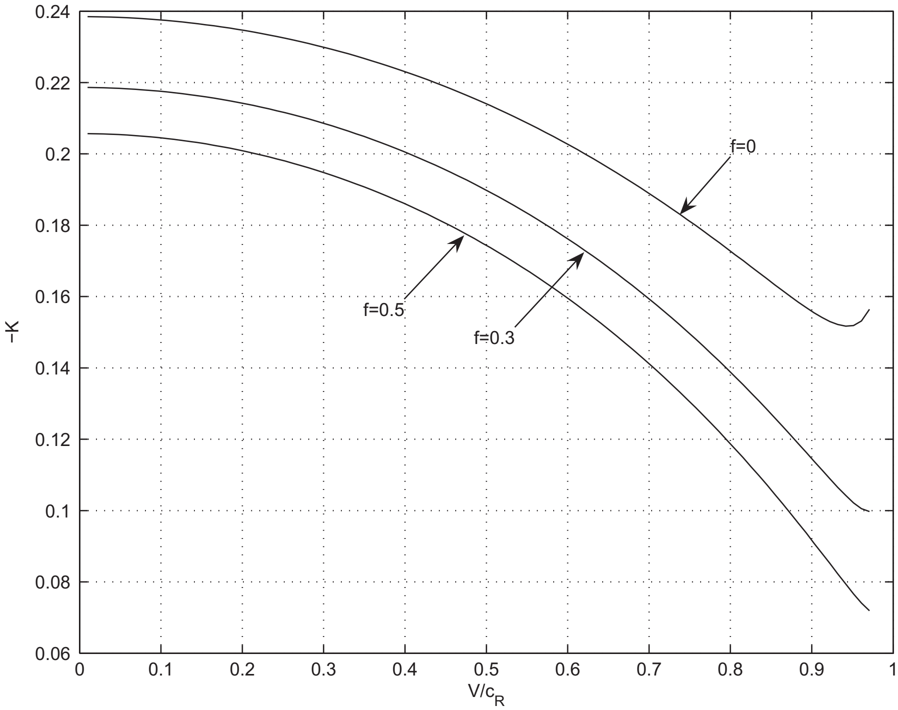

Fig. 4. The stress singularity factor K vs. the parameter V/c_R for the friction coefficients $f=0$, 0.3, and 0.5 when $\nu_+=0.3$ and the loading has the form (4.20).

zone regardless of whether there is friction in the contact zone ($f>0$) or not ($f=0$). The function $\sigma_{22}(x_1,0)$ is always bounded if x_1 approaches the tip of the slip zone along the interface. The tangential component vanishes if $f=0$ and has an integrable singularity at the tip if $x_1 \to Vt+b$ along the slip zone. When x_1 approaches the tip along the interface the traction component σ_{12} has always an integrable singularity.

Determine next the stress singularity factor for the traction σ_{12} (the corresponding σ_{22}-factor vanishes)

$$K = \lim_{x_1 - Vt \to b^+} [2\pi(x_1 - Vt - b)]^q \sigma_{12}|_{x_2=0}. \tag{4.28}$$

From (4.27), it becomes

$$K = \frac{(2\pi)^{1+q}\mu_+(1-q)\Upsilon_0}{b^{2-q}|\Gamma(\frac{3}{2}+i\varepsilon)|^2\Gamma(q)}. \tag{4.29}$$

If there is no friction in the contact zone $0<x_1-Vt<b$, i.e. if $f=0$, then $q=\frac{1}{2}$, and the factor K becomes the K_{II} stress intensity factor,

$$K_{\mathrm{II}} = \left(\frac{\pi}{2b}\right)^{3/2} \frac{\mu_+[2\Upsilon_1^1 + (1-4\varepsilon^2)\Upsilon_2^1]}{|\Gamma(\frac{3}{2}+i\varepsilon)|^2}. \tag{4.30}$$

The coefficient $-K$ ($-K_{\mathrm{II}}$ for $f=0$) is plotted as a function of the ratio V/c_R in Fig. 4. The coefficient is negative and its absolute value decreases when the speed V increases. However, for speeds V close to the Rayleigh speed it attains its minimum and then grows.

4.4. Dynamic energy release rate

The dynamic energy release rate, G, is defined by (Freund, 1990)

$$G = \lim_{\delta\to 0}\left\{\frac{1}{V}\int_{\Gamma_\delta}\left[\sigma_{mj}n_j\frac{\partial u_m}{\partial t} + (U+T)Vn_1\right]ds\right\}, \tag{4.31}$$

where Γ_δ is a simple curve in the upper half-plane with a starting point $x_s \in (b+Vt, +\infty)$ and a terminal point $x_t \in (Vt, b+Vt)$, and $\delta = \max|b+Vt-z|$, $z\in\Gamma_\delta$. U and T are the stress work density and the kinetic energy density, respectively,

$$T=\frac{\rho_+}{2}\frac{\partial u_m}{\partial t}\frac{\partial u_m}{\partial t}=\frac{\mu_+V^2}{2b^2c_s^2}\left[\left(\frac{\partial u_1}{\partial\xi}\right)^2+\left(\frac{\partial u_2}{\partial\xi}\right)^2\right],$$

$$U=\frac{1}{2}\left[-\frac{v_+}{2\mu_+(1+v_+)}\delta_{mj}\delta_{kl}+\frac{1}{4\mu_+}(\delta_{mk}\delta_{jl}+\delta_{ml}\delta_{jk})\right]\sigma_{mj}\sigma_{kl}$$
$$=\frac{1}{2\mu_+(1+v_+)}\left[\frac{1}{2}\sigma_{11}^2-v_+\sigma_{11}\sigma_{22}+\frac{1}{2}\sigma_{22}^2+(1+v_+)\sigma_{12}^2\right], \tag{4.32}$$

