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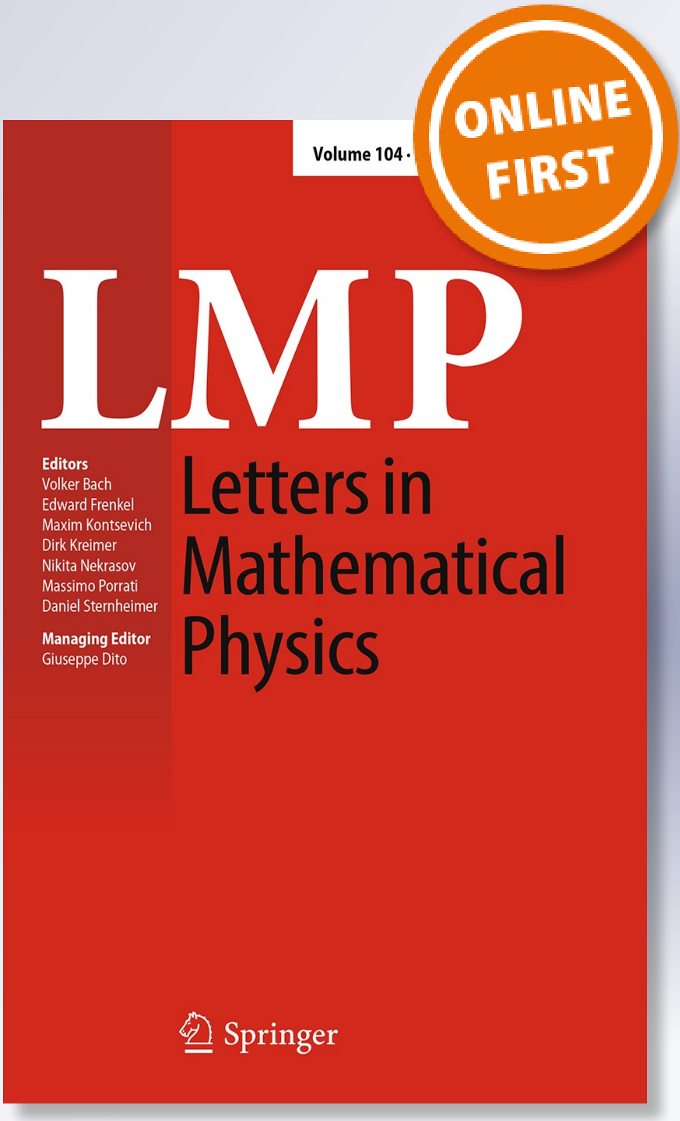

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The Baker–Akhiezer Function and Factorization of the Chebotarev–Khrapkov Matrix

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Abstract. A new technique is proposed for the solution of the Riemann–Hilbert problem with the Chebotarev–Khrapkov matrix coefficient $G(t)=\alpha_1(t)I+\alpha_2(t)Q(t)$, $\alpha_1(t),\alpha_2(t)\in H(L)$, $I=\text{diag}\{1,1\}$, $Q(t)$ is a 2×2 zero-trace polynomial matrix. This problem has numerous applications in elasticity and diffraction theory. The main feature of the method is the removal of essential singularities of the solution to the associated homogeneous scalar Riemann–Hilbert problem on the hyperelliptic surface of an algebraic function by means of the Baker–Akhiezer function. The consequent application of this function for the derivation of the general solution to the vector Riemann–Hilbert problem requires the finding of the ρ zeros of the Baker–Akhiezer function (ρ is the genus of the surface). These zeros are recovered through the solution to the associated Jacobi problem of inversion of abelian integrals or, equivalently, the determination of the zeros of the associated degree-ρ polynomial and solution of a certain linear algebraic system of ρ equations.

Mathematics Subject Classification. 30E25, 30F99, 45E.

Keywords. Riemann–Hilbert problem, Baker–Akhiezer function, Riemann surfaces.

1. Introduction

Many problems of elasticity [1,2,5,19,24], electromagnetic diffraction [3,4,7–9,12,16,21], and acoustic diffraction [6,18,25] require the solution of the vector Riemann–Hilbert problem (RHP) of the theory of analytic functions [27] $\mathbf{\Phi}^+(t)=G(t)\mathbf{\Phi}^-(t)+\mathbf{g}(t)$, $t\in L$, where L is either the whole real axis, or a finite segment, when the matrix $G(t)$ has the Chebotarev–Khrapkov (also known as Daniele–Khrapkov) structure [10,12,19],

$$G(t)=\alpha_1(t)I+\alpha_2(t)Q(t). \tag{1.1}$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are Hölder functions on L, $I=\text{diag}\{1,1\}$, and $Q(t)$ is a 2×2 zero-trace polynomial matrix. In the case $n=\deg f(z)\le 2$ ($\det Q(z)=h^2(z)f(z)$, and $f(z)$ has simple zeros only) the problem was solved in [19]. For a particular case of the matrix (1.1) and when $n=4$, the exact solution was derived in

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[12]. For any finite n, the vector problem is reduced [23] to a scalar RHP on a hyperelliptic surface of genus $\rho=[(n-1)/2]$. A theory of the RHP on compact Riemann surfaces and a constructive procedure for the solution of the associated Jacobi inversion problem was proposed in [28] (see also [29]). This technique was further developed and adjusted to specific needs of the RHPs on hyperelliptic surfaces arising in elasticity [5,24], diffraction theory in [6–8] and for symmetric vector RHPs in [3,4]. The method for the vector RHP with the coefficient (1.1) in the elliptic and hyperelliptic cases first factorizes the coefficient of the associated scalar RHP using the Weierstrass analogue of the Cauchy kernel. In general, that solution has an essential singularity at the infinite points of the surface due to unavoidable poles of the Weierstrass kernel. The next step of the procedure, the removal of the essential singularities, leads to the classical problem of the inversion of abelian integrals and, eventually, to the finding of the zeros of a certain degree-ρ polynomial.

The main goal of this paper was to develop a new factorization procedure for matrices of the form (1.1) based on the use of the Baker–Akhiezer function. The Baker–Akhiezer function plays an important role in the study of analytic properties of eigenfunctions of ordinary differential operators with periodic coefficients [13–15,17,20]. The representation of the Baker–Akhiezer function on a genus-ρ hyperelliptic surface $\mathcal{R}$

$$\mathcal{F}(P)=\mathrm{e}^{\Omega(P)}\frac{\theta(u_1(P)-\sigma_1+V_1^\circ,\ldots,u_\rho(P)-\sigma_\rho+V_\rho^\circ)}{\theta(u_1(P)-\sigma_1,\ldots,u_\rho(P)-\sigma_\rho)} \tag{1.2}$$

that we employ for the solution of the Wiener–Hopf matrix factorization problem was first written by A. R. Its in context of the finite gap solutions of the KdV equation [22]. Here, $P\in\mathcal{R}$, $\Omega(P)$ is an abelian integral of the second kind with zero A-periods and a certain prescribed polynomial growth at the infinite point of the surface $\mathcal{R}$, θ is the theta Riemann function, $u_1,\ldots,u_\rho$ form the canonical basis of abelian integrals of the first kind, $\sigma_j=k_j+u_j(P_1)+\cdots+u_j(P_\rho)$, P_j are simple poles of the Baker–Akhiezer function, k_j are the Riemann constants associated with the homology basis $\mathbf{a}_1$, $\mathbf{b}_1,\ldots,\mathbf{a}_\rho,\mathbf{b}_\rho$, and $V_j^\circ=(2\pi i)^{-1}\int_{\mathbf{b}_j}\mathrm{d}\Omega$, $j=1,\ldots,\rho$.

In Section 2, we state the vector RHP in the real axis with the matrix coefficient (1.1) and reduce it to a scalar RHP on a hyperelliptic surface $\mathcal{R}$ of the algebraic function $w^2=f(z)$. We derive a particular solution, $\chi_0(z,w)$, to the scalar RHP in Section 3. This solution satisfies the boundary condition but has inadmissible essential singularities at the two infinite points ∞_1 and ∞_2 of the surface. In Section 4, we construct the Baker–Akhiezer function (1.2) of the surface $\mathcal{R}$. This function is associated with an abelian integral of the second type with zero-A-periods used to remove the essential singularities and two Riemann θ-functions which serve to make the solution continuous through the B-cross sections. We find the Wiener–Hopf factors in terms of the functions $\chi_0(z,w)$ and $\mathcal{F}(P)$ and the general solution to the vector RHP in Section 5.

