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Subsonic propagation of a crack parallel to the boundary of a half-plane

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Abstract
A two-dimensional steady-state model problem on a half-plane with a semi-infinite crack propagating at constant speed parallel to the boundary of the half-plane is considered. The crack faces are subjected to normal and tangential loads, while the boundary of the half-plane is free of traction. The problem is formulated as an order-2 vector Riemann–Hilbert problem and then reduced to a system of singular integral equations in a semi-infinite segment with respect to the derivatives of the displacement jumps. The solution to the system of integral equations is represented in a series form in terms of an orthonormal basis of the associated Hilbert space, the orthonormal Jacobi polynomials. The coefficients of the expansions solve an infinite system of linear algebraic equations of the second kind. The stress intensity factors and the weight functions are determined and computed. The Griffith energy criterion is applied to derive a crack growth criterion in terms of the K_I- and K_{II}-stress intensity factors, the crack speed, the shear and dilatational waves speeds and the Griffith material constant.

Keywords
Griffith energy criterion, orthogonal polynomials, subsonic crack propagation, systems of singular integral equations

1. Introduction

The main motivation of this work is to study the boundary effects on a crack propagating parallel to the boundary of a solid. The static problem for a semi-infinite crack parallel to the boundary of a half-plane was analyzed by Zlatin and Khrapkov [1]. They transformed the model problem into an order-2 vector Riemann–Hilbert (RH) problem and managed to derive a closed-form solution by exact factorization of the matrix coefficient of the RH problem. The steady-state problem for a plane with a semi-infinite crack $-\infty < x_1 < 0, x_2 = 0$ driven by moving normal and tangential loads applied to the crack faces was considered by Craggs [2]. Because of the symmetry, the Craggs problem admits decoupling and can be solved in closed form by a variety of methods including the factorization method for a scalar RH problem, the Mellin transform method, which bypasses the RH problem, and the method of orthogonal polynomials. Many researchers analyzed different aspects of the Craggs model problem and considered its generalizations. Surveys of the results were given by Freund [3] and Broberg [4]. To our knowledge, no analytical solution for the steady-state problem on a coupled mode-I,II semi-infinite crack propagating parallel to the boundary of a half-plane is available in the literature.

In Section 2, we state the model problem for a half-plane $\{-\infty < x_1 < \infty, -\infty < x_2 < \delta\}$ $(\delta > 0)$ with a crack extending from $x_1 = -\infty, x_2 = 0$ to $x_1 = Vt, x_2 = 0$ and driven by normal and tangential components of traction applied to the crack faces. It is assumed that the loads move at the same speed V. By employing the method of integral transformations, we map the boundary-value problem for the governing system of partial

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differential equations to a vector RH problem with the matrix coefficient $\hat{G}(p) = a(p)I + b(p)S(p)$, where $a(p)$, $b(p)$ are Hölder functions, $I = \text{diag}\{1, 1\}$,

$$S(p) = \begin{pmatrix} 1 & -i\alpha_1 \tanh \alpha_0 p \\ i\alpha_2 \tanh \alpha_0 p & -1 \end{pmatrix}, \quad (1)$$

and α_0, α_1 and α_2 are real nonzero constants. At this stage, all our efforts availed us no results in factorizing the matrix $\hat{G}(p)$.

In Section 3, we transform the RH problem into a system of two singular integral equations first on the semi-infinite segment $(-\infty, 0)$ and then on the finite segment $(-1, 1)$. Its solution is found in the Hilbert space $L_{2,\rho}(-1, 1)$, $\rho = (1 + \zeta)^{1/2}(1 - \zeta)^{-1/2}$. By expanding the unknown functions in terms of an orthonormal basis of this space, we reduce the system of integral equations to an infinite system of linear algebraic equations of the second kind. Its solution is found by the reduction method.

In Section 4, we derive the stress intensity factors (SIFs) and the weight functions. In addition, we determine the energy released when the crack extends from x to $x + r$, and r is small. Then we apply the Griffith criterion and derive a Willis-type formula for a mode-I,II semi-infinite crack propagating along the boundary of a half-plane. The criterion of steady-state propagation of a semi-infinite crack in a plane was derived by Willis [5]. We also discuss some numerical results obtained for the SIFs, the weight functions and the Griffith criterion.

2. Formulation and a vector RH problem

A semi-infinite crack that occupies, at time t, the surface $S_2(t) = \{\mathbf{x} : -\infty < x_1 < Vt,\ x_2 = 0,\ |x_3| < \infty\}$ propagates in the direction parallel to the boundary of an elastic half-space $R^3_+ = \{\mathbf{x} : |x_1| < \infty,\ -\infty < x_2 < \delta,\ |x_3| < \infty\}$ $(\delta > 0)$. The boundary of the half-space is assumed to be free of traction. The speed V is constant and $V < c_R$, where c_R is the Rayleigh wave speed for the elastic solid whose density and the Lame constants are ρ, λ and μ, respectively. The banks of the crack are subjected to plane-strain loading

$$\sigma_{j2} = \sigma^\circ_{j2}(x_1 - Vt), \quad -\infty < x_1 - Vt < 0, \quad x_2 = 0^\pm, \quad j = 1, 2. \quad (2)$$

The system of equations for two-dimensional dynamic elasticity is the following

$$c_l^2 \Delta\phi - \frac{\partial^2 \phi}{\partial t^2} = 0, \quad c_s^2 \Delta\psi - \frac{\partial^2 \psi}{\partial t^2} = 0, \quad \mathbf{x} \in R^2_+ \setminus S_1(t), \quad (3)$$

where c_l and c_s are the longitudinal and shear wave speeds

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}}, \quad (4)$$

$S_1(t) = \{\mathbf{x} : -\infty < x_1 < Vt,\ x_2 = 0\}$, $R^2_+ = \{\mathbf{x} : |x_1| < \infty,\ -\infty < x_2 < \delta\}$, and ϕ and ψ are dynamic potentials such that the displacements u_1, u_2 and the stresses σ_{11}, σ_{12} are

$$u_1 = \frac{\partial\phi}{\partial x_1} + \frac{\partial\psi}{\partial x_2}, \quad u_2 = \frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_1},$$

$$\sigma_{12} = \mu\left(\frac{2\partial^2\phi}{\partial x_1 \partial x_2} - \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}\right),$$

$$\sigma_{22} = \lambda\Delta\phi + 2\mu\left(\frac{\partial^2\phi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_1 \partial x_2}\right). \quad (5)$$

The problem is steady state, and the advantage of introducing the moving coordinates $x = x_1 - Vt, y = x_2$ is evident. The governing equations (3) are simplified as

$$\alpha^2 \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad \beta^2 \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \quad (6)$$

where $\alpha = \sqrt{1 - V^2/c_l^2}$, $\beta = \sqrt{1 - V^2/c_s^2}$. On applying the Fourier transform

$$\hat{\phi}(p, y) = \int_{-\infty}^{\infty} \phi(x, y) e^{ipx} dx, \quad \hat{\psi}(p, y) = \int_{-\infty}^{\infty} \psi(x, y) e^{ipx} dx \tag{7}$$

to the differential equations (6), we find for $y < 0$,

$$\hat{\phi}(p, y) = C_0(p) e^{\alpha|p|y}, \quad \hat{\psi}(p, y) = D_0(p) e^{\beta|p|y}. \tag{8}$$

In the case $0 < y < \delta$

$$\hat{\phi}(p, y) = C_1(p) \cosh \alpha p y + C_2(p) \sinh \alpha p y,$$
$$\hat{\psi}(p, y) = D_1(p) \cosh \beta p y + D_2(p) \sinh \beta p y. \tag{9}$$

Here, the functions $C_j(p)$ and $D_j(p)$ ($j = 0, 1, 2$) are to be determined from the boundary conditions

$$\sigma_{xy} = \sigma_{yy} = 0, \quad |x| < \infty, \quad y = \delta,$$

$$\sigma_{xy} = \sigma_{12}^{\circ}(x), \quad \sigma_{yy} = \sigma_{22}^{\circ}(x), \quad x < 0, \quad y = 0^{\pm},$$

$$\frac{\partial u}{\partial x}(x, 0^+) - \frac{\partial u}{\partial x}(x, 0^-) = \chi_1(x), \quad \frac{\partial v}{\partial x}(x, 0^+) - \frac{\partial v}{\partial x}(x, 0^-) = \chi_2(x), \quad |x| < \infty, \tag{10}$$