$\mathbf{n}=(n_1,n_2)$ is the external (with respect to the positive direction on Γ_δ) normal to the curve Γ_δ, and

$$\sigma_{mj}n_j\frac{\partial u_m}{\partial t}=-\frac{V}{b}\left[\sigma_{11}n_1\frac{\partial u_1}{\partial\xi}+\sigma_{12}\left(n_1\frac{\partial u_2}{\partial\xi}+n_2\frac{\partial u_1}{\partial\xi}\right)+\sigma_{22}n_2\frac{\partial u_2}{\partial\xi}\right]. \tag{4.33}$$

Since the value of the integral in (4.31), the energy flux integral, is independent of the shape of Γ_δ it is convenient to choose the curve Γ_δ as an arc $\Gamma_\delta=\{z: z=b-\delta e^{i\theta}\}$, $-\pi<\theta<0$. Derive next the asymptotics of the stresses σ_{ij}, as $\zeta=1-(\delta/b)e^{i\theta}$ and $\delta\to0$. From the asymptotic relations (3.68) and (3.70) for the functions $\hat{\chi}_{mj}(\zeta)$ $(m,j=1,2)$, we obtain

$$\begin{pmatrix}\Phi''(\zeta_l)\\ \Psi''(\zeta_s)\end{pmatrix}\sim-\left(\frac{b}{\delta}\right)^q\frac{4\Upsilon_0e^{-iq\theta}}{D}\left[v_0\begin{pmatrix}-a_s\\ \dfrac{a}{2}\end{pmatrix}+v_1e^{-i\pi q}\begin{pmatrix}\dfrac{ia}{2}+fa_s\\ -ia_l-\dfrac{fa}{2}\end{pmatrix}\right],$$
$$\delta\to0,\quad -\pi<\arg(1-\zeta)<0, \tag{4.34}$$

where

$$v_0=\frac{\Gamma(2-q)}{|\Gamma(\frac{3}{2}+i\varepsilon)|^2},\quad v_1=\frac{p_2\Gamma(q)}{|\Gamma(\frac{1}{2}+i\varepsilon+q)|^2}. \tag{4.35}$$

This brings us to the asymptotic relations for the stresses as $\delta\to0$

$$\sigma_{11}\sim-\frac{4\mu_+\delta^{-q}\Upsilon_0}{b^{2-q}D}\left\{2a_s(a_s^2-a_l^2)[v_0\cos q\theta-fv_1\cos q(\pi+\theta)]+v_1[a_l^2(a_s^2+1)-2a_sa_l+\frac{1}{2}(1-a_s^4)]\sin q(\pi+\theta)\right\},$$

$$\sigma_{12}\sim\frac{2\mu_+\delta^{-q}\Upsilon_0}{b^{2-q}}[-v_0\sin q\theta+fv_1\sin q(\pi+\theta)],$$

$$\sigma_{22}\sim\frac{2\mu_+\delta^{-q}\Upsilon_0}{b^{2-q}}v_1\sin q(\pi+\theta), \tag{4.36}$$

and the tangential derivatives of the displacements

$$\frac{\partial u_1}{\partial\zeta}\sim-\frac{2\delta^{-q}\Upsilon_0}{b^{1-q}D}\{(a-2)a_s[v_0\cos q\theta-v_1f\cos q(\pi+\theta)]+v_1(a-2a_sa_l)\sin q(\pi+\theta)\},$$

$$\frac{\partial u_2}{\partial\zeta}\sim\frac{2\delta^{-q}\Upsilon_0}{b^{1-q}D}\{(a-2a_sa_l)[-v_0\sin q\theta+v_1f\sin q(\pi+\theta)]+v_1a_l(a-2)\cos q(\pi+\theta)\}. \tag{4.37}$$

Substituting (4.36) and (4.37) into (4.31) we obtain that the dynamic energy release rate G vanishes unless $f=0$. In the case of frictionless contact $q=\frac{1}{2}$ and as $\delta\to0$,

$$\sigma_{11}\sim\frac{2\mu_+\Upsilon_0}{b^{3/2}\delta^{1/2}}v_2\cos\frac{\theta}{2},$$

$$\sigma_{22}\sim\frac{2\mu_+\Upsilon_0}{b^{3/2}\delta^{1/2}}v_1\cos\frac{\theta}{2},$$

$$\sigma_{12}\sim-\frac{2\mu_+\Upsilon_0}{b^{3/2}\delta^{1/2}}v_0\sin\frac{\theta}{2},$$

$$\frac{\partial u_1}{\partial\zeta}\sim\frac{\Upsilon_0}{(b\delta)^{1/2}}v_3\cos\frac{\theta}{2},$$

$$\frac{\partial u_2}{\partial\zeta}\sim-\frac{\Upsilon_0}{(b\delta)^{1/2}}v_4\sin\frac{\theta}{2}, \tag{4.38}$$

where

$$\nu_2 = \frac{1}{D}\{[4a_s a_l + a_s^4 - 1 - 2a_l^2(a_s^2+1)]\nu_1 + 4a_s(a_l^2 - a_s^2)\nu_0\},$$

$$\nu_3 = -\frac{2}{D}[a_s(a-2)\nu_0 + (a - 2a_s a_l)\nu_1],$$

$$\nu_4 = \frac{2}{D}[(a - 2a_s a_l)\nu_0 + a_l(a-2)\nu_1]. \tag{4.39}$$

By evaluating the integral in (4.31) and using (5.18) we obtain

$$G = \frac{1}{\mu_+}AK_{\mathrm{II}}^2, \tag{4.40}$$

where

$$A = \frac{1}{16\nu_0^2}\left\{(\nu_2-\nu_0)\nu_3 - (\nu_0+\nu_1)\nu_4 + \nu_0^2 - \frac{\nu_1^2+\nu_2^2-2\nu_+\nu_1\nu_2}{2(1+\nu_+)} - \frac{V^2(\nu_3^2-\nu_4^2)}{4c_s^2}\right\}. \tag{4.41}$$