2. Scalar RHP on a Riemann Surface Associated with the Chebotarev–Khrapkov Matrix

Motivated by numerous applications in acoustics, electromagnetic theory, fluid mechanics and elasticity, we assume that the Riemann–Hilbert contour, L, is the whole real axis which splits the plane of a complex variable z into two half-planes, $D^+: \operatorname{Im} z > 0$ and $D^-: \operatorname{Im} z < 0$. Let $G(t)$ be a 2×2 matrix which is nonsingular in L and whose structure is

$$G(t) = \begin{pmatrix} \alpha_1(t) + \alpha_2(t) l_0(t) & \alpha_2(t) l_1(t) \\ \alpha_2(t) l_2(t) & \alpha_1(t) - \alpha_2(t) l_0(t) \end{pmatrix}, \tag{2.1}$$

where $\alpha_1(t), \alpha_2(t) l_j(t) \in \hat{H}(L)$, $j = 0, 1, 2$, $l_0(t)$, $l_1(t)$ and $l_2(t)$ are polynomials, and $\hat{H}(L)$ is the class of all Hölder functions $\alpha(t)$ in any finite interval in L which tend to a definite limit $\alpha(\infty)$ as $t \to \pm\infty$. For large t, they satisfy the condition $|\alpha(t) - \alpha(\infty)| < C|t|^{-\mu}$, $\mu > 0$, $C > 0$. Without loss of generality assume that $\det G(\infty) = 1$. Let $\mathbf{g}(t)$ be an order-2 $\hat{H}$-vector function on L such that $\mathbf{g}(\infty)$ is the zero-vector. Consider the following RHP.

Given $G(t)$ and $\mathbf{g}(t)$ find two vectors, $\mathbf{\Phi}^+(z)$ and $\mathbf{\Phi}^-(z)$, analytic in the domains D^+ and D^-, respectively, bounded at infinity, $\hat{H}$-continuous up to the contour L and satisfying the boundary condition

$$\mathbf{\Phi}^+(t) = G(t)\mathbf{\Phi}^-(t) + \mathbf{g}(t), \quad t \in L. \tag{2.2}$$

Denote $l_0^2(z) + l_1(z) l_2(z) = h^2(z) f(z)$ and $f(z) = z^n + \varepsilon_1 z^{n-1} + \ldots + \varepsilon_n$. All zeros, $r_1, r_2, \ldots, r_n$, of $f(z)$ are simple, while the zeros of the polynomial $h(z)$, $p_1, p_2, \ldots, p_l$, have multiplicity $m_1, m_2, \ldots, m_l$, respectively, and $m_1 + m_2 + \ldots + m_l = N$. Some or all zeros of the polynomial $l_0^2(z) + l_1(z) l_2(z)$ may have an odd multiplicity $2m_i + 1 \geq 3$. In this case the i-th zero is counted as a simple zero of $f(z)$ and an order-m_i zero of the polynomial $h(z)$. Assume that none of the zeros of $f(z)$ and $h(z)$ falls in the contour L (we refer to [1] otherwise). In addition, we assume that n is even, $n = 2\rho + 2$ (this is true for all known applications of the problem (2.2) with the matrix coefficient (2.1) to elasticity and diffraction theory). This implies $\deg[l_0^2(z) + l_1(z) l_2(z)] = 2N + 2\rho + 2$. Denote $\deg l_j(z) = \delta_j$, $j = 0, 1, 2$, and for simplicity, accept that $0 \leq \delta_0 \leq N + \rho + 1$ and $0 \leq \delta_j \leq 2N + 2\rho + 2$, $j = 1, 2$ $(\delta_1 + \delta_2 \leq 2N + 2\rho + 2)$.

Choose a single branch of $f^{1/2}(z)$ in the plane cut along simple smooth disjoint arcs $\gamma_1 = r_1 r_2$, $\gamma_2 = r_3 r_4, \ldots, \gamma_{\rho+1} = r_{2\rho+1} r_{2\rho+2}$ such that $f^{1/2}(z) \sim z^{\rho+1}$, $z \to \infty$. The functions

$$\lambda_1(t) = \alpha_1(t) + \alpha_2(t) h(t)\sqrt{f(t)}, \quad \lambda_2(t) = \alpha_1(t) - \alpha_2(t) h(t)\sqrt{f(t)} \tag{2.3}$$

are the eigenvalues of the matrix $G(t)$, and their product $\alpha_1^2(t) - \alpha_2^2(t) h^2(t) f(t)$ is the determinant of $G(t)$. To pursue the Wiener–Hopf factorization of $G(t)$, we split it as

$$G(t) = T(t)\Lambda(t)[T(t)]^{-1}, \tag{2.4}$$

where $\Lambda(t)=\text{diag}\{\lambda_1(t),\lambda_2(t)\}$,

$$T(z)=\begin{pmatrix} 1 & 1 \\ -\frac{l_0(z)-h(z)\sqrt{f(z)}}{l_1(z)} & -\frac{l_0(z)+h(z)\sqrt{f(z)}}{l_1(z)} \end{pmatrix}, \tag{2.5}$$

and reduce the problem of matrix factorization to a scalar RHP on a Riemann surface [23]. First, we introduce two new vectors, $\boldsymbol{\psi}(z)=(\psi_1(z),\psi(z))$ and $\mathbf{g}^\circ(t)=(g_1^\circ(t),g_2^\circ(t))$,

$$\boldsymbol{\psi}(z)=[T(z)]^{-1}\boldsymbol{\Phi}(z), \quad \mathbf{g}^\circ(t)=[T(t)]^{-1}\mathbf{g}(t), \tag{2.6}$$

where

$$[T(z)]^{-1}=\begin{pmatrix} \frac{l_0(z)}{2h(z)\sqrt{f(z)}}+\frac{1}{2} & \frac{l_1(z)}{2h(z)\sqrt{f(z)}} \\ -\frac{l_0(z)}{2h(z)\sqrt{f(z)}}+\frac{1}{2} & -\frac{l_1(z)}{2h(z)\sqrt{f(z)}} \end{pmatrix}. \tag{2.7}$$

The components of the vector $\boldsymbol{\psi}(z)$ are expressed through the components of the vector $\boldsymbol{\Phi}(z)$ as

$$\begin{aligned} \psi_1(z)&=\frac{1}{2}\left[1+\frac{l_0(z)}{h(z)\sqrt{f(z)}}\right]\Phi_1(z)+\frac{l_1(z)}{2h(z)\sqrt{f(z)}}\Phi_2(z),\\ \psi_2(z)&=\frac{1}{2}\left[1-\frac{l_0(z)}{h(z)\sqrt{f(z)}}\right]\Phi_1(z)-\frac{l_1(z)}{2h(z)\sqrt{f(z)}}\Phi_2(z). \end{aligned} \tag{2.8}$$

Similar formulas can be written for the components of the vectors $\mathbf{g}^\circ(t)$ and $\mathbf{g}(t)$. The new functions $\psi_1(z)$ and $\psi_2(z)$ may grow at infinity if $\delta_1>N+\rho+1$. Let $\delta=\max\{0,\delta_1-N-\rho-1\}$. Then since the functions $\Phi_1(z)$ and $\Phi_2(z)$ are bounded as $z\to\infty$, we have $|\psi_j(z)|<c_j|z|^\delta$, $z\to\infty$, $c_j=\text{const}$, $j=1,2$.

Due to continuity of the vector $\boldsymbol{\Phi}(z)$ through the branch cuts γ_j $(j=1,2,\ldots,\rho+1)$, we have $T^+(t)\boldsymbol{\psi}^+(t)=T^-(t)\boldsymbol{\psi}^-(t)$, $t\in\gamma_j$. This implies that the components of the vector $\boldsymbol{\psi}(z)$ satisfy the following Riemann–Hilbert boundary conditions:

$$\begin{aligned} &\psi_1^+(t)=\psi_2^-(t), \quad \psi_2^+(t)=\psi_1^-(t), \quad t\in\gamma_j, \quad j=1,2,\ldots,\rho+1,\\ &\psi_j^+(t)=\lambda_j(t)\psi_j^-(t)+g_j^\circ(t), \quad t\in L, \quad j=1,2, \end{aligned} \tag{2.9}$$

and may have poles $p_1,p_2,\ldots,p_l$ of multiplicity $m_1,m_2,\ldots,m_l$ at the zeros of the polynomial $h(z)$.