$u(x_1 - Vt, x_2) = u_1(x_1, x_2, t)$, $v(x_1 - Vt, x_2) = u_2(x_1, x_2, t)$, $\sigma_{xy}(x_1 - Vt, x_2) = \sigma_{12}(x_1, x_2, t)$, and $\sigma_{yy}(x_1 - Vt, x_2) = \sigma_{22}(x_1, x_2, t)$. The functions $\chi_1(x)$, $\chi_2(x)$ vanish for $x > 0$ and are unknown otherwise. The Fourier-transformed boundary conditions read

$$\frac{d^2\hat{\psi}}{dy^2} + p^2\hat{\psi} - 2ip\frac{d\hat{\phi}}{dy} = 0, \quad \frac{c_l^2}{c_s^2}\frac{d^2\hat{\phi}}{dy^2} + p^2\left(2 - \frac{c_l^2}{c_s^2}\right)\hat{\phi} + 2ip\frac{d\hat{\psi}}{dy} = 0, \quad y = \delta,$$

$$\frac{d^2\hat{\psi}}{dy^2} + p^2\hat{\psi} - 2ip\frac{d\hat{\phi}}{dy} = \Sigma_1^+(p) + \Sigma_1^-(p), \quad y = 0^{\pm},$$

$$\frac{c_l^2}{c_s^2}\frac{d^2\hat{\phi}}{dy^2} + p^2\left(2 - \frac{c_l^2}{c_s^2}\right)\hat{\phi} + 2ip\frac{d\hat{\psi}}{dy} = \Sigma_2^+(p) + \Sigma_2^-(p), \quad y = 0^{\pm},$$

$$\left(\frac{d\hat{\psi}}{dy} - ip\hat{\phi}\right)_{y=0^+} - \left(\frac{d\hat{\psi}}{dy} - ip\hat{\phi}\right)_{y=0^-} = \frac{X_1^-(p)}{p},$$

$$\left(\frac{d\hat{\phi}}{dy} + ip\hat{\psi}\right)_{y=0^+} - \left(\frac{d\hat{\phi}}{dy} + ip\hat{\psi}\right)_{y=0^-} = \frac{X_2^-(p)}{p}, \tag{11}$$

where the functions $\Sigma_1^-(p)$, $\Sigma_2^-(p)$ are known,

$$\Sigma_j^-(p) = \int_{-\infty}^{0} \sigma_{j2}^{\circ}(x, 0) e^{ipx} dx, \quad j = 1, 2, \tag{12}$$

while the other functions

$$\Sigma_1^+(p) = \frac{1}{\mu}\int_0^{\infty} \sigma_{xy}(x, 0) e^{ipx} dx, \quad \Sigma_2^+(p) = \frac{1}{\mu}\int_0^{\infty} \sigma_{yy}(x, 0) e^{ipx} dx,$$

$$X_j^-(p) = i\int_{-\infty}^{0} \chi_j(x) e^{ipx} dx, \quad j = 1, 2, \tag{13}$$

are to be determined. Six out of the eight equations in (11) determine the unknown functions $C_j(p)$, $D_j(p)$ ($j = 0, 1, 2$). The other two equations constitute the following vector RH problem

$$\begin{pmatrix} \Sigma_1^+(p) \\ \Sigma_2^+(p) \end{pmatrix} = G(p) \begin{pmatrix} X_1^-(p) \\ X_2^-(p) \end{pmatrix} - \begin{pmatrix} \Sigma_1^-(p) \\ \Sigma_2^-(p) \end{pmatrix}, \quad -\infty < p < +\infty, \tag{14}$$

where $G(p)$ is a Hermitian matrix

$$G(p) = \begin{pmatrix} g_{11}(p) & ig_{12}(p) \\ -ig_{12}(p) & g_{22}(p) \end{pmatrix}, \tag{15}$$

whose entries are given by

$$g_{11}(p) = \frac{e^{-(\alpha+\beta)\delta|p|}}{2\beta(1-\beta^2)} \left[R_1 \sinh(\alpha+\beta)\delta p - R_2 \sinh(\alpha-\beta)\delta p + \frac{2d(p)\,\mathrm{sgn}\,p}{R_1} \right],$$

$$g_{12}(p) = \frac{4R_2(1+\beta^2)}{R_1(1-\beta^2)} e^{-(\alpha+\beta)\delta|p|} \sinh^2 \frac{(\alpha-\beta)\delta p}{2},$$

$$g_{22}(p) = \frac{e^{-(\alpha+\beta)\delta|p|}}{2\alpha(1-\beta^2)} \left[R_1 \sinh(\alpha+\beta)\delta p + R_2 \sinh(\alpha-\beta)\delta p + \frac{2d(p)\,\mathrm{sgn}\,p}{R_1} \right],$$

$$d(p) = R_1^2 \sinh^2 \frac{(\alpha+\beta)\delta p}{2} - R_2^2 \sinh^2 \frac{(\alpha-\beta)\delta p}{2},$$

$$R_1 = (1+\beta^2)^2 - 4\alpha\beta, \quad R_2 = (1+\beta^2)^2 + 4\alpha\beta. \tag{16}$$

It can be shown that the eigenvalues of the matrix $G(p)$, $\lambda_1(p)$ and $\lambda_2(p)$, do not satisfy the condition $\lambda_1(p)\lambda_2(p) > 0$ for all $p \in (-\infty, \infty)$, and the matrix $G(p)$ cannot be factorized such that $G(p) = g_0(p)G_0(p)$, where $g_0(p)$ is a scalar, and the matrix $G_0(p)$ is positive definite for all real p. The vector RH problem may be rewritten as

$$\begin{pmatrix} \Sigma_1^+(p) \\ \Sigma_2^+(p) \end{pmatrix} = \hat{G}(p) \begin{pmatrix} \beta^{-1}X_1^-(p) \\ \alpha^{-1}X_2^-(p) \end{pmatrix} - \begin{pmatrix} \Sigma_1^-(p) \\ \Sigma_2^-(p) \end{pmatrix}, \quad -\infty < p < +\infty, \tag{17}$$

where the new matrix coefficient has the structure

$$\hat{G}(p) = a(p)I + b(p)S(p),$$

$$a(p) = \frac{e^{-(\alpha+\beta)\delta|p|}}{2(1-\beta^2)} \left[R_1 \sinh(\alpha+\beta)\delta p + \frac{2d(p)\,\mathrm{sgn}\,p}{R_1} \right],$$

$$b(p) = -\frac{R_2}{2(1-\beta^2)} e^{-(\alpha+\beta)\delta|p|} \sinh(\alpha-\beta)\delta p,$$

$$S(p) = \begin{pmatrix} 1 & -\frac{4i\alpha}{R_1}(1+\beta^2)\tanh\frac{1}{2}(\alpha-\beta)\delta p \\ \frac{4i\beta}{R_1}(1+\beta^2)\tanh\frac{1}{2}(\alpha-\beta)\delta p & -1 \end{pmatrix}, \tag{18}$$

and I is the unit 2×2 matrix. The matrix $\hat{G}(p)$ does not admit a Wiener–Hopf factorization by the methods currently available in the literature.