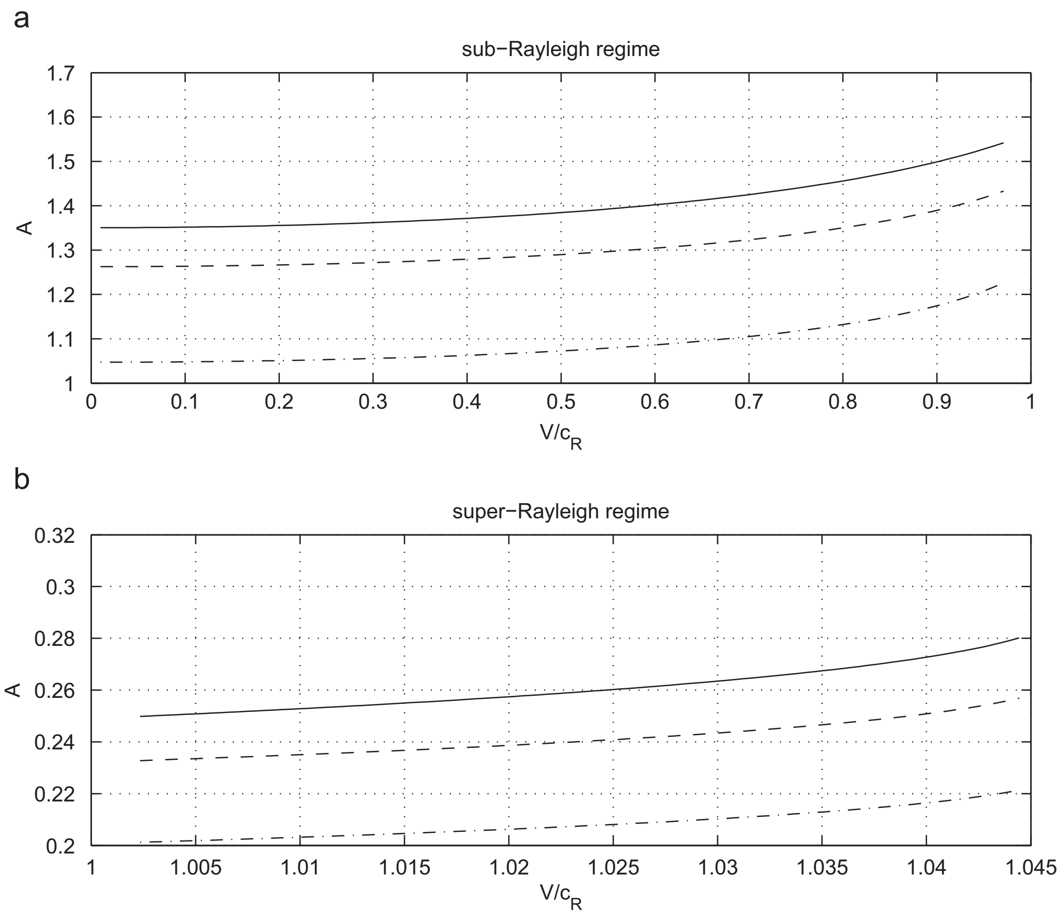

Fig. 5. The dimensionless dynamic energy release rate $A = \mu_+ G/K_{\mathrm{II}}^2$ vs. the parameter V/c_R when $f = 0$ for the Poisson ratios $\nu_+ = 0.1$ (–), 0.3 (– –), and 0.5 (–.); (a) the sub-Rayleigh regime and (b) the super-Rayleigh regime.

Here A is a function of only two dimensionless parameters, ν_+ and V/c_R. Computations show that this function is positive. As a function of the parameter V/c_R, it increases. For some values of the Poisson's ratio ν_+, it is plotted in Fig. 5a.

4.5. Normal displacement

To verify the sign of the normal displacement for $x_1<Vt$, find first the asymptotics of $\partial u_2/\partial\zeta$ as $\zeta=-\xi$ and $\xi\to 0^+$. Upon using the asymptotic relation for the matrix $X(\zeta)$

$$X(\zeta)\sim(-\zeta)^{q-1}p_2\begin{pmatrix} f & f \\ 1 & 1\end{pmatrix}+\begin{pmatrix} 1 & q \\ 0 & 0\end{pmatrix}-\frac{(-\zeta)^q p_2}{q}\begin{pmatrix} f(q^2+|\alpha+q-1|^2) & fq^2 \\ q^2+|\alpha+q-1|^2 & q^2\end{pmatrix},\quad \zeta\to 0^-, \tag{4.42}$$

because $\Upsilon_1^0+\Upsilon_2^0=0$ and $\alpha+q-1=-\frac{1}{2}+\varepsilon$, formula (4.2) gives

$$\begin{pmatrix} \Phi''(\zeta_l) \\ \Psi''(\zeta_s)\end{pmatrix}\sim-\frac{2(1-q)\Upsilon_1^0}{D}\begin{pmatrix} -2a_s \\ a\end{pmatrix}+\frac{2p_2(\frac{1}{4}+\varepsilon^2)^2\Upsilon_1^0(-\zeta)^q}{qD}\begin{pmatrix} -2a_sf-ia \\ af+2ia_l\end{pmatrix},\quad \zeta\to 0^-. \tag{4.43}$$

Combining this formula with (4.1) allows us to determine the asymptotics of the normal displacement as $\zeta\to 0^-$

$$\frac{\partial u_2}{\partial\zeta}\sim k_V(-\zeta)^q,\quad \zeta\to 0^-, \tag{4.44}$$

where

$$k_V=-\frac{a_lp_2(2-a)(1+4\varepsilon^2)\Upsilon_1^0}{2bqD}. \tag{4.45}$$

The computations show that this factor is always negative for the sub-Rayleigh regime and therefore the displacement u_2 is positive at least in a small left neighborhood of the point $x_1=Vt$. For example, for the loading (4.20), $\nu_+=0.3$, $V=0.5c_R$, the factor $k_V=-0.044953$ for $f=0$ and $k_V=-0.044385$ for $f=0.3$.