We wish to reformulate (2.9) as a scalar RHP on a Riemann surface. Let $\mathcal{R}$ be the two-sheeted Riemann surface of the algebraic function $w^2=f(z)$ formed by gluing two copies, $\mathbb{C}_1$ and $\mathbb{C}_2$, of the extended complex plane $\mathbb{C}\cup\infty$ along the cuts γ_j $(j=1,2,\ldots,\rho+1)$ such that

$$w=\begin{cases} \sqrt{f(z)}, & z\in\mathbb{C}_1,\\ -\sqrt{f(z)}, & z\in\mathbb{C}_2, \end{cases} \tag{2.10}$$

is a single-valued function on the surface $\mathcal{R}$. Here, $\sqrt{f(z)}$ is the branch chosen before. Let $\mathbf{a}_j$, $\mathbf{b}_j$ $(j=1,2,\ldots,\rho)$ be a homology basis of the genus-ρ surface $\mathcal{R}$

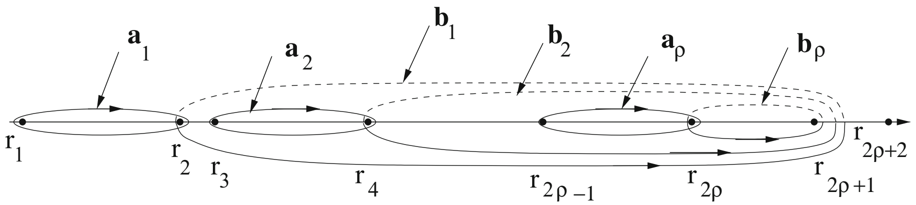

Figure 1. The canonical cross sections $\mathbf{a}_j$ and $\mathbf{b}_j$, $j=1,\dots,\rho$.

(Figure 1). Denote $\mathcal{L}=L_1\cup L_2$ the contour on the surface $\mathcal{R}$ with $L_j\subset\mathbb{C}_j$ ($j=1,2$) being two copies of the contour L. With each pair of the functions (ψ_1,ψ_2), (λ_1,λ_2) and (g_1°, g_2°) we associate the following functions on the surface $\mathcal{R}$:

$$\Psi(z,w)=\psi_j(z),\quad (z,w)\in\mathbb{C}_j,$$
$$\lambda(t,\xi)=\lambda_j(t),\quad g^*(t,\xi)=g_j^\circ(t),\quad (t,\xi)\in L_j,\quad j=1,2,\quad \xi=w(t). \tag{2.11}$$

The function $\Psi(z,w)$ may have simple poles at the branch points of the surface $\mathcal{R}$, $r_1,r_2,\dots,r_{2\rho+2}$ (recall [26] that a branch point r_j of the Riemann surface $\mathcal{R}$ is called an order-l_j pole of the function $\Psi(z,w)$ if $\Psi(z,w)\sim A_j\zeta^{-l_j}$, $\zeta\to 0$, $A_j=$ const, and $\zeta=(z-r_j)^{1/2}$ is a local uniformizing parameter of the point r_j). We also assert that the function $\Psi(z,w)$ is continuous through the contours γ_j ($j=1,2,\dots,\rho+1$), and therefore the vector RHP (2.2) on the plane is equivalent to the following scalar RHP on the surface $\mathcal{R}$.

Find a piece-wise analytic function $\Psi(z,w)$ with the discontinuity contour $\mathcal{L}$, $\hat{H}$-continuous up to the contour $\mathcal{L}$, satisfying the boundary condition

$$\Psi^+(t,\xi)=\lambda(t,\xi)\Psi^-(t,\xi)+g^*(t,\xi),\quad (t,\xi)\in\mathcal{L}, \tag{2.12}$$

and having poles $p_1,p_2,\dots,p_l$ of multiplicity $m_1,m_2,\dots,m_l$ in both sheets of the surface $\mathcal{R}$ and simple poles at the branch points $r_1,r_2,\dots,r_{2\rho+2}$. In neighborhoods of the two infinite points $\infty_j=(\infty,(-1)^{j-1}\infty)$ of the surface $\mathcal{R}$ the function $\Psi(z,w)$ satisfies the inequality $|\Psi(z,w)|<c_j|z|^\delta$, $c_j=const$, $j=1,2$.

3. Solution with An Essential Singularity at the Infinite Points

We begin with factorization of the function $\lambda(t,\xi)$. For an analogue of the Cauchy kernel, we choose the Weierstrass kernel

$$\mathrm{d}W=\frac{w+\xi}{2\xi}\frac{\mathrm{d}t}{t-z} \tag{3.1}$$

and analyze the integral

$$\frac{1}{2\pi i}\int_{\mathcal{L}}\log\lambda(t,\xi)\mathrm{d}W=\frac{1}{4\pi i}\int_L[\log\lambda_1(t)+\log\lambda_2(t)]\frac{\mathrm{d}t}{t-z}$$
$$+\frac{w}{4\pi i}\int_L[\log\lambda_1(t)-\log\lambda_2(t)]\frac{\mathrm{d}t}{\sqrt{f(t)}(t-z)}. \tag{3.2}$$

Pick a point on L, z_0, and treat it as the starting point, z_0^+, of the contour L (it is convenient to take $z_0=0$). Let

$$\kappa_j=\operatorname{ind}\lambda_j(t)=\frac{1}{2\pi}[\arg\lambda_j(t)]|_L, \tag{3.3}$$

where $\operatorname{ind}\lambda_j(t)$ is the index of the function $\lambda_j(t)$, and $[\arg\lambda_j(t)]|_L$ is the increment of $\arg\lambda_j(t)$ as t traverses the contour L in the positive direction with z_0^+ being the starting point. Because of the continuity of the functions $\lambda_1(t)$ and $\lambda_2(t)$ in the contour L both numbers, κ_1 and κ_2, are integers. Fix branches of the logarithmic functions $\log\lambda_1(t)$ and $\log\lambda_2(t)$ by the condition $\arg\lambda_j(z_0^+)=\phi_j$, $0\le\phi_j<2\pi$.

Then at the terminal point z_0 of the contour L (to distinguish the terminal and starting points, we denote the former point as z_0^-), $\arg\lambda_j(z_0^-)=\phi_j+2\pi\kappa_j$. Analysis of the singular integrals in the right-hand side (3.2) implies

$$\begin{aligned}
&\frac{1}{4\pi i}\int_L[\log\lambda_1(t)+\log\lambda_2(t)]\frac{dt}{t-z}\sim\frac{\kappa_1+\kappa_2}{2}\log(z-z_0), \quad z\to z_0,\\
&\frac{w}{4\pi i}\int_L[\log\lambda_1(t)-\log\lambda_2(t)]\frac{dt}{\sqrt{f(t)}(t-z)}\sim\frac{\kappa_1-\kappa_2}{2}(-1)^{j-1}\log(z-z_0),\\
&\quad z\to z_0, \quad (z,w)\in\mathbb{C}_j.
\end{aligned} \tag{3.4}$$

Consequently, the integral in the left-hand side (3.2) has a logarithmic singularity at the point $(z_0,w(z_0))\in\mathcal{L}$ in both sheets of the surface

$$\frac{1}{2\pi i}\int_{\mathcal{L}}\log\lambda(t,\xi)dW\sim\kappa_j\log(z-z_0), \quad z\to z_0, \quad (z,w)\in\mathbb{C}_j, \quad j=1,2. \tag{3.5}$$

It is an easy matter to move the singularity from the contour $\mathcal{L}$ to the surface $\mathcal{R}\backslash\mathcal{L}$ by adding the extra term

$$I(z,w)=\sum_{m=1}^{2}\operatorname{sgn}\kappa_m\sum_{j=1}^{|\kappa_m|}\int_{q_{m0}}^{q_{mj}}dW. \tag{3.6}$$

Here, $q_{m0}q_{mj}\subset\mathbb{C}_m$ are smooth simple contours which do not intersect the contours L_m, $q_{mj}=(z_{mj},(-1)^{m-1}\sqrt{f(z_{mj})})\in\mathbb{C}_m\backslash L_m$, $j=1,2,\dots,|\kappa_m|$, are arbitrary fixed points, z_{mj} are their affixes, and $q_{m0}=(z_0,(-1)^{m-1}\sqrt{f(z_0)})\in L_m$, $m=1,2$. The function $\exp\{I(z,w)\}$ is continuous through the contour $\mathcal{L}$ except for the points q_{10} and q_{20} at which the integral $I(z,w)$ has logarithmic singularities. In addition, the integral $I(z,w)$ has logarithmic singularities at the internal points q_{mj},

$$\begin{aligned}
&I(z,w)\sim-\kappa_m\log(z-z_0), \quad z\to z_0, \quad (z,w)\in\mathbb{C}_m,\\
&I(z,w)\sim\operatorname{sgn}\kappa_m\log(z-z_{mj}), \quad z\to z_{mj}, \quad (z,w)\in\mathbb{C}_m,\\
&\quad j=1,\dots,|\kappa_m|, \quad m=1,2.
\end{aligned} \tag{3.7}$$

At the same time, the sum of the integral (3.2) and (3.6) does not have the singularity at the points $(z_0, \pm\sqrt{f(z_0)})$. Now, to factorize the function $\lambda(t,\xi)$, we use the function

$$\chi_0(z,w) = \exp\left\{\frac{1}{2\pi i}\int_{\mathcal{L}} \log\lambda(t,\xi)\mathrm{d}W + \sum_{m=1}^{2}\operatorname{sgn}\kappa_m \sum_{j=1}^{|\kappa_m|}\int_{q_{m0}}^{q_{mj}} \mathrm{d}W\right\}. \tag{3.8}$$

The function $\chi_0(z,w)$ satisfies the homogeneous boundary condition

$$\chi_0^+(t,\xi) = \lambda(t,\xi)\chi_0^-(t,\xi), \quad (t,\xi)\in\mathcal{L}, \tag{3.9}$$

does not have singularities at the points q_{10} and q_{20}, but has inadmissible essential singularities at the points ∞_1 and ∞_2. Also, it has simple zeros z_{mj} on the sheet $\mathbb{C}_m$ if $\kappa_m > 0$ and simple poles z_{mj} on $\mathbb{C}_m$ if $\kappa_m < 0$ $(j = 1,\ldots,|\kappa_m|,\ m = 1,2)$.