3. System of integral equations

In this section we aim to map the RH problem into a system of singular integral equations on a semi-infinite segment and develop an efficient numerical scheme for its solution. By applying the inverse Fourier transform to the RH problem, using the convolution theorem and evaluating the integrals, we find the following expressions valid for $-\infty < x < \infty$

$$\sigma_{xy}(x,0) = -\frac{\gamma}{\pi\beta} \left\{ \int_{-\infty}^{0} \left[\frac{1}{\xi - x} + k_{11}(\xi - x) \right] \chi_1(\xi) d\xi + \int_{-\infty}^{0} k_{12}(\xi - x) \chi_2(\xi) d\xi \right\},$$

$$\sigma_{yy}(x,0) = -\frac{\gamma}{\pi\alpha}\left\{\int_{-\infty}^{0} k_{21}(\xi - x)\chi_1(\xi)d\xi + \int_{-\infty}^{0}\left[\frac{1}{\xi - x} + k_{22}(\xi - x)\right]\chi_2(\xi)d\xi\right\}, \tag{19}$$

where

$$k_{11}(x) = \frac{R(1-R)x}{2[x^2+4\alpha^2\delta^2]} - \frac{R(1+R)x}{2[x^2+4\beta^2\delta^2]} + \frac{(R^2-1)x}{x^2+(\alpha+\beta)^2\delta^2},$$

$$k_{12}(x) = \frac{4R(1+\beta^2)\beta}{R_1}\left(\frac{\alpha}{x^2+4\alpha^2\delta^2} + \frac{\beta}{x^2+4\beta^2\delta^2} - \frac{\alpha+\beta}{x^2+(\alpha+\beta)^2\delta^2}\right),$$

$$k_{21}(x) = -\frac{\alpha}{\beta}k_{12}(x),$$

$$k_{22}(x) = -\frac{R(1+R)x}{2[x^2+4\alpha^2\delta^2]} + \frac{R(1-R)x}{2[x^2+4\beta^2\delta^2]} + \frac{(R^2-1)x}{x^2+(\alpha+\beta)^2\delta^2},$$

$$R = \frac{R_2}{R_1}, \quad \gamma = \frac{\mu R_1}{2(1-\beta^2)}. \tag{20}$$

For $-\infty < x < 0$, the traction vector components $\sigma_{xy}(x,0)$ and $\sigma_{yy}(x,0)$ are known (see (10)). Slightly rearranged, the expressions (19) give the resulting system of singular integral equations

$$\int_{-\infty}^{0}\left[\frac{1}{\xi - x} + k_{11}(\xi - x)\right]\chi_1(\xi)d\xi + \int_{-\infty}^{0} k_{12}(\xi - x)\chi_2(\xi)d\xi = -\pi f_1(x), \quad -\infty < x < 0,$$

$$\int_{-\infty}^{0} k_{21}(\xi - x)\chi_1(\xi)d\xi + \int_{-\infty}^{0}\left[\frac{1}{\xi - x} + k_{22}(\xi - x)\right]\chi_2(\xi)d\xi = -\pi f_2(x), \quad -\infty < x < 0, \tag{21}$$

where

$$f_1(x) = \frac{\beta}{\gamma}\sigma_{12}^{\circ}(x), \quad f_2(x) = \frac{\alpha}{\gamma}\sigma_{22}^{\circ}(x). \tag{22}$$

For $\delta = \infty$ (the half-plane becomes a plane), the kernels $k_{ij}(x)$ $(i,j = 1,2)$ vanish, the system decouples, and its closed-form solution is

$$\chi_j(x) = \frac{1}{\pi\sqrt{-x}}\int_{-\infty}^{0}\frac{\sqrt{-\xi}f_j(\xi)d\xi}{\xi - x}, \quad -\infty < x < 0, \quad j = 1,2. \tag{23}$$

In the case of finite values of δ we propose an approximate scheme based on the method of orthogonal polynomials. First, we transform the system into another one on the segment $(-1,1)$. Introduce new variables and functions

$$\xi = \frac{\eta+1}{\eta-1}, \quad x = \frac{\zeta+1}{\zeta-1},$$

$$(1-\zeta)\tilde{\chi}_j(\zeta) = \chi_j(x), \quad (1-\zeta)\tilde{f}_j(\zeta) = f_j(x), \quad j = 1,2. \tag{24}$$

This brings us to the system

$$\int_{-1}^{1}\left[-\frac{1}{\eta-\zeta} + \frac{2k_{jj}(\xi - x)}{(1-\eta)(1-\zeta)}\right]\tilde{\chi}_j(\eta)d\eta$$

$$+\int_{-1}^{1}\frac{2k_{j3-j}(\xi - x)}{(1-\eta)(1-\zeta)}\tilde{\chi}_{3-j}(\eta)d\eta = -\pi\tilde{f}_j(\zeta), \quad -1 < \zeta < 1, \quad j = 1,2. \tag{25}$$

Let now $L_{2,\rho(\mu,\nu)}(a,b)$ be the Hilbert space of functions with the norm defined by

$$||\chi(x)|| = \left(\int_a^b |\chi(x)|^2\rho(\mu,\nu)dx\right)^{1/2}, \quad \rho(\mu,\nu) = (x-a)^{\mu}(b-x)^{\nu}. \tag{26}$$

As in the case $\delta = \infty$, we seek the solution, the functions $\chi_1(x)$ and $\chi_2(x)$, in the space $L_{2,\rho(-1/2,-3/2)}(0,\infty)$. At zero and infinity, the functions behave as

$$\chi_j(x) \sim C_{0j}x^{-1/2}, \quad x \to 0; \quad \chi_j(x) \sim C_{1j}x^{-3/2}, \quad x \to \infty \tag{27}$$

(C_{0j} and C_{1j} are nonzero constants). When written for the new functions $\tilde{\chi}_1(\zeta)$ and $\tilde{\chi}_2(\zeta)$, this results

$$\tilde{\chi}_j(\zeta) \sim \tilde{C}_{0j}(1+\zeta)^{-1/2}, \quad \zeta \to -1; \quad \tilde{\chi}_j(\zeta) \sim \tilde{C}_{1j}(1-\zeta)^{1/2}, \quad \zeta \to 1, \tag{28}$$

and the functions $\tilde{\chi}_1(\zeta)$ and $\tilde{\chi}_2(\zeta)$ belong to the space $L_{2,\rho(-1/2,1/2)}(-1,1)$. The functions $(1-\zeta)^{1/2}(1+\zeta)^{-1/2}\hat{P}_n^{1/2,-1/2}(\zeta)$ form an orthonormal basis of the space $L_{2,\rho(-1/2,1/2)}(-1,1)$. Here, $\hat{P}_n^{1/2,-1/2}(\zeta)$ are the orthonormal Jacobi polynomials. Therefore, the functions $\tilde{\chi}_1(\zeta)$ and $\tilde{\chi}_2(\zeta)$ can be expanded in terms of the basis functions as

$$\tilde{\chi}_j(\zeta) = (1-\zeta)^{1/2}(1+\zeta)^{-1/2}\sum_{m=0}^{\infty} a_m^{(j)}\hat{P}_m^{1/2,-1/2}(\zeta), \quad j=1,2, \quad -1<\zeta<1, \tag{29}$$

where the coefficients $a_n^{(j)}$ are to be determined. The original functions $\chi_1(x)$ and $\chi_2(x)$ may be put into the form

$$\chi_j(x) = \frac{2}{(1-x)\sqrt{-x}}\sum_{m=0}^{\infty} a_m^{(j)}\hat{P}_m^{1/2,-1/2}\left(\frac{x+1}{x-1}\right), \quad j=1,2, \quad -\infty<x<0. \tag{30}$$

By employing next the spectral relation

$$\int_{-1}^{1}\left(\frac{1-\eta}{1+\eta}\right)^{1/2}\frac{\hat{P}_m^{1/2,-1/2}(\eta)d\eta}{\eta-\zeta} = -\pi\hat{P}_m^{-1/2,1/2}(\zeta), \quad -1<\zeta<1, \tag{31}$$

and the orthonormality of the polynomials,

$$\int_{-1}^{1}\left(\frac{1+\zeta}{1-\zeta}\right)^{1/2}\hat{P}_m^{-1/2,1/2}(\zeta)\hat{P}_n^{-1/2,1/2}(\zeta)d\zeta = \delta_{mn}, \quad n,m=0,1,\dots \tag{32}$$

(δ_{mn} is the Kronecker symbol), we transform the system of integral equations into an infinite algebraic system

$$a_n^{(j)} + \sum_{m=0}^{\infty}[c_{nm}^{(j,j)}a_m^{(j)} + c_{nm}^{(j,3-j)}a_m^{(3-j)}] = b_n^{(j)}, \quad n=0,1,\dots;j=1,2. \tag{33}$$