To verify that the displacement u_2 is positive as $x_2=0$ for all $x_1<Vt$, we need the exact formula for u_2. It follows from (4.1), (3.45) and (3.55) that the tangential derivative of the displacement u_2 on the line $x_2=0$ and $x_1<Vt$ ($\zeta=\xi<$) is given by

$$\frac{\partial u_2}{\partial\zeta}=\frac{2a_l(2-a)}{bD}\begin{cases} p_2[\hat\chi_{21}(\zeta)\hat\Upsilon_1(\zeta)+\hat\chi_{22}(\zeta)\hat\Upsilon_2(\zeta)], & -1<\zeta<0, \\ 2\sqrt{a_s}\,\mathrm{Im}\{q_1[\hat\psi_{11}(\zeta)\hat\Upsilon_1(\zeta)+\hat\psi_{12}(\zeta)\hat\Upsilon_2(\zeta)]\}, & -\infty<\zeta<-1,\end{cases} \tag{4.46}$$

where

$$\hat\Upsilon_j(\zeta)=i\Upsilon_j(\zeta),\quad \mathrm{Im}\,\hat\Upsilon_j(\zeta)=0,\quad \zeta<0. \tag{4.47}$$

Computations show that the displacement u_2 determined by integrating formula (4.46) is positive for $\xi<0$, $x_2=0$.

5. Super-Rayleigh regime

In this case the parameter γ is the same as in the subsonic case, namely, $\gamma=2-q$. The eigenvalues of the matrix A_1, $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$, are positive. By formula (3.22), the parameters α and β become now $\alpha=-\rho-i\varepsilon+k_\alpha$, $\beta=-\rho+i\varepsilon+k_\beta$. The parameter ρ is recovered from Eq. (3.39). As in the sub-Rayleigh case, we choose $\rho=q$. Since the expressions $\alpha+\rho$ and $\beta+\rho$ are different from those in the subsonic case, the asymptotic expressions of the matrices $X(\zeta)$ and $[X(\zeta)]^{-1}$ as $\zeta\to\infty$ are different as well,

$$X(\zeta)\sim E'(-\zeta)^{i\varepsilon}+\overline{E'}(-\zeta)^{-i\varepsilon},$$

$$[X(\zeta)]^{-1}\sim E''(-\zeta)^{-i\varepsilon}+\overline{E''}(\zeta)^{i\varepsilon},\quad \zeta\to\infty, \tag{5.1}$$

where E' and E'' are 2×2 rank-1 matrices with non-zero constant elements. Here the integers k_α and k_β are taken to be zero. Thus, in the super-Rayleigh regime, the parameters α, β, γ, and ρ are

$$\alpha=-q-i\varepsilon,\quad \beta=-q+i\varepsilon,\quad \gamma=2-q,\quad \rho=q. \tag{5.2}$$

As in the sub-Rayleigh case, we specify the parameters p_j and q_j ($j=1,2$)

$$p_1=1,\quad p_2=\frac{\sqrt{a_s}\Gamma(2-q)\Gamma(1+q+i\varepsilon)}{\varepsilon(\varepsilon^2+1)(\sqrt{a_l}+if\sqrt{a_s})\Gamma(q)\Gamma(i\varepsilon-q)},\quad \mathrm{Im}\,p_2=0,$$

$$q_1=\frac{\Gamma(2-q)\Gamma(2i\varepsilon)}{(\sqrt{a_l}+if\sqrt{a_s})\Gamma(i\varepsilon-q)\Gamma(2+i\varepsilon)},\quad q_2=\overline{q_1}. \tag{5.3}$$

Because of the asymptotics (5.1) of the matrix $X(\zeta)$ the traction components σ_{12} and σ_{22} and the tangential derivatives of the displacement components have the following asymptotics at infinity ($x_2 = 0^+$):

$$\sigma_{m2} \sim \frac{1}{\xi}(c'_m \cos\varepsilon\ln\xi + c''_m \sin\varepsilon\ln\xi), \quad \xi \to +\infty,$$

$$\frac{\partial u_m}{\partial \xi} \sim \frac{1}{\xi}(\hat{c}'_m \cos\varepsilon\ln\xi + \hat{c}''_m \sin\varepsilon\ln\xi), \quad \xi \to -\infty, \tag{5.4}$$

where c'_m, c''_m, $\hat{c}'_m$, and $\hat{c}''_m$ $(m = 1, 2)$ are non-zero real constants. Notice that the asymptotic relations (5.4) are different from the ones in the sub-Rayleigh case given by (4.5) and (4.6).