4. Baker–Akhiezer Function

Our aim is to quench the essential singularities at the infinite points of the function $\chi_0(z,w)$ by employing the Baker–Akhiezer function, $\mathcal{F}(z,w)$, on the genus-ρ surface $\mathcal{R}$ associated with the function $\chi_0(z,w)$. The function $\mathcal{F}(z,w)$ has to satisfy the following two conditions:

(i) it is meromorphic everywhere on $\mathcal{R}$ except at the points ∞_1 and ∞_2,
(ii) the function $\chi_0(z,w)\mathcal{F}(z,w)$ is bounded at the points ∞_1 and ∞_2.

Setting

$$\chi_0(z,w) = e^{\beta_0(z) + w\beta_1(z)}, \tag{4.1}$$

where

$$\beta_0(z) = \frac{1}{4\pi i}\int_L [\log\lambda_1(t) + \log\lambda_2(t)]\frac{\mathrm{d}t}{t-z} + \frac{1}{2}\sum_{m=1}^{2}\operatorname{sgn}\kappa_m\sum_{j=1}^{|\kappa_m|}\int_{z_0}^{z_{mj}}\frac{\mathrm{d}t}{t-z},$$
$$\beta_1(z) = \frac{1}{4\pi i}\int_L [\log\lambda_1(t) - \log\lambda_2(t)]\frac{\mathrm{d}t}{\sqrt{f(t)}(t-z)} + \frac{1}{2}\sum_{m=1}^{2}\operatorname{sgn}\kappa_m\sum_{j=1}^{|\kappa_m|}\int_{q_{m0}}^{q_{mj}}\frac{\mathrm{d}t}{\xi(t-z)}, \tag{4.2}$$

we study the behavior of the function $\chi_0(z,w)$ at the infinite points. For the branch $\sqrt{f(z)}$ chosen we have

$$\sqrt{f(z)} = \sqrt{\prod_{j=1}^{2\rho+2}(z-r_j)} = z^{\rho+1}\sum_{m=0}^{\infty} c_m z^{-m}, \tag{4.3}$$

Here,

$$c_0=1,\quad c_1=\frac{(-1/2)_1}{1!}\sum_{j=1}^{2\rho+2} r_j,$$
$$c_2=\frac{(-1/2)_2}{2!}\sum_{j=1}^{2\rho+2} r_j^2+\frac{[(-1/2)_1]^2}{(1!)^2}\sum_{j=1}^{2\rho+2} r_j\sum_{m=1,m\neq j}^{2\rho+2} r_m, \tag{4.4}$$
$$c_3=\frac{(-1/2)_3}{3!}\sum_{j=1}^{2\rho+2} r_j^3+\frac{(-1/2)_1(-1/2)_2}{1!2!}\sum_{j=1}^{2\rho+2} r_j^2\sum_{m=1,m\neq j}^{2\rho+2} r_m,\dots,$$

where $(a)_m=a(a+1)\dots(a+m-1)$ is the factorial symbol. By virtue of (4.2)

$$\beta_1(z)=\sum_{j=0}^{\infty}\frac{\tilde{c}_j}{z^{j+1}}, \tag{4.5}$$

where

$$\tilde{c}_j=-\frac{1}{4\pi i}\int_L[\log\lambda_1(t)-\log\lambda_2(t)]\frac{t^j\,\mathrm{d}t}{\sqrt{f(t)}}-\frac{1}{2}\sum_{m=1}^{2}\operatorname{sgn}\kappa_m\sum_{j=1}^{|\kappa_m|}\int_{q_{m0}}^{q_{mj}}\frac{t^j\,\mathrm{d}t}{\xi}, \tag{4.6}$$

and, therefore, as $z\to\infty$,

$$\sqrt{f(z)}\beta_1(z)=z^\rho\sum_{m=0}^{\infty}\frac{d_m}{z^m},\quad d_m=\sum_{k=0}^{m}c_k\tilde{c}_{m-k}. \tag{4.7}$$

This brings us to the expansion of the function $\chi_0(z,w)$ at the infinite points

$$\chi_0(z,w)=\exp\{(-1)^{j-1}M(z)+O(1)\},\quad (z,w)\to\infty_j,\quad (z,w)\in\mathbb{C}_j,\quad j=1,2, \tag{4.8}$$

where

$$M(z)=d_0z^\rho+d_1z^{\rho-1}+\dots+d_{\rho-1}z. \tag{4.9}$$

Our next step is to construct a special abelian integral of the second kind,

$$\Omega(P)=\int_{P_0}^{P}\mathrm{d}\Omega,\quad P_0=(r_{2\rho+2},0),\quad P=(z,w). \tag{4.10}$$

Determine $\Omega(P)$ by the following properties:

(a) $\Omega(P)\sim(-1)^jM(z),\ P\to\infty_j\in\mathbb{C}_j,\ j=1,2,$
(b) $\int_{\mathbf{a}_j}\mathrm{d}\Omega=0,\quad j=1,2,\dots,\rho.$

We seek the abelian differential $\mathrm{d}\Omega$ in the form

$$\mathrm{d}\Omega = \frac{e_0 z^{2\rho} + e_1 z^{2\rho-1} + \cdots + e_{2\rho}}{w} \mathrm{d}z, \tag{4.11}$$

where the coefficients e_j are to be determined. We wish to exploit this formula to study the behavior of the integral $\Omega(P)$ at the infinite points. Because of (4.3), we have

$$\mathrm{d}\Omega = (-1)^{j-1} \left(\tilde{e}_0 z^{\rho-1} + \tilde{e}_1 z^{\rho-2} + \cdots + \tilde{e}_{\rho-1} + \frac{\tilde{e}_\rho}{z} + \cdots \right) \mathrm{d}z, \tag{4.12}$$

where $\tilde{e}_m$ are defined recursively by

$$\tilde{e}_m = -\sum_{k=1}^{m} \tilde{e}_{m-k} c_k + e_m, \quad m = 0, 1, \ldots, \rho. \tag{4.13}$$

By integrating (4.12), we determine the asymptotic expansion of the abelian integral $\Omega(P)$

$$\Omega(P) = (-1)^{j-1} \left(\frac{\tilde{e}_0 z^\rho}{\rho} + \frac{\tilde{e}_1 z^{\rho-1}}{\rho - 1} + \cdots + \tilde{e}_{\rho-1} z + \tilde{e}_\rho \log z - \frac{\tilde{e}_{\rho+1}}{z} - \ldots \right) + K_j,$$
$$P \to \infty_j \in \mathbb{C}_j, \quad j = 1, 2, \tag{4.14}$$

where K_1 and K_2 are constants. On satisfying the property (a) of the integral $\Omega(P)$, we find the coefficients $\tilde{e}_0, \ldots, \tilde{e}_\rho$

$$\tilde{e}_0 = -\rho d_0, \quad \tilde{e}_1 = -(\rho - 1) d_1, \quad \tilde{e}_2 = -(\rho - 2) d_2, \ \ldots, \ \tilde{e}_{\rho-1} = -d_{\rho-1}, \quad \tilde{e}_\rho = 0. \tag{4.15}$$

Due to (4.13) we can express the coefficients e_m $(m = 0, 1, \ldots, \rho)$ through $\tilde{e}_m$

$$e_m = \tilde{e}_m + \sum_{k=1}^{m} \tilde{e}_{m-k} c_k, \quad m = 0, 1, \ldots, \rho. \tag{4.16}$$

The remaining coefficients $e_{\rho+1}, e_{\rho+2}, \ldots, e_{2\rho}$ in the representation (4.11) of the abelian differential are fixed by solving the system of ρ linear algebraic equations

$$\sum_{m=\rho+1}^{2\rho} U_{jm} e_m = \hat{d}_j, \quad j = 1, 2, \ldots, \rho, \tag{4.17}$$

which follows from the property (b) of the integral $\Omega(P)$. Here,

$$\hat{d}_j = -\sum_{m=0}^{\rho} U_{jm} e_m, \quad U_{jm} = \int_{\mathbf{a}_j} \frac{z^{2\rho-m}}{w(z)} \mathrm{d}z. \tag{4.18}$$

This completes the construction of the abelian integral $\Omega(P)$.