Here,

$$c_{nm}^{(j,l)} = \frac{2}{\pi}\int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{(1-\zeta)^3}}\frac{\hat{P}_n^{-1/2,1/2}(\zeta)\hat{P}_m^{1/2,-1/2}(\eta)}{\sqrt{1-\eta^2}}k_{jl}(\xi-x)d\eta d\zeta,$$
$$b_n^{(j)} = -\int_{-1}^{1}\sqrt{\frac{1+\zeta}{(1-\zeta)^3}}\hat{P}_n^{-1/2,1/2}(\zeta)f_j(x)d\zeta. \tag{34}$$

In order to compute the coefficients $c_{nm}^{(j,l)}$ we rearrange the integrands as

$$\sqrt{\frac{1+\zeta}{(1-\zeta)^3}}\frac{k_{jj}(\xi-x)}{\sqrt{1-\eta^2}} = \sqrt{\frac{1+\zeta}{1-\zeta}}\sqrt{\frac{1-\eta}{1+\eta}}\hat{k}_{jj}(\eta,\zeta), \quad j=1,2,$$
$$\sqrt{\frac{1+\zeta}{(1-\zeta)^3}}\frac{k_{12}(\xi-x)}{\sqrt{1-\eta^2}} = \sqrt{1-\zeta^2}\sqrt{\frac{1-\eta}{1+\eta}}\hat{k}_{12}(\eta,\zeta), \tag{35}$$

where

$$\hat{k}_{jj}(\eta,\zeta) = (R^2-1)\Sigma_1(\eta,\zeta;\alpha+\beta) + (-1)^j[r_j\Sigma_1(\eta,\zeta;2\beta) - r_{3-j}\Sigma_1(\eta,\zeta;2\alpha)], \quad j=1,2,$$

$$\hat{k}_{12}(\eta,\zeta) = \frac{2R(1+\beta^2)\beta}{R_1}[\Sigma_0(\eta,\zeta;2\alpha) + \Sigma_0(\eta,\zeta;2\beta) - 2\Sigma_0(\eta,\zeta;\alpha+\beta)],$$

$$\hat{k}_{21}(\eta,\zeta) = -\frac{\alpha}{\beta}\hat{k}_{12}(\eta,\zeta),$$

$$r_1 = \frac{R}{2}(1+R), \quad r_2 = \frac{R}{2}(1-R), \quad \Sigma_0(\eta,\zeta;\varepsilon) = \frac{\varepsilon(1-\eta)}{4(\eta-\zeta)^2 + [\varepsilon\delta(1-\eta)(1-\zeta)]^2},$$

$$\Sigma_1(\eta,\zeta;\varepsilon) = \frac{2(\zeta-\eta)}{4(\eta-\zeta)^2 + [\varepsilon\delta(1-\eta)(1-\zeta)]^2}, \tag{36}$$

and apply the Gaussian type quadrature formulas [6]

$$\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}f(\zeta)d\zeta = \frac{4\pi}{2M+1}\sum_{j=1}^{M}\cos^2\phi_j f(\cos 2\phi_j),$$

$$\int_{-1}^{1}\sqrt{1-\zeta^2}f(\zeta)d\zeta = \frac{\pi}{M+1}\sum_{j=1}^{M}\sin^2 2\psi_j f(\cos 2\psi_j), \tag{37}$$

where

$$\phi_j = \frac{(2j-1)\pi}{2(2M+1)}, \quad \psi_j = \frac{j\pi}{2(M+1)} \tag{38}$$

and M is the number of abscissas. Using the connection between the Chebyshev and the orthonormal Jacobi polynomials

$$\hat{P}_n^{-1/2,1/2}(\zeta) = \sqrt{\frac{2}{\pi(\zeta+1)}}T_{2n+1}\left(\sqrt{\frac{\zeta+1}{2}}\right),$$

$$\hat{P}_n^{1/2,-1/2}(\zeta) = \frac{1}{\sqrt{\pi}}U_{2n}\left(\sqrt{\frac{\zeta+1}{2}}\right), \tag{39}$$

we derive ultimately

$$c_{nm}^{(j,j)} = \frac{32(-1)^m}{(2M+1)^2}\sum_{j=1}^{M}\cos\phi_j\cos(2n+1)\phi_j\sum_{s=1}^{M}\cos\phi_s\cos(2m+1)\phi_s\hat{k}_{jj}(-\cos 2\phi_s, \cos 2\phi_j),$$

$$c_{nm}^{(1,2)} = \frac{32(-1)^m}{(2M+1)(M+1)}\sum_{j=1}^{M}\sin^2\psi_j\cos\psi_j\cos(2n+1)\psi_j$$

$$\times\sum_{s=1}^{M}\cos\phi_s\cos(2m+1)\phi_s\hat{k}_{12}(-\cos 2\phi_s, \cos 2\psi_j). \quad c_{nm}^{(2,1)} = -\frac{\alpha}{\beta}c_{nm}^{(1,2)}. \tag{40}$$

The integrals $b_n^{(j)}$ can be written in the form

$$b_n^{(j)} = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{0}\frac{1}{\sqrt{1-x}}T_{2n+1}\left(\sqrt{\frac{-x}{1-x}}\right)f_j(x)dx. \tag{41}$$

Their convergence is guaranteed if $f_j(x) \in L_1(0,A)$ for any finite $A > 0$, and $f_j(x) = o(|x|^{-1/2})$, $x \to -\infty$.

Show finally that if $\delta \to \infty$, then the solution of the infinite system tends to the closed-form solution (23) for the whole plane. When $\delta \to \infty$, then $c_{nm}^{(j,l)} \to 0$ and $a_m^{(j)} \to b_m^{(j)}$, that is if $\delta = \infty$, then

$$\tilde{\chi}_j(\zeta) = -\sqrt{\frac{1-\zeta}{1+\zeta}} \int_{-1}^{1} f_j(\xi) \sqrt{\frac{1+\eta}{(1-\eta)^3}} \Lambda(\eta,\zeta) d\eta, \tag{42}$$

where

$$\Lambda(\eta,\zeta) = \sum_{m=0}^{\infty} \hat{P}_m^{-1/2,1/2}(\eta) \hat{P}_m^{1/2,-1/2}(\zeta). \tag{43}$$

To summarize this series, we employ the relations (39) and also the formula

$$\lim_{q \to 1^-} \sum_{m=0}^{\infty} q^{2m+1} \sin(2m+1)x = \frac{1}{2\sin x}. \tag{44}$$

This gives us

$$\Lambda(\eta,\zeta) = \frac{1}{\pi(\eta-\zeta)}, \tag{45}$$

and therefore,

$$\chi_j(x) = -\frac{1}{\pi} \sqrt{\frac{(1-\zeta)^3}{1+\zeta}} \int_{-1}^{1} f_j(\xi) \sqrt{\frac{1+\eta}{(1-\eta)^3}} \frac{d\eta}{\eta-\zeta}, \tag{46}$$

where $x = (\zeta+1)/(\zeta-1)$, $\xi = (\eta+1)/(\eta-1)$. This formula, when rearranged, coincides with (23).