To define the length of the contact zone, we use the condition (4.12), the same as in the sub-Rayleigh regime. Notice, however, that in the super-Rayleigh if l is not a root of Eq. (4.12), then case

$$\lim_{\xi \to 0^+} \sigma_{22}\xi^{1-q}\left[\lim_{\xi \to 0^-} \frac{\partial u_2}{\partial \xi}(-\xi)^{1-q}\right]^{-1} = \frac{\mu_+ c_s^2 D \sin \pi q}{b a_l V^2} > 0, \quad c_R < V < c_s. \tag{5.5}$$

The condition (4.12) guarantees the boundedness of the traction σ_{22} at the point $\xi = 0$ and also the smoothness of the profile of the crack in the vicinity of the point $\xi = 0$. Thus, the parameter l is the root of the transcendental equation (4.15), where the parameters α, β, γ, p_j, and q_j $(j = 1, 2)$ are given by (5.2) and (5.3). Computations are implemented for the loading (4.20) when $h = b_2/b_1 = 0.5$, $\nu_+ = 0.3$ and the friction coefficients $f = 0$, 0.3, and 0.5. The numerical tests show that in the super-Rayleigh regime the length of the contact zone $b = lb_1$ is not a monotonic function (Fig. 6). It tends to infinity as $V \to c_R$ and $V \to c_s$, attains its minimum, b_{min}, at a certain speed $V \in (c_R, c_s)$ and apart from a small (left) neighborhood of the point c_R, $b_V \ll b_{min}$, where b_V is the length of the contact zone in the sub-Rayleigh regime.

Determine next the stress singularity factor K and the dynamic energy release rate. Similarly to the sub-Rayleigh case, we find the asymptotics of the functions $\Phi''(\zeta_l)$ and $\Psi''(\zeta_s)$ as $\zeta \to 1$

$$\begin{pmatrix} \Phi''(\zeta_l) \\ \Psi''(\zeta_s) \end{pmatrix} \sim -\frac{2\Upsilon_0(1-\zeta)^{-q}}{D}\begin{pmatrix} -2\nu_0 a_s + \nu_1 e^{-i\pi q}(ia + 2fa_s) \\ \nu_0 a - \nu_1 e^{-i\pi q}(2ia_l + fa) \end{pmatrix}, \quad \zeta \to 1, \tag{5.6}$$

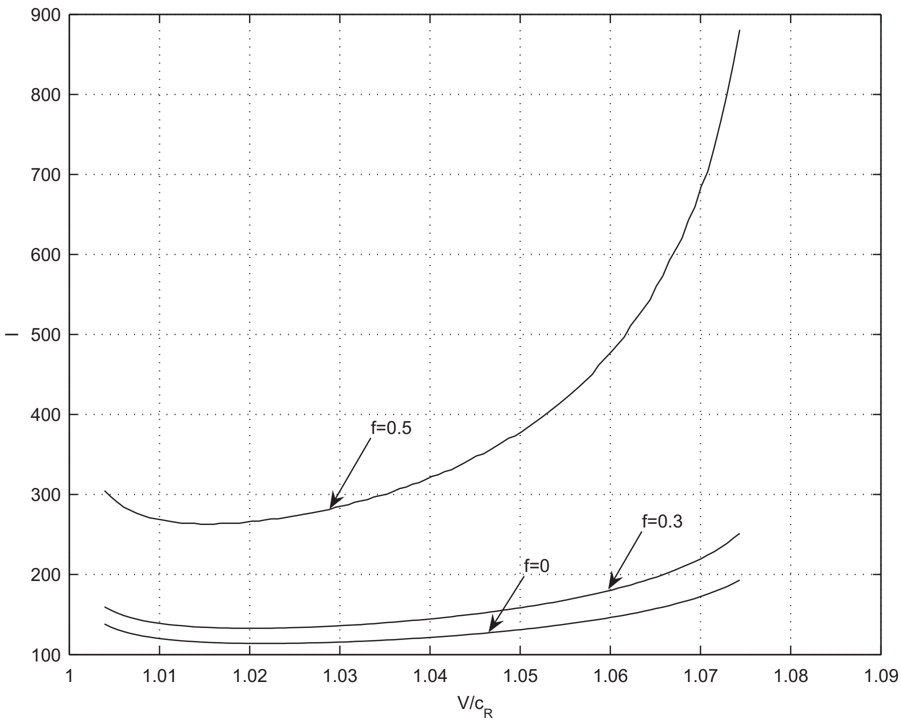

Fig. 6. The dimensionless length of the friction zone $l = b/b_1$ vs. the parameter V/c_R in the super-Rayleigh regime for the friction coefficients $f = 0$, 0.3, and 0.5 when $b_2/b_1 = 0.5$, $\nu = 0.3$, and $\sigma_0/\tau_0 = -0.1$.

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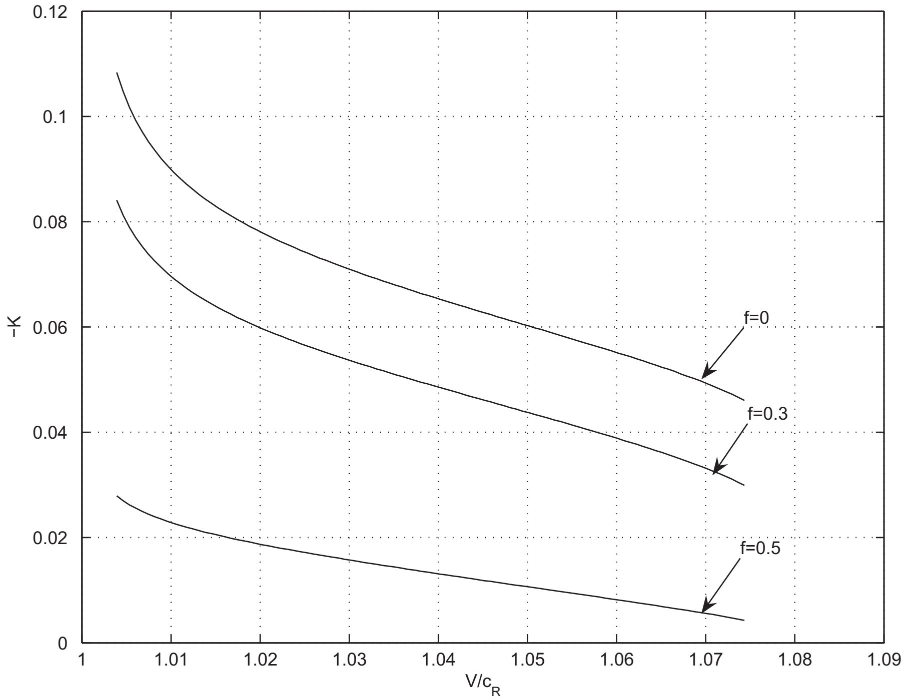