It becomes evident that the product $\chi_0(z,w)\exp\{\Omega(P)\}$ is bounded as $P\to\infty_j\in\mathbb{C}_j$, $j=1,2$. This function is continuous through the cross sections $\mathbf{a}_j$ of the surface $\mathcal{R}$ because of the zero A-periods and discontinuous through the cross sections $\mathbf{b}_j$ $(j=1,2,\dots,\rho)$ due to the non-zero B-periods of the integral $\Omega(P)$. Our efforts will now be directed towards annihilating the jumps $\exp\{V_m\}$,

$$V_m=\int_{\mathbf{b}_m}\mathrm{d}\Omega,\quad m=1,2,\dots,\rho, \tag{4.19}$$

of the function $\exp\{\Omega(P)\}$ through the cross sections $\mathbf{b}_m$, $m=1,2,\dots,\rho$.

Let $\mathrm{d}\omega_j$ $(j=1,2,\dots,\rho)$ be the canonical basis of Abelian differentials of the first kind

$$\mathrm{d}\omega_j=\frac{c_j^{(1)}z^{\rho-1}+c_j^{(2)}z^{\rho-2}+\dots+c_j^{(\rho)}}{w}\mathrm{d}z, \tag{4.20}$$

where the constants $c_j^{(k)}$ $(k,j=1,2,\dots,\rho)$ are chosen such that

$$\int_{\mathbf{a}_k}\mathrm{d}\omega_j=\delta_{jk}. \tag{4.21}$$

Denote the B-periods of the basis $\mathrm{d}\omega_j$ by

$$B_{jk}=\int_{\mathbf{b}_k}\mathrm{d}\omega_j. \tag{4.22}$$

The matrix $B=(B_{jk})$ $(j,k=1,2,\dots,\rho)$ is symmetric and $\operatorname{Im}B$ is positive definite. The principal tool we shall use to suppress the discontinuities of $\exp\{\Omega(P)\}$ is the Riemann θ-function

$$\theta(\mathbf{s}(P))=\theta(s_1(P),s_2(P),\dots,s_\rho(P)) \tag{4.23}$$

defined by

$$\theta(\mathbf{s}(P))=\sum_{n_1,\dots,n_\rho=-\infty}^{\infty}\exp\left\{\sum_{j=1}^{\rho}\sum_{k=1}^{\rho}B_{jk}n_jn_k+2\pi i\sum_{j=1}^{\rho}n_js_j(P)\right\}. \tag{4.24}$$

Because of the positive definiteness of the matrix $\operatorname{Im}B$ the series converges for all $\mathbf{s}(P)$. The θ-function has periods $\mathbf{n}=(n_1,n_2,\dots,n_\rho)$, n_j are integers, and quasi-periods $\mathbf{B}_j=(B_{j1},B_{j2},\dots,B_{j\rho})$, $j=1,2,\dots,\rho$,

$$\begin{aligned}&\theta(s_1+n_1,\dots,s_\rho+n_\rho)=\theta(s_1,\dots,s_\rho),\\&\theta(s_1+B_{j1},\dots,s_\rho+B_{j\rho})=\exp\{-\pi iB_{jj}-2\pi is_j\}\theta(s_1,\dots,s_\rho).\end{aligned} \tag{4.25}$$

Introduce next the function

$$\mathcal{F}_0(P)=\frac{\theta(u_1(P)-\sigma_1+V_1^\circ,\dots,u_\rho(P)-\sigma_\rho+V_\rho^\circ)}{\theta(u_1(P)-\sigma_1,\dots,u_\rho(P)-\sigma_\rho)}. \tag{4.26}$$

Here, $V_j^\circ = (2\pi i)^{-1} V_j$ and $u_j(P)$ are the integrals

$$u_j(P) = \int_{P_0}^{P} d\omega_j, \quad j = 1, 2, \ldots, \rho, \tag{4.27}$$

which form the canonical basis of abelian integrals of the first kind. It is convenient to choose P_0 as the branch point $r_{2\rho+2}$. The numbers σ_j are chosen to be

$$\sigma_j = \sum_{m=1}^{\rho} u_j(P_m) + k_j, \quad j = 1, \ldots, \rho, \tag{4.28}$$

where P_m $(m = 1, 2, \ldots, \rho)$ are some arbitrary distinct fixed points on $\mathcal{R}$ say, on $\mathbb{C}_1$, $P_m = (\zeta_m, \sqrt{f(\zeta_m)})$, such that the θ-functions in (4.26) are not identically zero. The parameters k_j in (4.28) are the Riemann constants which, for the hyperelliptic surface $\mathcal{R}$ and for the homology basis chosen, can be taken as (see for example [8])

$$k_j = -\frac{j}{2} + \frac{1}{2} \sum_{m=1}^{\rho} B_{jm}. \tag{4.29}$$

The function $\mathcal{F}_0(P)$ has ρ simple poles $P_1, P_2, \ldots, P_\rho$ [11, p. 303] lying in the first sheet and ρ simple zeros which may lie on either sheet of the surface. Call these zeros $Q_j = (t_j, w_j)$, where $w_j = \sqrt{f(t_j)}$ if $Q_j \in \mathbb{C}_1$ and $w_j = -\sqrt{f(t_j)}$ if $Q_j \in \mathbb{C}_2$, $j = 1, 2, \ldots, \rho$. The position of these zeros is unknown a priori, and without loss of generality these zeros are assumed to be simple. The function $\mathcal{F}_0(P)$ is continuous through the cross sections $\mathbf{a}_j$ and discontinuous through the cross sections $\mathbf{b}_j$, $j = 1, \ldots, \rho$. Due to (4.25) its jumps are $\exp\{-V_j\}$. This implies that the function $\mathcal{F}(z, w) = \exp\{\Omega(P)\}\mathcal{F}_0(P)$ is meromorphic on $\mathcal{R}$ (it is continuous through the loops $\mathbf{b}_j$). The set of singularities of the function $\mathcal{F}(z, w)$ comprises the two infinite points ∞_1 and ∞_2 and ρ simple poles $P_m = (\zeta_m, \sqrt{f(\zeta_m)}) \in \mathbb{C}_1$, $m = 1, 2, \ldots, \rho$. Therefore,

$$\mathcal{F}(P) = e^{\Omega(P)} \frac{\theta(u_1(P) - \sigma_1 + V_1^\circ, \ldots, u_\rho(P) - \sigma_\rho + V_\rho^\circ)}{\theta(u_1(P) - \sigma_1, \ldots, u_\rho(P) - \sigma_\rho)} \tag{4.30}$$

is the Baker–Akhiezer function of the surface $\mathcal{R}$ with the homology basis $\mathbf{a}_j$, $\mathbf{b}_j$ $(j = 1, \ldots, \rho)$ associated with the abelian integral $\Omega(P)$ and the poles $P_1, \ldots, P_\rho$.