4. Near-tip field

4.1. SIFs

To evaluate the SIFs

$$K_I = \lim_{x \to 0^+} \sqrt{2\pi x} \sigma_{yy}(x,0), \quad K_{II} = \lim_{x \to 0^+} \sqrt{2\pi x} \sigma_{xy}(x,0), \tag{47}$$

we analyze the asymptotics of the stresses $\sigma_{xy}(x,0)$, $\sigma_{yy}(x,0)$ as $x \to 0^+$. From (19), we have

$$\sigma_{yy}(x,0) \sim -\frac{\gamma}{\pi\alpha} \int_{-\infty}^{0} \frac{\chi_2(\xi) d\xi}{\xi - x}, \quad x \to 0^+. \tag{48}$$

On making the substitutions (24) from (29), we derive as $x \to 0^+$ ($\zeta \to -1^-$)

$$\sigma_{yy}(x,0) \sim \frac{\gamma(1-\zeta)}{\pi\alpha} \sum_{m=0}^{\infty} a_m^{(2)} \int_{-1}^{1} \sqrt{\frac{1-\eta}{1+\eta}} \hat{P}_m^{1/2,-1/2}(\eta) \frac{d\eta}{\eta-\zeta}. \tag{49}$$

We now employ the following relation for the Jacobi polynomials

$$\int_0^1 \tau^{\alpha_1}(1-\tau)^{\alpha_2} P_n^{\alpha_1,\alpha_2}(1-2\tau) \frac{d\tau}{\tau-t} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2+n+1)}{\Gamma(\alpha_1+\alpha_2+n+1)} F(1+n, -\alpha_1-\alpha_2-n, 1-\alpha_1; t)$$
$$+ \frac{\Gamma(-\alpha_1)\Gamma(1+\alpha_1+n)}{n!} (-t)^{\alpha_1} F(1+\alpha_1+n, -\alpha_2-n, 1+\alpha_1; t), \quad t \notin (0,1). \tag{50}$$

For the orthonormal Jacobi polynomials $\hat{P}_n^{1/2,-1/2}(\eta)$ this relation can be simplified, and we obtain

$$\sigma_{yy}(x,0) \sim \frac{2\gamma x^{-1/2}}{\sqrt{\pi}\alpha} \sum_{m=0}^{\infty} (-1)^m a_m^{(2)}, \quad x \to 0^+. \tag{51}$$

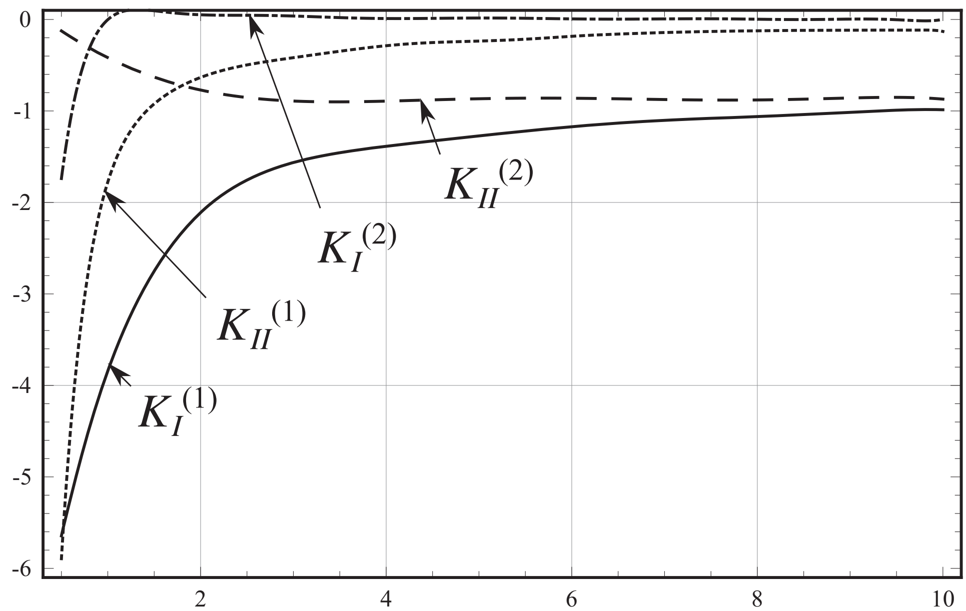

Figure 1. The SIFs versus δ for $V/c_R = 0.5$: $K_I^{(1)}$ and $K_{II}^{(1)}$ are the SIFs for the case $\sigma_{22}^{\circ}(x) = 1$, $-1 < x < 0$, $\sigma_{22}^{\circ}(x) = 0$, $x < -1$, and $\sigma_{12}^{\circ}(x) = 0$, $x < 0$; $K_I^{(2)}$ and $K_{II}^{(2)}$ are the SIFs for the case $\sigma_{12}^{\circ}(x) = 1$, $-1 < x < 0$, $\sigma_{12}^{\circ}(x) = 0$, $x < -1$, and $\sigma_{22}^{\circ}(x) = 0$, $x < 0$.

On comparing this asymptotic relation with the definition of the SIF K_I, we derive the final formula for K_I,

$$K_I = \frac{2\sqrt{2}\gamma}{\alpha}\sum_{m=0}^{\infty}(-1)^m a_m^{(2)}. \tag{52}$$

Similarly,

$$K_{II} = \frac{2\sqrt{2}\gamma}{\beta}\sum_{m=0}^{\infty}(-1)^m a_m^{(1)}. \tag{53}$$

For computations, we take the Poisson ratio $\nu = 0.3$. The Rayleigh speed c_R is defined explicitly [7] as

$$c_R = c_s\sqrt{s_*}, \quad s_* = \frac{1}{3}(8 - R_+ - R_-), \quad R_\pm = \left(\frac{45\kappa_0}{2} - 404 \pm \frac{3\sqrt{3R_*}}{2}\right)^{1/3},$$

$$R_* = -14656 + 2768\kappa_0 - 181\kappa_0^2 + 4\kappa_0^3, \quad \kappa_0 = 8(2-\nu)/(1-\nu) \in (16, 24). \tag{54}$$

The way in which the SIFs approach their asymptotic values as $\delta \to \infty$ is seen in Figure 1: the factors $K_I^{(1)}$ and $K_{II}^{(2)}$ tend to K_I° and K_{II}°, respectively, while the other two factors, $K_I^{(2)}$ and $K_{II}^{(1)}$ tend to zero. Here,

$$K_I^{\circ} = \frac{2\sqrt{2}\gamma}{\alpha}\sum_{m=0}^{\infty}(-1)^m b_m^{(2)}, \quad K_{II}^{\circ} = \frac{2\sqrt{2}\gamma}{\beta}\sum_{m=0}^{\infty}(-1)^m b_m^{(1)}. \tag{55}$$

In Figure 1, the crack propagation speed V is chosen to be $V = 0.5c_R$. The factors $K_I^{(1)}$ and $K_{II}^{(1)}$ denote the SIFs for the case when $\sigma_{22}^{\circ}(x) = 1$ for $-1 < x < 0$, and $\sigma_{22}^{\circ}(x) = 0$ otherwise, and $\sigma_{12}^{\circ}(x) = 0$ for all $x < 0$. The factors $K_I^{(2)}$ and $K_{II}^{(2)}$ are the SIFs for the case $\sigma_{12}^{\circ}(x) = 1$ for $-1 < x < 0$, and $\sigma_{12}^{\circ}(x) = 0$, $x < -1$, and $\sigma_{22}^{\circ}(x) = 0$ for all $x < 0$. As $\delta \to 0$, the absolute values of all the factors except for $K_{II}^{(2)}$ are growing. The SIF $K_{II}^{(2)}$ approaches zero as the distance between the crack and the boundary of the half-plane tends to zero.

When the crack is close to the surface ($\delta = 2$) and the crack propagation speed V is growing toward the Rayleigh speed c_R, the magnitudes of all the SIFs apart from $K_I^{(1)}$ are growing. The factor $K_I^{(1)}$ is decreasing as V/c_R approaches 1. This is shown in Figure 2.