Fig. 7. The stress singularity factor K vs. the parameter V/c_R in the super-Rayleigh case for the friction coefficients $f=0$, 0.3, and 0.5 when $\nu_+=0.3$ and the loading has the form (4.20).

where

$$v_0 = \frac{\Gamma(2-q)}{|\Gamma(2+i\varepsilon)|^2}, \quad v_1 = \frac{p_2\Gamma(q)}{|\Gamma(1+i\varepsilon+q)|^2},$$

$$\Upsilon_0 = \Gamma(q+1)[(q+1)\Upsilon_1^1 + (q-\varepsilon^2)\Upsilon_2^1]. \tag{5.7}$$

Exploiting this asymptotics for the coefficient (4.28) we have

$$K = \frac{2^{1+q}\pi^q\mu_+}{b^{2-q}}\Upsilon_0 v_0 \sin\pi q. \tag{5.8}$$

Fig. 7 shows the effect on the stress singularity factor $-K>0$ of changing the speed V normalized to c_R. The coefficient $-K$ decreases as the speed V increases from $V=c_R$ to $V=c_s$.

As in the sub-Rayleigh case, we compute the dynamic energy release rate G when $f=0$ (if the contact is not frictionless, then the parameter G vanishes). The formula for G coincides with that for G in the sub-Rayleigh regime. Note, however, that in the representation (4.41) for the parameter A, the parameters v_0 and v_1 should be replaced by the parameters defined in (5.7). Computations implemented for different values of Poisson's ratio and speeds show that the dynamic energy release rate G is positive. As a function of V/c_R, the dimensionless dynamic energy release rate $A=\mu_+G/K_{\mathrm{II}}^2$ is plotted in Fig. 5b.

Finally, we determine the normal displacement for $x_1<Vt$ $(\xi<0)$, $x_2=0$,

$$u_2|_{x_2=0} = \int_0^{\xi} \left.\frac{\partial u_2}{\partial \xi}\right|_{x_2=0} d\xi, \tag{5.9}$$

The derivative $\partial u_2/\partial\xi|_{x_2=0}$ is given by (4.46) and (4.47), where

$$\hat{\Upsilon}_j(\zeta) = \frac{1}{2\pi}\int_{-1/l}^{-h/l}[X^+(\xi_0)]^{-1}\frac{g(\xi_0)\,d\xi_0}{\xi_0-\xi}, \quad \xi \notin \left(-\frac{1}{l}, -\frac{h}{l}\right),$$

$$\hat{\Upsilon}_j(\zeta) = \frac{i}{2}[X^+(\xi)]^{-1}g(\xi) + \frac{1}{2\pi}P.V.\int_{-1/l}^{-h/l}[X^+(\xi_0)]^{-1}\frac{g(\xi_0)\,d\xi_0}{\xi_0-\xi}, \quad \xi \in \left(-\frac{1}{l}, -\frac{h}{l}\right). \tag{5.10}$$

The principal value of the singular integral in (5.10) can be evaluated by an n-point Gauss type quadrature formula

$$\int_{\delta_1}^{\delta_2}\frac{F(\tau)\,d\tau}{\tau-\xi} = \frac{\pi}{n}\sum_{j=1}^{n}F_0(\tau_j,\xi), \quad \delta_1<\xi<\delta_2, \tag{5.11}$$