5. Vector RHP

5.1. MATRIX FACTORIZATION IN TERMS OF THE BAKER–AKHIEZER FUNCTION

We are interested in factorizing the matrix $G(t)$ in terms of the function $\mathcal{F}(z, w)$. In other words, we wish to express two matrices $X^+(t)$ and $X^-(t)$ through the Baker–Akhiezer function such that

$$G(t) = X^+(t)[X^-(t)]^{-1}, \quad t \in L, \tag{5.1}$$

where $X(z)=X^{\pm}(z)$, $z\in D^{\pm}$, and $X^{+}(z)$ and $X^{-}(z)$ are analytic and nonsingular everywhere in D^{+} and D^{-}, respectively, apart from at most a finite number of points where they may have poles or where $\det X(z)=0$. Let $\chi(z,w)$ be a nontrivial solution to the following homogeneous RHP problem on the surface $\mathcal{R}$:

Find a piece-wise meromorphic function $\chi(z,w)$ with the discontinuity contour $\mathcal{L}$, $\hat{H}$-continuous up to the contour $\mathcal{L}$ except for a finite number of poles and satisfying the boundary condition

$$\chi^{+}(t,\xi)=\lambda(t,\xi)\chi^{-}(t,\xi),\quad (t,\xi)\in\mathcal{L}. \tag{5.2}$$

Then the matrix of factorization $X(z)$ can be expressed exclusively through the function $\chi(z,w)$ and the matrix $Y(z,w)$ given by

$$Y(z,w)=\frac{1}{2}\left[I+\frac{1}{h(z)w}Q(z)\right],\quad Q(z)=\begin{pmatrix} l_0(z) & l_1(z)\\ l_2(z) & -l_0(z)\end{pmatrix},\quad I=\mathrm{diag}\{1,1\}, \tag{5.3}$$

in the form [6,23]

$$X(z)=\chi(z,w)Y(z,w)+\chi(z,-w)Y(z,-w). \tag{5.4}$$

It is a simple matter to verify that

$$\begin{aligned} &[X(z)]^{-1}=\frac{Y(z,w)}{\chi(z,w)}+\frac{Y(z,-w)}{\chi(z,-w)},\\ &Q^2(z)=h^2(z)f(z)I,\quad Y^2(z,w)=Y(z,w),\quad Y(z,w)Y(z,-w)=0, \end{aligned} \tag{5.5}$$

and because of (5.2)

$$\begin{aligned} X^{+}(t)[X^{-}(t)]^{-1}&=\frac{1}{2}[\lambda_1(t)+\lambda_2(t)]I+\frac{1}{2h(t)\sqrt{f(t)}}[\lambda_1(t)-\lambda_2(t)]Q(t)\\ &=G(t),\quad t\in L. \end{aligned} \tag{5.6}$$

We assert that the function $\chi_0(z,w)\mathcal{F}(z,w)$ meets the boundary condition (5.2), and it is bounded at the infinite points ∞_1 and ∞_2 (the Baker–Akhiezer function $\mathcal{F}(z,w)$ annihilates the essential singularities of the function $\chi_0(z,w)$ at the infinite points). Thus, the function

$$\begin{aligned} \chi(z,w)&=\chi_0(z,w)\mathcal{F}(z,w)\\ &=\mathrm{e}^{\beta_0(z)+w\beta_1(z)+\Omega(P)}\frac{\theta(u_1(P)-\sigma_1+V_1^{\circ},\ldots,u_\rho(P)-\sigma_\rho+V_\rho^{\circ})}{\theta(u_1(P)-\sigma_1,\ldots,u_\rho(P)-\sigma_\rho)} \end{aligned} \tag{5.7}$$

is a meromorphic solution to the scalar RHP (5.2) on the surface $\mathcal{R}$, and the matrix (5.4) generates Wiener–Hopf matrix factors of the matrix $G(t)$.

5.2. GENERAL SOLUTION TO THE SCALAR RHP ON THE RIEMANN SURFACE

To derive the general solution to the vector RHP (2.2), we solve the scalar RHP on the Riemann surface $\mathcal{R}$ (2.12). On employing the factorization (5.2) of the function $\lambda(t,\xi)$, we write

$$\frac{\Psi^+(t,\xi)}{\chi^+(t,\xi)} = \frac{\Psi^-(t,\xi)}{\chi^-(t,\xi)} + \frac{g^*(t,\xi)}{\chi^+(t,\xi)}, \quad (t,\xi)\in\mathcal{L}. \tag{5.8}$$

Since $g^*(t,\xi)[\chi^+(t,\xi)]^{-1}$ is an $\hat{H}$-continuous function on the surface $\mathcal{R}$, due to the Sokhotski–Plemelj formulas it admits a representation in terms of the limit values of the Weierstrass integral

$$F(z,w) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{g^*(t,\xi)}{\chi^+(t,\xi)}\mathrm{d}W, \quad (z,w)\in\mathcal{R}\backslash\mathcal{L}, \tag{5.9}$$

as follows:

$$F^+(t,\xi) - F^-(t,\xi) = \frac{g^*(t,\xi)}{\chi^+(t,\xi)}, \quad (t,\xi)\in\mathcal{L}. \tag{5.10}$$

The integral (5.9) can be conveniently written as

$$F(z,w) = F_1(z) + wF_2(z), \tag{5.11}$$

where

$$F_1(z) = \frac{1}{4\pi i}\int_{\mathcal{L}} \frac{g^*(t,\xi)\mathrm{d}t}{\chi^+(t,\xi)(t-z)}, \quad F_2(z) = \frac{1}{4\pi i}\int_{\mathcal{L}} \frac{g^*(t,\xi)\mathrm{d}t}{\xi(t)\chi^+(t,\xi)(t-z)}. \tag{5.12}$$

Consequently, we may replace the boundary condition (5.8) by

$$\frac{\Psi^+(t,\xi)}{\chi^+(\xi)} - F^+(t,\xi) = \frac{\Psi^-(t,\xi)}{\chi^-(\xi)} - F^-(t,\xi), \quad (t,\xi)\in\mathcal{L}, \tag{5.13}$$

and apply the Liouville theorem to obtain

$$\Psi(z,w) = \chi(z,w)[F(z,w) + R(z,w)], \quad (z,w)\in\mathcal{R}, \tag{5.14}$$

where $R(z,w)$ is a rational function on the surface $\mathcal{R}$,

$$R(z,w) = R_1(z) + wR_2(z), \tag{5.15}$$

and $R_1(z)$ and $R_2(z)$ are rational functions in the z-plane. The function $\Psi(z,w)$ has poles at the points with affixes $p_1, p_2, \ldots, p_l$ of multiplicity $m_1, m_2, \ldots, m_l$, respectively, lying in both sheets of the surface. Therefore, the rational function $R(z,w)$ has also poles of the same multiplicity at these points. In addition, due to ρ simple zeros $Q_j = (t_j, w_j)$ $(j = 1, 2, \ldots, \rho)$ of the Baker–Akhiezer function (these zeros are to be determined) the function $R(z,w)$ may have simple poles at these points. If $\kappa_m > 0$ $(m = 1, 2)$, the function $R(z,w)$ has simple poles at the points $q_{mj} = (z_{mj}, (-1)^{m-1}\sqrt{f(z_{mj})}) \in \mathbb{C}_m\backslash L_m$. Otherwise, if $\kappa_m \le 0$ $(m = 1, 2)$, the function $R(z,w)$ is bounded at the points q_{mj}. Also, the function $R(z,w)$ may have simple poles at the branch points $r_1, r_2, \ldots, r_{2\rho+2}$, the poles of the function $\Psi(z,w)$. Since $|\Psi(z,w)| < c_j|z|^\delta$ as $z\to\infty_m$, $m = 1, 2$, $\delta = \max\{0, \delta_1 - N - \rho - 1\}$, the functions $R_1(z)$ and $R_2(z)$ may have poles of order $\delta_1 - N - \rho - 1$ and $\delta_1 - N - 2\rho - 2$, respectively, at the infinite point. The most general form of the rational functions $R_1(z)$ and $R_2(z)$ with the poles described is given by

$$R_1(z)=C_0+\sum_{j=1}^{\delta_1-N-\rho-1} M'_j z^j+\sum_{j=1}^{\rho}\frac{C_j w_j}{z-t_j}+\sum_{k=1}^{l}\sum_{j=1}^{m_k}\frac{D'_{kj}}{(z-p_k)^j}$$
$$+\sum_{m=1}^{2}(-1)^{m-1}\sum_{j=1}^{\kappa_m}\frac{E_{mj}\sqrt{f(z_{mj})}}{z-z_{mj}},$$
$$R_2(z)=\sum_{j=0}^{\delta_1-N-2\rho-2} M''_j z^j+\sum_{j=1}^{\rho}\frac{C_j}{z-t_j}+\sum_{j=1}^{2\rho+2}\frac{K_j}{z-r_j}+\sum_{k=1}^{l}\sum_{j=1}^{m_k}\frac{D''_{kj}}{(z-p_k)^j}$$
$$+\sum_{m=1}^{2}\sum_{j=1}^{\kappa_m}\frac{E_{mj}}{z-z_{mj}}, \tag{5.16}$$

where M'_j $(j=1,2,\dots,\delta_1-N-\rho-1)$, M''_j $(j=0,1,\dots,\delta_1-N-2\rho-2)$, C_j, $(j=0,1,\dots,\rho)$, K_j $(j=1,2,\dots,2\rho+2)$, D'_{kj}, D''_{kj} $(j=1,2,\dots,m_k;\ k=1,2,\dots,l)$, and E_{mj} $(j=1,2,\dots,\kappa_m,\ m=1,2)$ are arbitrary constants. In total, the rational function $R(z,w)$ possesses κ free constants, and κ is defined by

$$\kappa=\begin{cases}2\delta_1+\tilde\kappa+1, & N+2\rho+2\le\delta_1\le 2N+2\rho+2,\\ \delta_1+N+2\rho+\tilde\kappa+2, & N+\rho+2\le\delta_1\le N+2\rho+1,\\ 2N+3\rho+\tilde\kappa+3, & 0\le\delta_1\le N+\rho+1,\end{cases}$$
$$\tilde\kappa=\max\{0,\kappa_1\}+\max\{0,\kappa_2\}. \tag{5.17}$$

Analysis of formulas (5.15) and (5.16) shows that the function $R(z,w)$ has simple poles at the points $Q_j=(t_j,w_j)$ and removable singularities at the points $(t_j,-w_j)\in\mathcal{R}$ $(j=1,2,\dots,\rho)$. Also, if $\kappa_m>0$, it has simple poles at the points $q_{mj}\in\mathbb{C}_m$, $q_{mj}=(z_{mj},(-1)^{m-1}\sqrt{f(z_{mj})})$ and removable singularities at the points $(z_{mj},(-1)^m\sqrt{f(z_{mj})}\in\mathbb{C}_{3-m}$ $(j=1,2,\dots,\kappa_m;\ m=1,2)$.