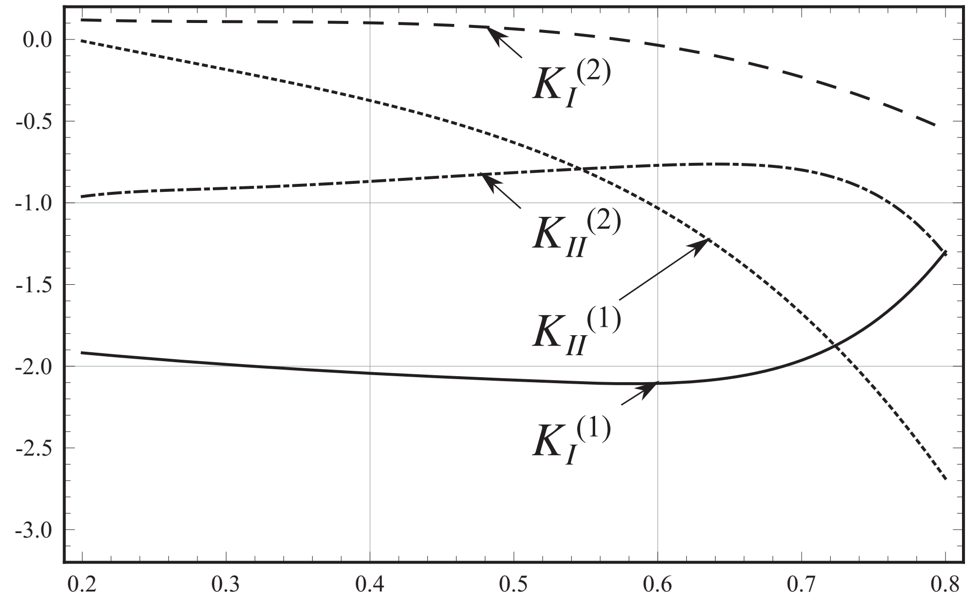

Figure 2. The SIFs versus V/c_R for $\delta = 2$: $K_I^{(1)}$ and $K_{II}^{(1)}$ are the SIFs for the case $\sigma_{22}^{\circ}(x) = 1, -1 < x < 0, \sigma_{22}^{\circ}(x) = 0, x < -1$, and $\sigma_{12}^{\circ}(x) = 0, x < 0$; $K_I^{(2)}$ and $K_{II}^{(2)}$ are the SIFs for the case $\sigma_{12}^{\circ}(x) = 1, -1 < x < 0, \sigma_{12}^{\circ}(x) = 0, x < -1$, and $\sigma_{22}^{\circ}(x) = 0, x < 0$.

4.2. Weight functions

Introduce next the weight functions $W_{I,I}(x)$, $W_{I,II}(x)$, $W_{II,I}(x)$ and $W_{II,II}(x)$ as follows

$$K_I = \int_{-\infty}^{0} W_{I,I}(\xi)\sigma_{22}^{\circ}(\xi)d\xi + \int_{-\infty}^{0} W_{I,II}(\xi)\sigma_{12}^{\circ}(\xi)d\xi,$$
$$K_{II} = \int_{-\infty}^{0} W_{II,I}(\xi)\sigma_{22}^{\circ}(\xi)d\xi + \int_{-\infty}^{0} W_{II,II}(\xi)\sigma_{12}^{\circ}(\xi)d\xi. \quad (56)$$

To determine the weight functions, we use the method introduced in Antipov [8]. The pair $W_{I,I}(x)$, $W_{II,I}(x)$ can be found if loading

$$\sigma_{12}(\xi, 0) = 0, \quad \sigma_{22}(\xi, 0) = \delta(\xi - x), \quad -\infty < \xi < 0, \quad (57)$$

is applied. Here, $-\infty < x < 0$ and $\delta(x)$ is the Dirac delta function. For this case, the coefficients $b_n^{(1)}$, $b_n^{(2)}$ can explicitly be computed

$$b_n^{(1)} = 0, \quad b_n^{(2)} = -\frac{\alpha}{\gamma}\frac{\sqrt{-x}}{1-x}\hat{P}_n^{-1/2,1/2}\left(\frac{x+1}{x-1}\right). \quad (58)$$

It is useful to express the coefficients $b_n^{(2)}$ through the Chebyshev polynomials. From (39)

$$b_n^{(2)} = -\frac{\alpha}{\gamma\sqrt{\pi(1-x)}}T_{2n+1}\left(\sqrt{\frac{x}{x-1}}\right). \quad (59)$$

With this relation, we now split $a_n^{(j)}$ as

$$a_n^{(j)} = b_n^{(j)} + \tilde{a}_n^{(j)}, \quad j = 1, 2, \quad (60)$$

and therefore the weight function $W_{I,I}$ can be represented in the form

$$W_{I,I}(x) = W_{I,I}^{\circ}(x) + \tilde{W}_{I,I}(x), \quad (61)$$

where

$$W_{I,I}^{\circ}(x) = \frac{2\sqrt{2}\gamma}{\alpha}\sum_{m=0}^{\infty}(-1)^m b_m^{(2)}, \quad \tilde{W}_{I,I}(x) = \frac{2\sqrt{2}\gamma}{\alpha}\sum_{m=0}^{\infty}(-1)^m \tilde{a}_m^{(2)}, \quad (62)$$

The first term can be computed explicitly

$$W^{\circ}_{I,I}(x) = -\sqrt{\frac{2}{-\pi x}}, \tag{63}$$

while the elevation of the function $\tilde{W}_{I,I}(x)$ and also the function

$$W_{II,I}(x) = \frac{2\sqrt{2}\gamma}{\beta}\sum_{m=0}^{\infty}(-1)^m \tilde{a}_m^{(1)}, \tag{64}$$

requires the solution of the infinite system

$$\tilde{a}_n^{(j)} + \sum_{m=0}^{\infty}[c_{nm}^{(j,j)}\tilde{a}_m^{(j)} + c_{nm}^{(j,3-j)}\tilde{a}_m^{(3-j)}] = \tilde{b}_n^{(j)}, \quad n = 0, 1, \ldots; \ j = 1, 2, \tag{65}$$

where

$$\tilde{b}_n^{(1)} = 0, \quad \tilde{b}_n^{(2)} = -\sum_{m=0}^{\infty} c_{nm}^{(2,2)} b_m^{(2)}. \tag{66}$$

Similarly, the pair $W_{I,II}(x)$, $W_{II,II}(x)$ may be written as

$$W_{I,II}(x) = \frac{2\sqrt{2}\gamma}{\alpha}\sum_{m=0}^{\infty}(-1)^m \tilde{a}_m^{(2)},$$

$$W_{II,II}(x) = -\sqrt{\frac{2}{-\pi x}} + \frac{2\sqrt{2}\gamma}{\beta}\sum_{m=0}^{\infty}(-1)^m \tilde{a}_m^{(1)}, \tag{67}$$

where the coefficients $\tilde{a}_m^{(1)}$, $\tilde{a}_m^{(2)}$ form the solution of the infinite system (65) with the right-hand side given by

$$\tilde{b}_n^{(1)} = -\sum_{m=0}^{\infty} c_{nm}^{(1,1)} b_m^{(1)}, \quad \tilde{b}_n^{(2)} = 0. \tag{68}$$

For computations of the weight functions, we choose $\nu = 0.3$ and $x = -1$. When V is fixed ($V < c_R$) and $\delta \to \infty$, the weight functions $W_{I,I}$ and $W_{II,II}$ tend to $-\sqrt{2/\pi}$, while the other two functions, $W_{I,II}$ and $W_{II,I}$, vanish, see Figure 3. The magnitudes of all the weight functions grow as $\delta \to 0$ and V is fixed ($V = 0.5c_R$ in Figure 3).

The weight function curves in Figure 4 for $\delta = 1$, $\nu = 0.3$, $x = -1$ show that the weight function $W_{I,I}$ decreases as $V \to c_R$. The other functions may also decrease when the normalized speed V/c_R is close to 1. Our numerical scheme becomes less reliable when V approaches the critical speed c_R.