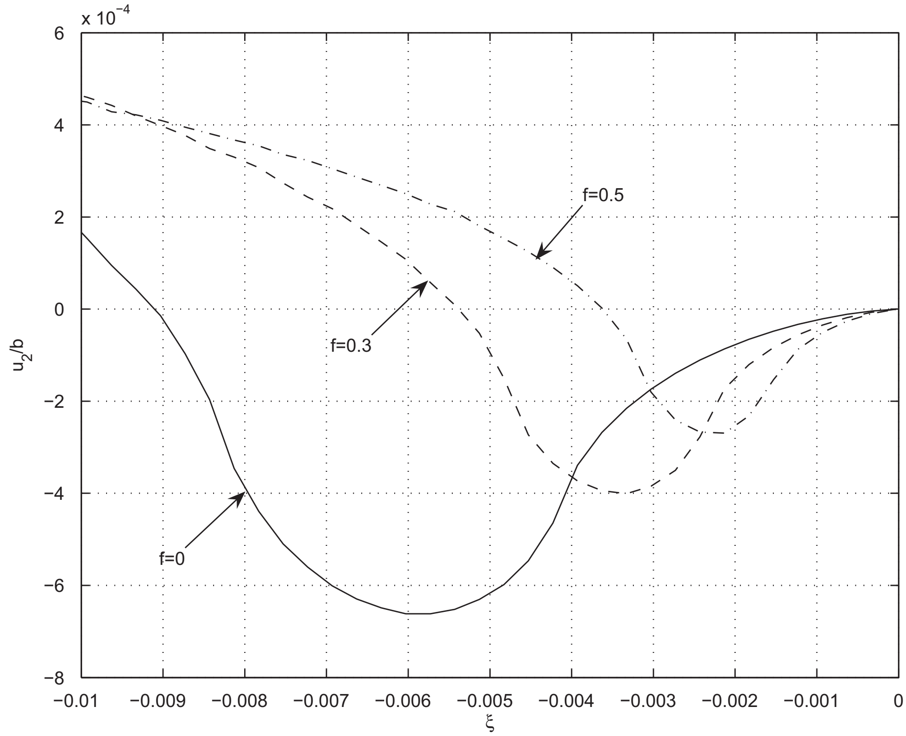

Fig. 8. The dimensionless normal displacement u_2/b as a function of ξ in the super-Rayleigh case for $\nu_+ = 0.3$, the loading (4.20) when $\mu_+^{-1}\tau^o = 1$, $V = \frac{1}{2}(c_R + c_s)$ and the friction coefficients $f = 0$, 0.3, and 0.5. The length of the slip zone $l = b/b_1$ is defined from the condition of smoothness of the crack profile at $\xi = 0$ ($l = 120.737$ for $f = 0$).

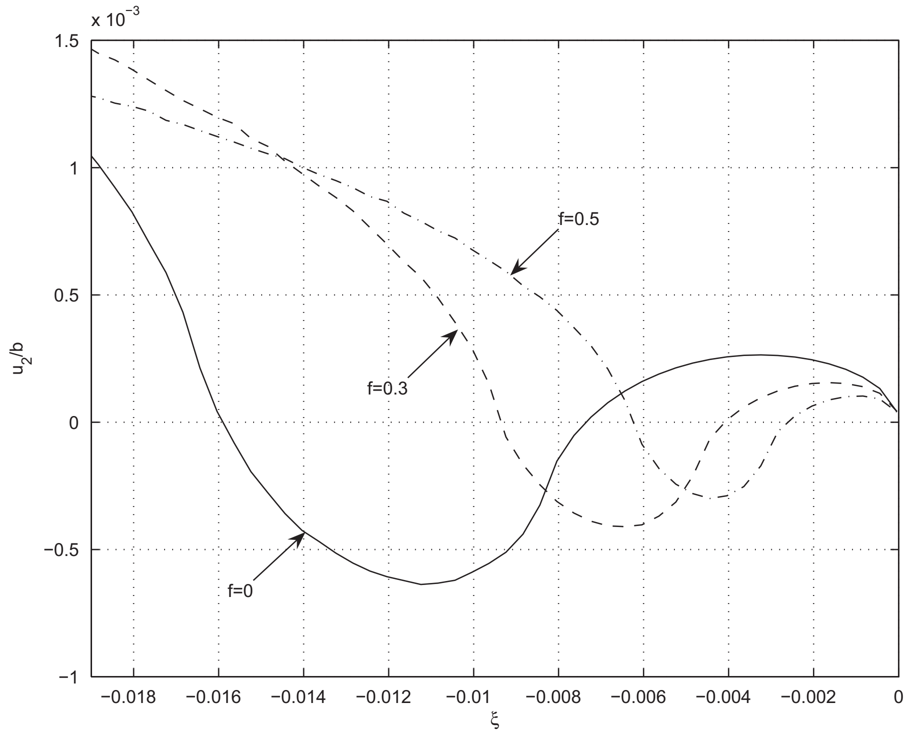

Fig. 9. The dimensionless normal displacement u_2/b as a function of ξ in the super-Rayleigh case for $\nu_+ = 0.3$, the loading (4.20) when $\mu_+^{-1}\tau^o = 1$, $V = \frac{1}{2}(c_R + c_s)$ and the friction coefficients $f = 0$, 0.3, and 0.5. The length of the slip zone is chosen to be $l_0 = \frac{1}{2}l$, l is defined by (4.15).

where

$$\tau_j = \frac{1}{2}(\delta_1 + \delta_2) - \frac{1}{2}(\delta_1 - \delta_2)\cos\frac{(j-\frac{1}{2})\pi}{n},$$

$$F_0(\tau, \xi) = \frac{1}{\tau - \xi}[F(\tau)\sqrt{(\delta_2 - \tau)(\tau - \delta_1)} - F(\xi)\sqrt{(\delta_2 - \xi)(\xi - \delta_1)}]. \tag{5.12}$$

The numerical results (Fig. 8) show that for the super-Rayleigh regime the normal component u_2 of the displacement is positive everywhere in the opening zone $x_1 < Vt$ apart from a small (left) neighborhood of the point $x_1 = Vt$ ($\xi = 0$). The length of this zone decreases when the friction coefficient grows and attains its maximum as $f = 0$. It turns out that it is impossible to overcome the presence of this segment even if we admit that the profile of the crack is not smooth at the point $\xi = 0$: this zone just moves to the interior of the crack. In Fig. 9 we plot the normal displacement for the case of an arbitrary length of the slip zone (it is chosen to be a half of that defined by the condition (4.15)). The numerical results show (see Fig. 9) that in a small neighborhood of the point $\xi = 0$ the displacement u_2 is positive. This is consistent with the result (5.5) obtained for any value of l except for the one which solves Eq. (4.15). In the case of an arbitrary contact zone length, in addition to the presence of the negative displacement zone, the crack profile is not smooth at the point $\xi = 0$, and both traction components, σ_{22} and σ_{12}, are singular when $\xi \to 0^+$. Therefore, we discard this singular solution with an arbitrary length of the contact zone as a possible model for the super-Rayleigh regime.