Owing to the poles of the function $\chi(z,w)$ and the structure of the functions $F(z,w)$ and $R(z,w)$ we may expect that the function $\Psi(z,w)$ possesses some poles unacceptable for the solution to the RHP (2.12). Such singularities have to be removed. Due to the simple poles of the Baker–Akhiezer function and, therefore, the poles of the function $\chi(z,w)$ at the points $P_1,P_2,\dots,P_\rho$ lying in the first sheet the function $\Psi(z,w)$ has simple poles at these points. We put

$$F(z,w)+R(z,w)=0,\quad (z,w)=(\zeta_j,\sqrt{f(\zeta_j)}),\quad j=1,2,\dots,\rho, \tag{5.18}$$

and the points P_j $(j=1,\dots,\rho)$ become removable singularities.

If $\kappa_m<0$ $(m=1,2)$, then the function $\chi(z,w)$ has $-\kappa_m$ simple poles at the points $q_{mj}\in\mathbb{C}_m$. For the purpose of removing these poles, we request

$$F(z,w)+R(z,w)=0,\quad (z,w)=(z_{mj},(-1)^{m-1}\sqrt{f(z_{mj})}),$$
$$j=1,2,\dots,-\kappa_m,\quad m=1,2. \tag{5.19}$$

If $0 \le \delta_1 \le N+\rho+1$, then the function $\Psi(z,w)$ has to be bounded at infinity. However, due to the function w in the representations (5.11) and (5.15) it has order-ρ poles at the points ∞_1 and ∞_2. Expand the function $F_2(z)+R_2(z)$ in a neighborhood of the infinite point

$$F_2(z)+R_2(z)=\frac{v_1}{z}+\cdots+\frac{v_\rho}{z^\rho}+\frac{v_{\rho+1}}{z^{\rho+1}}+\cdots. \tag{5.20}$$

These poles become removable singularities of the function $\Psi(z,w)$ if and only if

$$v_1=v_2=\cdots=v_\rho=0. \tag{5.21}$$

In the case $N+\rho+2\le\delta_1\le N+2\rho$, we have to have $|\Psi(z,w)|<c_j|z|^{\delta_1-N-\rho-1}$ as $z\to\infty_m$, $m=1,2$. However, the function $\Psi(z,w)$ found has poles of order ρ at the points ∞_1 and ∞_2. Since $1\le\delta_1-N-\rho-1\le\rho-1$, to have the asymptotics required, we have to put

$$v_1=v_2=\cdots=v_{2\rho+N-\delta_1+1}=0. \tag{5.22}$$

In the case $N+2\rho+1\le\delta_1\le 2N+2\rho+2$, the function $\Psi(z,w)$ has the asymptotics we need without any extra conditions.

Denote $\hat{\kappa}=\max\{0,-\kappa_1\}+\max\{0,-\kappa_2\}$. We have $2\rho+\hat{\kappa}$, $3\rho+N-\delta_1+\hat{\kappa}+1$ and $\rho+\hat{\kappa}$ conditions for the free constants in the cases $0\le\delta_1\le N+\rho+1$, $N+\rho+2\le\delta_1\le N+2\rho$ and $N+2\rho+1\le\delta_1\le 2N+2\rho+2$, respectively. If these conditions are fulfilled, then the function $\Psi(z,w)$ given by (5.14) is the general solution to the RHP (2.12).

5.3. ZEROS OF THE BAKER–AKHIEZER FUNCTION

To complete the procedure presented, we have to determine the points Q_j ($j=1,\ldots,\rho$), the zeros of the Baker–Akhiezer function (4.30), or, equivalently, the zeros of the θ-function (without loss of generality we may assume that it is not identically equal to zero)

$$\theta(u_1(P)-\sigma_1+V_1^\circ,\ldots,u_\rho(P)-\sigma_\rho+V_\rho^\circ). \tag{5.23}$$

We need to know not only the affixes t_j of these zeros, but also to identify the sheet of the surface in which they are located to determine the rational function $R(z,w)$. On setting

$$\sigma_m-V_m^\circ=\sum_{j=1}^{\rho}u_m(Q_j)+k_m \quad \text{(modulo the periods)}, \quad m=1,2,\ldots,\rho, \tag{5.24}$$

we obtain that the points Q_j are the zeros of the function $\mathcal{F}(P)$ indeed. The system (5.24) can equivalently be written as the Jacobi problem of inversion of abelian integrals:

Find ρ points on the surface $\mathcal{R}$, $Q_1, Q_2, \ldots, Q_\rho$, and 2ρ integers, $\mu_1, \mu_2, \ldots, \mu_\rho$ and $\nu_1, \nu_2, \ldots, \nu_\rho$, such that

$$\sum_{j=1}^{\rho} \int_{P_0}^{Q_j} d\omega_m + \sum_{j=1}^{\rho} \nu_j B_{mj} + \mu_m = \hat{\sigma}_m - k_m, \quad m = 1, 2, \ldots, \rho, \tag{5.25}$$

where $\hat{\sigma}_m = \sigma_m - V_m^\circ$.

This problem reduces [28] to the system of symmetric algebraic equations

$$t_1^m + t_2^m + \cdots + t_\rho^m = \tau_m, \quad m = 1, 2, \ldots, \rho, \tag{5.26}$$

where τ_m are known and given in terms of the residues at the infinite points [28] or the two zeros of the surface [8] of functions expressible in terms of the θ-function. The system may be converted into the problem of determination of ρ zeros of an associated order-ρ polynomial. The integers ν_m are found by solving the linear system [8]

$$\sum_{j=1}^{\rho} \nu_j \operatorname{Im} B_{mj} = \operatorname{Im} b_m, \quad m = 1, 2, \ldots, \rho, \tag{5.27}$$

while the integers μ_m are defined by

$$\mu_m = \operatorname{Re} b_m - \sum_{j=1}^{\rho} \nu_j \operatorname{Re} B_{mj}, \quad m = 1, 2, \ldots, \rho, \tag{5.28}$$

explicitly. Here,

$$b_m = \hat{\sigma}_m - k_m - \sum_{j=1}^{\rho} u_m(Q_j). \tag{5.29}$$

There are 2^ρ points on the surface $\mathcal{R}$ which have affixes defined by the ρ zeros of the polynomial associated with the system (5.26). However, there is one and only one set of points $\{Q_1, \ldots, Q_\rho\}$ which have the affixes $t_1, \ldots, t_\rho$, respectively, such that all the numbers $\nu_1, \ldots, \nu_\rho$ and $\mu_1, \ldots, \mu_\rho$ defined by (5.27) and (5.28) are integers.

5.4. GENERAL SOLUTION TO THE VECTOR RHP

Having derived the solution to the scalar RHP on the surface $\mathcal{R}$ (2.12), we can now determine and examine the solution to the original vector RHP (2.2). From (2.6) and (2.5), we express the components of the vector $\mathbf{\Phi}(z)$, $\Phi_1(z)$ and $\Phi_2(z)$, as

$$\begin{aligned} \Phi_1(z) &= \psi_1(z) + \psi_2(z), \\ \Phi_2(z) &= -\frac{l_0(z)}{l_1(z)}[\psi_1(z) + \psi_2(z)] + \frac{h(z)\sqrt{f(z)}}{l_1(z)}[\psi_1(z) - \psi_2(z)], \quad z \in \mathbb{C}, \end{aligned} \tag{5.30}$$

where $\psi_m(z)=\Psi(z,w)$, $(z,w)\in\mathbb{C}_m$, $m=1,2$. We have obtained the solution of the RHP (2.12) in the class of functions having the poles $p_1,\ldots,p_l$ of multiplicity $m_1,\ldots,m_l$, respectively, due to the presence of the polynomial $h(z)$ in (2.8) and its zeros at these points. However, the solution to the original RHP (2.2), the vector $\mathbf{\Phi}(z)$, has to be analytic at these points. This can be achieved by introducing the following N conditions

$$\lim_{z\to p_j}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\{(z-p_j)^{m_j-k}[\psi_1(z)+\psi_2(z)]\}=0,\quad k=0,1,\ldots,m_j-1,\\ j=1,2,\ldots,l. \tag{5.31}$$

If these conditions are satisfied, then the functions $\Phi_1(z)$ and $\Phi_2(z)$ are analytic at the poles of the functions $\psi_1(z)$ and $\psi_2(z)$ (the zeros of $h(z)$).