4.3. Griffith criterion

Following Willis [5] consider the potential energy δU released when the crack $S_1(t) = \{\mathbf{x} : -\infty < x_1 < Vt, \ x_2 = 0\}$ extends to $S_1(t) + \delta S_1(t) = \{\mathbf{x} : -\infty < x_1 < Vt + r, \ x_2 = 0\}$, where r is small. The energy δU may be expressed as

$$\delta U = \frac{1}{2}\int_0^r \{\sigma_{xy}(x,0)\delta[u](x) + \sigma_{yy}(x,0)\delta[v](x)\}dx. \tag{69}$$

Here, $[u] + \delta[u]$, $[v] + \delta[v]$ are the displacement jumps related to the extended crack, and

$$\sigma_{xy} \sim \frac{K_{II}}{\sqrt{2\pi x}}, \quad \sigma_{yy} \sim \frac{K_I}{\sqrt{2\pi x}}, \quad x \in (0, r), \quad r \to 0^+. \tag{70}$$

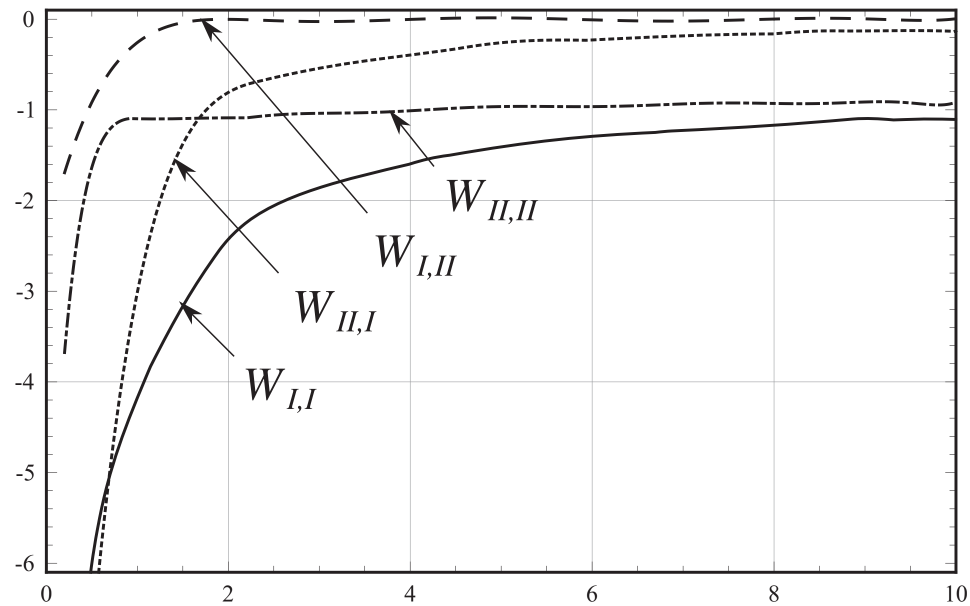

Figure 3. The weight functions $W_{j,l}(x)$ $(j, l = I, II)$ versus δ for $V/c_R = 0.5, x = -1$.

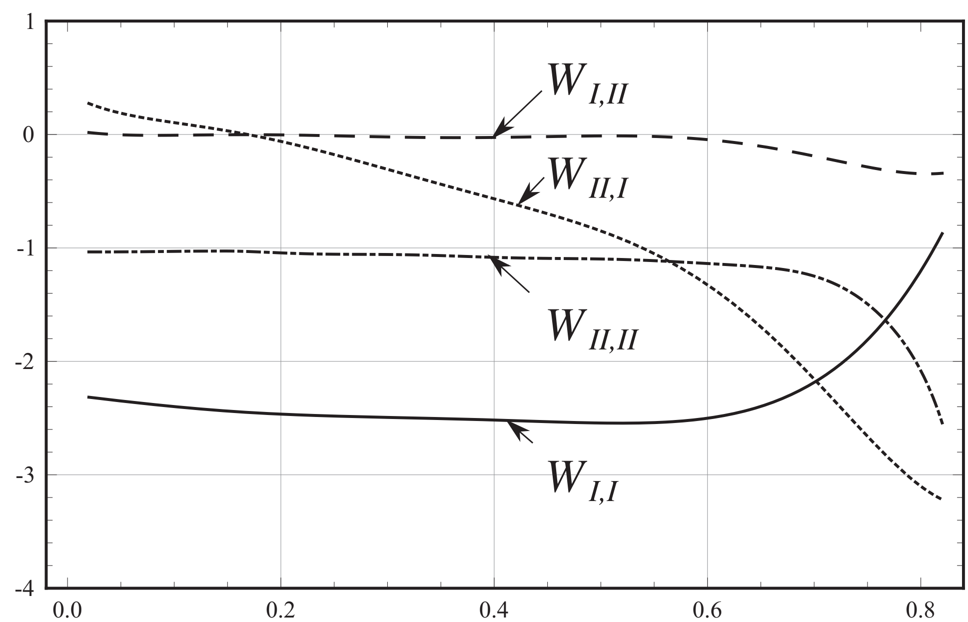

Figure 4. The weight functions $W_{j,l}(x)$ $(j, l = I, II)$ versus V/c_R for $\delta = 1, x = -1$.

To find asymptotic expansions for $\delta[u]$, $\delta[v]$, we integrate (30) and fix the constant of integration by assuring that the displacement jumps vanish at the crack tip. For the displacement jumps $[u]$ and $[v]$, we have

$$\begin{pmatrix} u(x,0^+) - u(x,0^-) \\ v(x,0^+) - v(x,0^-) \end{pmatrix} = 2\sum_{m=0}^{\infty} \begin{pmatrix} a_m^{(1)} \\ a_m^{(2)} \end{pmatrix} \int_0^x \hat{P}_m^{1/2,-1/2}\left(\frac{\xi+1}{\xi-1}\right) \frac{d\xi}{(1-\xi)\sqrt{-\xi}}, \quad x < 0. \tag{71}$$

The displacement jumps vanish at the point $x = 0$ and tend to the values

$$-2\sum_{m=0}^{\infty} \begin{pmatrix} a_m^{(1)} \\ a_m^{(2)} \end{pmatrix} \int_{-\infty}^0 \hat{P}_m^{1/2,-1/2}\left(\frac{\xi+1}{\xi-1}\right) \frac{d\xi}{(1-\xi)\sqrt{-\xi}} \tag{72}$$

as $x \to -\infty$. These values are finite and, in general, nonzero.

Since $\hat{P}_m^{1/2,-1/2}(-1) = \pi^{-1/2}(-1)^m$, for small negative x,

$$[v](x) = -4\sqrt{\frac{-x}{\pi}}\sum_{m=0}^{\infty}(-1)^m a_m^{(2)} + O(|x|^{3/2}), \quad x \to 0^-. \tag{73}$$

When the crack extends to $x = r$, due to the relation (52),

$$[v](x) = -\sqrt{\frac{2(r-x)}{\pi}}\frac{\alpha K_I}{\gamma} + O\{(r-x)^{3/2}\}, \quad x \to r. \tag{74}$$

Similarly,

$$[u](x) = -\sqrt{\frac{2(r-x)}{\pi}}\frac{\beta K_{II}}{\gamma} + O\{(r-x)^{3/2}\}, \quad x \to r. \tag{75}$$

Now, on substituting formulas (70), (74) and (75) into (69), we obtain

$$\delta U \sim -\frac{1}{4\gamma}(\alpha K_I^2 + \beta K_{II}^2)r, \quad r \to 0^+. \tag{76}$$

This formula can be written in terms of the Rayleigh function

$$D(V) = 4\sqrt{\left(1 - \frac{V^2}{c_l^2}\right)\left(1 - \frac{V^2}{c_s^2}\right)} - \left(2 - \frac{V^2}{c_s^2}\right)^2 \tag{77}$$

as

$$\delta U \sim \frac{V^2 r}{2c_s \mu D(V)}(\alpha K_I^2 + \beta K_{II}^2), \quad r \to 0^+. \tag{78}$$

According to the Griffith criterion, the crack starts propagating if the energy δU is equal to or greater than the increase in the surface energy $2Tr$, $\delta U \geq 2Tr$, where T is the Griffith material constant. This criterion may be represented in terms of the SIFs in the form

$$\sqrt{1 - \frac{V^2}{c_l^2}}K_I^2 + \sqrt{1 - \frac{V^2}{c_s^2}}K_{II}^2 \geq \frac{4Tc_s^2 \mu D(V)}{V^2}. \tag{79}$$

Notice, that if $\delta = \infty$, and $\sigma_{12}^{\circ}(x) = 0$ for all $x < 0$, then the inequality (79) coincides with the criterion

$$K_I^2 \geq \frac{4Tc_s^2 \mu D(V)}{V^2\sqrt{1 - V^2/c_l^2}} \tag{80}$$

obtained by Willis [5]. For finite values of δ, even when the tangential component of loading vanishes, the SIF $K_{II} \neq 0$, and both factors, K_I and K_{II}, are involved in the Griffith crack propagation criterion.