Let now the contact zone be defined by the condition (4.15) and the contact be frictionless ($f = 0$). In this case, the traction components and the tangential derivative of the normal displacement vanish at the point $\xi = 0$ while the tangential derivative of the tangential displacement is bounded and non-zero. The normal displacement is negative for $-\xi_* < \xi < 0$. We emphasize that in comparison to the slip zone length l, the length ξ_* is very small. For the parameters chosen, $l = 120.737$ and $\xi_* = 0.0091$. The magnitude of the normal displacement in this zone is also small: the displacement u_2 far away from the negative zone is of order 10^{-2} while the minimum of u_2/b in $(-\xi_*, 0)$ is -6.62×10^{-4}. The interval $(-\xi_*, 0)$ can be interpreted as a zone where a part of the mode-I, II crack becomes an adhesion zone. Thus, the whole crack can be split into the following three parts: a leading mode-II crack ($Vt < x_1 < Vt + b$), a small adhesion zone ($-b\xi_* + Vt < x_1 < Vt$) and a trailed semi-infinite crack ($-\infty < x_1 < -b\xi_* + Vt$). This interpretation is similar to the "mother–daughter" mechanism proposed by Abraham and Gao (2000) and Gao et al. (2001) for modeling of a mode-I, II crack propagating at super-Rayleigh speed.

6. Conclusions

The main contribution of this work is the exact solution to the model problem on a semi-infinite crack $-\infty < x_1 < Vt + bt$ with a finite friction zone ahead propagating subsonically along the interface between an elastic half-plane and a rigid half-plane. Mathematically, the problem is equivalent to a vector Riemann–Hilbert problem with a piece-wise constant 2×2 matrix coefficient. The matrix has three points of discontinuity, $0, 1$, and ∞. We have factorized the matrix and solved the problem in terms of Kummer's solutions to the associated hypergeometric equation. The sub-Rayleigh ($V < c_R$) and super-Rayleigh ($c_R < V < c_s$) ranges have been analyzed. For both cases, we have derived a transcendental equation whose solution defines the length of the friction zone not prescribed in the model. Asymptotic relations for the stresses and displacements at the ends of the crack and the friction zone have also been obtained. In addition, we have found expressions for the stress singularity factor at the end of the friction zone and the dynamic energy release rate G. It turns out that if the contact in the slip zone is frictionless then the energy release rate G is positive and the stress intensity factor K_{II} does not vanish not only for the sub-Rayleigh regime but also for the super-Rayleigh range of speeds. There are two main differences in the solutions for these two regimes. First, the traction components and the tangential derivatives of the displacements as $\xi \to +\infty$ and $\xi \to -\infty$, respectively, behave as

$$\xi^{-\eta}(C'\cos\varepsilon\ln\xi + C''\sin\varepsilon\ln\xi),$$

where C' and C'' are non-zero constants. The exponent $\eta = \frac{3}{2}$ in the sub-Rayleigh case and it is different from the one in the super-Rayleigh regime which is $\eta = 1$.

Second, in the sub-Rayleigh case, the normal displacement is positive everywhere in the opening zone while in the super-Rayleigh regime, there is a small interval $(-b\xi_* + Vt, Vt)$ in the opening zone $(-\infty, Vt)$ where the normal displacement is negative. This may mean that between the leading mode-II crack $(Vt, b + Vt)$ and the trailing semi-infinite crack $(-\infty, -b\xi_* + Vt)$ there is a cohesive zone $(-b\xi_* + Vt, Vt)$. Modeling of the boundary conditions in the cohesive zone requires an additional study.

We have shown that for both regimes, the singularity in stress at the tip $\xi = 1$ ($x_1 = Vt + b, x_2 = 0$) of the slip zone has order $(1-\xi)^{-q}$, where q is a real parameter and $0 < q \le \frac{1}{2}$. This parameter depends on the dimensionless parameters V/c_s, V/c_l, and the friction coefficient f. The displacements and the traction components are monotonic near the slip tip. The normal traction component σ_{22} is bounded if x_1 approaches the slip tip along the interface. The parameter $q = \frac{1}{2}$ if the contact is frictionless and $0 < q < \frac{1}{2}$ otherwise. Except for the particular case $f = 0$ the flux of energy into the crack tip is zero, and the Griffith energy balance criterion cannot be applied. As in the case of intersonic propagation of a shear crack in a homogeneous medium (Antipov et al., 2004), it is possible to develop a Barenblatt type fracture criterion (Barenblatt, 1962) and use a Dugdale type linear cohesion model (Dugdale, 1960) for subsonic propagation of a mode-I, II interfacial crack with a Coulomb friction zone ahead. Such a model has to remove the singularity at the crack tip. To achieve this we should preserve the type of the boundary condition in the contact zone and assume that the length of the cohesion zone is unknown a priori:

$$\sigma_{12} - f\sigma_{22} = \begin{cases} Y, & 0 < u_1 < \delta_0, \\ 0, & \delta_0 < u_1 < \infty, \end{cases}$$

where Y and δ_0 are certain parameters. The function $\sigma_{12} - f\sigma_{22}$ vanishes everywhere in the contact zone apart from the interval $(Vt + b - L, Vt + b)$. The length L of the cohesion zone is defined from the condition $2u_1(Vt + b - L) = 0$. The stresses are bounded at the point $x_1 = Vt + b$. In the case of frictionless contact, it is possible to apply both the Griffith and Dugdale–Barenblatt criteria as it has been done by Willis (1967) for subsonic crack propagation in a homogeneous medium.

This technique may be naturally extended to two distinct 2d problems. The first one concerns a semi-infinite crack with a friction zone ahead propagating subsonically along the interface between two elastic half-planes with different elastic constants. The second problem models a wedge cutting a plane along a straight interface when there are three points of discontinuity in the boundary conditions.

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