Let $\tilde{p}_j$ be order-$\tilde{m}_j$ zeros ($j=1,2,\ldots,\tilde{l}$) of the function $l_1(z)$, $\tilde{m}_1+\cdots+\tilde{m}_{\tilde{l}}=\delta_1$. These zeros are poles of the same multiplicity of the function $\Phi_2(z)$ in (5.30). To remove these poles, we require

$$\lim_{z\to \tilde{p}_j}\frac{\mathrm{d}^k}{\mathrm{d}z^k}[(z-\tilde{p}_j)^{\tilde{m}_j-k}\Phi_2(z)]=0,\quad k=0,1,\ldots,\tilde{m}_j-1,\quad j=1,2,\ldots,\tilde{l}. \tag{5.32}$$

Finally, we need to guarantee that the functions $\Phi_1(z)$ and $\Phi_2(z)$ are bounded at infinity. Analyze first the case $0\le\delta_1\le N+\rho$. Since $|\psi_j(z)|\le c_j$ $(j=1,2)$ as $z\to\infty$, it follows from (5.30) that the function $\Phi_1(z)$ is bounded. Expand the function $\Phi_2(z)$ in a neighborhood of the infinite point

$$\Phi_2(z)=\tilde{v}_{N+\rho-\delta_1+1}z^{N+\rho-\delta_1+1}+\cdots+\tilde{v}_1 z+\tilde{v}_0+\cdots. \tag{5.33}$$

On putting

$$\tilde{v}_1=\tilde{v}_2=\ldots=\tilde{v}_{N+\rho-\delta_1+1}. \tag{5.34}$$

we remove the growth of the function $\Phi_2(z)$. As $\delta_1=N+\rho+1$, the function $\Phi_2(z)$ is bounded unconditionally.

Consider now the case $N+\rho+2\le\delta_1\le 2N+2\rho+2$. It follows from (5.30) and the asymptotics of the functions $\psi_1(z)$ and $\psi_2(z)$ that at the infinite point the function $\Phi_2(z)$ is bounded, while the function $\Phi_1(z)$ has a pole of order $\delta_1-N-\rho-1$. Let

$$\psi_1(z)+\psi_2(z)=\hat{v}_{\delta_1-N-\rho-1}z^{\delta_1-N-\rho-1}+\cdots+\hat{v}_1 z+\hat{v}+\cdots,\quad z\to\infty. \tag{5.35}$$

On satisfying the conditions

$$\hat{v}_1=\hat{v}_2=\cdots=\hat{v}_{\delta_1-N-\rho-1}=0 \tag{5.36}$$

we obtain the function $\Phi_1(z)$ bounded at the infinite point.

We now summarize the results.

Theorem. Let $G(t)$ *be a nonsingular* 2×2 *matrix*

$$G(t)=\begin{pmatrix}\alpha_1(t)+\alpha_2(t)l_0(t) & \alpha_2(t)l_1(t)\\ \alpha_2(t)l_2(t) & \alpha_1(t)-\alpha_2(t)l_0(t)\end{pmatrix}, \tag{5.37}$$

where $\alpha_1(t),\alpha_2(t)l_j(t)\in\hat{H}(L)$, $j=0,1,2$, $l_0(t)$, $l_1(t)$ *and* $l_2(t)$ *are polynomials, and* L *is the real axis. Denote* $l_0^2(z)+l_1(z)l_2(z)=h^2(z)f(z)$, $\delta_1=\deg l_1(z)$ *and* $N=\deg h(z)$. *Assume that* $2\rho+2=\deg f(z)$, *the zeros of the polynomial* $f(z)$ *are simple, and none of the zeros of* $f(z)$ *and* $h(z)$ *fall in the contour* L.

Let κ_1 *and* κ_2 *be the integers defined by* $\kappa_j=\operatorname{ind}\lambda_j(t)$, $t\in L$, *where* $\lambda_1(t)$ *and* $\lambda_2(t)$ *are the eigenvalues of* $G(t)$, $\lambda_j=\alpha_1-(-1)^j\alpha_2 h\sqrt{f}$, $j=1,2$. *Denote* $\tilde{\kappa}=\max\{0,\kappa_1\}+\max\{0,\kappa_2\}$.

Then the functions (5.30) *possess* κ *arbitrary constants*

$$\kappa=\begin{cases}2\delta_1+\tilde{\kappa}+1, & N+2\rho+2\le\delta_1\le 2N+2\rho+2,\\ \delta_1+N+2\rho+\tilde{\kappa}+2, & N+\rho+2\le\delta_1\le N+2\rho+1,\\ 2N+3\rho+\tilde{\kappa}+3, & 0\le\delta_1\le N+\rho+1,\end{cases} \tag{5.38}$$

which have to satisfy $\kappa'=\kappa-\kappa_1-\kappa_2-2$ *additional conditions* (5.18), (5.19), (5.21), (5.22), (5.31), (5.32), (5.34) *and* (5.37). *If* $\kappa_1+\kappa_2\ge -2$, *then the solution to the problem* (2.2) *exists, has* $\kappa_1+\kappa_2+2$ *free constants and is defined by* (5.30). *Otherwise, the solution does not exist. If however the vector* $\mathbf{g}(t)$ *satisfies* $-\kappa_1-\kappa_2-2$ *conditions which guarantee that all the additional conditions are fulfilled, then the solution exists and it is unique.*

6. Conclusions

We have proposed a new technique for deriving Wiener–Hopf factors of the Chebotarev–Khrapkov matrix $G(t)=\alpha_1(t)I+\alpha_2(t)Q(t)$, $\alpha_1(t),\alpha_2(t)Q(t)\in\hat{H}(L)$, $Q(t)$ is a 2×2 zero-trace polynomial matrix. The method has been applied to solve the vector RHP $\mathbf{\Phi}^+(t)=G(t)\mathbf{\Phi}^-(t)+\mathbf{g}(t)$, $t\in L$. The known technique [6,23] first reduces the vector problem to a scalar RHP on the Riemann surface $\mathcal{R}$ of the algebraic function $w^2=f(z)$, $\det Q(z)=h^2(z)f(z)$. Then, it finds a function $\chi_0(z,w)$ which factorizes the coefficient of the RHP on the surface and allows for essential singularities at the infinite points of $\mathcal{R}$. These singularities are removed by solving a certain Jacobi problem of inversion of hyperelliptic integrals. At this stage, a meromorphic solution is derived. The inadmissible poles due to the technique applied are removed afterwards. In contrast with this method, the technique we have developed hinges on the derivation of the Baker–Akhiezer function widely used in the theory of integrable systems. This procedure quenches the essential singularities by constructing a special abelian integral of the second type $\Omega(P)$. It has zero A-periods, and the principal part of the function $\exp\{\Omega(P)\}$ at the infinite points is derived according to the behavior of the function $\chi_0(z,w)$ at the infinite points. The consequent use of the quotient of two Riemann θ-functions serves to annihilate the discontinuity of the function $\exp\{\Omega(P)\}$ due to the nonzero B-periods of the integral $\Omega(P)$. The product of the function $\exp\{\Omega(P)\}$ and the quotient of the two θ-functions forms the Baker–Akhiezer function $\mathcal{F}(P)$, while the product of $\chi_0(z, w)$ and $\mathcal{F}(P)$ forms a solution of the homogeneous scalar RHP on the surface. It does not have essential singularities and is a meromorphic function in $\mathcal{R}$ with a finite number of prescribed poles. This gives Wiener–Hopf factors of $G(t)$ and does not require the solution of a Jacobi inversion problem. For the general solution of the vector RHP, however, the solution of the associated Jacobi inversion problem is unavoidable. This is because the Baker–Akhiezer function has ρ zeros (ρ is the genus of the surface $\mathcal{R}$), and their location cannot be prescribed. At the stage of application of the generalized Liouville theorem, the zeros of the Baker–Akhiezer function are needed for determination of the rational vector in the general solution. This information can be recovered by stating and solving the corresponding Jacobi inversion problem.

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