Another way to represent the crack propagation criterion is to rewrite inequality (79) as

$$H(K_I, K_{II}, V/c_s, V/c_l) \geq \mu T, \tag{81}$$

where

$$H = \frac{\alpha K_I^2 + \beta K_{II}^2}{4(c_s/V)^2 D(V)}. \tag{82}$$

Figure 5 shows the results of calculations of the function H versus δ for $V/c_R = 0.5$ and some loads. It is seen that H rapidly advances as the distance between the crack and the half-plane boundary decreases.

The dependence of H on the normalized crack speed V/c_R when $\delta = 2$ is plotted in Figure 6. The function $H \to \infty$ as $V/c_R \to 0$ and it grows as the crack speed V approaches the Rayleigh speed. The curves in Figure 6 are reminiscent of the graph of modulus of cohesion $k_c(V)$ versus V/c_s [9] in the Barenblatt-type criterion for intersonic shear crack propagation.

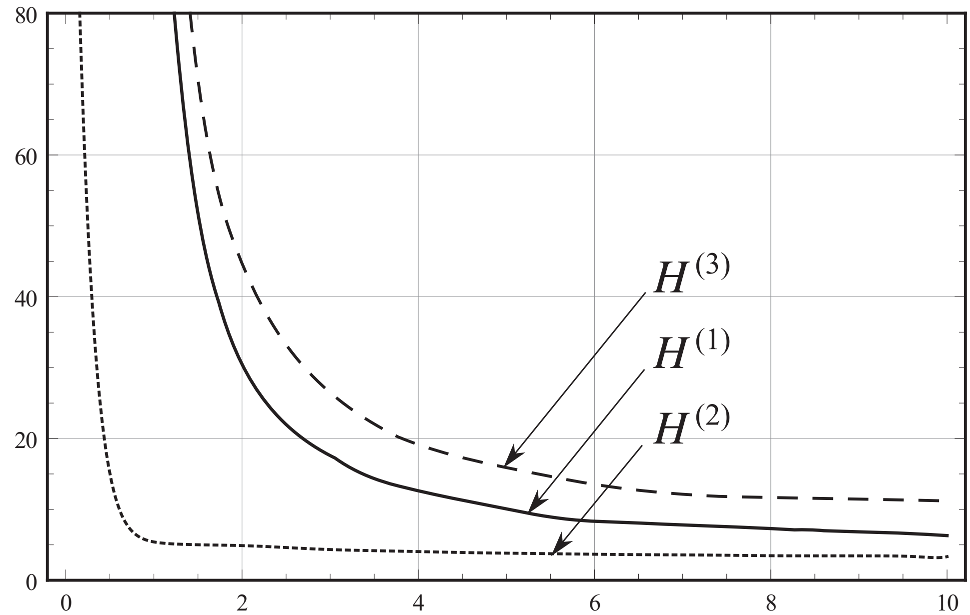

Figure 5. The function H versus δ for $V/c_R = 0.5$, $x = -1$ and loading $\sigma^\circ_{22} = \delta(\xi - x)$, $\sigma^\circ_{12} = 0$: $H^{(1)}$; $\sigma^\circ_{22} = 0$, $\sigma^\circ_{12} = \delta(\xi - x)$: $H^{(2)}$; $\sigma^\circ_{22} = \delta(\xi - x)$, $\sigma^\circ_{12} = \delta(\xi - x)$: $H^{(3)}$.

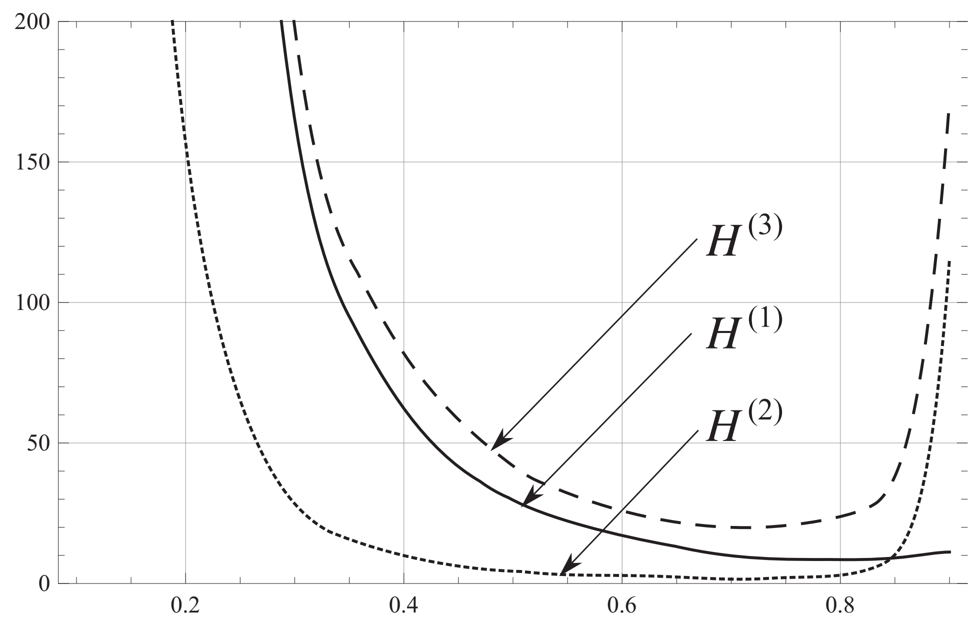

Figure 6. The function H versus V/c_R for $\delta = 2$, $x = -1$ and loading $\sigma^\circ_{22} = \delta(\xi - x)$, $\sigma^\circ_{12} = 0$: $H^{(1)}$; $\sigma^\circ_{22} = 0$, $\sigma^\circ_{12} = \delta(\xi - x)$: $H^{(2)}$; $\sigma^\circ_{22} = \delta(\xi - x)$, $\sigma^\circ_{12} = \delta(\xi - x)$: $H^{(3)}$.

5. Conclusions

We have analyzed a two-dimensional steady-state problem on propagation of a semi-infinite crack in a half-plane. The crack is subjected to normal and tangential loads applied to its faces, and it propagates at speed V along the half-plane boundary free of traction. The boundary of the half-plane breaks the symmetry of the problem, and, in contrast to the problem for a plane, the modes I and II are coupled. We have deduced an order-2 vector RH problem associated with the model. The coefficient is a Hermitian matrix that cannot be factorized in a closed form. We have reduced the problem to a system of two singular integral equations with respect to the derivatives of the displacement jumps. The method of orthogonal polynomials has been employed for its solution. The unknown functions have been expanded in terms of the orthonormal Jacobi polynomials. The coefficients of the expansions have been determined from an infinite system of linear algebraic equations of the second kind.

We have derived formulas for the SIFs K_I and K_{II} and the weight functions $W_{I,I}$, $W_{I,II}$, $W_{II,I}$ and $W_{II,II}$. By determining the energy δU released when the crack extends to a small distance, we applied the Griffith criterion and established that the crack starts propagating when $H \geq \mu T$, where

$$H = \frac{\sqrt{1-(V/c_l)^2}K_I^2 + \sqrt{1-(V/c_s)^2}K_{II}^2}{4(c_s/V)^2 D(V)},$$

$D(V)$ is the Rayleigh function, c_s, c_l are the shear and longitudinal waves speeds, μ is the shear modulus and T is the Griffith material constant. We have computed the SIFs, the weight functions and the function H for different V/c_R and δ (c_R is the Rayleigh speed, and δ is the distance between the half-plane boundary and the crack). It has been found that H grows to infinity when the distance δ between the crack and the half-plane boundary decreases while the crack speed does not vary. The function H monotonically decreases as δ grows. When the distance δ is fixed, H, as a function of V/c_R, attains its minimum in the interval $(0, 1)$ and grows as V/c_R approaches the points 0 and 1.

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Conflict of interest

None declared.

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